Heat extraction from the ground by horizontal pipes : a mathematical analysis

Claesson, Johan; Dunand, Alain

1983

Document D1:1983

Heat extraction from the ground by horizontal pipes

A mathematical analysis

Johan Claesson
Alain Dunand

Swedish Council for Building Research

D1:1983

HEAT EXTRACTION FROM THE GROUND BY HORIZONTAL PIPES

A mathematical analysis

Johan Claesson
Department of Mathematical Physics, Lunds university, Sweden

Alain Dunand
Institut de Mècanique, Grenoble, France

This document refers to research grant 791305-0 from the Swedish Council for Building Research to Department of Mathematical Physics, Lunds university, Sweden

D1:1983

ISBN 91-540-3851-0
Swedish Council for Building Research, Stockholm, Sweden

Spångbergs Tryckerier AB, Stockholm 1983

FOREWORD

This report is the result of a cooperation between Institut de Mécanique in Grenoble and the department of mathematical physics in Lund. The main work was done during a half-year research visit in Grenoble by Johan Claesson. The visit was financed by a grant from the French research council CNRS. A great deal of support and help has been provided by the research leader Georges Vachaud in Grenoble. On the Swedish side the work has been supported by the Swedish Building Research Council BFR.

Contents | Page

Appendices

1. INTRODUCTION

A heat pump that is used for the heating of a building relies on a suitable heat source in the surroundings. A possibility that has attracted considerable interest during the last years is to extract the heat from the ground via horizontal pipes which are buried at a certain depth. Intensive studies of these systems are in progress.

Water with a temperature below the ground around the pipe is circulated in the pipes. A heat flux to the pipe is obtained. The ground around the pipe is cooled. This will in turn induce a thermal recharge process from the ground surface and warmer ground further away. This thermal recharge process determines the long-term heat extraction potential of the system. The thermal process in the ground is analysed in this study.

A heat pump has a certain time-varying heat requirement during the annual heating cycle. The temperature of the heat carrier fluid that circulates through the pipes and on the cold side of the heat pump determines, in interaction with the temperature field in the ground around the pipe, the heat extraction rate at each time. The basic question is now: What temperatures must be imposed on the fluid in order to obtain the required heat uptake? This study will provide answers and guide-lines for different situations. The temperature of the fluid should not become too low for several reasons. The efficiency of the heat pump decreases with decreasing temperature on the cold side. The environmental impact and in particular freezing of the ground will impose restrictions on the cooling of the ground around the pipes.

The analyses which are presented in this study are based on analytical solutions of the heat conduction equation in the ground. The aim is to provide basic mathematical methods to assess the heat extraction potential in different situations. The mathematical derivations are given in the appendices. Many numerical examples are considered.

The dynamical, multidimensional thermal process in the ground with its coupling to the heat extraction strategies is often quite complicated. It is therefore important to start the analysis with simple basic situations. These are then put together to represent more complex cases. A deeper understanding of the processes is obtained in this way.

The case when the ground freezes is not considered in this study. Rapid hourly or shorter temperature and heat extraction fluctuations at the pipe are not dealt with. The starting point of the analysis is a prescribed heat extraction rate which may vary on a time-scale from a few hours to years. The ground is assumed to consist of a homogeneous material. The case of two soil layers and the effects of ground water and infiltration are however also discussed. The temperature at the ground surface is a given function of time. The ground surface may have a constant thermal resistance. The case with variable resistance due for example to snow of changing depth is not dealt with.

The basic principle of superposition is expounded in chapter 2. By this the thermal process is separated into different basic ones. The steady-state component is discussed at length in chapter 4. Heat extraction pulses and superposition of pulses are discussed in chapter 6, while periodic variations are analysed in chapter 7. Next chapter is devoted to the effects of ground water flow and infiltration. Finally temperature variations along the pipes, pipe end effects, thermal influence on the surroundings, and the influence between pipes for different pipe configurations are discussed. The various formulas are summarized in chapter 11. The study is ended by various conclusions and a summary of results.

2. SUPERPOSITION PRINCIPLE

The complex thermal process in the ground may be considered as a superposition of more elementary ones, if the heat conduction equation and the boundary conditions are linear. This superposition technique will be used throughout this study. It will therefore be discussed here in some detail.

The superposition is not valid if there is freezing in the ground. This case is therefore excluded throughout this study. The second basic requirement is that the boundary condition at the ground surface is of a linear type as given by formula 5.1.1. More refined conditions at the ground surface such as a variable thermal surface resistance due to snow or other climatic conditions such as the wind velocity cannot be accounted for. A time-independent ground water flow or infiltration is allowed, while time variations of mass flows render the use of superposition invalid. The thermal properties of the ground are allowed to be different in different parts.

Figure 2.1 illustrates the superposition principle for a vertical cross-section in the ground with a single pipe. The temperature process may be regarded as the sum of two other ones as shown in the figure.

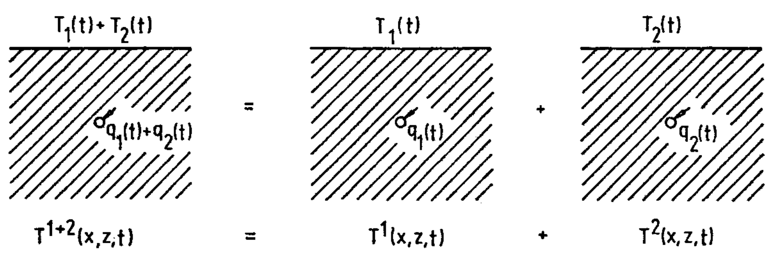


Figure 2.1. Superposition of two temperature processes.

The temperature process T^{1+2} is the sum of the two processes T^1 and T^2, which are defined by the boundary temperatures $T_1(t)$ and

$T_2(t)$ and by the heat extraction rates $q_1(t)$ and $q_2(t)$ respectively. The superposed process 1+2 has the ground surface temperature $T_1(t)+T_2(t)$ and the heat extraction rate $q_1(t)+q_2(t)$. Care must be observed concerning the initial condition at t=0. The initial temperatures are of course also to be superimposed:

$$T^{1+2}(x,z,0) = T^1(x,z,0) + T^2(x,z,0) \tag{2.1}$$

Figure 2.2 shows a case of superposition for two pipes in the ground. The ground surface temperature is $T_0(t)$ and the heat extraction rates are $q_1(t)$ and $q_2(t)$. The thermal process may be regarded as a sum of three more elementary cases.

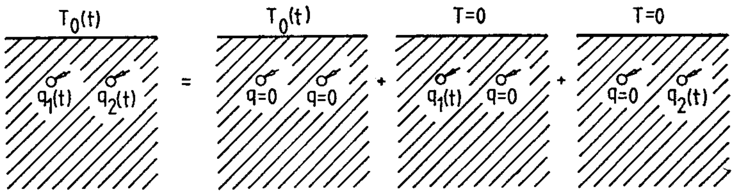


Figure 2.2. Superposition of three basic thermal processes.

The first process accounts for the boundary temperature $T_0(t)$. The other two processes are then to have zero boundary temperature. The heat extraction rates are put equal to zero at both pipes. This case therefore represents the ordinary temperature field without heat extraction pipes. The second case accounts for the heat extraction of the left pipe. The heat extraction at the other pipe is put equal to zero. The second case therefore represents the heat extraction $q_1(t)$ with a single pipe, when the ground surface temperature is zero. The third case accounts in the same way for the other pipe. The initial temperatures at t=0 are also to be superimposed. If one of the three cases has the initial temperature of the original left case in Figure 2.2, then the other two cases to the right are to have zero intial temperatures in the ground.

The thermal process due to a certain heat extraction q(t) to a pipe may by superposition be regarded as a sum of two extraction rates $q=q_1+q_2$. Figure 2.3 shows a case when a heat extraction pulse during a time $t_1 \leq t \leq t_2$ is regarded as a sum of two simpler step pulses.

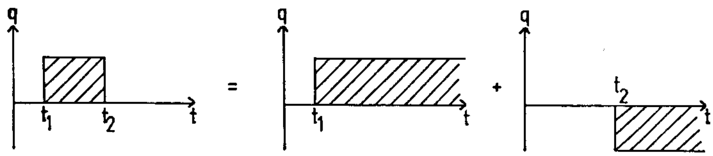


Figure 2.3. Superposition of two step pulses into a finite pulse.

A time-varying heat extraction may be regarded as a sum of finite pulses of the type that is shown to the left in Figure 2.3. These finite pulses may in turn be regarded as sums of step pulses of the type shown to the right in Figure 2.3. So a time-varying heat extraction is by superposition given by the sum of a number of elementary step pulses. The basic problem to solve is then the thermal process due to a step pulse of heat extraction of a single pipe in a ground with zero ground surface temperature. This is done in chapter 6.

Another important case is the steady-state heat extraction. Chapter 4 is devoted to this. Instead of regarding the heat extraction q(t) as a sum of finite pulses one may use a Fourier representation of q(t). Such periodic heat extraction rates are analysed in chapter 7.

The analyses of this study are systematically made with prescribed extraction rates q(t). One may instead start with prescribed fluid temperatures in the pipe and then compute the ensuing extraction rates. This is however not to be recommended, since it leads to a far more complicated analysis. This is due to the fact that the superposition now requires that the fluid temperatures at certain pipes are zero. This means in general that there are singularities at these pipes. One cannot isolate the processes of different pipes from each other any more. The present analysis with prescribed extraction rates is therefore much simpler.

3. TIME SCALES

The heat extraction and the thermal process in the ground involve quite different time scales from hourly fluctuations to the annual cycle and even variations from year to year. A clear appreciation of the time scales of the fundamental processes that are involved is of great use in the understanding of these heat extraction systems. The different time scales will lead to different types of analyses.

Let us first consider the basic situation of a constant heat extraction rate q for $t > 0$ to a single pipe at a depth D below the ground surface. The quantity q is the heat extraction rate per unit length along the buried pipe. The (outer) radius of the pipe is R. The thermal conductivity of the ground is λ (W/m^oC) and the diffusivity a (m^2/s). See Figure 3.1.

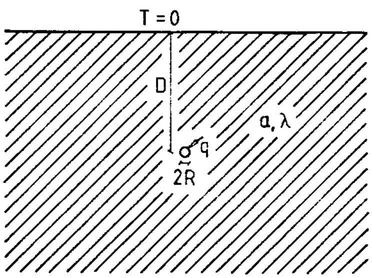


Figure 3.1. Constant heat extraction q to a single pipe from a starting time t=0.

The pure effect of the extraction q is considered. So the boundary temperature at the ground surface is T=0. The temperature of the ground at the starting time is also zero. The solution of this problem is given in chapter 6.

A characteristic time for this process is:

$$t_D = \frac{2D^2}{a} \qquad (3.1)$$

This time t_D is of fundamental importance so it will be used throughout the study.

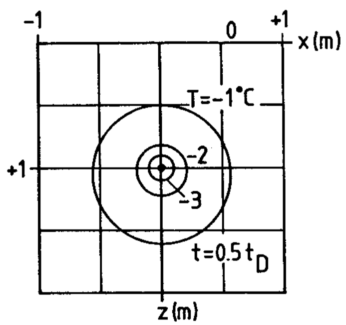
We take the following data:

$$D = 1 \text{ m} \qquad R = 0.02 \text{ m}$$
$$\lambda = 1.5 \text{ W/m}^{o}\text{C} \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \qquad (3.2)$$
$$q = 10 \text{ W/m}$$

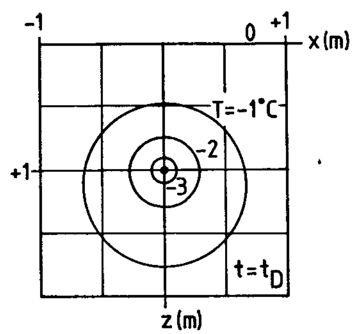
The characteristic time t_D is then:

$$t_D = \frac{2 \cdot 1^2}{0.75 \cdot 10^{-6}} \text{ s} = 30.9 \text{ days} \simeq 1 \text{ month} \qquad (3.3)$$

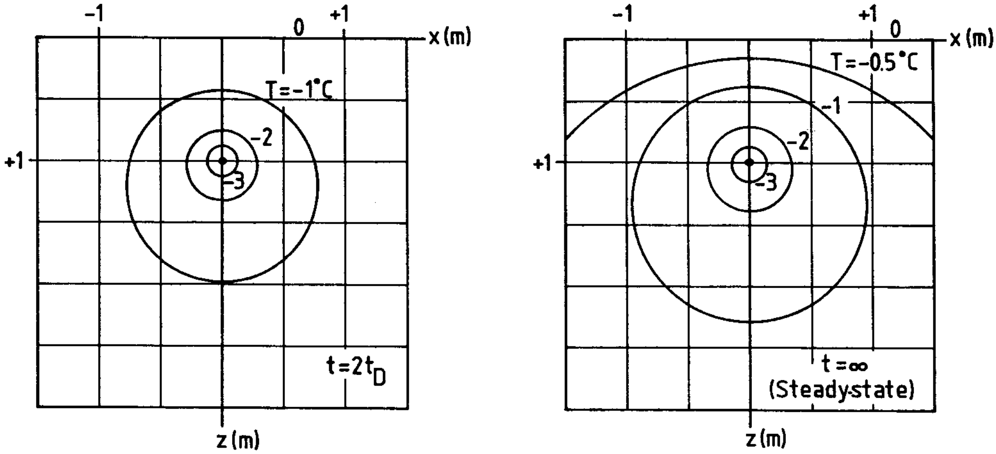
The temperature fields of this particular case are shown in Figure 3.2 for four different times $t = 0.5 \cdot t_D$, t_D, $2 \cdot t_D$, ∞. The last time gives the steady-state situation. It should be remembered that the given temperature field is the one that is due to the heat extraction. A complete picture will require the superposition of other contributions. In particular there is always a contribution from the temperature at the ground surface; cf Figure 2.2.



