Existence, Uniqueness, and Causality Theorems for Wave Propagation in Stratified, Temporally Dispersive, Complex Media

Rikte, Sten

Published in:
SIAM Journal on Applied Mathematics

DOI:
10.1137/S0036139994279190

1997

SIAM J. APPL. MATH.
Vol. 57, No. 5, pp. 1373–1389, October 1997

008

EXISTENCE, UNIQUENESS, AND CAUSALITY THEOREMS FOR WAVE PROPAGATION IN STRATIFIED, TEMPORALLY DISPERSIVE, COMPLEX MEDIA*

STEN RIKTE†

Abstract. A mixed initial-boundary value problem for a nonlocal, hyperbolic equation is analyzed with respect to unique solubility and causality. The regularity of the step response and impulse response (the Green functions) is investigated, and a wave front theorem is proved. The problem arises, e.g., at time-varying, electromagnetic, plane wave excitation of stratified, temporally dispersive, bi-isotropic or anisotropic slabs. Concluding, the problem is uniquely solvable, strict causality holds, and a well-defined wave front speed exists. This speed is independent of dispersion and excitation, and depends on the nondispersive properties of the medium only.

Key words. causality, dispersive, bi-isotropic, chiral

AMS subject classifications. 35L40, 35Q60, 78A40

PII. S0036139994279190

1. Introduction.

During the 1980s and the 1990s, pulse propagation in stratified, temporally dispersive slabs has been studied extensively [1, 4, 5, 9, 10, 12, 14, 16, 17, 18]. Related inverse scattering problems have been addressed as well [4, 5, 8, 12, 16, 19]. Since all media are dispersive to some degree, these investigations are of principal interest. Temporal dispersion in linear, time-invariant media is modeled by time convolution in the constitutive relations; see Hopkinson [11] for an early reference and Karlsson and Kristensson [13] for a modern treatment. The propagation of mechanical pulses in dispersive, viscoelastic media has been treated by Ammicht, Corones, and Krueger [1], Karlsson [12], and Corones and Karlsson [5]. Electromagnetic pulse propagation in dispersive, nonmagnetic, isotropic media was discussed by Beezley and Krueger [4] and later by Kristensson [16]. Both these problems lead to scalar wave equations involving memory terms. Using standard techniques, these wave equations can be reduced to hyperbolic systems of two coupled, first-order, integro-differential equations.

Attention has also been paid to the interaction between electromagnetic fields and dispersive, bi-isotropic media [17, 18] and dispersive, anisotropic media [9]. The bi-isotropic medium is isotropic but has a constitutive coupling between the electric and magnetic fields. Consequently, it is characterized by four susceptibility functions. In the anisotropic medium, there is no constitutive coupling between the electric and magnetic fields. However, the medium is not isotropic, and in the extreme case, it is characterized by 18 susceptibility functions. The bi-anisotropic medium, which is the most general complex medium, involves at most 36 susceptibility functions. Since the longitudinal field components in Maxwell's equations can be eliminated using resolvent operators, wave propagation in dispersive, complex slabs leads to hyperbolic systems of four coupled, first-order, integro-differential equations.

In the analysis of the wave propagation problems above, two different, but related, methods are employed, namely the invariant imbedding technique and the Green

*Received by the editors December 28, 1994; accepted for publication (in revised form) March 25, 1996. This research was supported by a grant from the Swedish Research Council for Engineering Sciences.

http://www.siam.org/journals/siap/57-5/27919.html

†Department of Electromagnetic Theory, Lund Institute of Technology, P.O. Box 118, S-221 00 Lund, Sweden (sten@teorel.lth.se).

functions approach. Both methods are based on wave splitting and Duhamel's principle. In the imbedding method, a one-parameter family of scattering problems related to the original problem is studied. In the Green functions method, the internal fields are related to the excitation at the boundary via a propagation operator of convolution type. The Green functions are simply the classical contributions to this impulse response or fundamental solution. Both methods rest upon the assumption that—in the weak sense—the propagation problem has a unique, well-behaved solution in each bounded time interval. Furthermore, it is surmised that strict causality holds for wave propagation in dispersive media. The attribute "weak" refers to that the integro-differential equations are to be integrated along the characteristics (defined by the nondispersive properties of the medium). This is also the appropriate measure at numerical evaluation. The concept of strict causality refers to that the speed of the wave front—which is undefined at this point—is lower than or equal to the speed given by the nondispersive properties of the medium [21].

Unique solubility of hyperbolic systems modeling wave propagation in nondispersive media is discussed by Courant and Hilbert [7] and by Ayoubi [2]. The first uniqueness and causality results for homogeneous, dispersive, isotropic media were reported by Sommerfeld [22]. The most general results in this field have been obtained by Roberts [21]. In this reference, plane wave incidence on the stratified, dispersive, nonmagnetic, and optically impedance-matched isotropic slab is discussed. However, the presented theorems can be applied to other (scalar) problems also. The existence of a unique, well-behaved, weak solution to this wave propagation problem in each bounded time interval is established. Furthermore, strict causality is verified, and a theorem concerning the regularity of the wave front is presented and proved. As a consequence of this wave front theorem, a well-defined wave front speed can be introduced also at wave propagation in dispersive, isotropic media. As expected, this wave front speed is independent of the dispersive properties of the medium as well as the incident plane wave.

The questions of existence, uniqueness, and causality at wave propagation in dispersive, complex media have not been properly attended to. In the present paper, the theorems in [21] are generalized to a mixed initial-boundary value problem for a nonlocal hyperbolic equation, which covers electromagnetic pulse propagation in large classes of stratified, dispersive, bi-anisotropic media and which may be applicable to other wave propagation problems also. In addition, the Green functions equations are derived, discussed, and proved uniquely solvable. Due to the close relationship between the imbedding method and the Green functions method, it is conjectured that the imbedding equations can be proved uniquely solvable by referring to theorems presented in this paper.

The investigations in, e.g., [8, 9, 17, 19, 18] suggest the study of the following mixed initial-boundary value problem, defined in the product set $(x, s) \in (0,1) \times \mathbb{R}$:

$$
\begin{cases}
\begin{pmatrix} (\partial_x + \partial_s)\boldsymbol{e}^+(x,s) \\ (\partial_x - \partial_s)\boldsymbol{e}^-(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \boldsymbol{e}^+(x,s) \\ \boldsymbol{e}^-(x,s) \end{pmatrix} + \displaystyle\int_{-\infty}^{s} \mathbf{a}(x, s-s') \begin{pmatrix} \boldsymbol{e}^+(x,s') \\ \boldsymbol{e}^-(x,s') \end{pmatrix} ds', \\
\boldsymbol{e}^{\pm}(x,s) = \mathbf{0}, \quad s \le 0, \\
t_0 \boldsymbol{e}^i(s) = \boldsymbol{e}^+(+0,s) - r_0 \boldsymbol{e}^-(+0,s), \quad t_0 + r_0 = 1, \\
\boldsymbol{e}^-(1-0,s) = r_1 \boldsymbol{e}^+(1-0,s).
\end{cases}
\tag{1.1}
$$

The independent variables s and x are the travel-time coordinates for time and slab depth, respectively. For each ordered pair (x, s), $\boldsymbol{e}^i(s)$, $\boldsymbol{e}^{\pm}(x, s) \in M_{2\times 1}(\mathbb{R})$ and $\mathbf{a}(x, s)$, $\mathbf{b}(x) \in M_{4\times 4}(\mathbb{R})$, where $M_{m\times n}(\mathbb{R})$ is the linear space over $\mathbb{R}$ consisting of the $m \times n$ matrices with real entries. The real numbers r_0, t_0, and r_1 are due to optical impedance mismatch at the slab walls, $x = 0$ and $x = 1$, respectively.

The functions $\mathbf{a}$ and $\mathbf{b}$ and the optical reflection coefficients r_0 and r_1 (viewed from the slab) depend on the properties of the complex medium. Temporal dispersion is modeled by time convolution with the kernel $\mathbf{a}$, for which one has $\mathbf{a}(x, s) = \mathbf{0}$ for all $s < 0$. The vector fields $\boldsymbol{e}^{\pm}$ have been obtained by (optical) wave splitting. A wave splitting is a change of the dependent variables, such that in the simple media outside the complex slab, the split vector fields $\boldsymbol{e}^+$ and $\boldsymbol{e}^-$ represent the right-going and the left-going waves, respectively; see, e.g., [8, 9, 17, 19, 18] or section 5. Throughout space, the sum $\boldsymbol{e}^+ + \boldsymbol{e}^-$ and the difference $\boldsymbol{e}^+ - \boldsymbol{e}^-$ represent the total electric and magnetic fields, respectively; see, e.g., section 5. The second relation in (1.1) shows that the slab is initially quiescent; therefore, $\int_{-\infty}^{s}$ can be substituted for $\int_0^s$ in the first one. The third and forth relations are boundary conditions. The incident electric field $\boldsymbol{e}^i$ at the front wall is initially quiescent: $\boldsymbol{e}^i(s) = \mathbf{0}$ for all $s < 0$. Clearly, there is no incoming field from the right. Observe also that the metal-backed slab is not excluded ($r_1 = -1$).

The matrix notation in (1.1) is appropriate for the wave propagation and scattering problems referred to above and is employed throughout this paper. Every vector (in the plane) is identified with its column vector representation in the usual basis (i.e., as a 2×1 matrix), and is typed in italic boldface. Quadratic matrices are typed in roman boldface.

In section 2, the weak canonical problem is examined with respect to unique solubility and regularity. In section 3, it is proved that the general problem (1.1) is uniquely solvable in the weak sense in each bounded time interval and that strict causality holds. Furthermore, a theorem concerning wave fronts is given, which implies that the speed of the wave front is precisely one. In section 4, the Green functions equations are derived and proved to be uniquely solvable. In addition, it is shown that the solution to (1.1) can be written in the form

$$\begin{aligned} \boldsymbol{e}^{\pm}(x, s) = &\int_0^{s-x} \mathbf{g}^{\pm}(x, s - s')\boldsymbol{e}^i(s')\, ds' \\ &+ \sum_{k=k^{\pm}}^{\infty} [\mathbf{u}^{\pm}(x, \pm x + 2k)]\boldsymbol{e}^i(s \mp x - 2k), \end{aligned} \tag{1.2}$$

where $k^+ = 0$, $k^- = 1$, and $[\mathbf{u}^{\pm}(x, s)]$ denotes the jump in $\mathbf{u}^{\pm}(x, s)$ at (x, s). The matrix-valued functions $\mathbf{u}^{\pm}(x, s)$ and $\mathbf{g}^{\pm}(x, s)$, which depend on the properties of the complex medium only, are the canonical solutions and the Green functions, respectively. Finally, in section 5, (1.1) is derived for a large class of bi-isotropic media.

2. The weak canonical problem. In this section, linearly polarized excitation with the Heaviside step is considered. Two theorems concerning this canonical problem are presented and commented upon. The proofs can be found in Appendix A.

Theorem 2.1 states that—in the weak sense—the canonical problem is uniquely solvable in each bounded time interval. The proof is similar to the one given in the scalar, dispersive case [21]. The basic idea of the proof is the repeated use of the Banach fixed-point theorem [23]. A similar method of proving unique solubility of hyperbolic integro-differential equations has been employed also by Beezley [3].