3.2a



3.2b



3.2c

3.2d

Figure 3.2. Temperature fields due to a constant heat extraction in the case of 3.2. t_D = 1 month.

The displacement of the isotherm T = -1 ^{0}C is shown in Figure 3.3.

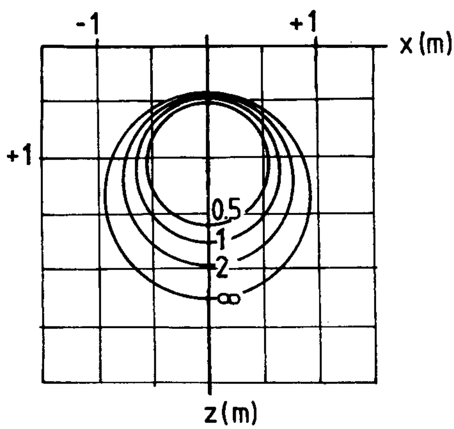


Figure 3.3. The displacement of the isotherm T = -1 ^{0}C in the case (3.2) for t = $0.5t_D$, t_D, $2t_D$ and ∞.

The temperature profiles in a vertical cut through the pipe are shown in greater detail in Figure 3.4.

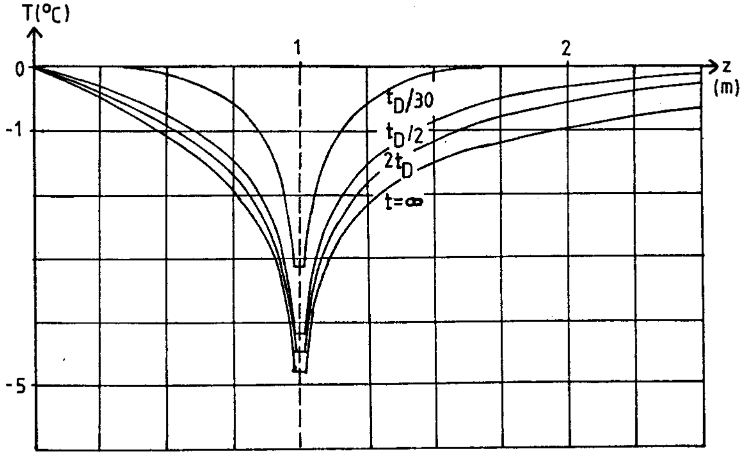


Figure 3.4. Temperature profiles along a vertical cut through the pipe. Data according to 3.2.

The temperature T_R at the pipe radius is of particular importance. One shall have this temperature at the pipe in order to obtain the prescribed constant heat extraction rate q. Figure 3.5 shows T_R.

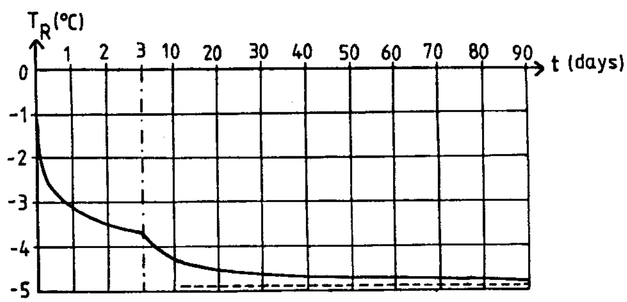


Figure 3.5. Heat extraction temperature T_R at the (outer) pipe radius as a function of time. Data according to 3.2.

Another basic thermal process is the temperature recovery after the termination of the heat extraction. The heat extraction is

$$q(t) = \begin{cases} q & t < 0 \\ 0 & t > 0 \end{cases} \qquad (3.4)$$

The data of 3.2 are used. We take a very long extraction pulse, so that the temperature becomes steady-state for $t < 0$. The pulse is terminated at t=0, when the temperature field is given by 3.2 d. The process for $t > 0$ gives the thermal recovery after the heat extraction.

Figure 3.6 a shows the temperature profiles on the vertical line through the pipe (x=0). The displacement of the isotherm $T = -0.5\ ^{0}C$ is shown in Figure 3.6 b. The pipe temperature $T_R(t)$ is given in Figure 3.7.

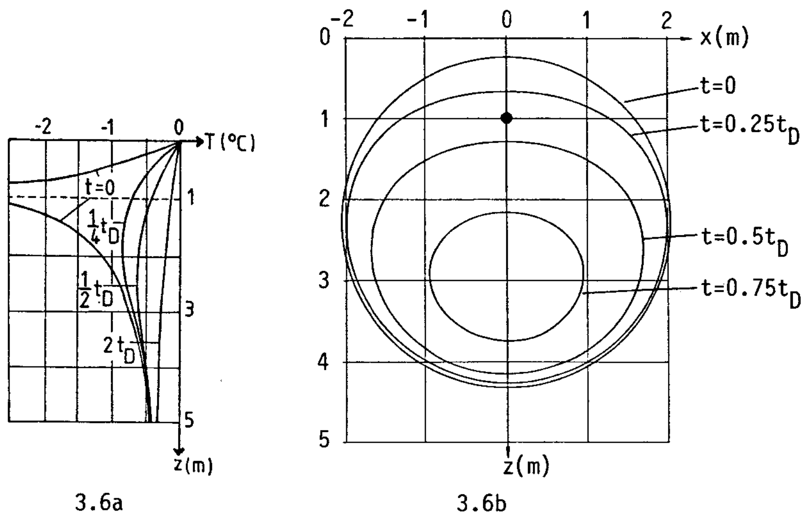


Figure 3.6. Thermal recovery after a pulse 3.4. Data from 3.2.
a: Temperatures along x=0. b: Evolution of isotherm $T = -0.5\ ^{0}C$.

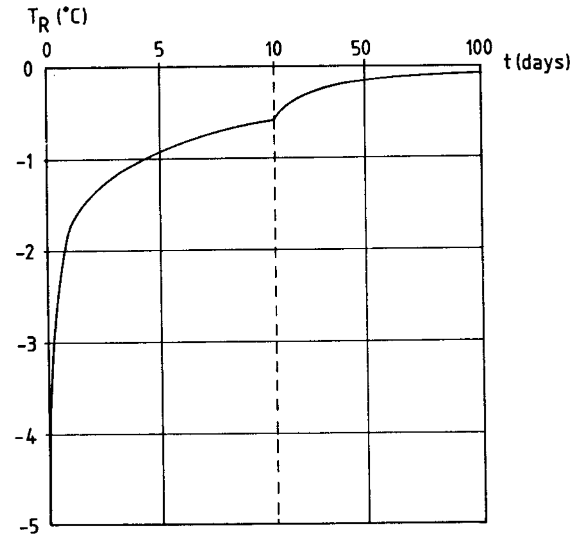


Figure 3.7. Pipe temperature T_R during thermal recovery after heat extraction. Data according to 3.2 and 3.3.

The recovery at the pipe is as we see from figure 3.7 very rapid in the beginning. We have after an infinite pulse:

$$T_R(t_D/30) \cong 0.36\ T_R(0)$$

$$T_R(t_D/3) \cong 0.1\ T_R(0) \qquad (3.5)$$

$$T_R(t_D) \cong 0.05\ T_R(0)$$

4. STEADY-STATE HEAT EXTRACTION

This chapter is devoted to a rather extensive study of the steady-state heat extraction from one or several pipes. The steady-state case may seem to be quite far from real, dynamical heat extraction situations. But the time scale t_D of obtaining more or less steady-state conditions is often smaller than the extraction period. The steady-state heat extraction is then a base load contribution to the total thermal process. The applicability and importance of the steady-state contribution is quite wide.

4.1 One pipe in the ground

The considered case of steady-state heat extraction by a single pipe in the ground is shown in Figure 4.1.

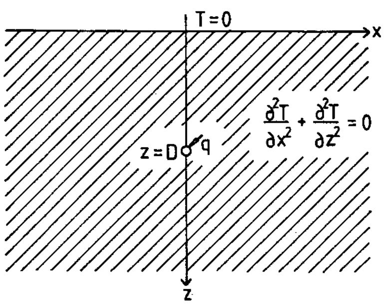


Figure 4.1. Steady-state heat extraction by a single pipe in the ground.

The pipe has its center at (x,z) = (0,D). The rate of heat extraction from the ground to the pipe is q per unit length of the pipe (W/m). A negative value of q means that heat is flowing from the pipe to the ground.

The ground is assumed to be homogeneous with a thermal conductivity λ (W/mK). The steady-state temperature T(x,z) shall satisfy the Laplace equation. See Figure 4.1. The temperature at the ground-surface z=0 is zero. The superimposed effect of the real temperature variation at the ground surface is discussed in chapter 5. The present solution T(x,z) represents the additional temperature field due to the heat sink q. The thermal process is assumed to be two-dimensional in the (x,z)-plane perpendicular to the pipe. Three-dimensional effects will be discussed in chapter 9. Modifications due to ground water flow and water infiltration will be discussed in chapter 8.

The temperature due to a single line sink in an infinite homogeneous medium with a thermal conductivity λ is:

$$\frac{q}{2\pi\lambda} \ln(r) \qquad (4.1.1)$$

Here r is the distance to the line sink.

In the present case of Figure 4.1 the medium is semi-infinite. The temperature is to be zero at the boundary z=0. This will be satisfied if we imagine that a mirror line sink with the opposite strength -q is placed in (0,-D).

The temperature T consists of two terms of the type 4.1.1. We get the basic solution:

$$T(x,z) = \frac{q}{2\pi\lambda} \ln\left(\frac{\sqrt{x^2+(z-D)^2}}{\sqrt{x^2+(z+D)^2}}\right) \qquad (4.1.2)$$

The nominator of the argument of the logarithm is the distance from (x,z) to the line sink at (0,D), while the denominator is the distance to the mirror sink at (0,-D).

Let R be the outer radius of the pipe. The periphery of the pipe at the soil is given by

$$x^2 + (z-D)^2 = R^2 \qquad (4.1.3)$$

The distance from a point on the pipe periphery to the line sink +q at (0,D) is of course R, while the distance to the mirror line sink varies from 2D-R to 2D+R. But the radius R will always be much smaller than the depth D:

$$R \ll D \qquad (4.1.4)$$

The assumption 4.1.4 will be used throughout this study. The minute variation of the denominator in the logarithm of 4.1.2 is then negligible. We have the approximation:

$$\sqrt{x^2 + (z+D)^2} \simeq 2D \quad \text{for} \quad x^2 + (z-D)^2 = R^2 \qquad (4.1.5)$$

This type of approximation will be used throughout this study.

Let T_R denote the temperature at the pipe periphery 4.1.3. Then we have from 4.1.2, 3 and 5:

$$T_R = \frac{q}{2\pi\lambda} \cdot \ln\left(\frac{R}{2D}\right) \qquad (4.1.6)$$

The logarithm and the extraction temperature T_R are of course negative, since heat is extracted from the ground to the pipe.

Formula 4.1.6 may be written in the following way:

$$-T_R = q \cdot \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) \qquad (4.1.7)$$

The driving temperature difference between the ground surface and the pipe periphery is $0-T_R$. The ensuing heat flux is q. The second factor of 4.1.7 is therefore a <u>thermal resistance</u> between the pipe and the ground surface. We may write

$$\boxed{0 - T_R = m \cdot q} \qquad (4.1.8)$$

Here m is the thermal resistance of the ground per unit length of the pipe:

$$\boxed{m = \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right)} \qquad (4.1.9)$$

The dimension of m is K/(W/m). The thermal resistance m represents the necessary driving temperature for unit heat extraction rate.

The quantity $2\pi\lambda m$, where m is the thermal resistance per unit pipe length, is dimensionless. We will call it the thermal resistance factor. In particular we have from 4.1.9. for a single pipe in the ground:

$$2\pi\lambda m = \ln\left(\frac{2D}{R}\right) \tag{4.1.10}$$

It is given in Table 4.1. The relatively slow variation with the quotient R/D is note-worthy. A twenty-fold decrease of R/D from 0.1 to 0.005 will only double the thermal resistance.

R/D	0.001	0.005	0.01	0.02	0.05	0.07	0.1	0.2
$2\pi\lambda m$	7.60	5.99	5.30	4.61	3.69	3.35	3.00	2.30

Table 4.1. Thermal resistance factor for a single pipe.

The inverse of the thermal resistance is a heat transfer coefficient:

$$\boxed{q = \frac{1}{m}\cdot(0-T_R)} \tag{4.1.11}$$

The quantity 1/m (W/mK) gives the steady-state heat flux per unit pipe length for a unit driving temperature difference.

The thermal conductivity of soils varies between, say, 0.8 and 2 W/mK, while the heat capacity C normally is about $2\cdot10^6$ J/m^{3}K. As a reference case we will take the following data for the soil:

$$\lambda = 1.5 \text{ W/mK} \qquad a = \frac{\lambda}{C} = 0.75\cdot10^{-6} \text{ m}^2\text{/s} \tag{4.1.12}$$

$$(C = 2\cdot10^6 \text{ J/m}^3\text{K})$$

For the pipe in the ground we take the following reference case:

$$D = 1\ m \qquad R = 0.02\ m \qquad q = 10\ W/m \tag{4.1.13}$$

For this reference case 4.1.12-13 we have the characteristic time t_D (3.1):

$$t_D = \frac{2D^2}{a} \simeq 1\ \text{month} \tag{4.1.14}$$

The thermal resistance factor is:

$$\ln\left(\frac{2 \cdot 1}{0.02}\right) = 4.61 \tag{4.1.15}$$

The thermal resistance is from 4.1.10:

$$m = \frac{1}{2\pi \cdot 1.5} 4.61 = 0.49\ Km/W \tag{4.1.16}$$

The pipe temperature is from 4.1.8:

$$T_R = -0.49 \cdot 10 \simeq -5\ ^oC \tag{4.1.17}$$

So in order to obtain a steady-state heat flux q=10 W/m a temperature of -5 ^oC is to be maintained at the pipe perphery. It must be remembered that we are talking about the temperature contribution due to the heat extraction. If the natural temperature at the pipe depth is, say, +7 ^oC, then the real pipe temperature is +7 - 5 ^oC = +2 ^oC.

The thermal resistance between the fluid in the pipe and the ground at the outer periphery of the pipe will require that a still lower temperature is maintained in the fluid. This is discussed in section 4.3.

The complete steady-state temperature field of the reference case 4.1.12-13 is shown in Figure 4.2.

4.2 Thermal influence region

The thermal influence region around the heat extraction pipe is of interest in order to assess the effect on other heat extraction pipes. The temperature change from natural conditions is also of interest from an environmental point of view. It should be observed that the steady-state represents essentially the greatest thermal impact on the surrounding ground except for the immediate vicinity of the pipe, where dynamical effects are dominating. Cf Figure 3.2 a-d.

The temperature field of reference case 4.1.12-13 is shown in Figures 3.2 d and 4.2-4.

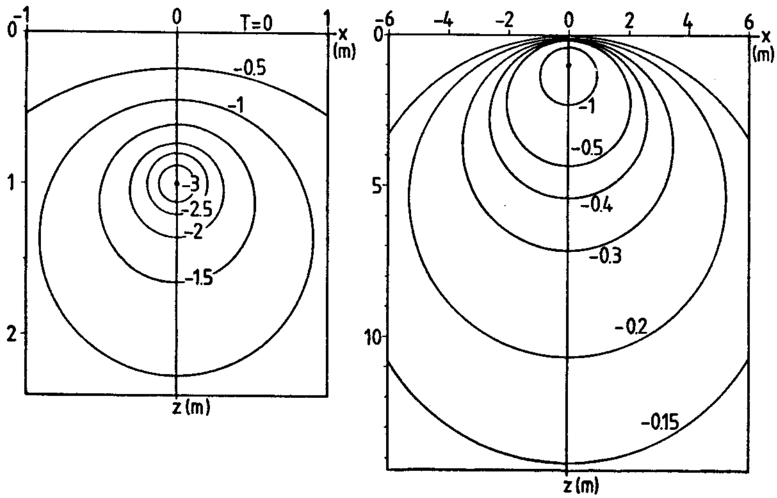


Figure 4.2. Stead-state temperature field around a single heat extraction pipe. Reference case 4.1.12-13.

The isotherms of the temperature field 4.1.2 are circles. The center and the radius of a certain isotherm T are given by

$$(0,\ D\cdot\coth(\frac{2\pi\lambda|T|}{q})) \quad \text{and} \quad \frac{D}{\sinh(\frac{2\pi\lambda|T|}{q})} \qquad (4.2.1)$$

respectively.

The temperature profile in the vertical cut x=0, z>0 is shown in Figure 4.3. The steep temperature gradient near the pipe is noteworthy. The temperature increases from $-5^{o}C$ to $-2.5^{o}C$ within the first 20 centimeters.

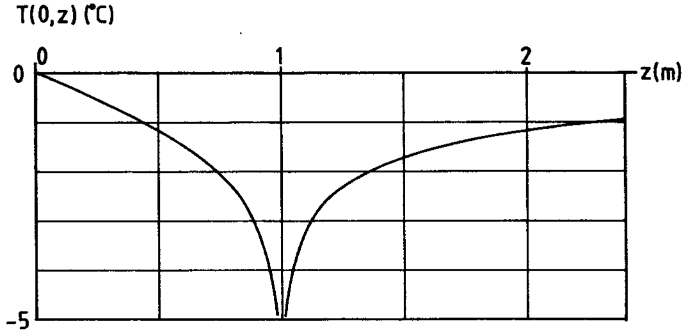


Figure 4.3. Temperature profile in the cut x=0 for reference case 4.1.12-13.

The temperature profile at the pipe depth z=1 m is shown in Figure 4.4.

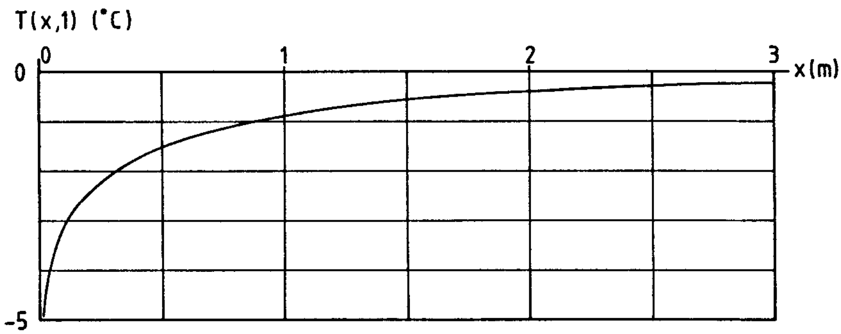


Figure 4.4. Temperature profile at the pipe depth z=1 m for reference case 4.1.12-13.

The temperatures at the depth z=D are of particular interest, since they give the influence on other pipes at the same depth. We have from 4.1.2:

$$T(x,D) = \frac{q}{2\pi\lambda}\cdot\ln\left(\frac{x}{\sqrt{x^2+4D^2}}\right) = \frac{q}{2\pi\lambda}\cdot\alpha \qquad (4.2.2)$$

The second factor α gives the relative temperature at a distance x from the pipe. It is given in table 4.2.

x/D	0.01	0.02	0.05	0.1	0.5	1	1.5	2	3	4	5
α	-5.3	-4.6	-3.7	-3.0	-1.4	-0.8	-0.5	-0.3	-0.2	-0.11	-0.07

x/D	7	10	20	50
α	-0.04	-0.02	-0.005	-0.001

Table 4.2. Relative temperature at the depth z=D for a single pipe according to 4.2.2.

The following simpler expression for the temperature 4.1.2 may be used at a certain distance from the pipe:

$$T(x,z) \simeq -\frac{q}{\pi\lambda}\cdot\frac{zD}{x^2+z^2} \qquad (\sqrt{x^2+z^2} > 3D) \qquad (4.2.3)$$

The error in the formula is only a few percent. For the reference case 4.1.12-13 we get from 4.2.3 for example:

$$\begin{aligned} &x = 0,\ z = 5\ \text{m} && T = -0.42\ ^{\circ}\text{C} \\ &x = 5\ \text{m},\ z = 1\ \text{m} && T = -0.08\ ^{\circ}\text{C} \qquad (4.2.4) \\ &x = 5\ \text{m},\ z = 5\ \text{m} && T = -0.21\ ^{\circ}\text{C} \end{aligned}$$

These temperatures shall be compared to the pipe temperature $T_R = -5\ ^{\circ}C$ (4.1.17).

4.3 Fluid-solid thermal resistance at the pipe

The pipe temperature T_R that we have discussed so far is the temperature in the soil at the outer periphery of the pipe. There is always a certain thermal resistance between the fluid in the pipe and this outer periphery in the soil. The fluid temperature T_f must therefore be lower than T_R in order to sustain the heat flux q over this thermal resistance.

Let m_p (Km/W) denote the total thermal resistance at the pipe, per unit pipe length, between the fluid and the surrounding soil. Then we have the relation:

$$\boxed{T_R - T_f = q \cdot m_p} \tag{4.3.1}$$

Adding formulas 4.1.8 and 4.3.1 we have:

$$\boxed{0 - T_f = q \cdot (m + m_p)} \tag{4.3.2}$$

The total thermal resistance between the fluid in the pipe and the ground surface is given by the sum $m+m_p$.

Let us first consider the steady-state heat flux over an annulus with an inner radius R_1 and an outer one R_2. Let as usual q be the heat flux per unit length and λ the thermal conductivity of the annulus material. The temperature difference is $T_2 - T_1$. See Figure 4.5.

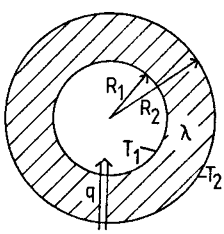


Figure 4.5. Steady-state heat flux through an annulus.

For this case we have the well-known formula:

$$T_2 - T_1 = q \cdot \frac{\ln(\frac{R_2}{R_1})}{2\pi\lambda} \tag{4.3.3}$$

The second factor defines the thermal resistance of the annulus:

$$m_{12} = \frac{1}{2\pi\lambda} \ln(\frac{R_2}{R_1}) \tag{4.3.4}$$

It should be remembered that the thermal resistance refers to a unit length of the annulus or of the pipe.

There are three contributions to the total pipe resistance m_p. The first part m_{pf} is the fluid and boundary layer resistance between the bulk of the fluid and the inner pipe wall. The second part m_{pw} is the thermal resistance of the annulus of the pipe wall itself. The third part m_{ps} is the contact resistance between the outer side of the pipe wall and the bulk soil at the radius R. We have:

$$m_p = m_{pf} + m_{pw} + m_{ps} \tag{4.3.5}$$

The pipe-wall resistance is given by an expression of type 4.3.4. Let λ_p be the thermal conductivity of the pipe wall material and R_- be the inner radius of the pipe. Then we have from 4.3.4:

$$m_{pw} = \frac{1}{2\pi\lambda_p} \ln(\frac{R}{R_-}) \tag{4.3.6}$$

The first contribution m_{pf} may be obtained from standard works on heat transfer. See for example [1]. The thermal resistance will depend on the fluid velocity. It turns out to be quite small in the present applications with turbulent flow in the pipe. We will neglect this term:

$$m_{pf} \simeq 0 \tag{4.3.7}$$

The third contribution, the contact resistance between the pipe and the soil, must be measured.

Let us consider some numerical examples. The soil resistance, i.e. the resistance between the ground surface and the outer periphery of the pipe, of reference case 4.1.12-13 was (4.1.16):

$$m = 0.49 \text{ Km/W} \tag{4.3.8}$$

From [1] we get the fluid-to-pipe-wall resistance for two fluid velocities:

$$v_f = 0.1 \text{ m/s} \qquad m_{pf} \simeq 0.019 \text{ Km/W} \tag{4.3.9A}$$

$$v_f = 1 \text{ m/s} \qquad m_{pf} \simeq 0.003 \text{ Km/W} \tag{4.3.9B}$$

Let us assume that the thickness of the wall is 3 mm:

$$R = 0.020 \text{ m} \qquad R_{-} = 0.017 \text{ m}$$

The pipe-wall resistance depends on the thermal conductivity of the wall. We may have for example:

$$\text{Polyethene: } \lambda_p = 0.40 \text{ W/mK} \qquad m_{pw} = 0.06 \text{ Km/W} \tag{4.3.10A}$$

$$\text{PVC} \quad : \lambda_p = 0.17 \text{ W/mK} \qquad m_{pw} = 0.15 \text{ Km/W} \tag{4.3.10B}$$

The values of 4.3.9 and 4.3.10 shall be compared to the soil resistance 4.3.8. We see that the contribution m_{pf} is indeed negligible in accordance with 4.3.7. The pipe wall resistance may be quite important as the values of 4.3.10 show. It is clearly important to avoid pipe materials with low thermal conductivity.

The contact resistance m_{ps} between the pipe and the ground must be carefully considered. Let us as an illustration of the dangers assume that the contact resistance correspond to a gap of air of 1 mm around the pipe. The thermal conductivity of air is 0.024 W/mK. The thermal resistance of the air gap is then (4.3.3):

$$m_{1\text{mm of air}} = \frac{1}{2\pi\cdot 0.024}\cdot \ln\left(\frac{0.021}{0.020}\right) = 0.32 \text{ (Km/W)} \tag{4.3.11}$$

This would give a thermal resistance which is 65% of that of the soil (4.3.8).

The fluid temperature of our reference example is now from 4.3.2, 5, 7, 8, 9A and 10A:

$$0 - T_f = 10 \cdot (0.49 + 0.019 + 0.06 + 0) = 5.7^{o}C$$

The fluid shall thus in this particular example be kept $5.7^{o}C$ below the undisturbed soil temperature at the pipe in order to obtain the required steady-state heat flux q=10 W/m.

The soil region near the pipe may have another thermal conductivity than the rest of the soil due to changes in moisture content.