Theorem 2.2 shows that the regularity of the solution to the canonical problem is increased with the regularity of the memory function $\mathbf{a}$. This admits the definition of the Green functions employed in, e.g., [8, 9, 17, 19, 18].

In the theorems and the proofs below, the following definitions and facts are employed: if $A \subset \mathbb{R}^d$ is an open set, the real linear space consisting of all functions $\mathbf{f}{:}A \to M_{m\times n}(\mathbb{R})$ with *bounded and continuous* derivatives up to order k in A is denoted by $\mathcal{C}^k_{m\times n}(A)$. This function space is complete furnished with the norm

$$\|\mathbf{f}\| = \max_{i,j} \|f_{i,j}\|_\infty = \max_{i,j}(\sup_{\boldsymbol{x}\in A} |f_{i,j}(\boldsymbol{x})|), \tag{2.1}$$

where $f_{i,j}$ are the components of $\mathbf{f}$. The class $\mathcal{C}_{m\times n}(\overline{A})$ is defined analogously. The product space $\mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$ over the real numbers, equipped with the norm $\|(\cdot,\cdot\cdot)\| = \max(\|\cdot\|, \|\cdot\cdot\|)$, where the norm $\|\cdot\|$ is defined in (2.1), is also a Banach space. Convergence in these norm-topologies is called uniform. By straightforward generalization of a theorem in real analysis, one can prove that, if the sequence $(\mathbf{f}_j)_{j=1}^\infty \in \mathcal{C}^1_{m\times n}(A) \times \mathcal{C}^1_{m\times n}(A)$ converges pointwise to $\mathbf{f}$ in A, and if $(\partial_i \mathbf{f}_j)_{j=1}^\infty \in \mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$ converges uniformly to $\mathbf{g}$ in A, then $\mathbf{g} \in \mathcal{C}_{m\times n}(A) \times \mathcal{C}_{m\times n}(A)$, $\mathbf{f}$ is differentiable in A with respect to the ith coordinate, and $\partial_i \mathbf{f} = \mathbf{g}$ in A. Furthermore, recall that a function f on a Banach space $(B, \|\cdot\|)$ is called a contraction if there exists a nonnegative number $r < 1$ such that $\|f(x) - f(y)\| \le r\|x - y\|$ for all points x and y in B, and that the Banach fixed-point theorem under these circumstances guarantees that f has a unique fixed point in B; i.e., there exists precisely one point $x \in B$ such that $f(x) = x$. Finally, the Heaviside step function is denoted by H, and $\mathbf{I}$ is the 2×2 identity matrix.

In the absence of axial symmetry, two directions of polarization of the incident field must be considered in (1.1). It is appropriate to treat these two canonical problems together; therefore, a matrix-valued step response (the canonical solutions) $\mathbf{u}^\pm$ is introduced in Theorem 2.1 below. Geometrical quantities defined in Theorem 2.1 or its proof are illustrated in Figure 1. The main theorem of this section is Theorem 2.1.

THEOREM 2.1 (weak canonical problem). *Let the given functions* $\mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I} \times \mathbb{R}_+)$ *and* $\mathbf{b} \in \mathcal{C}_{4\times 4}(\mathbb{I})$*, where* $\mathbb{I} = (0, 1)$ *and* $\mathbb{R}_+ = (0, \infty)$*, be decomposed into* $\mathcal{C}_{2\times 2}$*-blocks according to*

$$\mathbf{a}(x,s) = \begin{pmatrix} \mathbf{a}_{11}(x,s) & \mathbf{a}_{12}(x,s) \\ \mathbf{a}_{21}(x,s) & \mathbf{a}_{22}(x,s) \end{pmatrix}, \quad \mathbf{b}(x) = \begin{pmatrix} \mathbf{b}_{11}(x) & \mathbf{b}_{12}(x) \\ \mathbf{b}_{21}(x) & \mathbf{b}_{22}(x) \end{pmatrix}, \quad (x,s) \in \mathbb{I} \times \mathbb{R}_+.$$

Define trapezoids by $Q_{2n} = \{(x,s) \in \mathbb{I} \times \mathbb{R}_+ {:} 0 < s < x + 2n\}$ *and unions of line segments by* $L^\pm = \cup_{k=0}^\infty \{(x, \pm x + 2k) \in \mathbb{I} \times \mathbb{R}_+\}$ *and* $L = L^+ \cup L^-$*. Furthermore, let* r_1*,* r_0*, and* t_0 *be given real numbers. Then, for every integer* $n \ge 0$*, the initial-boundary value problem defined in the product set* $\mathbb{I} \times \mathbb{R}_+$ *by*

$$\begin{cases} \begin{pmatrix} (\partial_x + \partial_s)\mathbf{u}^+(x,s) \\ (\partial_x - \partial_s)\mathbf{u}^-(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \mathbf{u}^+(x,s) \\ \mathbf{u}^-(x,s) \end{pmatrix} + \int_0^s \mathbf{a}(x, s-s') \begin{pmatrix} \mathbf{u}^+(x,s') \\ \mathbf{u}^-(x,s') \end{pmatrix} ds', \\ \mathbf{u}^\pm(x,0) = \mathbf{0}, \\ t_0 \mathbf{I} H(s) = \mathbf{u}^+(+0,s) - r_0 \mathbf{u}^-(+0,s), \\ \mathbf{u}^-(1-0,s) = r_1 \mathbf{u}^+(1-0,s) \end{cases} \tag{2.2}$$

has a unique solution $\mathbf{u}^\pm \in \mathcal{C}_{2\times 2}(Q_{2n} \setminus L)$ *in the weak sense in* Q_{2n}*, i.e., integrated along the characteristics. Thus,* $(\partial_x \pm \partial_s)\mathbf{u}^\pm$ *are understood as derivatives with respect*

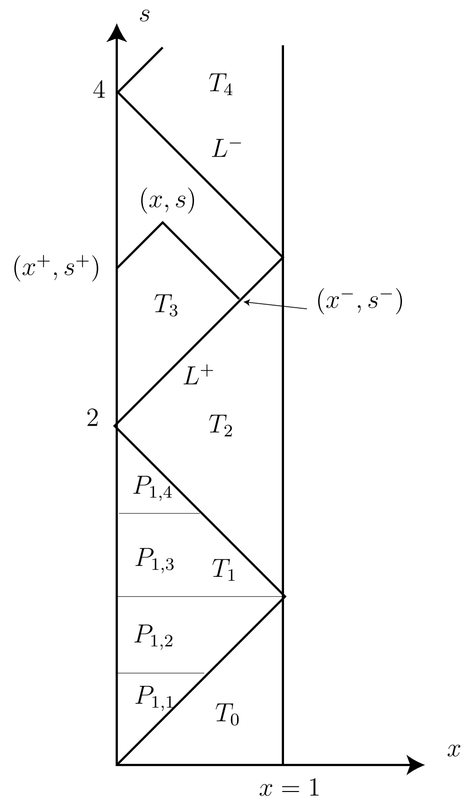

FIG. 1. *Geometrical quantities defined in Theorem* 2.1 *and its proof.* L^+ (L^-) *is the union of the open line segments of the boundaries of the triangles* T_n *with positive (negative) slopes. The trapezoids* Q_{2n} *and the triangles* T_n *are related to each other through the relations* $Q_0 = T_0$ *and* $Q_{2n} \setminus L = T_{2n} \cup T_{2n-1} \cup Q_{2n-2} \setminus L$.

to the vectors $(1, \pm 1)$*, respectively. The solution equals zero in* Q_0*, and for every* $j \in \{0, 1, \ldots, 2n\}$*, the restrictions of* $\mathbf{u}^{\pm}$ *to* T_j *can be extended continuously to* $\overline{T_j}$*, where the open triangle* T_j *is given by* Q_0 *if* $j = 0$*, by* $\{(x, s) \in \mathbb{I} \times \mathbb{R}_+ : x + j - 1 < s < -x + j + 1\}$ *if* j *is an odd integer, and by* $\{(x, s) \in \mathbb{I} \times \mathbb{R}_+ : -x + j < s < x + j\}$ *if* $j > 0$ *is even. Moreover,* $\mathbf{u}^{\pm}$ *have jump discontinuities across* $L^{\pm}$*, respectively. The jumps in* $\mathbf{u}^{\pm}$ *at the point* $(x, s) \in L^{\pm}$*, defined by* $[\mathbf{u}^{\pm}(x, s)] := \mathbf{u}^{\pm}(x, s+0) - \mathbf{u}^{\pm}(x, s-0)$*, satisfy the following ordinary differential equations, where* $\boldsymbol{\beta}^+ := \mathbf{b}_{11}$ *and* $\boldsymbol{\beta}^- := \mathbf{b}_{22}$:

$$\frac{d}{dx}[\mathbf{u}^{\pm}(x, \pm x + 2k)] = \left[\frac{d}{dx}\mathbf{u}^{\pm}(x, \pm x + 2k)\right] = \boldsymbol{\beta}^{\pm}(x)[\mathbf{u}^{\pm}(x, \pm x + 2k)] \tag{2.3}$$

for $x \in \mathbb{I}$*. At the boundary, the jumps are coupled to one another as*

$$\begin{cases} [\mathbf{u}^+(+0, 0)] = t_0 \mathbf{I}, \qquad [\mathbf{u}^+(+0, 2k)] = r_0[\mathbf{u}^-(+0, 2k)], \quad k > 0, \\ [\mathbf{u}^-(1-0, 2k-1)] = r_1[\mathbf{u}^+(1-0, 2k-1)], \quad k > 0. \end{cases} \tag{2.4}$$

Finally, if $[\mathbf{u}^+(+0, 2k)] \neq \mathbf{0}$*, then* $[\mathbf{u}^+(x, x + 2k)]$ *is nonsingular for each* $x \in \mathbb{I}$*, and, if* $[\mathbf{u}^-(1-0, 2k-1)] \neq \mathbf{0}$*, then* $[\mathbf{u}^-(x, -x + 2k)]$ *is nonsingular for each* $x \in \mathbb{I}$*. In particular,* $\mathbf{u}^+$ *and* $\mathbf{u}^+ \pm \mathbf{u}^-$*, but not* $\mathbf{u}^-$*, are discontinuous across the line* $s = x$*.*

Note that if the matrices $(\mathbf{b}_{11}(x))_{0\le x\le 1}$ all commute, a closed form expression for the jumps in $\mathbf{u}^+$ can be obtained: $[\mathbf{u}^+(x,x+2k)] = \exp\left(\int_0^x \mathbf{b}_{11}(x')dx'\right)[\mathbf{u}^+(+0,2k)]$. Analogously, if the matrices $(\mathbf{b}_{22}(x))_{0\le x\le 1}$ all commute, integration of (2.3) yields $[\mathbf{u}^-(x,-x+2k)] = \exp\left(\int_1^x \mathbf{b}_{22}(x')dx'\right)[\mathbf{u}^-(1-0,2k-1)]$. Note also that if $\mathbf{I}$ is replaced by $\mathbf{0}$ in Theorem 2.1, then the solution $\mathbf{u}_0^{\pm}$ to (2.2) is identically zero. This follows from the uniqueness assertion, since $\mathbf{u}^{\pm}$ and $\mathbf{u}^{\pm}+\mathbf{u}_0^{\pm}$ both solve (2.2), if $\mathbf{u}^{\pm}$ is the solution in Theorem 2.1. More generally, it follows that the unique weak solution to (2.2) subject to the input $\sum c_i H_{s_i}$ instead of H, where $c_i \in \mathbb{R}$, $s_i \in \mathbb{R}_+$, and $H_{s_i}(s) = H(s-s_i)$, is given by $\sum c_i \mathbf{u}_{s_i}^{\pm}$, where $\mathbf{u}_{s_i}^{\pm}(x,s) = \mathbf{u}^{\pm}(x,s-s_i)$.

The regularity of the canonical solutions $\mathbf{u}^{\pm}$ is now discussed further. First, by necessity, $\mathbf{u}^{\pm}$ are continuous on the broken lines $L^{\mp}$, respectively, i.e., $\mathbf{u}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n}\setminus L^{\mp})$. Second, by the hyperbolic integro-differential equation (2.2), the derivatives $(\partial_x \pm \partial_s)\mathbf{u}^{\pm}$ are as regular as $\mathbf{u}^{\pm}$, i.e., $(\partial_x \pm \partial_s)\mathbf{u}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n}\setminus L)$. The existence of the classical time derivatives

$$\mathbf{g}^{\pm} := \partial_s \mathbf{u}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n}\setminus L) \tag{2.5}$$

—which is crucial for the Green functions formulation—is, however, not guaranteed by Theorem 2.1. The requirement (2.5) can be met by increasing the regularity of the memory function $\mathbf{a}(x,s)$. The second theorem of this section is as follows.

THEOREM 2.2. *If, in the foregoing theorem,* $\mathbf{a}$ *is differentiable with respect to* s *in* $\mathbb{I}\times\mathbb{R}_+$, *and if* $\partial_s\mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I}\times\mathbb{R}_+)$, *then* $\mathbf{u}^{\pm} \in \mathcal{C}^1_{2\times 2}(Q_{2n}\setminus L)$, *and the restrictions of the partial derivatives of* $\mathbf{u}^{\pm}$ *to* T_j *can be extended continuously to* $\overline{T_j}$ *for each* j.

According to Theorem 2.2, the step response $\mathbf{u}^{\pm}$ is differentiable in $Q_{2n}\setminus L$ provided that the dispersion model is regular enough. Consequently, the contributions $\mathbf{g}^{\pm}$ to the impulse response are well defined.

The functions $\mathbf{g}^{\pm}$ have finite jump discontinuities across L, and the jumps can easily be computed in terms of the jumps in $\mathbf{u}^{\pm}$. For instance, if $(x,s)\in L^-$, then $[\partial_x\mathbf{u}^+(x,s)] - [\partial_s\mathbf{u}^+(x,s)] = \mathbf{0}$. This is due to the fact that $\mathbf{u}^+$ is continuous across L^- and the directional derivative $\partial_x - \partial_s$ is along L^-. On the other hand, by (2.2), one obtains $[\partial_x\mathbf{u}^+(x,s)] + [\partial_s\mathbf{u}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]$ at $(x,s)\in L^-$, whence

$$[\mathbf{g}^+(x,s)] = [\partial_x\mathbf{u}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]/2, \quad (x,s)\in L^-. \tag{2.6}$$

In particular, $\mathbf{u}^+$ is, in general, not differentiable on L^-. Analogously, one obtains

$$[\mathbf{g}^-(x,s)] = -[\partial_x\mathbf{u}^-(x,s)] = -\mathbf{b}_{21}(x)[\mathbf{u}^+(x,s)]/2, \quad (x,s)\in L^+. \tag{2.7}$$

Consequently, $\mathbf{u}^-$ is, in general, not differentiable on L^+. Furthermore, $\mathbf{u}^{\pm}$ might not be differentiable on $L^{\pm}$, respectively, even if $[\mathbf{u}^{\pm}] = \mathbf{0}$. To see this, integrate (2.2) along both sides of the characteristics, differentiate with respect to s, and subtract. In the limit, these operations yield

$$\begin{cases} [\mathbf{g}^+(x,x+2k)] = [\mathbf{g}^+(+0,2k)] + \displaystyle\int_0^x \mathbf{b}_{11}(x')[\mathbf{g}^+(x',x'+2k)]\,dx' \\ \qquad\qquad + \displaystyle\int_0^x \left(\mathbf{a}_{11}(x',+0) - \mathbf{b}_{12}(x')\mathbf{b}_{21}(x')/2\right)[\mathbf{u}^+(x',x'+2k)]\,dx', \\ [\mathbf{g}^-(x,-x+2k)] = [\mathbf{g}^-(1-0,2k-1)] + \displaystyle\int_1^x \mathbf{b}_{22}(x')[\mathbf{g}^-(x',-x'+2k)]\,dx' \\ \qquad\qquad + \displaystyle\int_1^x \left(\mathbf{a}_{22}(x',+0) + \mathbf{b}_{21}(x')\mathbf{b}_{12}(x')/2\right)[\mathbf{u}^-(x',-x'+2k)]\,dx', \end{cases}$$

where (2.6) and (2.7) have also been employed. This equation can be solved. By a well-known theorem in real analysis, $[\mathbf{g}^{\pm}(\cdot,\pm\cdot+2k)]\in\mathcal{C}^1_{2\times 2}(\mathbb{I})$ and

$$\frac{d}{dx}[\mathbf{g}^{\pm}(x,\pm x+2k)]=\boldsymbol{\beta}^{\pm}(x)[\mathbf{g}^{\pm}(x,\pm x+2k)]+\boldsymbol{\alpha}^{\pm}(x)[\mathbf{u}^{\pm}(x,\pm x+2k)], \tag{2.8}$$

where $\boldsymbol{\alpha}^{+}(x)=\mathbf{a}_{11}(x,+0)-\mathbf{b}_{12}(x)\mathbf{b}_{21}(x)/2$, $\boldsymbol{\alpha}^{-}(x)=\mathbf{a}_{22}(x,+0)+\mathbf{b}_{21}(x)\mathbf{b}_{12}(x)/2$, $\boldsymbol{\beta}^{+}(x)=\mathbf{b}_{11}(x)$, and $\boldsymbol{\beta}^{-}(x)=\mathbf{b}_{22}(x)$ for $x\in\mathbb{I}$. If $[\mathbf{u}^{+}(+0,2k)]\neq\mathbf{0}$, one obtains

$$\begin{aligned}[]
[\mathbf{g}^{+}(x,x+2k)]&=[\mathbf{u}^{+}(x,x+2k)][\mathbf{u}^{+}(+0,2k)]^{-1}[\mathbf{g}^{+}(+0,2k)]\\
&\quad+[\mathbf{u}^{+}(x,x+2k)]\int_0^x[\mathbf{u}^{+}(x',x'+2k)]^{-1}\boldsymbol{\alpha}^{+}(x')[\mathbf{u}^{+}(x',x'+2k)]\,dx',
\end{aligned}\tag{2.9}$$

where the results in Theorem 2.1 have been used. If $[\mathbf{u}^{-}(1-0,2k-1)]\neq\mathbf{0}$, then

$$\begin{aligned}[]
[\mathbf{g}^{-}(x,-x+2k)]&=[\mathbf{u}^{-}(x,-x+2k)][\mathbf{u}^{-}(1-0,2k-1)]^{-1}[\mathbf{g}^{-}(1-0,2k-1)]\\
&\quad+[\mathbf{u}^{-}(x,-x+2k)]\int_1^x[\mathbf{u}^{-}(x',-x'+2k)]^{-1}\boldsymbol{\alpha}^{-}(x')[\mathbf{u}^{-}(x',-x'+2k)]\,dx'.
\end{aligned}$$

At the boundary, the jumps in $\mathbf{g}^{\pm}$ are related to each other as

$$\begin{cases}
[\mathbf{g}^{+}(+0,0)]=r_0[\mathbf{g}^{-}(+0,0)]=-\dfrac{r_0}{2}\mathbf{b}_{21}(+0)[\mathbf{u}^{+}(+0,0)]=-\dfrac{r_0t_0}{2}\mathbf{b}_{21}(+0),\\
\displaystyle\sum_{j=+,-}[\mathbf{g}^{-}(1-0,2k-1)]_j=r_1\sum_{j=+,-}[\mathbf{g}^{+}(1-0,2k-1)]_j,\quad k>0,\\
\displaystyle\sum_{j=+,-}[\mathbf{g}^{+}(+0,2k)]_j=r_0\sum_{j=+,-}[\mathbf{g}^{-}(+0,2k)]_j,\quad k>0,
\end{cases}\tag{2.10}$$

where the subscript $+(-)$ indicates that the jump across $L^{+}(L^{-})$ is referred to. The jumps in the x-derivatives of $\mathbf{u}^{\pm}$ across $L^{\pm}$, respectively, are of less interest, but can be computed by (2.3), once the jumps in the s-derivatives have been calculated.

From the above results, it is possible to make statements about the regularity of the canonical solutions on $L^{\pm}$ in the partial mismatch cases (1) $r_1=0$ and (2) $r_1\neq 0$ and $r_0=0$, which are of special interest. In both cases, $\mathbf{u}^{+}$ is discontinuous across the line segment $s=x$, while $\mathbf{u}^{-}$ is continuous, but not differentiable, across this line segment.

(1) $\mathbf{u}^{-}$ is continuous across the line segment $s=2-x$ but not differentiable, while $\mathbf{u}^{+}$ is differentiable across this line segment. Across the line segment $s=2+x$, $\mathbf{u}^{+}$ is continuous but, in general, not differentiable, while $\mathbf{u}^{-}$ is differentiable. If also $r_0=0$, $\mathbf{u}^{\pm}$ are both differentiable on this line segment. On the rest of L, $\mathbf{u}^{\pm}$ are both differentiable.

(2) $\mathbf{u}^{-}$ is discontinuous across the line segment $s=2-x$, while $\mathbf{u}^{+}$ is continuous but not differentiable. Across the line segment $s=2+x$, $\mathbf{u}^{+}$ is continuous but not differentiable, and $\mathbf{u}^{-}$ is differentiable. Across the line segment $s=4-x$, $\mathbf{u}^{-}$ is continuous but not differentiable, and $\mathbf{u}^{+}$ is differentiable. On the rest of L, $\mathbf{u}^{\pm}$ are both differentiable.

3. The full propagation problem. In this section, the results in the preceding section are extended to a more general input $\boldsymbol{e}^i$. Two theorems are given. Theorem 3.1 shows that there exists a unique, well-behaved, weak solution to the general propagation problem (1.1). The solution is given explicitly in terms of the canonical solutions $\mathbf{u}^{\pm}$ and the excitation $\boldsymbol{e}^i$ at the front wall (Duhamel's integral). Theorem 3.2 shows that the notion of wave front speed is well defined also for wave propagation in dispersive, complex media.

The first theorem of this section is as follows.