Let us assume that a cylindrical region $R < r < R_1$ around the pipe has the thermal conductivity λ_1 instead of λ. The thermal resistance of this region is given by an expression of type 4.3.3. If R_1 is much less than D, then 4.1.9 can be used for the remaining soil resistance:

$$m = \frac{1}{2\pi\lambda_1} \ln\left(\frac{R_1}{R}\right) + \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R_1}\right) \qquad (4.3.12)$$

More generally, λ_1 can be a function of the distance from the pipe: $\lambda_1 = \lambda_1(r)$, $R \leq r \leq R_1$. Then the thermal resistance is:

$$m = \frac{1}{2\pi} \int_R^{R_1} \frac{dr}{\lambda_1(r) r} + \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R_1}\right) \qquad (4.3.13)$$

4.4 Ground surface thermal resistance

The boundary condition at the ground surface has until now been that the boundary temperature is given. The heat extraction part of the thermal process has then as boundary condition zero temperature (T(x,0) = 0); cf. Figure 2.2. The contribution from the boundary temperature is discussed in chapter 5.

A more realistic boundary condition is to have a contact resistance at the ground surface and a given temperature above the contact layer. See 5.1.1.

Let α_s (W/m^2K) be the heat transfer coefficient at the ground surface. We assume that α_s is a given constant. The boundary condition for the steady-state heat extraction part of the heat transfer process is then:

$$-\lambda \frac{\partial T}{\partial z} = \alpha_s(0-T) \qquad z=0 \qquad (4.4.1)$$

The problem of Figure 3.1. is apart from this unchanged.

The solution to this new problem, when there is a thermal resistance at the ground surface, is derived in appendix 1. The important quantity is the temperature T_R at the pipe radius. We have from A1.13:

$$-T_R = \frac{q}{2\pi\lambda} \{\ln\left(\frac{2D}{R}\right) + 2\, e^{\frac{2D\alpha_s}{\lambda}} \cdot E_1\left(\frac{2D\alpha_s}{\lambda}\right)\} \qquad (4.4.2)$$

The first part on the right-hand side gives the previous case with zero resistance at the ground surface ($\alpha_s = \infty$). See 4.1.9.

The thermal resistance factor is now with the notation of 4.1.8.

$$2\pi\lambda m = \ln\left(\frac{2D}{R}\right) + g_s(D\alpha_s/\lambda) \qquad (4.4.3)$$

The function g_s gives the increase of the thermal resistance factor due to the thermal resistance at the ground surface:

$$g_s = 2\, e^{\frac{2D\alpha_s}{\lambda}} \cdot E_1\left(\frac{2D\alpha_s}{\lambda}\right) \qquad (4.4.4)$$

Here $E_1(s)$ is the so-called exponential integral. It is given in a table in [2A]. The function $g_s(s)$ is given in table 4.3.

$\frac{D\alpha_s}{\lambda}$	0.5	1	2	3	4	5	10
g_s	1.19	0.72	0.41	0.29	0.22	0.18	0.10

Table 4.3. Contribution from a ground surface resistance to the thermal resistance factor according to 4.4.2-4.

Let us consider the reference case 4.1.12-13. The thermal resistance factor was (4.1.15):

$$2\pi\lambda m = 4.61 \tag{4.4.5}$$

We consider two cases:

$$\begin{aligned} \frac{D\alpha_s}{\lambda} &= 5 \quad : \quad g_s = 0.18 \qquad \frac{0.18}{4.61} = 0.04 \\ \frac{D\alpha_s}{\lambda} &= 1 \quad : \quad g_s = 0.75 \qquad \frac{0.75}{4.61} = 0.16 \end{aligned} \tag{4.4.6}$$

The case $\frac{D\alpha_s}{\lambda} = 1$ means that the thermal resistance $1/\alpha_s$ of the ground surface is equal to that of a soil layer with the thickness D. The second case therefore corresponds to a very high surface resistance. The first case $1/\alpha_s = D/5\lambda$ is more realistic since the resistance corresponds to a soil layer of D/5 = 0.2 m.

The second extreme case gives an increase of thermal resistance due to the ground surface of 16%, while the first more normal case only gives 4% increase. The conclusion of this is that the effect of the ground surface resistance is quite small in the present applications.

It may at first sight be surprising that the effect is so small. The reason is that the major part of the temperature fall from the pipe to the ground surface occurs close to the pipe. Thermal resistances close to the pipe are therefore quite important, while a change further away at the ground surface is of minor importance.

The thermal resistance at the ground surface will be neglected in the following. This means that we have the simple boundary condition T=0 at z=0 for the heat extraction part of our problem.

4.5 Two pipes

Figure 4.6 shows the present case of steady-state heat extraction for two parallel pipes in the ground.

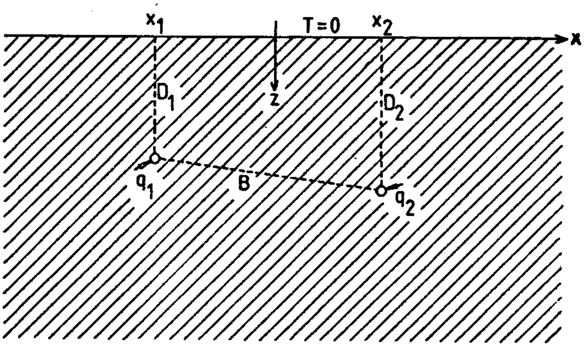


Figure 4.6. Steady-state heat extraction by two pipes in the ground.

The pipes lie at the points (x_1, D_1) and (x_2, D_2). The distance between the pipes is B. The heat extraction rates are q_1 and q_2 respectively.

The steady-state temperature in the ground is obtained from the one-pipe solution 4.1.2 by superposition. We have:

$$\begin{aligned} T(x,z) &= \frac{q_1}{2\pi\lambda}\cdot \ln\left(\frac{\sqrt{(x-x_1)^2+(z-D_1)^2}}{\sqrt{(x-x_1)^2+(z+D_1)^2}}\right) \\ &+ \frac{q_2}{2\pi\lambda}\cdot \ln\left(\frac{\sqrt{(x-x_2)^2+(z-D_2)^2}}{\sqrt{(x-x_2)^2+(z+D_2)^2}}\right) \end{aligned} \tag{4.5.1}$$

The thermal influence region is obtained directly by superposition of the single-pipe influence as discussed in section 4.2.

Let us assume that the two pipes have the same outer radius R. The pipe periphery temperatures become with the use of approximations of type 4.1.5, where the temperature variation around the pipe periphery due to another pipe is neglected:

$$\begin{aligned} T_{R1} &= \frac{q_1}{2\pi\lambda}\ln\left(\frac{R}{2D_1}\right) + \frac{q_2}{2\pi\lambda}\ln\left(\frac{B}{B_-}\right) \\ T_{R2} &= \frac{q_2}{2\pi\lambda}\ln\left(\frac{R}{2D_2}\right) + \frac{q_1}{2\pi\lambda}\ln\left(\frac{B}{B_-}\right) \end{aligned} \tag{4.5.2}$$

Here B_- is the distance between one of the pipes and the mirror one of the other:

$$B_- = \sqrt{(x_1-x_2)^2 + (D_1+D_2)^2} = \sqrt{B^2 + 4D_1D_2} \quad (4.5.3)$$

Let us introduce the following notations:

$$m_1 = \frac{1}{2\pi\lambda} \ln \left(\frac{2D_1}{R}\right) \qquad m_2 = \frac{1}{2\pi\lambda} \ln \left(\frac{2D_2}{R}\right)$$
$$m_{12} = \frac{1}{2\pi\lambda} \ln \left(\frac{\sqrt{B^2+4D_1D_2}}{B}\right) \quad (4.5.4)$$

The quantities m_1 and m_2 are single-pipe soil resistances according to 4.1.9. The quantity m_{12} represents the interaction between the pipes. It tends to zero, when B tends to infinity. The resistances 4.5.4 all refer to a unit length of the pipes.

The equations 4.5.2 may now be written:

$$-T_{R1} = q_1 \cdot m_1 + q_2 \cdot m_{12}$$
$$-T_{R2} = q_2 \cdot m_2 + q_1 \cdot m_{12} \quad (4.5.5)$$

The fluid temperatures in the pipes are denoted T_{f1} and T_{f2}, while the pipe resistances are m_{p1} and m_{p2}. Then we have by definition:

$$T_{R1} - T_{f1} = q_1 \cdot m_{p1}$$
$$T_{R2} - T_{f2} = q_2 \cdot m_{p2} \quad (4.5.6)$$

Adding 4.5.5 and 4.5.6 gives the relation between fluid temperatures and extraction rates:

$$-T_{f1} = q_1 \cdot m_{t1} + q_2 \cdot m_{12}$$
$$-T_{f2} = q_2 \cdot m_{t2} + q_1 \cdot m_{12} \quad (4.5.7)$$

Here we have introduced the notation:

$$m_{t1} = m_1 + m_{p1} \qquad m_{t2} = m_2 + m_{p2} \quad (4.5.8)$$

The resistance m_{ti} is the total thermal resistance for a single pipe between the fluid in the pipe and the ground surface.

The fluid temperature variation along a pipe is rather small for normal fluid velocitites. The temperature difference between two pipes is also normally rather small. An important special case is therefore, when the two fluid temperatures are equal:

$$T_{f1} = T_{f2} = T_f \tag{4.5.9}$$

The equations 4.5.7 and 9 define a relation between T_f and q_1 and between T_f and q_2. We get for pipe 1:

$$-T_f = q_1 \frac{m_{t1} \cdot m_{t2} - m_{12}^2}{m_{t2} - m_{12}} = q_1 \cdot (m_{t1} + m_{12} \frac{m_{t1} - m_{12}}{m_{t2} - m_{12}}) \tag{4.5.10}$$

The expression for pipe 2 is of course analogous. We note that the ratio between the heat extraction rates becomes:

$$\frac{q_1}{q_2} = \frac{m_{t2} - m_{12}}{m_{t1} - m_{12}} \tag{4.5.10'}$$

The extraction rates are equal, when the resistances m_{t1} and m_{t2} are equal.

The total heat extraction rate from the two pipes per unit length is q_1+q_2. We get from 4.5.10 and 10':

$$-T_f = (q_1+q_2) \cdot m_{1+2} \text{ ,} \tag{4.5.11}$$

where

$$m_{1+2} = \frac{m_{t1} \cdot m_{t2} - m_{12}^2}{m_{t1} + m_{t2} - 2m_{12}} \tag{4.5.12}$$

The quantity m_{1+2} is the total thermal resistance between the fluid in the two pipes and the ground surface.

An important case is when the two pipes lie at the same depth as shown in Figure 4.7.

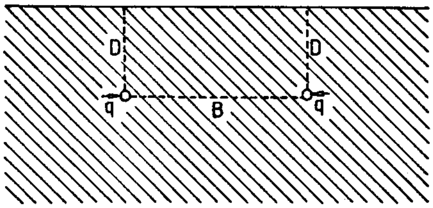


Figure 4.7. Two pipes at the same depth.

We assume that the process is symmetrical with respect to the two pipes:

$$T_{f1} = T_{f2} = T_f \qquad m_{p1} = m_{p2} = m_p \tag{4.5.13}$$

Then we have:

$$m_{t1} = m_{t2} = m_p + \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right)$$

$$q_1 = q_2 = q \tag{4.5.14}$$

The relation between the heat extraction rate q and the fluid temperature T_f for two pipes at the depth D, is from 4.5.10, 14 and 4:

$$-T_f = q\cdot m_p + q\cdot \frac{1}{2\pi\lambda}\left(\ln\left(\frac{2D}{R}\right)+g'\right) \tag{4.5.15}$$

The factor g' is given by

$$g' = \ln\left(\frac{\sqrt{B^2+4D^2}}{B}\right) \tag{4.5.16}$$

It represents the influence of the other pipe. It is given in Table 4.4. These values are to be added to the value of ln(2D/R) as given by Table 4.1.

B/D	0.1	0.25	0.5	0.75	1	1.5	2	3	5	10
g'	3.00	2.09	1.42	1.05	0.80	0.51	0.35	0.18	0.07	0.02

Table 4.4. Contribution g' of a second pipe at the same depth to the thermal resistance factor according to 4.5.15-16.

Let us consider the following example with data from the reference case 4.1.13.

$$R/D = 0.02 : \ln(\frac{2D}{R}) = 4.61$$

B/D	0.25	0.5	1	2
g'	2.09	1.42	0.80	0.35
g'/4.61	0.45	0.31	0.17	0.08

(4.5.17)

The second pipe increases the thermal resistance factor from 4.61 to 4.61+g'. For example, a second pipe at a distance of B=D meter increases the resistance factor with 17%. We see that the second pipe must lie quite close in order to have a significant influence.

Another way to represent the effect of the influence between the pipes is to compare the two pipes with two independent pipes. The total heat extraction is compared for the same fluid temperature T_f. The heat extraction q_1+q_2 from the two pipes is given by 4.5.11 and 12. The heat extraction 2q from two independent single pipes is given by 4.3.2. The ratio η is thus:

$$\eta = \frac{q_1+q_2}{2q} = \frac{-T_f}{m_{1+2}} \cdot \frac{m+m_p}{2(-T_f)} = \frac{m+m_p}{2m_{1+2}} \qquad (4.5.18)$$

For two pipes at the same depth have from 4.5.12-16:

$$\eta = \frac{2\pi\lambda m_p+\ln(2D/R)}{2\pi\lambda m_p+\ln(2D/R)+g'} \qquad (4.5.19)$$

As an example we take:

$$\frac{R}{D} = 0.02 \qquad m_p = 0 \qquad (4.5.20)$$

The extraction ratio η is then a function of B/D only. It is given in Table 4.5.