THEOREM 3.1. *Let* $\mathbf{a}$, $\mathbf{b}$, r_1, r_0, t_0, $L^{\pm}$, Q_{2n}, *and* $\mathbf{u}^{\pm}$ *be as in Theorem* 2.1. *Let the vector* $\boldsymbol{e}^i:\mathbb{R}_+ \to M_{2\times 1}(\mathbb{R})$ *be continuously differentiable with bounded derivative with exception for at most a finite number of points,* $0 \leq s_1 < \cdots < s_p$, *where it is undefined. Finally, let* $\Gamma = \Gamma^+ \cup \Gamma^-$, *where* $\Gamma^{\pm} = \cup_{k=1}^{p} \{(x,s) \in \mathbb{I} \times \mathbb{R}_+ : (0, s_k) + L^{\pm}\}$. *Then, for every integer* $n \geq 0$, *the initial-boundary value problem* (1.1), *defined in* $\mathbb{I} \times \mathbb{R}_+$, *has a unique solution* $\boldsymbol{e}^{\pm} \in \mathcal{C}_{2\times 1}(Q_{2n} \setminus \Gamma^{\pm})$ *in the weak sense in* Q_{2n}, *i.e., integrated along the characteristics. Thus,* $\partial_x \pm \partial_s$ *are interpreted as derivatives with respect to the vectors* $(1, \pm 1)$, *respectively. The solution is given explicitly by*

$$\boldsymbol{e}^{\pm}(x,s) = \partial_s \int_0^{s-x} \mathbf{u}^{\pm}(x, s-s')\boldsymbol{e}^i(s')\, ds'. \tag{3.1}$$

If $\boldsymbol{e}^i(s) = \mathbf{0}$ *when* $s < s_0$ *for some arbitrary nonnegative number* s_0, *then* $\boldsymbol{e}^{\pm} = \mathbf{0}$ *in* Q_{s_0}. *If* $\partial_s \mathbf{a}$ *exists in* $\mathbb{I} \times \mathbb{R}_+$ *and* $\partial_s \mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I} \times \mathbb{R}_+)$, *then* $\boldsymbol{e}^{\pm} \in \mathcal{C}^1_{2\times 1}(Q_{2n} \setminus \Gamma)$.

Proof. The Cauchy convergence principle guarantees the existence of $\boldsymbol{e}^i(s_j \pm 0) := \lim \boldsymbol{e}^i(s)$, as $s \to s_j \pm 0$, at each discontinuity point s_j, since, e.g., $\boldsymbol{e}^i(s') - \boldsymbol{e}^i(s'') = \int_{s''}^{s'} \frac{d}{ds}\boldsymbol{e}^i(s)\, ds$, $s_j < s'' < s' < s_{j+1}$, has the limit $\mathbf{0}$, when $s', s'' \searrow s_j$. Thus, $\boldsymbol{e}^i$ has a finite jump discontinuity at the point s_j, and the jump in $\boldsymbol{e}^i$ at s_j is defined as $[\boldsymbol{e}^i(s_j)] := \boldsymbol{e}^i(s_j + 0) - \boldsymbol{e}^i(s_j - 0)$. A solution to the problem (1.1) is immediately obtained by a straightforward extension of Duhamel's principle (see [7]):

$$\boldsymbol{e}^{\pm}(x,s) = \sum_{k=1}^{p} \mathbf{u}^{\pm}(x, s-s_k)[\boldsymbol{e}^i(s_k)] + \int_0^{\infty} \mathbf{u}^{\pm}(x, s-s') \left\{ \frac{d}{ds'}\boldsymbol{e}^i \right\}(s')ds'$$

for all $(x,s) \in Q_{2n} \setminus \Gamma$, where $\{\frac{d}{ds}\boldsymbol{e}^i\}(s)$ denotes the classical derivative of $\boldsymbol{e}^i$ at s. Use of the fact that $\mathbf{u}^{\pm}(x,s) = \mathbf{0}$ when $x > s$ yields the desired result (3.1). The solution inherits the regularity of $\mathbf{u}^{\pm}$ and $\boldsymbol{e}^i$. Consequently, $\boldsymbol{e}^{\pm} \in \mathcal{C}_{2\times 1}(Q_{2n} \setminus \Gamma^{\pm})$ and $\boldsymbol{e}^{\pm} = \mathbf{0}$ in Q_{s_0} if $\boldsymbol{e}^i(s) = \mathbf{0}$ when $s < s_0$. Moreover, the solution (3.1) is the only solution in the weak sense. For let $(\boldsymbol{e}^+, \boldsymbol{e}^-)$ be the difference between two such solutions. Clearly, the matrix-valued functions $\begin{pmatrix}\boldsymbol{e}^{\pm} & \mathbf{0}\end{pmatrix}$, where the second column is the zero vector, solve the canonical problem in Theorem 2.1, with $\mathbf{I}$ replaced by $\mathbf{0}$, and since the solution of this problem is unique, $(\boldsymbol{e}^+, \boldsymbol{e}^-)$ is zero. Finally, the last sentence in the theorem holds according to Theorem 2.2 (any (first) derivative of Duhamel's integral (3.1) may act upon $\mathbf{u}^{\pm}$). The proof is finished. $\square$

Recall that the vector fields $\boldsymbol{e}^+ \pm \boldsymbol{e}^-$ are essentially the electric and magnetic fields. By Theorem 3.1, $\boldsymbol{e}^+ \pm \boldsymbol{e}^- \in \mathcal{C}_{2\times 1}(Q_{2n} \setminus \Gamma)$. Furthermore, $\boldsymbol{e}^+ \pm \boldsymbol{e}^- = \mathbf{0}$ in Q_{s_0} if $\boldsymbol{e}^i(s) = \mathbf{0}$ for all $s < s_0$. Consequently, the speed of the wave front is ≤ 1; that is, *strict causality holds for wave propagation in dispersive, complex media.*

Theorem 3.2 below asserts the existence of a well-defined wave front inside the dispersive, complex medium, whenever the incident field has a well-defined front edge. The proof uses the fact that this already has been established for the step response $\mathbf{u}^+$; see Theorem 2.1. The main consequence of Theorem 3.2 is that *the speed of the wave front is precisely one, independent of the dispersion kernel* $\mathbf{a}$ *and the excitation* $\boldsymbol{e}^i$. The second theorem of this section is as follows.

THEOREM 3.2 (wave front speed). *Let, in the preceding theorem,* $\boldsymbol{e}^i$ *have the following additional property: there is a number* $\delta > 0$ *such that the restriction of* $\boldsymbol{e}^i$

to $(0,\delta)$ *is continuously differentiable and* $\boldsymbol{e}^i(s) \neq \mathbf{0}$ *for all* $0 < s < \delta$. *Then, for each* $x \in (0,1)$, *there is a number* s, $x < s < x+\delta$, *such that* $\boldsymbol{e}^+(x,s) \neq \mathbf{0}$, *where* $(\boldsymbol{e}^+, \boldsymbol{e}^-)$ *is the unique solution to the problem* (1.1) *given by* (3.1). *The same statement is true for the vector fields* $\mathbf{e}^+ \pm \boldsymbol{e}^-$.

Proof. Put $\boldsymbol{e}^i = (e_1^i, e_2^i)$ and choose a real number δ_0, $0 < \delta_0 < \delta$, such that both e_1^i and e_2^i do not change sign in the interval $0 < s < \delta_0$. In particular, this implies that at least one of the terms $\int_0^{\delta_0} e_1^i(s')\,ds'$, $\int_0^{\delta_0} e_2^i(s')\,ds'$ is nonzero. Assume, contrary to the hypothesis of the theorem, that there is a point $x \in (0,1)$ such that $\boldsymbol{e}^+(x, s+x) = \mathbf{0}$ for all $s \in (0,\delta)$. By Theorem 2.1 in the previous section, $\det(\mathbf{u}^+(x, x+0)) \neq 0$, and since $\mathbf{u}^+(x,x+0)$ is a continuous extension, there is a number $\delta_1 > 0$ such that

$$\det \begin{pmatrix} u_{11}^+(x, x+s_1) & u_{12}^+(x, x+s_2) \\ u_{21}^+(x, x+s_3) & u_{22}^+(x, x+s_4) \end{pmatrix} \neq 0 \tag{3.2}$$

for all s_1, s_2, s_3, s_4 such that $0 < s_1, s_2, s_3, s_4 < \delta_1$. It is not a restriction to assume that $\delta_0 < \delta_1$. Equation (3.1) implies that $\mathbf{0} = \partial_s \int_0^s \mathbf{u}^+(x, x+s-s')\boldsymbol{e}^i(s')\,ds'$, $0 < s < \delta_0$, so that $\mathbf{0} = \int_0^s \mathbf{u}^+(x, x+s-s')\boldsymbol{e}^i(s')\,ds'$, $0 < s < \delta_0$. The mean value theorem of integral calculus asserts that there are positive real numbers $\delta_2, \delta_3, \delta_4, \delta_5$, such that $\delta_2, \delta_3, \delta_4, \delta_5 < \delta_0$ and

$$\begin{cases} u_{11}^+(x, x+\delta_0-\delta_2) \displaystyle\int_0^{\delta_0} e_1^i(s')\,ds' + u_{12}^+(x, x+\delta_0-\delta_3) \displaystyle\int_0^{\delta_0} e_2^i(s')\,ds' = 0, \\ u_{21}^+(x, x+\delta_0-\delta_4) \displaystyle\int_0^{\delta_0} e_1^i(s')\,ds' + u_{22}^+(x, x+\delta_0-\delta_5) \displaystyle\int_0^{\delta_0} e_2^i(s')\,ds' = 0. \end{cases}$$

Equation (3.2) implies that this system of equations has the trivial solution only; i.e., $\int_0^{\delta_0} e_1^i(s')\,ds' = 0$ and $\int_0^{\delta_0} e_2^i(s')\,ds' = 0$, which contradicts the second sentence of the proof. Thus, there exists a number s, $x < s < x+\delta$, such that $\boldsymbol{e}^+(x,s) \neq \mathbf{0}$.

Analogously, since $\det(\mathbf{u}^+(x,x+0) \pm \mathbf{u}^-(x,x+0)) \neq 0$ for each $x \in \mathbb{I}$, and $\mathbf{u}^+(x,x+0) \pm \mathbf{u}^-(x,x+0)$ are continuous extensions, there is a number $\delta_6 > 0$ such that

$$\det \begin{pmatrix} u_{11}^+(x,x+s_1) \pm u_{11}^-(x,x+s_1) & u_{12}^+(x,x+s_2) \pm u_{12}^-(x,x+s_2) \\ u_{21}^+(x,x+s_3) \pm u_{21}^-(x,x+s_3) & u_{22}^+(x,x+s_4) \pm u_{22}^-(x,x+s_4) \end{pmatrix} \neq 0$$

for all s_1, s_2, s_3, s_4 such that $0 < s_1, s_2, s_3, s_4 < \delta_6$. An investigation similar to the one above shows that for each $x \in (0,1)$, there exists a number s, $x < s < x+\delta$, such that $\boldsymbol{e}^+(x,s) \pm \boldsymbol{e}^-(x,s) \neq \mathbf{0}$. The proof is finished. □

4. The Green functions. As mentioned in section 1, the results of this paper have already been used by the scientific community in a number of papers on direct and inverse scattering in dispersive, complex media [8, 9, 17, 19, 18]. In these articles, the Green functions equations are employed rather than the canonical functions equations (2.2). For completeness, the Green functions equations are now derived and proved to be uniquely solvable in the weak sense. Throughout the section, the memory function $\mathbf{a}(x,s)$ is assumed to be as regular as stated in Theorem 2.2.

There are slight variations among different authors in definitions of the Green functions $\mathbf{g}(x,s)$. In this article, the Green functions are defined as the classical time derivatives of the canonical functions $\mathbf{u}^{\pm}(x,s)$, that is, by (2.5). This definition is closely related to the original one by Krueger and Ochs [20].

The relationship between the split vector fields $\boldsymbol{e}^{\pm}$ and the Green functions $\mathbf{g}^{\pm}$ and the excitation $\boldsymbol{e}^{i}$ is obtained by performing the differentiation in Duhamel's integral (3.1). The final result is given by (1.2). Consequently, the Green functions are the classical contributions to the impulse response.

The properties of the Green functions are given by the following theorem.

THEOREM 4.1. *Let* $\mathbf{a}$, $\mathbf{b}$, r_1, r_0, t_0, $L^{\pm}$, L, Q_{2n}, T_k, *and* $\mathbf{u}^{\pm}$ *be as in Theorem* 2.2. *Then, for each integer* $n \geq 0$, *the integro-differential equation defined in* $\mathbb{I} \times \mathbb{R}_+$ *by*

(4.1)
$$\begin{cases} \begin{pmatrix} (\partial_x + \partial_s)\mathbf{g}^+(x,s) \\ (\partial_x - \partial_s)\mathbf{g}^-(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \mathbf{g}^+(x,s) \\ \mathbf{g}^-(x,s) \end{pmatrix} + \int_0^s \mathbf{a}(x, s-s') \begin{pmatrix} \mathbf{g}^+(x,s') \\ \mathbf{g}^-(x,s') \end{pmatrix} ds' \\ \quad + \sum_{m=1}^{\infty} \begin{pmatrix} \mathbf{a}_{11}(x, s-x-2m+2) & \mathbf{a}_{12}(x, s+x-2m) \\ \mathbf{a}_{21}(x, s-x-2m+2) & \mathbf{a}_{22}(x, s+x-2m) \end{pmatrix} \begin{pmatrix} [\mathbf{u}^+(x, x+2m-2)] \\ [\mathbf{u}^-(x, -x+2m)] \end{pmatrix}, \\ \mathbf{g}^{\pm}(x,0) = \mathbf{0}, \\ \mathbf{g}^+(+0,s) = r_0 \mathbf{g}^-(+0,s), \quad \mathbf{g}^-(1-0,s) = r_1 \mathbf{g}^+(1-0,s), \\ [\mathbf{g}^+(x,s)] = \mathbf{b}_{12}(x)[\mathbf{u}^-(x,s)]/2, \quad (x,s) \in L^-, \\ [\mathbf{g}^-(x,s)] = -\mathbf{b}_{21}(x)[\mathbf{u}^+(x,s)]/2, \quad (x,s) \in L^+ \end{cases}$$

has a unique solution, $\mathbf{g}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n} \setminus L) = \mathcal{C}_{2\times 2}(\cup_{k=0}^{2n} T_k)$, *in the weak sense of line integration along the characteristics within each triangle* T_k, $k \leq 2n$. *Thus, the derivatives are interpreted as derivatives with respect to the vectors* $(1, \pm 1)$, *respectively. The solution is given by the Green functions defined by* (2.5). *The finite jumps in* $\mathbf{g}^{\pm} = \partial_s \mathbf{u}^{\pm}$ *across* $L^{\pm}$, *respectively, are given by* (2.8), *and the jump conditions on the boundary, by* (2.10).

As an immediate consequence of (2.9) and (2.10), the "initial values" of the Green functions become

$$\begin{cases} \mathbf{g}^+(x, x+0) = \mathbf{Q}_0^+(x) \left(-r_0 \mathbf{b}_{21}(+0)/2 + \int_0^x \mathbf{Q}_0^+(x')^{-1} \boldsymbol{\alpha}^+(x') \mathbf{Q}_0^+(x')\, dx' \right), \\ \mathbf{g}^-(x, x+0) = -\mathbf{b}_{21}(x) \mathbf{Q}_0^+(x)/2, \end{cases}$$

where $\mathbf{Q}_0^+(x) = \mathbf{u}^+(x, x+0)$ for $x \in \mathbb{I}$.

Observe that knowledge of the unique existence of a weak solution to the Green functions equations in the sense of Theorem 4.1 is sufficient for numerical purposes [8, 9, 17, 19]. Notice also that the finite jumps in $\mathbf{g}^{\pm}$ across $L^{\mp}$, respectively, cannot be obtained from the integro-differential equation (4.1).

Proof of Theorem 4.1. By Theorem 2.2, the Green functions (2.5) are well defined. Line integration of the canonical equations (2.2) along the characteristics within any triangle T_k, $k \leq 2n$, followed by differentiation with respect to s shows that the Green functions constitute a weak solution to the problem (4.1). Thus, the existence of a solution is proved. The jump conditions are direct consequences of (2.6) and (2.7). Suppose that there is another weak solution to the problem (4.1). The difference between these solutions then satisfies (2.2) with input $\mathbf{0}$ instead of $\mathbf{I}$. By uniqueness in Theorem 2.1, this difference is zero; consequently, there is a unique weak solution within each triangle T_k. The proof is finished. ∎

Finally, the regularity of the Green functions across L in the mismatch cases, (1) $r_1 = 0$ and (2) $r_1 \neq 0$ and $r_0 = 0$, is commented upon. It is easy to obtain the explicit expressions for the jumps in $\mathbf{g}^{\pm}$ across L by combining various formulas in this paper; therefore, a quantitative discussion is sufficient.

The "initial values" show that $\mathbf{g}^\pm$ are both discontinuous across the line segment $s = x$ even in the optically impedance-matched case.

(1) $\mathbf{g}^-$ is discontinuous across the line segment $s = 2 - x$, while $\mathbf{g}^+$ is continuous. Across the line segment $s = 2 + x$, $\mathbf{g}^+$ is discontinuous, and $\mathbf{g}^-$ is continuous. If also $r_0 = 0$, $\mathbf{g}^\pm$ are both continuous on this line segment. On the rest of L, $\mathbf{g}^\pm$ are both continuous.

(2) $\mathbf{g}^\pm$ are both discontinuous across the line segment $s = 2 - x$. Across the line segment $s = 2+x$, $\mathbf{g}^+$ is discontinuous, but $\mathbf{g}^-$ is continuous. Across the line segment $s = 4 - x$, $\mathbf{g}^-$ is discontinuous, and $\mathbf{g}^+$ is continuous. On the rest of L, $\mathbf{g}^\pm$ are both continuous.

5. A bi-isotropic example. In this section, electromagnetic pulse propagation at normal incidence on the stratified, dispersive, bi-isotropic slab is discussed. It is demonstrated that this problem reduces to the mixed initial-boundary value problem (1.1); consequently, all of the theorems and results presented in this article apply.

The bi-isotropic slab is located between the planes $x_3 = 0$ and $x_3 = d$. The media outside the slab are assumed to be simple—homogeneous, isotropic, and without dispersion. However, the medium to the right of the slab might differ from the medium to the left. The slab is excited by a transient, transverse plane wave, which is incident from the left. The incident electric field at the front wall, $x_3 = 0$, which is denoted by $\boldsymbol{E}^i(t)$ at the time t, is supposed to be quiescent before a finite time T_1, i.e., $\boldsymbol{E}^i(t) = \mathbf{0}$ for all times $t < T_1$. Moreover, it is assumed to be continuously differentiable with bounded derivative except for at most a finite number of points, $t_1 < \cdots < t_p$, where it is undefined. The set of all incident electric fields with these properties forms a linear space over the real numbers.

The constitutive relations of the bi-isotropic medium at the time t and at the point $\boldsymbol{r} \equiv (x_1, x_2, x_3) \equiv x_1\widehat{\mathbf{x_1}} + x_2\widehat{\mathbf{x_2}} + x_3\widehat{\mathbf{x_3}}$ are defined by the following relation between the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ on one hand, and the electric and magnetic flux densities, $\boldsymbol{D}$ and $\boldsymbol{B}$, respectively, on the other:

$$(5.1)\qquad \begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon(x_3)\left(\boldsymbol{E}(\boldsymbol{r},t) + (\chi_{ee} * \boldsymbol{E})(\boldsymbol{r},t)\right) + c(x_3)^{-1}(\chi_{em} * \boldsymbol{H})(\boldsymbol{r},t), \\ \boldsymbol{B}(\boldsymbol{r},t) = c(x_3)^{-1}(\chi_{me} * \boldsymbol{E})(\boldsymbol{r},t) + \mu(x_3)\left(\boldsymbol{H}(\boldsymbol{r},t) + (\chi_{mm} * \boldsymbol{H})(\boldsymbol{r},t)\right), \end{cases}$$

where, e.g., $(\chi_{ee} * \boldsymbol{E})(\boldsymbol{r},t) = \int_{-\infty}^t \chi_{ee}(x_3, t - t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'$. It is understood that the slab is initially quiescent, i.e., there is a time T, such that $\boldsymbol{E}(\boldsymbol{r},t) = \mathbf{0}$ for all $t \le T$, and similarly for the magnetic field $\boldsymbol{H}(\boldsymbol{r},\cdot)$. Therefore, $\int_{-\infty}^t$ can be substituted for $\int_T^t$ in the convolutions above. The positive functions $\epsilon(x_3)$ and $\mu(x_3)$ are the nondispersive parts of the permittivity and permeability, respectively, and $c(x_3) := (\mu(x_3)\epsilon(x_3))^{-1/2}$. All of the integral kernels $\chi_{ee}(x_3,t)$, $\chi_{em}(x_3,t)$, $\chi_{me}(x_3,t)$, and $\chi_{mm}(x_3,t)$ have the same unit, s^{-1}. These functions are referred to as the susceptibility kernels of the medium. Clearly, the kernels χ_{ee} and χ_{mm} model the ordinary dispersive effects, while the chirality, $(\chi_{em} - \chi_{me})/2$, and the nonreciprocity, $(\chi_{em} + \chi_{me})/2$, are the characteristic properties of the bi-isotropic medium. The medium is reciprocal if $\chi_{em} + \chi_{me} = 0$; see [13].

The medium is expected to be stratified with respect to depth, i.e., $\epsilon(x_3)$ and $\mu(x_3)$ depend on the spatial variable x_3 only, whereas the susceptibility kernels depend on x_3 and the time t only. The functions ϵ and μ are continuously differentiable with bounded derivatives in the interval $(0,d)$, and the susceptibility kernels and their first and second time derivatives are assumed to be bounded and continuous functions in $(x_3,t) \in (0,d) \times (0,\infty)$. The susceptibility kernels are equal to zero when $t < 0$; see [13].