B/D	0.05	0.1	0.2	0.4	0.6	0.8	1.0	1.2	1.4
η	0.56	0.61	0.67	0.74	0.79	0.82	0.85	0.87	0.89

B/D	1.6	1.8	2.0	2.25	2.5	3	4	5	7	10
η	0.91	0.92	0.93	0.94	0.95	0.96	0.98	0.98	0.992	0.996

Table 4.5. Heat extraction with two pipes according to Figure 4.7 relative to that of two independent pipes. Data according to 4.5.20.

The function η of Table 4.5 is shown in Figure 4.9 together with the case of two pipes in the same ditch.

It is also shown in Figure 4.14, where η is given for different number of pipes.

Another interesting case is when the two pipes are buried in the same ditch at the depth D and D+B respectively. See Figure 4.8.

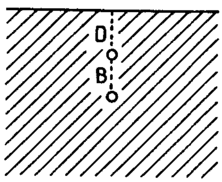


Figure 4.8. Two pipes which have been buried in the same ditch.

The thermal resistances are from 4.5.4 and 8:

$$
\begin{aligned}
m_{t1} &= m_{p1} + \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) \\
m_{t2} &= m_{p2} + \frac{1}{2\pi\lambda} \ln\left(\frac{2B+2D}{R}\right) \qquad (4.5.21) \\
m_{12} &= \frac{1}{2\pi\lambda} \ln\left(\frac{B+2D}{B}\right)
\end{aligned}
$$

The total heat extraction from these two pipes are given by 4.5.11, 12 and 21.

This case offers an illustrative optimization problem. The total flux q_1+q_2 of 4.5.11 is to be as high as possible for a fixed T_f. The thermal resistance m_{1+2} of 4.5.12 is to be minimized. Let us first consider the case, when the ditch depth B+D is fixed, while D is varying. The extraction rate increases with decreasing D, so the upper pipe is just to be placed as close to the ground surface as possible.

The interesting optimization occurs on the other hand, when the position of the upper pipe is fixed, while the lower one is varying. It is clear that the lower pipe should not be to close to the upper one or to deep in the ground. There must exist an intermediate optimum. The resistance m_{1+2} is to be minimized for varying B, while D is kept fixed.

We will study this optimum for the following particular data:

$$R = 0.02 \text{ m} \qquad D = 1 \text{ m}$$
$$\lambda = 1.5 \text{ J/ms}^{\circ}\text{C} \qquad m_{p1} = m_{p2} = 0 \qquad (4.5.22)$$

Then we get from 4.5.12:

$$m_{t1} = 0.49 \text{ msK/J}$$

B(m)	0.1	0.25	0.5	0.75	1	2	3	4	5	6	10	50
m_{1+2}	0.41	0.37	0.34	0.33	0.32	0.31	0.30	0.30	0.30	0.30	0.31	0.32
$\frac{m_{t1}}{m_{1+2}}$	1.20	1.33	1.44	1.50	1.53	1.59	1.61	1.62	1.62	1.61	1.61	1.53

Table 4.6. Thermal resistance m_{1+2} of example 4.5.22. The third line gives the increase of heat extraction when the second, lower pipe is introduced.

The quotient m_{t1}/m_{1+2} gives the increase of heat extraction for a fixed fluid temperature T_f, when the second, lower pipe is introduced. The thermal resistance m_{1+2} has in the present case a minimum for

B = 4.8 m. The two pipes will in this optimal case deliver 62% more heat than the single upper pipe alone. It should be remembered that in the present discussion we fix the temperature and compare extraction rates.

The table shows for example that a pipe at the depth 1 m and the second at the depth 2 m will deliver 53% more heat than the upper pipe alone.

There are some important conclusions to be drawn from Table 4.6. The increase of heat extraction capacity increases rapidly in the beginning when B is small. There is a gain of 44% for B = 0.5 m and of 53% for B = 1 m, when we compare with a single upper pipe. The increase of the gain is then rather small up to the maximum 62% for B = 4.8 m. The maximum is extremely flat. The variation of the extraction rate is below 3% when the lower pipe lies between 2 and 20 meters of depth. It is also note-worthy that this theoretical optimum lies so deep as 5 m.

Let us also compare the two pipes in one ditch with two independent pipes at the depth D. The data 4.5.20, which contains the case 4.5.22, are again used. The heat extraction ratio η is given by 4.5.18. The values of the lowest line of Table 4.6 are to be halved since we are comparing with two pipes. The result is shown in Figure 4.9, which also shows the previous case with two pipes at the same depth.

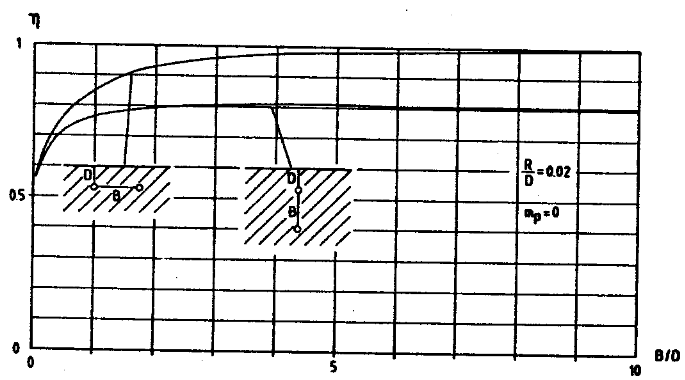


Figure 4.9. Heat extraction by two pipes compared to that of two independent pipes.

4.6 Three pipes

We shall here only consider the case when the three pipes lie at the same depth. We also assume that the two distances between the pipes are equal. See Figure 4.10.

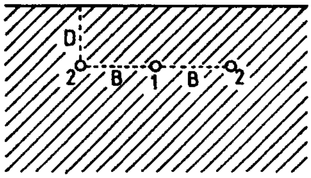


Figure 4.10. Considered case of steady-state heat extraction by three pipes.

The heat extraction rate of the central pipe is q_1. The outer pipes are by assumption thermally equal with the extraction rate q_2. The temperature field has three contributions of type 4.1.2.

The fluid temperatures of the central pipe and the outer ones become:

$$-T_{f1} = q_1 \cdot (m_p + m) + q_2 \cdot (m_{12} + m_{12})$$
$$-T_{f2} = q_2 \cdot (m_p + m + m_{22}) + q_1 \cdot m_{12} \qquad (4.6.1)$$

The pipe resistance m_p is discussed in section 4.3, while m is the single-pipe soil resistance 4.1.9. The coupling resistances between pipes 1 and 2 and between the outer pipes become:

$$m_{12} = \frac{1}{2\pi\lambda} \cdot \ln\left(\frac{\sqrt{B^2+4D^2}}{B}\right)$$
$$m_{22} = \frac{1}{2\pi\lambda} \cdot \ln\left(\frac{\sqrt{4B^2+4D^2}}{2B}\right) \qquad (4.6.2)$$

We now assume that the central pipe and the outer pipes have the same fluid temperature T_f. The ratio between q_1 and q_2 is then from 4.6.1:

$$\frac{q_1}{q_2} = 1 - \frac{m_{12}-m_{22}}{m_p+m-m_{12}} \tag{4.6.3}$$

The central pipe extracts less heat than the outer ones so the ratio is less than 1.

For the total thermal resistance between the three pipes and the ground surface we have:

$$-T_f = (q_1+2q_2)\cdot m_{2+1+2}$$
$$m_{2+1+2} = \frac{(m_p+m)(m_p+m+m_{22})-2m_{12}^2}{3(m_p+m)+m_{22}-4m_{12}} \tag{4.6.4}$$

We use the data 4.5.20 again:

$$R/D = 0.02 \qquad m_p = 0 \tag{4.6.5}$$

We compare the heat extraction q_1+2q_2 with that of three single pipes for the same extraction temperature T_f:

$$\eta = \frac{q_1+2q_2}{3q} = \frac{-T_f}{m_{2+1+2}} \cdot \frac{m+m_p}{3(-T_f)} = \frac{(m+m_p)/3}{m_{2+1+2}} \tag{4.6.6}$$

The result is given in Table 4.7 and shown in Figure 4.14.

B/D	0.05	0.1	0.2	0.4	0.6	0.8	1	1.2	1.4
η	0.41	0.46	0.53	0.62	0.69	0.74	0.78	0.81	0.84

1.6	1.8	2	2.25	2.5	3	4	5	7	10
0.86	0.88	0.90	0.91	0.92	0.94	0.96	0.98	0.99	0.994

Table 4.7. Heat extraction with three pipes relative to that of three independent pipes. Data according to 4.6.5.

We see again that there is a considerable gain, when B is increased for small B. The heat extraction of three pipes with a spacing of 0.2 D is 53% of that of three free pipes. The extraction increases to 78%, when the distance is increased to B = D. The gain after, say, B = 2D is marginal.

The ratio 4.6.3 of fluxes becomes in the present example for B = D:

$$\frac{q_1}{q_2} = 0.88$$

4.7 Four pipes

Figure 4.11 shows the next case to be studied.

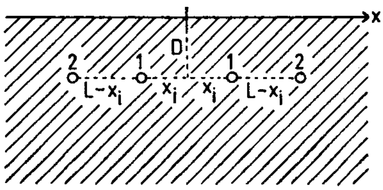


Figure 4.11. Considered case with four pipes.

The four pipes lie at the depth D. They lie symmetrically with respect to the z-axis. The distance between the outer pipes is 2L. The distance $2x_i$ between the inner pipes is variable $0 < x_i < L$.

The steady-state heat extraction rates are q_1 and q_2 for inner and outer pipes respectively. The temperature field is a sum of four terms of type 4.1.2. The fluid temperatures become:

$$\begin{aligned} -T_{f1} &= q_1(m_p+m+m_{11}) + q_2(m_{12}+m'_{12}) \\ -T_{f2} &= q_2(m_p+m+m_{22}) + q_1(m_{12}+m'_{12}) \end{aligned} \qquad (4.7.1)$$

The coupling resistances are:

$$m_{11} = \frac{1}{2\pi\lambda}\ln\left(\frac{\sqrt{4x_i^2+4D^2}}{2x_i}\right) \qquad m_{22} = \frac{1}{2\pi\lambda}\ln\left(\frac{\sqrt{4L^2+4D^2}}{2L}\right)$$

$$m_{12} = \frac{1}{2\pi\lambda}\ln\left(\frac{\sqrt{(L-x_i)^2+4D^2}}{L-x_i}\right) \quad m'_{12} = \frac{1}{2\pi\lambda}\ln\left(\frac{\sqrt{(L+x_i)^2+4D^2}}{L+x_i}\right) \qquad (4.7.2)$$

When the inner and outer temperatures are equal, we get from 4.7.1 the following ratio between the extraction rates:

$$\frac{q_1}{q_2} = \frac{m_p+m+m_{22}-m_{12}-m'_{12}}{m_p+m+m_{11}-m_{12}-m'_{12}} \qquad (4.7.3)$$

The total thermal resistance between the four pipes and the ground is from 4.7.1 and 4.7.3:

$$-T_f = (2q_1+2q_2)m_{2+1+1+2}$$

$$m_{2+1+1+2} = \frac{1}{2}\cdot\frac{(m_p+m+m_{11})(m_p+m+m_{22})-(m_{12}+m'_{12})^2}{2m_p+2m+m_{11}+m_{22}-2m_{12}-2m'_{12}} \qquad (4.7.4)$$

Let us compare the system of four pipes with four single pipes. The ratio between the total heat extraction for the same fluid temperature is from 4.3.2 and 4.7.4:

$$\eta = \frac{2q_1+2q_2}{4q} = \frac{-T_f}{m_{2+1+1+2}}\cdot\frac{1}{4}\cdot\frac{m+m_p}{-T_f} = \frac{(m+m_p)/4}{m_{2+1+1+2}} \qquad (4.7.5)$$

The ratio η is a function of the dimensionless variables λm_p, x_i/D, L/D and R/D. We consider again the particular case:

$$\frac{R}{D} = 0.02 \qquad m_p = 0 \qquad (4.7.6)$$

Figure 4.12 shows η as a function of x_i/D for some values of L/D.

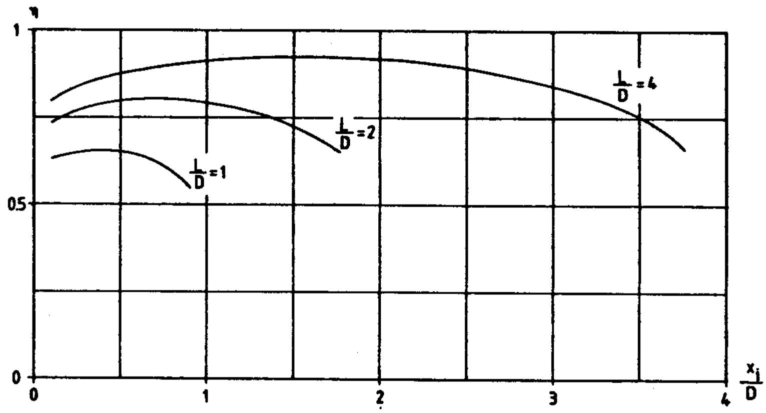


Figure 4.12. Heat extraction 4.7.5 of four pipes according to Figure 4.11 relative to that of four independent pipes. $m_p = 0$, $R/D = 0.02$.