The electromagnetic field satisfies the source-free Maxwell equations:

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \quad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \quad \nabla \cdot \boldsymbol{D} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{B} = \mathbf{0}. \tag{5.2}$$

Transverse solutions, independent of the transverse coordinates (x_1, x_2), are sought. In other words, the solution to the propagation problem can be written in the form $\boldsymbol{E}(\boldsymbol{r}, t) = \widehat{\mathbf{x_1}} E_1(x_3, t) + \widehat{\mathbf{x_2}} E_2(x_3, t)$ throughout space. Observe that it is not necessary to assume that the 3-components of the vector fields vanish inside the bi-isotropic medium; the independence of the spatial variables (x_1, x_2) and the Maxwell equations (5.2) imply that D_3 and B_3 are both constant, and by the continuity at the walls, they are both equal to zero throughout space. The constitutive relations and the associative law for causal convolutions then imply that both $E_3(x_3, \cdot)$ and $H_3(x_3, \cdot)$ satisfy the equation $f + (\chi_{ee} + \chi_{mm} + \chi_{ee} * \chi_{mm} - \chi_{em} * \chi_{me}) * f = 0$, which is a linear Volterra integral equation of the second kind, and therefore has the unique continuous solution $f = 0$; see [15]. One arrives at the same conclusion if f has the regularity described in the third paragraph of this section.

With the transverse ansatz above, the Maxwell equations (5.2) can be written

$$\partial_3 \boldsymbol{E} = \partial_t (\mathbf{J}\boldsymbol{B}), \qquad \partial_3 (\mathbf{J}\boldsymbol{H}) = \partial_t \boldsymbol{D}, \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.3}$$

where a compact matrix notation, pertinent to the analysis of the propagation of electromagnetic waves in the bi-isotropic slab, has been introduced. Put $\boldsymbol{\chi}_{ee} \colon= \chi_{ee}\mathbf{I}$, $\boldsymbol{\chi}_{me} \colon= \chi_{me}\mathbf{J}$, $\boldsymbol{\chi}_{mm} \colon= \chi_{mm}\mathbf{I}$, $\boldsymbol{\chi}_{em} \colon= \chi_{em}\mathbf{J}$. By the constitutive relations (5.1), the flux densities $\boldsymbol{B}$ and $\boldsymbol{D}$ in (5.3) are eliminated, and a hyperbolic integro-differential equation in the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ is obtained:

$$\begin{aligned} \partial_3 \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} &= \frac{\eta'}{\eta} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} \\ &\quad + c^{-1} \partial_t \left(\begin{pmatrix} \boldsymbol{\chi}_{me} * & \mathbf{I} + \boldsymbol{\chi}_{mm} * \\ \mathbf{I} + \boldsymbol{\chi}_{ee} * & -\boldsymbol{\chi}_{em} * \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix} \right), \end{aligned} \tag{5.4}$$

where $\eta \colon= \sqrt{\mu/\epsilon}$ is a locally defined optical wave impedance. Next, the optical wave splitting,

$$\begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} = \mathbf{P} \begin{pmatrix} \boldsymbol{E} \\ \eta \mathbf{J}\boldsymbol{H} \end{pmatrix}, \qquad \mathbf{P} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{P}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}, \tag{5.5}$$

is adopted. The wave splitting technique is a well-established method for solving direct and inverse scattering problems. For a recent survey of the technique, the reader is referred to [6]. Recent contributions to the solution of direct and inverse scattering problems in dispersive, complex media can be found in [8, 9, 19, 18].

The form of the matrix $\mathbf{P}^{-1}$ shows that the electric field is the sum of the split vector fields, $\boldsymbol{E}^{\pm}$, and that the magnetic field is proportional to the difference (with a matrix as proportionality constant). Outside the slab, $\boldsymbol{E}^{\pm}$ represent the general right-going and left-going waves. More precisely, $\boldsymbol{E}^{\pm}(x_3, \cdot)$ are the incident and reflected electric fields at position x_3, respectively, if $x_3 < 0$. Analogously, $\boldsymbol{E}^{-}(x_3, \cdot) = \mathbf{0}$, and $\boldsymbol{E}^{+}(x_3, \cdot)$ is the transmitted electric field at position x_3, if $x_3 > d$.

The wave splitting and the continuity of (the tangential components of) the magnetic and electric fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at the boundary yield

$$\begin{aligned}\boldsymbol{E}^r(t) &= \frac{2\eta(-0)}{\eta(+0)+\eta(-0)}\boldsymbol{E}^-(+0,t) + \frac{\eta(+0)-\eta(-0)}{\eta(+0)+\eta(-0)}\boldsymbol{E}^i(t),\\ \boldsymbol{E}^t(t) &= \frac{2\eta(d+0)}{\eta(d+0)+\eta(d-0)}\boldsymbol{E}^+(d-0,t),\end{aligned}$$

where $\boldsymbol{E}^r(t)$ is the electric field of the reflected transverse plane wave at the front wall and $\boldsymbol{E}^t(t)$ is the electric field of the transmitted transverse plane wave at the back wall, both evaluated at time t. Thus, once the functions $\boldsymbol{E}^-(+0,\cdot)$ and $\boldsymbol{E}^+(d-0,\cdot)$ are known, the direct scattering problem is solved. In the second formula, the fact that there is no incident field from the right has been used. Another consequence of the wave splitting and the boundary conditions is

$$\begin{cases}\boldsymbol{E}^-(d-0,t) = r_1\boldsymbol{E}^+(d-0,t),\\ t_0\boldsymbol{E}^i(t) = \boldsymbol{E}^+(+0,t) - r_0\boldsymbol{E}^-(+0,t),\end{cases}\tag{5.6}$$

where

$$r_1 = \frac{\eta(d+0)-\eta(d-0)}{\eta(d+0)+\eta(d-0)}, \quad r_0 = \frac{\eta(-0)-\eta(+0)}{\eta(-0)+\eta(+0)}, \quad \text{and} \quad t_0 + r_0 = 1.$$

The hyperbolic integro-differential equation for the split vector fields $\boldsymbol{E}^\pm$ is easily obtained from the wave equation (5.4) and the wave splitting (5.5). The result,

$$\begin{aligned}&\begin{pmatrix}(\partial_3 + c^{-1}\partial_t)\boldsymbol{E}^+\\ (\partial_3 - c^{-1}\partial_t)\boldsymbol{E}^-\end{pmatrix} = \frac{\eta'}{2\eta}\begin{pmatrix}\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\boldsymbol{E}^+\\ \boldsymbol{E}^-\end{pmatrix} + \frac{1}{2c}\partial_t\left(\boldsymbol{\chi} * \begin{pmatrix}\boldsymbol{E}^+\\ \boldsymbol{E}^-\end{pmatrix}\right),\\ &\boldsymbol{\chi} := \begin{pmatrix}-\boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & -\boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me}\\ \boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & \boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me}\end{pmatrix},\end{aligned}\tag{5.7}$$

is clearly equivalent to the Maxwell equations for the bi-isotropic medium. According to the second and third paragraphs of this section, there is a time $T_0 := \min(T_1, T)$ such that

$$\begin{cases}\boldsymbol{E}^i(t) = \mathbf{0}, & t < T_0,\\ \boldsymbol{E}^\pm(x_3,t) = \mathbf{0}, & (x_3,t) \in (0,d)\times(-\infty,T_0].\end{cases}\tag{5.8}$$

Introduce travel-time coordinates, (x,s), by

$$s(t) = \frac{t-T_0}{t_{slab}}, \qquad x(x_3) = \frac{1}{t_{slab}}\int_0^{x_3}\frac{dx_3'}{c(x_3')}, \qquad t_{slab} = \int_0^d \frac{dx_3'}{c(x_3')},$$

and put $\mathbf{e}^\pm(x,s) := \boldsymbol{E}^\pm(x_3(x),t(s))$ and $\mathbf{e}^i(s) := \boldsymbol{E}^i(t(s))$. By these substitutions of variables, (5.7), (5.6), and (5.8) are transformed into the nonlocal hyperbolic initial-boundary value problem (1.1), where the functions $\mathbf{a}$ and $\mathbf{b}$ are defined by

$$\begin{cases}\mathbf{a}(x,s) = \dfrac{t_{slab}}{2}\partial_s\boldsymbol{\chi}\left(x_3(x), t_{slab}s\right), & (x,s)\in\mathbb{I}\times\mathbb{R}_+ \equiv (0,1)\times(0,\infty),\\ \mathbf{b}(x) = \dfrac{t_{slab}}{2}\boldsymbol{\chi}(x_3(x),0) + \dfrac{d}{dx}\ln\sqrt{\dfrac{\eta(x_3(x))}{\eta_0}}\begin{pmatrix}\mathbf{I} & -\mathbf{I}\\ -\mathbf{I} & \mathbf{I}\end{pmatrix}, & x\in\mathbb{I},\end{cases}$$

and η_0 is the wave impedance in vacuum. From the third paragraph of this section, it is clear that $\mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I}\times\mathbb{R}_+)$ is differentiable with respect to time s, $\partial_s \mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I}\times\mathbb{R}_+)$, and $\mathbf{b} \in \mathcal{C}_{4\times 4}(\mathbb{I})$, so that all theorems in the previous sections are applicable to this wave propagation problem. Equation (5.7) reveals that the speed of the wave front is precisely $c(x_3)$ at the point x_3, as expected.

As a final remark, note that the solution to (1.1) in this bi-isotropic case is axially symmetric; i.e., if $\boldsymbol{e}^{\pm}$ is the solution corresponding to the input $\boldsymbol{e}^{i}$, and $\mathbf{R}$ is an arbitrary rotation matrix in the x_1-x_2-plane, then $\mathbf{R}\boldsymbol{e}^{\pm}$ is the solution corresponding to the input $\mathbf{R}\boldsymbol{e}^{i}$. This is not surprising since the constitutive relations for the bi-isotropic medium are isotropic. More generally, this happens for media such that $\mathbf{R}\mathbf{a}_{ij}\mathbf{R}^{-1} = \mathbf{a}_{ij}$ and $\mathbf{R}\mathbf{b}_{ij}\mathbf{R}^{-1} = \mathbf{b}_{ij}$, $1 \le i,j \le 2$, i.e., for media such that all the submatrices $\mathbf{a}_{ij}$ and $\mathbf{b}_{ij}$ of $\mathbf{a}$ and $\mathbf{b}$, respectively, defined by the decompositions in Theorem 2.1, commute with every rotation matrix $\mathbf{R}$.

Appendix. Proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. A necessary condition for the existence of a weak solution $\mathbf{u}^{\pm} \in \mathcal{C}_{2\times 2}(Q_{2n} \setminus L)$ in Q_{2n} to (2.2) is that $\mathbf{u}^{+}$ and $\mathbf{u}^{-}$ can be extended to continuous functions in $Q_{2n} \setminus L^{+}$ and $Q_{2n} \setminus L^{-}$, respectively. This fact is used below in the construction of the solution. Choose $T > 0$ such that $Tn_T = 2$ for some even integer n_T, and such that

$$b(T){:} = 2(1 + |r_1| + |r_0|)(\|\mathbf{b}\|T + \|\mathbf{a}\|T^2) < 1/2, \tag{A.1}$$

where the different norms $\|\cdot\|$ are defined in (2.1). Assume that the theorem holds in the set $\cup_{j=0}^{k-1} T_j$ for some k, where $0 \le k \le 2n$, and prove that it holds also in $\cup_{j=0}^{k} T_j$, by using the Banach fixed-point theorem n_T $(n_T/2)$ times if $k \ne 0$ $(k = 0)$. By (A.1), it will be clear that the method works for all k, including $k = 0$, and the theorem follows from the induction axiom. Consider the continuous map $\mathbf{f} \equiv (\mathbf{f}^{+}, \mathbf{f}^{-})$ on the Banach space $\mathcal{C}_{2\times 2}(T_k) \times \mathcal{C}_{2\times 2}(T_k)$ defined by

(A.2)

$$\begin{aligned}
(\mathbf{f}^{+}(\mathbf{u}^{+},\mathbf{u}^{-}))(x,s) &= (\mathbf{B}^{+}(\mathbf{u}^{+},\mathbf{u}^{-}))(x,s) \\
&\quad + \int_{x^{+}}^{x} \mathbf{b}_{11}(x')\,\mathbf{u}^{+}(x', s-x+x')\,dx' + \int_{x^{+}}^{x} \mathbf{b}_{12}(x')\,\mathbf{u}^{-}(x', s-x+x')\,dx' \\
&\quad + \int_{x^{+}}^{x} \left(\int_{-\infty}^{s-x+x'} \mathbf{a}_{11}(x', s-x+x'-s'')\,\mathbf{u}^{+}(x',s'')\,ds'' \right) dx' \\
&\quad + \int_{x^{+}}^{x} \left(\int_{-\infty}^{s-x+x'} \mathbf{a}_{12}(x', s-x+x'-s'')\,\mathbf{u}^{-}(x',s'')\,ds'' \right) dx', \\
(\mathbf{f}^{-}(\mathbf{u}^{+},\mathbf{u}^{-}))(x,s) &= (\mathbf{B}^{-}(\mathbf{u}^{+},\mathbf{u}^{-}))(x,s) \\
&\quad + \int_{x^{-}}^{x} \mathbf{b}_{21}(x')\,\mathbf{u}^{+}(x', s-x'+x)\,dx' + \int_{x^{-}}^{x} \mathbf{b}_{22}(x')\,\mathbf{u}^{-}(x', s-x'+x)\,dx' \\
&\quad + \int_{x^{-}}^{x} \left(\int_{-\infty}^{s-x'+x} \mathbf{a}_{21}(x', s-x'+x-s'')\,\mathbf{u}^{+}(x',s'')\,ds'' \right) dx' \\
&\quad + \int_{x^{-}}^{x} \left(\int_{-\infty}^{s-x'+x} \mathbf{a}_{22}(x', s-x'+x-s'')\,\mathbf{u}^{-}(x',s'')\,ds'' \right) dx',
\end{aligned}$$

where it is agreed that $\mathbf{u}^{\pm}(x,s)$ for points $(x,s) \in \cup_{j=0}^{k-1} T_j$ attain the values computed in the previous steps. This map is induced by line-integration along the characteristics of (2.2), and the points $(x^{\pm}, s^{\pm})$, where $s^{\pm} < s$, are the points where the straight lines emanating from (x, s) with slopes ± 1, respectively, cut the boundary of T_k, ∂T_k, and $\mathbf{B}^{\pm}(\mathbf{u}^+, \mathbf{u}^-)$ are the corresponding initial-boundary values of $\mathbf{f}^{\pm}(\mathbf{u}^+, \mathbf{u}^-)$ at these points; see Figure 1. For a fixed element $(\mathbf{u}^+, \mathbf{u}^-)$ in the domain of $\mathbf{f}$, these quantities are functions defined on T_k. If k is odd, they are given by

$$\text{(A.3)} \qquad \begin{cases} (x^+(x,s), s^+(x,s)) = (0, s-x), \\ (x^-(x,s), s^-(x,s)) = 2^{-1}(x+s-k+1, x+s+k-1) \in \partial T_k \cap \partial T_{k-1}, \\ (\mathbf{B}^-(\mathbf{u}^+, \mathbf{u}^-))(x,s) := \mathbf{u}^-(x^-, s^- - 0), \\ (\mathbf{B}^+(\mathbf{u}^+, \mathbf{u}^-))(x,s) := t_0 \mathbf{I} + r_0 (\mathbf{f}^-(\mathbf{u}^+, \mathbf{u}^-))(x^+ + 0, s^+), \end{cases}$$

and if k is even, they are

$$\text{(A.4)} \qquad \begin{cases} (x^+(x,s), s^+(x,s)) = 2^{-1}(k-s+x, k+s-x) \in \partial T_k \cap \partial T_{k-1}, \\ (x^-(x,s), s^-(x,s)) = (1, s+x-1), \\ (\mathbf{B}^+(\mathbf{u}^+, \mathbf{u}^-))(x,s) := \mathbf{u}^+(x^+, s^+ - 0), \\ (\mathbf{B}^-(\mathbf{u}^+, \mathbf{u}^-))(x,s) := r_1 (\mathbf{f}^+(\mathbf{u}^+, \mathbf{u}^-))(x^- - 0, s^-). \end{cases}$$

It is obvious that every element $(\mathbf{f}^+(\mathbf{u}^+, \mathbf{u}^-), \mathbf{f}^-(\mathbf{u}^+, \mathbf{u}^-))$ in the range of $\mathbf{f}$ can be extended to a function in $\mathcal{C}_{2\times 2}(\overline{T_k}) \times \mathcal{C}_{2\times 2}(\overline{T_k})$ in a natural way, and by the fourth formula in (A.3) and (A.4), it is clear that this extension satisfies the boundary values in (2.2). Furthermore, by the third condition in (A.4) and (A.3), it follows that $\mathbf{f}^{\pm}(\mathbf{u}^+, \mathbf{u}^-)$ are restrictions to T_k of continuous extensions of $\mathbf{u}^{\pm}$ from $\overline{T_{k-1}}$ to $\overline{T_{k-1}} \cup T_k$, respectively; see the first sentence of the proof. Differentiation of $\mathbf{f}^{\pm}(\mathbf{u}^+, \mathbf{u}^-)$ with respect to the vectors $(1, \pm 1)$, respectively, yields

$$\begin{pmatrix} (\partial_x + \partial_s)\mathbf{f}^+(\mathbf{u}^+, \mathbf{u}^-)(x,s) \\ (\partial_x - \partial_s)\mathbf{f}^-(\mathbf{u}^+, \mathbf{u}^-)(x,s) \end{pmatrix} = \mathbf{b}(x) \begin{pmatrix} \mathbf{u}^+(x,s) \\ \mathbf{u}^-(x,s) \end{pmatrix} + \int_{-\infty}^{s} \mathbf{a}(x, s-s') \begin{pmatrix} \mathbf{u}^+(x,s') \\ \mathbf{u}^-(x,s') \end{pmatrix} ds'.$$

Note that the derivatives of the boundary-value functions $\mathbf{B}^{\pm}(\mathbf{u}^+, \mathbf{u}^-)$ are zero. The theorem is essentially proved, if it can be shown that the map $\mathbf{f}$ has a unique fixed point $(\mathbf{u}^+, \mathbf{u}^-) \in \mathcal{C}_{2\times 2}(T_k) \times \mathcal{C}_{2\times 2}(T_k)$. Unfortunately, this cannot be accomplished in one step only; therefore, the following subdivision of T_k is introduced (see also Figure 1): $P_{k,j} := T_k \cap (\mathbb{I} \times (k-1+(j-1)T, k-1+jT))$, $j \in \{1, \ldots, n_T\}$.

Since $b(T) < 2^{-1}$, the Banach fixed-point theorem implies that $\mathbf{f}$ has a unique fixed point in $\mathcal{C}_{2\times 2}(P_{k,1}) \times \mathcal{C}_{2\times 2}(P_{k,1})$ if $k \neq 0$ and in $\mathcal{C}_{2\times 2}(P_{0,n_T/2+1}) \times \mathcal{C}_{2\times 2}(P_{0,n_T/2+1})$ if $k = 0$. In the latter case, the solution is obviously zero. That $\mathbf{f}$ actually is a contraction follows easily from (A.1)–(A.4):

$$\left\| \mathbf{f}(\mathbf{u}^+, \mathbf{u}^-) - \mathbf{f}(\mathbf{v}^+, \mathbf{v}^-) \right\| \leq b(T) \left\| (\mathbf{u}^+, \mathbf{u}^-) - (\mathbf{v}^+, \mathbf{v}^-) \right\|,$$

for all $(\mathbf{u}^+, \mathbf{u}^-), (\mathbf{v}^+, \mathbf{v}^-) \in \mathcal{C}_{2\times 2}(P_{k,j}) \times \mathcal{C}_{2\times 2}(P_{k,j})$.

In the next step, the procedure in the previous paragraph is repeated to show that $\mathbf{f}$ has a unique fixed point in the Banach space $\mathcal{C}_{2\times 2}(P_{k,2}) \times \mathcal{C}_{2\times 2}(P_{k,2})$ if $k \neq 0$ (and in $\mathcal{C}_{2\times 2}(P_{0,n_T/2+2}) \times \mathcal{C}_{2\times 2}(P_{0,n_T/2+2})$ if $k = 0$), at which the restriction of $(\mathbf{u}^+, \mathbf{u}^-)$ to $P_{k,1}$ ($P_{0,n_T/2+1}$) in (A.2) is the unique solution obtained in the first step. Clearly, $\mathbf{u}^{\pm}$ become continuous on the part of the horizontal line $s = k-1+T$ ($s = T$) that is contained in T_k (T_0). It takes $n_T - 2$ ($n_T/2 - 2$) other steps to show that the map $\mathbf{f}$ has a unique fixed point $(\mathbf{u}^+, \mathbf{u}^-) \in \mathcal{C}_{2\times 2}(T_k) \times \mathcal{C}_{2\times 2}(T_k)$, which is equal to zero if $k = 0$.