The relative extraction rate η has, as a function of x_i, a maximum, which is very flat. In fact we have:

$\frac{L}{D} = 1:\quad \eta_{max} = 0.65 \quad \text{for} \quad \frac{x_i}{D} = 0.40$

$\eta \simeq 0.65 \quad \text{for} \quad 0.21 < \frac{x_i}{D} < 0.55$

$\frac{L}{D} = 2:\quad \eta_{max} = 0.80 \quad \text{for} \quad \frac{x_i}{D} = 0.72$

$\eta \simeq 0.80 \quad \text{for} \quad 0.51 < \frac{x_i}{D} < 0.93$

$\frac{L}{D} = 4:\quad \eta_{max} = 0.92 \quad \text{for} \quad \frac{x_i}{D} = 1.38$

$\eta \simeq 0.92 \quad \text{for} \quad 1.10 < \frac{x_i}{D} < 1.69$

As long as the pipes are not too close to each other it does not matter much where the inner pipes are placed.

The maximal η is in the three cases obtained with an accuracy of two digits for the case of equal spacing between the pipes ($x_i = L/3$).

Figure 4.13 shows the case with equal spacing B between the pipes.

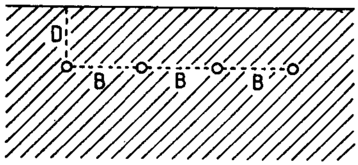


Figure 4.13. Four pipes with equal spacing B.

This case is obtained if we take L = 1.5B and x_i = 0.5B in 4.7.2.

The relative heat extraction η is given by 4.7.5. Table 4.9 gives η as a function of B/D for the particular case 4.7.6.

B/D	0.05	0.1	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
η	0.33	0.38	0.45	0.56	0.63	0.69	0.74	0.78	0.81	0.84

B/D	1.8	2	2.25	2.5	3	4	5	7	10
η	0.86	0.88	0.89	0.91	0.93	0.96	0.97	0.98	0.99

Table 4.9. Heat extraction with four pipes accoring to Figure 4.13 relative to four independent pipes. Data according to 4.7.6.

Figure 4.14 shows the relative heat extraction rate η for two, three and four pipes at the depth D. The spacing between the pipes is B. The values of η are taken from Tables 4.5, 4.7 and 4.9 respectively.

The result for N=6, N=10 and N=∞ are also shown. The limit with an infinite number of pipes is discussed in section 4.9.

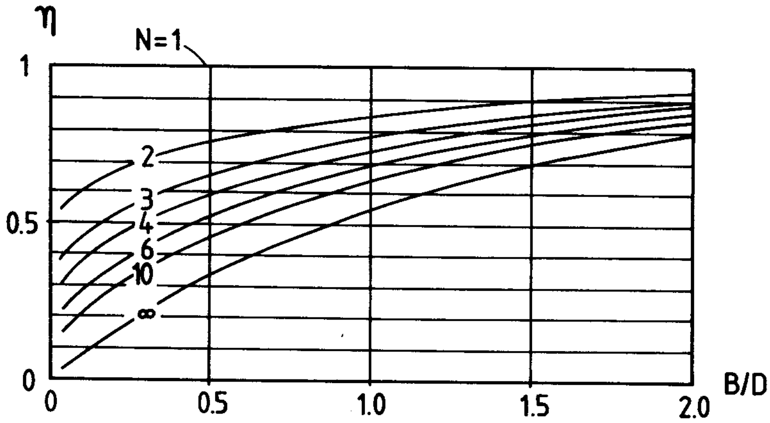


Figure 4.14. Steady-state heat extraction from N pipes compared to N independent pipes. The pipes lie at a depth D. The spacing between the pipes is B. R/D=0.02 , $m_p=0$.

4.8 N pipes

The general formulas for steady-state heat extraction by N pipes are now easy to give. Let q_i be the heat extraction rate of pipe i, which lies at $x=x_i$, $z=D_i$. The steady-state temperature is a sum of N contributions of type 4.1.2:

$$T(x,z) = \sum_{i=1}^{N} \frac{q_i}{2\pi\lambda} \ln\left(\frac{\sqrt{(x-x_i)^2+(z-D_i)^2}}{\sqrt{(x-x_i)^2+(z+D_i)^2}}\right) \qquad (4.8.1)$$

The soil resistance of pipe i alone is

$$m_i = \frac{1}{2\pi\lambda} \ln\left(\frac{2D_i}{R}\right) \qquad (4.8.2)$$

The radius R could without problems be different for different pipes: $R \to R_i$. The distance between pipe i and pipe j is:

$$B_{ij} = \sqrt{(x_i-x_j)^2 + (D_i-D_j)^2} \qquad (4.8.3)$$

The distance between pipe i and the mirror of pipe j is:

$$\sqrt{(x_i-x_j)^2 + (D_i+D_j)^2} = \sqrt{B_{ij}^2 + 4D_iD_j} \qquad (4.8.4)$$

The coupling resistance between i and j is then:

$$m_{ij} = \frac{1}{2\pi\lambda} \ln\left(\frac{\sqrt{B_{ij}^2+4D_iD_j}}{B_{ij}}\right) \qquad (4.8.5)$$

The fluid temperature of pipe i, which has the pipe resistance m_{pi}, is now:

$$-T_{fi} = q_i(m_i+m_{pi}) + \sum_{\substack{j=1\\j\neq i}}^{N} q_j\, m_{ij} \qquad i = 1, 2, \ldots N \qquad (4.8.6)$$

This is a linear equation system between the fluid temperatures T_{fi} and the heat fluxes q_i. An important particular case is when the fluid temperatures are essentially equal:

$$T_{fi} = T_f \qquad i = 1, 2, \ldots N \qquad (4.8.7)$$

The heat fluxes are proportional to T_f. They are obtained by solving 4.8.6.-7. The cases considered in the previous sections led to two equations for which the solution is simple to write down. The general case for higher N is simple to solve with a computer.

4.9 Infinite array of pipes

The extreme case of an infinite array of pipes is of interest, since it gives a limit for many pipes. The case is shown in Figure 4.15.

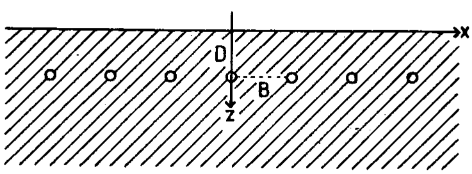


Figure 4.15. Infinite array of pipes.

The pipes lie at the depth z=D. The distance between the pipes is B. The heat extraction rate q is the same for all pipes. The temperature at the surface z=0 is zero.

The well-known solution of this problem is:

$$T(x,z) = \frac{q}{4\pi\lambda}\cdot\ln\left(\frac{\cosh(\frac{2\pi(D-z)}{B})-\cos(\frac{2\pi x}{B})}{\cosh(\frac{2\pi(D+z)}{B})-\cos(\frac{2\pi x}{B})}\right) \tag{4.9.1}$$

Our particular interest is the temperature at the pipe radius $x^2+(z-D)^2 = R^2$. There is a single pipe contribution of the type ln(R). The remaining part represents the contribution from the other pipes and from the mirror pipes at z = -D. The variation around the pipe periphery of these contributions are as in the previous discussions neglected, since R is much smaller than D and B. We get after some manipulations from 4.9.1 the temperature at the pipe radius:

$$T_R = -\frac{q}{2\pi\lambda}\ln(\frac{B}{\pi R}\sinh(\frac{2\pi D}{B})) \tag{4.9.2}$$

The thermal resistance of the soil between one of the pipes and the ground is then:

$$m = \frac{1}{2\pi\lambda}\ln(\frac{B}{\pi R}\sinh(\frac{2\pi D}{B})) = \frac{1}{2\pi\lambda}\{\ln(\frac{2D}{R}) + f(\frac{B}{D})\}, \tag{4.9.3}$$

where

$$f(\frac{B}{D}) = \ln(\frac{B}{2\pi D}\cdot\sinh(\frac{2\pi D}{B})) \tag{4.9.4}$$

The part ln(2D/R) is the thermal resistance factor 4.1.10 of a single pipe. The second part f(B/D) in 4.9.3 gives the influence of the other pipes in the infinite array. The function f(B/D) is given in Table 4.10.

B/D	0.1	0.25	0.5	0.75	1.0	1.5	2	4	10
f	58.0	21.2	9.34	5.56	3.75	2.06	1.30	0.35	0.06

Table 4.10. The function 4.9.4 which gives the influence of surrounding pipes in an infinite array.

Let us consider an example:

$$\frac{R}{D} = 0.02 \quad : \quad \ln(\frac{2D}{R}) = 4.61$$

$$\frac{B}{D} = 1 \quad : \quad f(\frac{B}{D}) = 3.75$$

The infinite array increases the thermal resistance factor from 4.61 to 4.61 + 3.75; i.e. with 81%.

Let η as usual denote the ratio between the heat extraction of a pipe in the infinite array and that of a free pipe. Then we have from 4.9.3 and 4.3.2:

$$\eta = \frac{2\pi\lambda m_p + \ln(\frac{2D}{R})}{2\pi\lambda m_p + \ln(\frac{2D}{R}) + f(\frac{B}{D})} \tag{4.9.5}$$

A particular case of 4.9.5 is shown in Figure 4.14.

4.10 Influence between pipes

We will in this section illustrate somewhat further the influence on the pipe temperature from adjacent pipes. Consider a pipe at a depth D. There is an array of N pipes to the right. These pipes lie at the same depth. The spacing between the pipes is B. See Figure 4.16.

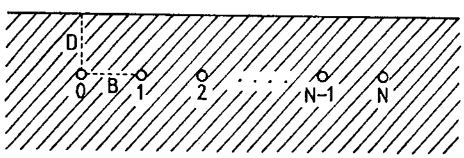


Figure 4.16. Influence on a pipe 0 from an array of N pipes.

We assume for simplicity that the heat extraction rate q is the same for all N+1 pipes. This will not exactly be the case, when the fluid temperatures are to be equal. But we have seen that the difference between the extraction rates are relatively small.

The coupling resistance between pipe 0 and pipe j is from 4.8.5:

$$m_{0j} = \frac{1}{2\pi\lambda} \ln\left(\frac{\sqrt{(jB)^2+4D^2}}{jB}\right) \tag{4.10.1}$$

The temperature of pipe 0 is then from 4.8.6 with q_i=q:

$$-T_{f0} = q \cdot m_p + \frac{q}{2\pi\lambda} \{\ln(\frac{2D}{R}) + h_N(\frac{B}{D})\} \tag{4.10.2}$$

Here h_N is given by

$$h_N(\frac{B}{D}) = \sum_{j=1}^{N} \ln\left(\frac{\sqrt{(jB)^2+4D^2}}{jB}\right) \tag{4.10.3}$$

The sum h_N represents the influence of N adjacent pipes. This term is to be compared to the thermal resistance factor ln(2D/R) of the pipe itself.

The function f(B/D) of 4.9.4 represented the influence of an infinite array to the right and to the left. We therefore have:

$$\lim_{N\to\infty} h_N(\frac{B}{D}) = \frac{1}{2} f(\frac{B}{D}) \tag{4.10.4}$$

The function f is given by 4.9.4. The function h_N is shown in Figure 4.17.

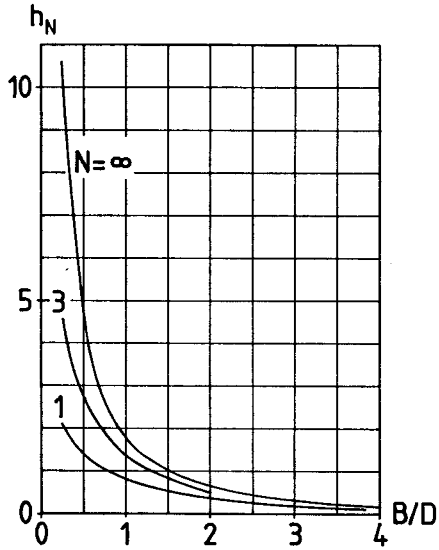
Let us take the case

$$\frac{R}{D} = 0.02 \quad : \quad \ln(\frac{2D}{R}) = 4.61$$

The values of Figure 4.17 shall then be compared to 4.61. We have for example:

$$\frac{B}{D} = 1 \ , \quad N = 2 \quad : \quad h_N = 1.15 \qquad \frac{1.15}{4.61} = 0.25$$

The two pipes increases the thermal resistance with 25%. As another example we take a pipe with two pipes to the left and three pipes to the right. This gives two contributions which are to be added.



		B/D				
		0.25	0.5	1	2	4
N	1	2.09	1.42	0.80	0.35	0.11
	2	3.50	2.22	1.15	0.46	0.14
	3	4.55	2.73	1.34	0.51	0.16
	4	5.36	3.08	1.45	0.54	0.16
	5	5.99	3.33	1.52	0.56	0.17
	6	6.50	3.51	1.57	0.57	0.17
	7	6.92	3.65	1.61	0.58	0.17
	8	7.27	3.76	1.64	0.59	0.18
	9	7.56	3.85	1.67	0.60	0.18
	10	7.80	3.93	1.69	0.60	0.18
	20	9.08	4.28	1.78	0.63	0.18
	∞	10.6	4.67	1.88	0.65	0.18

Figure 4.17. The function $h_N(B/D)$ (4.10.3), which represents the approximate influence of an array of N pipes on one side according to Figure 4.16.

$$\frac{B}{D} = 1 \qquad N = 2 \;:\; h_N = 1.15 \qquad \frac{1.15 + 1.34}{4.61} = 0.54$$

$$N = 3 \;:\; h_N = 1.34$$

There is an increase of 54% for the thermal resistance.

We note from Figure 4.17 that the influence of surrounding pipes is quite small for $B/D > 2$, and considerable for $B/D < 0.5$.

4.11 A bundle of pipes

Sometimes a few pipes are put together in a bundle and buried at the same depth in a ditch. The heat extraction potential would however be increased, if the pipes are brought apart from each other. We shall in this section illustrate how much there is to be gained.

Figure 4.18 shows two pipes directly in contact with each other and at a distance B from each other. We assume that B is reasonably large compared to the radius R, but small compared to the depth D.

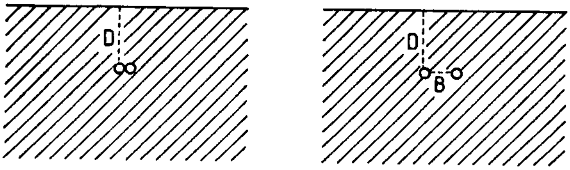


Figure 4.18. Two pipes in direct contact (left) and at a moderate distance B (right).

Let q_b be the steady-state heat extraction from the two pipes, when they lie together. The previous formulas cannot be used directly, since the outer boundary of the two pipes is not circular. But we can introduce an equivalent radius R_{eq}. The heat extraction from the two pipes in contact is then given by

$$-T_f = q_b\left(\frac{1}{2} m_p + \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R_{eq}}\right)\right) \qquad (4.11.1)$$

The pipe resistance between the soil and the fluid is halved since there are two pipes. The equivalent radius must satisfy:

$$R < R_{eq} < 2R \qquad (4.11.2)$$

Let us consider the example:

$$D = 1 \text{ m} \qquad R = 0.02 \text{ m}$$

$$\begin{aligned} R_{eq} &= R & \ln\left(\frac{2D}{R_{eq}}\right) &= 4.61 \\ R_{eq} &= \sqrt{2}\cdot R & \ln\left(\frac{2D}{R_{eq}}\right) &= 4.26 \\ R_{eq} &= 2R & \ln\left(\frac{2D}{R_{eq}}\right) &= 3.91 \end{aligned} \qquad (4.11.3)$$

We take

$$R_{eq} = \sqrt{2}\, R \qquad (4.11.4)$$

The error with this choice should not exceed a few percent.

The thermal resistances for the two pipes at the moderate distance B are given by 4.5.4. The two depths D_1 and D_2 are essentially equal to D independent of the relative positions of the two pipes, since B is assumed to be much smaller than D. Then we have with good approximation:

$$-T_f = q \cdot m_p + q \cdot \frac{1}{2\pi\lambda} \{\ln(\frac{2D}{R}) + \ln(\frac{2D}{B})\} \qquad (4.11.5)$$

The term B^2 was neglected compared to $4D^2$ in the last logarithm.

The quotient between the heat extraction rates is now from 4.11.1 and 5:

$$k_2 = \frac{2q}{q_b} = \frac{2\pi\lambda m_p + 2 \cdot \ln(\frac{2D}{\sqrt{2} \cdot R})}{2\pi\lambda m_p + \ln(\frac{2D}{R}) + \ln(\frac{2D}{B})} \qquad (4.11.6)$$

The function k_2 is given in Table 4.11 in a particular case. We note that there is a gain of the order of 10-20% for a moderate distance B.

Figure 4.19 shows three pipes either together or separated somewhat from each other. We assume in the latter case that the pipes form an equilateral triangle with the side B.

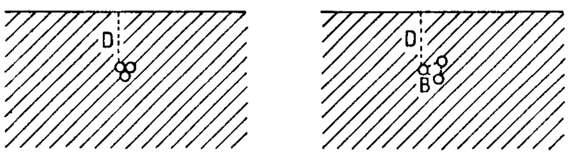


Figure 4.19. Three pipes in a bundle (left) or separated a distance B from each other.

The maximal distance for B is determined by the width of the ditch.

Let q_b be the heat extraction from the three pipes in the bundle. We have in analogy with 4.11.1:

$$-T_f = q_b \cdot \{\frac{1}{3} m_p + \frac{1}{2\pi\lambda} \ln(\frac{2D}{R_{eq}})\} \qquad (4.11.7)$$

We take

$$R_{eq} = \sqrt{3}\,R \tag{4.11.8}$$

The coupling resistance between two of the pipes, when they are at the distance B, is given by 4.8.5. The depth is essentially the same for all three pipes, since B is much smaller than D. The term B^2 can also be neglected compared to $4D^2$ in 4.8.5. Then we have for the heat extraction q for one of the three pipes (4.8.6):

$$-T_f = q\cdot m_p + q\cdot\frac{1}{2\pi\lambda}\{\ln(\frac{2D}{R}) + 2\cdot\ln(\frac{2D}{B})\} \tag{4.11.9}$$

The quotient between the heat extraction rates is now from 4.11.7-9:

$$k_3 = \frac{3q}{q_b} = \frac{2\pi\lambda m_p+3\cdot\ln(\frac{2D}{\sqrt{3}\cdot R})}{2\pi\lambda m_p+\ln(\frac{2D}{R})+2\cdot\ln(\frac{2D}{B})} \tag{4.11.10}$$

The ratio k_3 is given in a particular case in Table 4.11.

As a further illustration we shall compare the heat extraction rate of three pipes in the two cases of Figure 4.20.

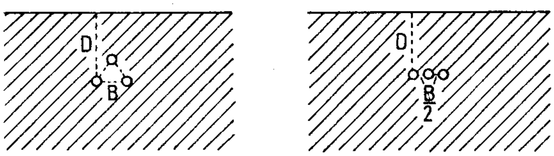


Figure 4.20. Comparison of heat extraction rates for three pipes in triangular (left) and linear (right) configurations.

The total space used has the linear extension B. The heat extraction in the linear case is given by 4.6.4 and in the triangular case by 4.11.9. Let us take:

$$\frac{R}{D} = 0.02 \qquad m_p = 0 \qquad \frac{B}{D} = 0.2 \tag{4.11.11}$$

We can use 4.6.7 and Table 4.7 (B/D = 0.1) in the linear case:

$$q_1 + 2q_2 = 3\eta \cdot q = 3 \cdot 0.46 \cdot \frac{-T_f}{\frac{1}{2\pi\lambda} \ln(\frac{2D}{R})} \qquad (4.11.12)$$

Here q is the heat extraction of the corresponding single pipe. From 4.11.9 we have for the heat extraction 3q in the triangular case:

$$3q = 3 \frac{-T_f}{\frac{1}{2\pi\lambda} \{\ln(\frac{2D}{R})+2\ln(\frac{2D}{B})\}} \qquad (4.11.13)$$

The quotient is

$$\frac{3q}{q_1+2q_2} = \frac{\ln(100)}{0.46(\ln(100)+2\ln(10))} = 1.09 \qquad (4.11.14)$$

The simple change from a linear to a triangular configuration gives in this example an increase of 9% for the heat extraction.

Let us finally consider the case of four pipes which are put either in a bundle or at the corners of a square with the side B. See Figure 4.21.

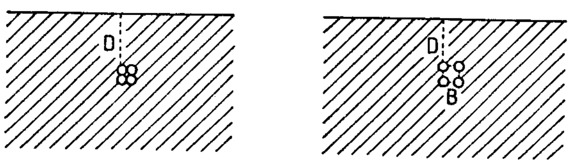


Figure 4.21. Comparison of heat extraction rates for four pipes in a bundle (left) or in a quadratic configuration (right).

Let q_b be the heat extraction rate of the bundle with four pipes. Then we have

$$-T_f = q_b \cdot \{\frac{1}{4} m_p + \frac{1}{2\pi\lambda} \ln(\frac{2D}{R_{eq}})\} \qquad (4.11.15)$$

We take

$$R_{eq} = 2R \qquad (4.11.16)$$

Any one of the four pipes has two pipes at a distance B and one pipe at a distance $\sqrt{2}$ B. The heat extraction rate is then with the argument of 4.11.5 and 9:

$$-T_f = q\,m_p + q\cdot\frac{1}{2\pi\lambda}\{\ln(\frac{2D}{R}) + 2\cdot\ln(\frac{2D}{B}) + \ln(\frac{2D}{\sqrt{2}B})\} \qquad (4.11.17)$$

The quotient between the heat extraction rates in the two cases of Figure 4.21 is from 4.11.15-17:

$$k_4 = \frac{4q}{q_b} = \frac{2\pi\lambda m_p + 4\cdot\ln(\frac{2D}{2R})}{2\pi\lambda m_p + \ln(\frac{2D}{R}) + 2\cdot\ln(\frac{2D}{B}) + \ln(\frac{2D}{\sqrt{2}\,B})} \qquad (4.11.18)$$

Let us as usual consider the case:

$$\frac{R}{D} = 0.02 \qquad m_p = 0 \qquad (4.11.19)$$

Table 4.11 gives the function k_4 for some values of B/D. It also gives the corresponding values k_2 and k_3 from 4.11.6 and 10.

	B/D 0.1	0.2	0.3
k_2	1.12	1.23	1.31
k_3	1.15	1.32	1.45
k_4	1.18	1.40	1.57

Table 4.11. Improvement of heat extraction when the pipes in a bundle are put apart. See Figures 4.18, 19 and 21.

It is quite clear from Table 4.14 that one shall always put the pipes as much apart as possible.

We see from 4.11.5, 9 and 17 that the thermal resistances and the formulas for the heat extraction rate of a number of pipes are simplified, when the pipes lie at moderate distances. Let us consider the general case with N pipes. The distances between the pipes are denoted B_{ij}. We assume:

$$R \ll B_{ij} \ll D \qquad (D_i \simeq D) \qquad (4.11.20)$$

The thermal resistances 4.8.2 and 5 are now with good accuracy

$$m_i = m = \frac{1}{2\pi\lambda}\ln\left(\frac{2D}{R}\right) \qquad m_{ij} = \frac{1}{2\pi\lambda}\ln\left(\frac{2D}{B_{ij}}\right) \qquad (4.11.21)$$

Secondly, we assume that the N pipes lie symmetrically relative to each other. The heat fluxes are then equal. Formula 4.8.6-7 becomes:

$$-T_f = q(m_p + m + \sum_{j\neq i} m_{ij}) \qquad (4.11.22)$$

The sum of coupling resistances is, in the case where the pipes lie in a symmetric pattern, independent of i. Formulas 4.11.5, 9 and 17 are special cases of 4.11.22.

The N pipes may be treated as a unit. Let m_N be the total thermal resistance between N pipes which are placed in a symmetric pattern at moderate distances (i.e. 4.11.20 is valid). Then we have from 4.11.22:

$$-T_f = Nq \cdot m_N$$

$$(4.11.23)$$

$$m_N = \frac{1}{N}\left[m_p + \frac{1}{2\pi\lambda}\{\ln\left(\frac{2D}{R}\right) + \sum_{j=2}^{N}\ln\left(\frac{2D}{B_{1j}}\right)\}\right]$$

4.12 Two layers of soil

The soil has in the cases discussed so far been of a single type with the thermal conductivity λ. The case with two layers of soil will be considered in this section.

The ground $z > 0$ consists of two layers $0 < z < H$ and $z > H$ with different thermal conductivities. We will only consider the steady-state heat extraction by a single pipe at a depth z=D. See Figure 4.22. The extraction pipe may lie in the lower stratum D>H or in the top layer D<H. The thermal conductivity in the layer where the pipe lies is λ.

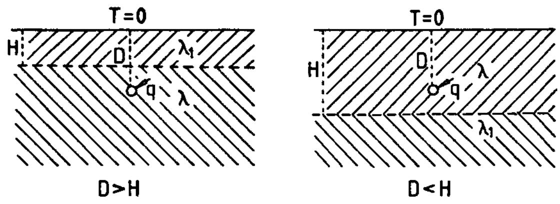


Figure 4.22. Steady-state heat extraction by a single pipe in a two-layered soil.

The conductivity in the other layer is denoted λ_1.

The temperature field and in particular the thermal resistance of the soil between the pipe and the ground are derived in appendix 2. We have from A2.16 and A2.22:

$$m = \frac{1}{2\pi\lambda} \{\ln(\frac{2D}{R}) + p(\frac{H}{D}, \sigma)\} \qquad (4.12.1)$$

The parameter σ is given by

$$\sigma = \frac{\lambda-\lambda_1}{\lambda+\lambda_1} \qquad (4.12.2)$$

The value of σ ranges from +1 to -1, when λ_1 varies from zero to infinity. The function p is given by the infinite sums A2:16 and A2:22 for D>H and D<H respectively. They give the influence due to the introduction of a different conductivity λ_1 in the second soil layer. The function p is shown in Figure 4.23.

The two extremes $\lambda_1=0$ and $\lambda_1=\infty$ are quite interesting in the case D<H, when the pipe lies in the top layer.