It remains to verify the statements concerning the jump-discontinuities. The solution to (2.2) is zero in Q_0, and by (A.3), $\mathbf{u}^-$ is continuous on the line $s = x$. Equation (A.2) then gives that $\mathbf{u}^+(x, x+0) = t_0\mathbf{I} + \int_0^x \mathbf{b}_{11}(x')\,\mathbf{u}^+(x', x'+0)\,dx'$ for all $x \in \mathbb{I}$, which is the required result for the function $x \to [\mathbf{u}^+(x,x)]$. Finally, if $\mathbf{Q}^+(x){:} = \mathbf{u}^+(x, x+0)$, $x \in \mathbb{I}$, then $\det \mathbf{Q}^+(0) \neq 0$, and basic matrix theory yields

$$
\begin{aligned}
&\frac{d}{dx}\det \mathbf{Q}^+(x) = \det\big(\boldsymbol{Q}_1'(x) \quad \boldsymbol{Q}_2(x)\big) + \det\big(\boldsymbol{Q}_1(x) \quad \boldsymbol{Q}_2'(x)\big)\\
&= \det\big(\mathbf{b}_{11}(x)\boldsymbol{Q}_1(x) \quad \boldsymbol{Q}_2(x)\big) + \det\big(\boldsymbol{Q}_1(x) \quad \mathbf{b}_{11}(x)\boldsymbol{Q}_2(x)\big) = \operatorname{tr}(\mathbf{b}_{11}(x))\det \mathbf{Q}^+(x)
\end{aligned}
$$

for all $x \in \mathbb{I}$, where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are the column vectors of $\mathbf{Q}^+$. Thus, $\det \mathbf{Q}^+(x) = \exp\big(\int_0^x \operatorname{tr}(\mathbf{b}_{11}(x'))\,dx'\big)\det \mathbf{Q}^+(0) \neq 0$, $x \in \mathbb{I}$, which proves that $\mathbf{Q}^+(x)$ is nonsingular for each $x \in \mathbb{I}$. The other results follow analogously. The proof is finished. $\square$

Proof of Theorem 2.2. Assume that for some k, $0 < k \leq 2n$, the theorem holds in $\cup_{j=0}^{k-1} T_j$ and prove the validity of the theorem in $\cup_{j=0}^{k} T_j$. The theorem then follows by induction in k, since no special consideration has to be made in the first step or depending on whether k is odd or even. The map $\mathbf{f}$ in the proof of Theorem 2.1 has a unique fixed point $(\mathbf{u}^+, \mathbf{u}^-) \in \mathcal{C}_{2\times 2}(T_k) \times \mathcal{C}_{2\times 2}(T_k)$. It must be shown that $\mathbf{u}^\pm$ actually belong to $\mathcal{C}^1_{2\times 2}(T_k)$. Since $(\partial_x \pm \partial_s)\mathbf{u}^\pm \in \mathcal{C}_{2\times 2}(T_k)$ by Theorem 2.1, it is sufficient to show that $\partial_s \mathbf{u}^\pm$ exist and belong to $\mathcal{C}_{2\times 2}(T_k)$. To this end, define recursively a sequence $(\mathbf{u}_j^+, \mathbf{u}_j^-)_{j=0}^\infty$ in $\mathcal{C}^1_{2\times 2}(P_{k,1}) \times \mathcal{C}^1_{2\times 2}(P_{k,1})$ by $(\mathbf{u}_{j+1}^+, \mathbf{u}_{j+1}^-) = (\mathbf{f}^+(\mathbf{u}_j^+, \mathbf{u}_j^-), \mathbf{f}^-(\mathbf{u}_j^+, \mathbf{u}_j^-))$, which is possible since $\partial_s \mathbf{a}$ exists in $\mathbb{I} \times \mathbb{R}_+$ and $\partial_s \mathbf{a} \in \mathcal{C}_{4\times 4}(\mathbb{I} \times \mathbb{R}_+)$. The proof of the Banach fixed-point theorem—this is actually the method of successive approximations, where the first element in the sequence can be chosen arbitrarily—implies that this sequence converges uniformly to $(\mathbf{u}^+, \mathbf{u}^-)$ in $P_{k,1}$, since

$$
\big\|(\mathbf{u}_j^+, \mathbf{u}_j^-) - (\mathbf{u}_i^+, \mathbf{u}_i^-)\big\| \leq b(T)\,\big\|(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - (\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\big\| \tag{A.5}
$$

for all $i, j > 0$ by (A.1)–(A.4). Similarly, these equations yield

$$
\begin{aligned}
\big\|\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-) - \partial_s(\mathbf{u}_i^+, \mathbf{u}_i^-)\big\| &\leq b(T)\,\big\|\partial_s(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - \partial_s(\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\big\|\\
&\quad + a(T)\,\big\|(\mathbf{u}_{j-1}^+, \mathbf{u}_{j-1}^-) - (\mathbf{u}_{i-1}^+, \mathbf{u}_{i-1}^-)\big\| \qquad \forall i, j > 0,
\end{aligned} \tag{A.6}
$$

where $a(T)$ is independent of k and $(\mathbf{u}_j^+, \mathbf{u}_j^-)_{j=0}^\infty$. Equations (A.5) and (A.6) imply that

$$
\begin{aligned}
&\big\|\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-) - \partial_s(\mathbf{u}_i^+, \mathbf{u}_i^-)\big\|\\
&\leq \frac{(2b(T))^{i-1}}{(1 - 2b(T))}\Big(\big\|\partial_s(\mathbf{u}_1^+, \mathbf{u}_1^-) - \partial_s(\mathbf{u}_0^+, \mathbf{u}_0^-)\big\| + 2a(T)\,\big\|(\mathbf{u}_1^+, \mathbf{u}_1^-) - (\mathbf{u}_0^+, \mathbf{u}_0^-)\big\|\Big)
\end{aligned}
$$

if $0 < i < j$; i.e., $(\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-))_{j=0}^\infty$ is a Cauchy sequence in $\mathcal{C}_{2\times 2}(P_{k,1}) \times \mathcal{C}_{2\times 2}(P_{k,1})$. Since this function space is complete, the sequence $(\partial_s(\mathbf{u}_j^+, \mathbf{u}_j^-))_{j=0}^\infty$ converges uniformly in $P_{k,1}$ to a bounded and continuous function $(\mathbf{v}^+, \mathbf{v}^-)$. By the third paragraph of section 2, it follows that $\partial_s(\mathbf{u}^+, \mathbf{u}^-)$ exists and equals $(\mathbf{v}^+, \mathbf{v}^-)$. Thus, $\mathbf{f}$ has a unique fixed point in $\mathcal{C}^1_{2\times 2}(P_{k,1}) \times \mathcal{C}^1_{2\times 2}(P_{k,1})$.

In the next step, the procedure in the previous paragraph is repeated to show that $\mathbf{f}$ has a unique fixed point in $\mathcal{C}^1_{2\times 2}(P_{k,2}) \times \mathcal{C}^1_{2\times 2}(P_{k,2})$. In this second step, we let the restriction of $(\mathbf{u}^+, \mathbf{u}^-)$ to $P_{k,1}$ be the unique solution obtained in the first step. Since $\partial_s \mathbf{u}^\pm$ exist and are continuous on both sides of the part of the horizontal line $s = k - 1 + T$ that is contained in T_k, and since, by construction, $(\partial_x \pm \partial_s)\mathbf{u}^\pm$ and

$\partial_x \mathbf{u}^{\pm}$ exist and are continuous on this part of the line, $\partial_s \mathbf{u}^{\pm}$ exist and are continuous here also. Just as in the proof of Theorem 2.1, it takes $n_T - 2$ similar steps to show that there is a unique $\mathcal{C}^1_{2\times 2} \times \mathcal{C}^1_{2\times 2}$-solution $(\mathbf{u}^+, \mathbf{u}^-)$ to (2.2) in $\cup_{j=0}^{k} T_j$. From the explicit form of the derivatives of the solution, it is clear that these functions can be extended to bounded and continuous functions in $\overline{T_k}$. The proof is finished. □

Acknowledgments. The author would like to thank Professor G. Kristensson for some useful advice concerning the presentation of this work.

REFERENCES

[1] E. Ammicht, J. P. Corones, and R. J. Krueger, *Direct and inverse scattering for viscoelastic media*, J. Acoust. Soc. Amer., 81 (1987), pp. 827–834.

[2] I. S. Ayoubi, *An Inverse Scattering Problem for Hyperbolic Systems of First-Order in Semi-Infinite Media*, Ph.D. thesis, Texas Tech University, Lubbock, TX, 1989.

[3] R. S. Beezley, *Electromagnetic Direct and Inverse Problems for Absorbing Media*, Ph.D. thesis, University of Nebraska, Lincoln, NE, 1985.

[4] R. S. Beezley and R. J. Krueger, *An electromagnetic inverse problem for dispersive media*, J. Math. Phys., 26 (1985), pp. 317–325.

[5] J. P. Corones and A. Karlsson, *Transient direct and inverse scattering for inhomogeneous viscoelastic media: Obliquely incident SH mode*, Inverse Problems, 4 (1988), pp. 643–660.

[6] J. P. Corones, G. Kristensson, P. Nelson, and D. L. Seth, eds., *Invariant Imbedding and Inverse Problems*, SIAM, Philadelphia, 1992.

[7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, II, Interscience, New York, 1962, pp. 512–515.

[8] J. Fridén, *Inverse scattering for anisotropic mirror image symmetric media*, Inverse Problems, 10 (1994), pp. 1133–1144.

[9] J. Fridén, G. Kristensson, and R. D. Stewart, *Transient electromagnetic wave propagation in anisotropic dispersive media*, J. Opt. Soc. Amer. A, 10 (1993), pp. 2618–2627.

[10] J. Fuks, A. Karlsson, and G. Larson, *Direct and inverse scattering for dispersive media*, Inverse Problems, 10 (1994), pp. 555–571.

[11] J. Hopkinson, *Residual charge of the leyden jar.—Dielectric properties of different glasses*, Philos. Trans. Roy. Soc. London, 167 (1877), pp. 599–626.

[12] A. Karlsson, *Inverse scattering for viscoelastic media using transmission data*, Inverse Problems, 3 (1987), pp. 691–709.

[13] A. Karlsson and G. Kristensson, *Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain*, J. Electro. Waves Appl., 6 (1992), pp. 537–551.

[14] A. Karlsson, G. Kristensson, and H. Otterheim, *Transient wave propagation in gyrotropic media*, in Invariant Imbedding and Inverse Problems, J. P. Corones, G. Kristensson, P. Nelson, and D. L. Seth, eds., SIAM, Philadelphia, 1992, pp. 77–89.

[15] R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin, Heidelberg, 1989.

[16] G. Kristensson, *Direct and inverse scattering problems in dispersive media—Green's functions and invariant imbedding techniques*, in Direct and Inverse Boundary Value Problems, R. Kleinman, R. Kress, and E. Martensen, eds., Methoden und Verfahren der Mathematischen Physik, Band 37, Matematisches Forschungsinstitut Oberwolfach, Germany, 1991, pp. 105–119.

[17] G. Kristensson and S. Rikte, *Scattering of transient electromagnetic waves in reciprocal bi-isotropic media*, J. Electro. Waves Appl., 6 (1992), pp. 1517–1535.

[18] G. Kristensson and S. Rikte, *The inverse scattering problem for a homogeneous bi-isotropic slab using transient data*, in Inverse Problems in Mathematical Physics, L. Päivärinta and E. Somersalo, eds., Springer-Verlag, Berlin, 1993, pp. 112–125.

[19] G. Kristensson and S. Rikte, *Transient wave propagation in reciprocal bi-isotropic media at oblique incidence*, J. Math. Phys., 34 (1993), pp. 1339–1359.

[20] R. J. Krueger and R. L. Ochs, *A Green's function approach to the determination of internal fields*, Wave Motion, 11 (1989), pp. 525–543.

[21] T. M. Roberts, *Causality theorems*, in Invariant Imbedding and Inverse Problems, J. P. Corones, G. Kristensson, P. Nelson, and D. L. Seth, eds., SIAM, Philadelphia, 1992, pp. 114–128.

[22] A. Sommerfeld, *Über die Fortpflanzung des Lichtes in dispergierenden Medien*, Ann. Phys., 44 (1914), pp. 177–202.

[23] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, I: *Fixed-Point Theorems*, Springer-Verlag, New York, 1986.