The case $\lambda_1=\infty$ implies that the temperature at the interface z=H is the same as at the ground surface. This would be the case, if there were a sufficiently strong ground water stream in the region z>H. We have from 4.12.1 and A2.23:

$$m = \frac{1}{2\pi\lambda} \{\ln(\frac{2D}{R}) + \ln(\frac{H}{\pi D}\cdot\sin(\frac{\pi D}{H}))\} \qquad (4.12.3)$$

(H>D, $\lambda_1=\infty$ or T=0 for z=H)

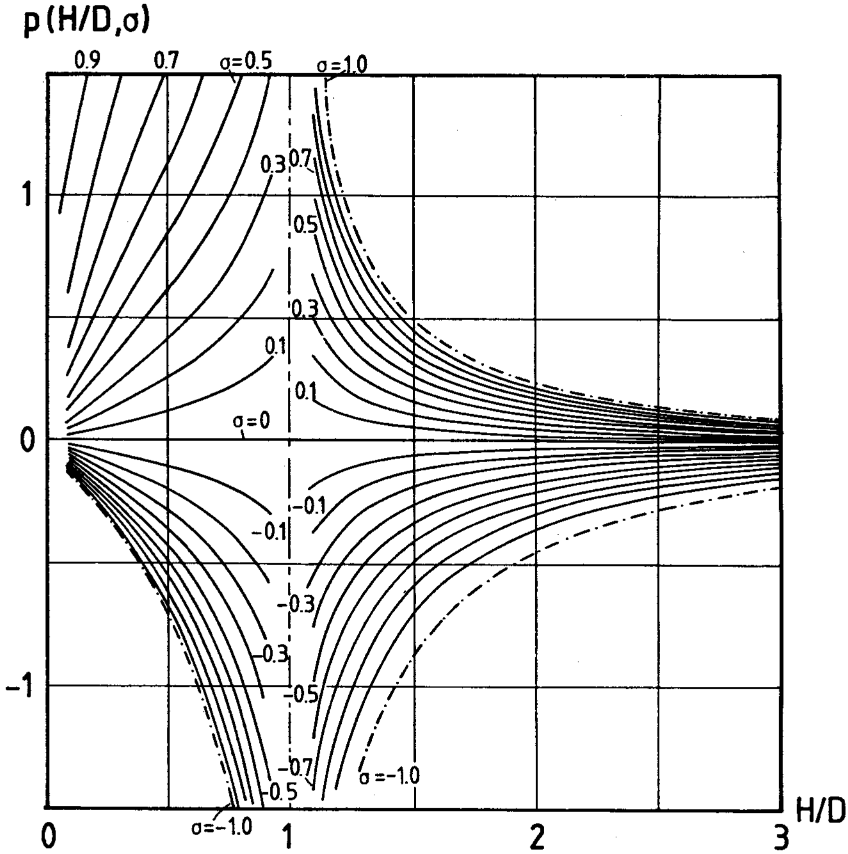


Figure 4.23. The function p, which represents the influence of a second soil layer with different thermal conductivity. Formulas 4.12.1-2.

In the other extreme $\lambda_1=0$ we have from 4.12.1 and A2.23:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) + \ln\left(\frac{2H}{\pi D}\tan\left(\frac{\pi D}{2H}\right)\right)\right\} \qquad (4.12.4)$$

$$(H>D,\ \lambda_1=0 \text{ or } \frac{\partial T}{\partial z} = 0 \text{ for } z=H)$$

Let us consider two examples. The first one is a granite bedrock which is covered by 1.5 m of a sandy soil. We take:

$$\lambda = 0.9 \text{ W/m}^{\circ}\text{C} \qquad \lambda_1 = 3.5 \text{ W/m}^{\circ}\text{C}$$

$$R = 0.02 \text{ m} \qquad H = 1.5 \text{ m} \tag{4.12.5}$$

$$0 < D < H$$

The thermal resistance of the soil is given by 4.12.1:

$$m = \frac{1}{2\pi \cdot 0.9} \{\ln(\frac{2D}{0.02}) + p(\frac{1.5}{D}, -0.6)\} \tag{4.12.6}$$

The thermal resistance 4.12.6 is shown in Figure 4.24.A. We note that there is a maximum for D=1.0 m. It is interesting to see that the thermal resistance decreases, when the depth D to the pipe is increased from D = 1.0 m. This effect is due to the much higher conductivity in granite. There is a considerable gain to put the pipe directly on a granite bedrock with its high thermal conductivity.

As the second example we consider a case with a ground water level at z=H. The upper, dry soil has a smaller conductivity:

$$R = 0.02\text{m} \qquad H = 1\text{m}$$

$$0 < D < 1: \qquad \lambda = 0.9 \text{ W/mK} \qquad \lambda_1 = 2.1 \text{ W/mK} \tag{4.12.7}$$

$$D > 1 \qquad \lambda = 2.1 \text{ W/mK} \qquad \lambda_1 = 0.9 \text{ W/mK}$$

The thermal resistance of the soil is given by 4.12.1. The result is shown in Figure 4.24.B. We see from the curve that it is better to put the pipe in the lower layer with the higher conductivity. The pipe should be placed at a certain distance from the upper layer. There is actually a local minimum for the thermal resistance at the depth z = 1.5 m.

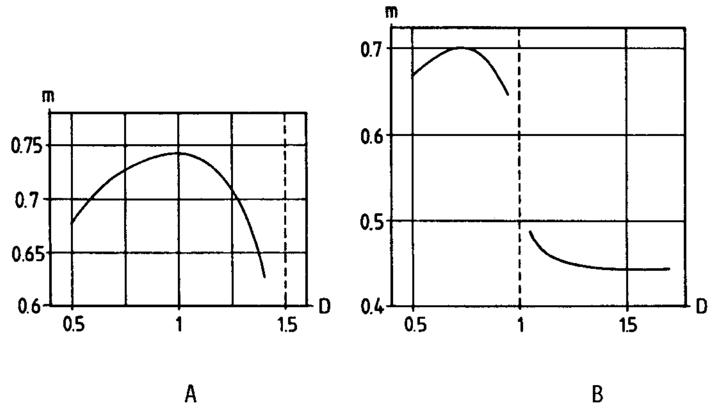


Figure 4.24. Thermal resistance in a two-layered soil as a function of pipe depth D.
A: Example 4.12.5. B: Example 4.12.7.

4.13 Heat flux at the ground surface

The heat extraction pipes change the natural temperature conditions in the ground. The thermal influence region has been discussed in section 4.2. Another important aspect of the thermal impact is the change of the heat flux at the ground surface. This influences for example the melting of the snow in springtime.

This section is devoted to a study of the additional steady-state heat flux at the ground surface due to the heat extraction pipes. Let $F_z(x)$ be this heat flux <u>into</u> the ground:

$$F_z(x) = -\lambda \left.\frac{\partial T}{\partial z}\right|_{z=0} \tag{4.13.1}$$

The temperature field of the single pipe is given by 4.1.2. The heat flux down through the ground surface due to this pipe is then with 4.1.2 and 4.13.1:

$$F_z(x) = \frac{q}{\pi D} \cdot \frac{D^2}{x^2+D^2} \tag{4.13.2}$$

The integrated flux from -x to x becomes:

$$F_I(x) = \int_{-x}^{x} F_z(x')dx' = \frac{2q}{\pi} \arctan(\frac{x}{D}) \tag{4.13.3}$$

The functions F_z and F_I are given in Table 4.12.

x/D	0	0.5	1	2	3	5	10	25
$\frac{\pi D}{q} \cdot F_z$	1	0.80	0.50	0.20	0.10	0.04	0.01	0.002
$\frac{1}{q} F_I$	0	0.30	0.50	0.70	0.80	0.87	0.94	0.97

Table 4.12. Heat flux F_z (4.13.2) and integrated heat flux F_I (4.13.3) at the ground surface for steady-state heat extraction with one pipe.

The maximum heat flux right above the pipe is $q/(\pi D)$. The relative value decreases to 0.5 for x=D and to 0.1 for x=3D. The integrated heat flux F_I tends of course to q as x tends to infinity. All heat is provided through the ground surface in steady-state. The idea that some of the extracted heat is furnished by geothermal heat from below is not correct.

The ground surface heat flux due to several pipes is obtained from 4.13.2 by superposition:

$$F_z(x) = \sum_{i=1}^{N} \frac{q_i}{\pi D_i} \frac{D_i^2}{(x-x_i)^2+D_i^2} \tag{4.13.4}$$

Let us consider an example with four pipes. For simplicity we assume that the heat extraction rates are equal:

$$\begin{aligned} N = 4 \qquad & q_1 = q_2 = q_3 = q_4 = q \\ & D_i = D = 1 \text{ m} \\ & x_1 = -3 \text{ m} \quad x_2 = -1 \text{ m} \\ & x_3 = +1 \text{ m} \quad x_4 = +3 \text{ m} \end{aligned} \tag{4.13.5}$$

The resulting heat flux is shown in Figure 4.25.

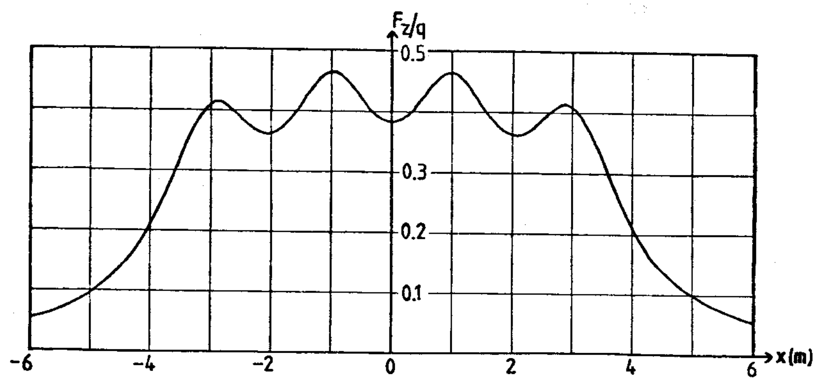


Figure 4.25. Heat flux at the ground surface due to four extraction pipes. Example 4.13.5.

5. EFFECT OF GROUND SURFACE TEMPERATURES

The thermal conditions and the temperature at the ground surface vary with time. This gives under natural conditions without heat extraction a certain temperature in the ground, which is a function of time and depth. This natural temperature field is, according to the superposition technique of chapter 2, to be added to the temperature fields due to heat extraction pipes. See Figure 2.2. This chapter is devoted to a brief discussion of the undisturbed, natural temperature field in the ground.

5.1 Boundary conditions at the ground surface

In general, the following type of boundary condition at the ground surface is assumed to be valid:

$$(T_s - T)\alpha_s = -\lambda \frac{\partial T}{\partial z} \qquad z=0 \qquad (5.1.1)$$

Here $T_s(t)$ is the temperature in the air at the ground surface. It is a given function of time. It may include a part that in an approximate way accounts for solar radiation. The heat transfer coefficient α_s of the surface is a given constant. The thermal resistance at the ground surface may often in our present heat extraction applications be neglected ($\alpha_s = \infty$). An example is given in section 4.4. The boundary condition at the ground surface is then:

$$T = T_s(t) \qquad z=0 \qquad (5.1.2)$$

The boundary condition 5.1.1 (and the special case 5.1.2) is a linear one. This is a prerequisite for the use of superposition. A more precise boundary condition would require a variable α_s. It depends for example on the wind velocity. The effect of snow cannot be considered. The radiation makes α_s dependent of the surface temperature. All these complications are however neglected. They are judged to be of minor importance at the depth D of the pipes, since D is of the order of one meter. The diurnal variations at the surface do not reach this depth. It is only the average conditions over days that matter at a depth of one meter.

5.2 Natural ground temperatures

The natural ground temperatures are determined by $T_s(t)$. The surface temperature and the ensuing thermal process in the ground may by superposition be regarded as a result of more elementary cases. We will here only consider two basic cases in order to illustrate what happens. For simplicity we use the simpler boundary condition 5.1.2.

The first case is a step change of the ground surface temperature at the time t=0:

$$T_s(t) = \begin{cases} T_0 & t < 0 \\ T_0+T_1 & t > 0 \end{cases} \qquad (5.2.1)$$

The solution of this well-known case is given in [3a].

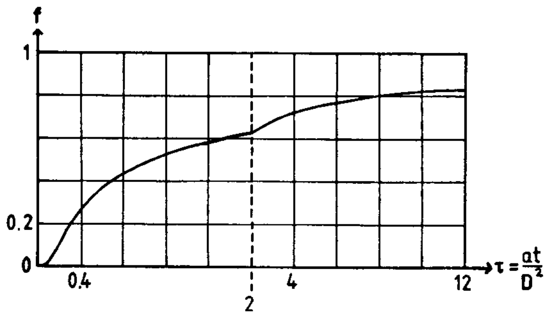
$$T(z,t) = \begin{cases} T_0 & t < 0 \\ T_0+T_1 \cdot \mathrm{erfc}\left(\frac{z}{\sqrt{4at}}\right) & t > 0 \end{cases} \qquad (5.2.2)$$

Here erfc denotes the complementary error function [2B].

The response at a depth z=D to a unit step change at the surface is given by:

$$f = \mathrm{erfc}\left(\frac{1}{2\sqrt{\tau}}\right) \qquad \tau = \frac{at}{D^2} \qquad (5.2.3)$$

This function is given in Figure 5.1.



τ	f
0.1	0.03
0.3	0.20
1	0.48
3	0.68
10	0.82
30	0.90
100	0.96

Figure 5.1. Response at a depth D to a unit step change of the ground surface temperature.

The time D^2/a gives the time scale of the response at the depth D to a change of the surface temperature. About 50% of the surface change is felt at z=D after the time $t = D^2/a$. Numerically we have for example:

$$a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \qquad \begin{array}{ll} D = 0.1 \text{ m} & D^2/a = 3.7 \text{ hours} \\ D = 0.5 \text{ m} & D^2/a = 3.9 \text{ days} \\ D = 1 \text{ m} & D^2/a = 15 \text{ days} \\ D = 2 \text{ m} & D^2/a = 2 \text{ months} \\ D = 4 \text{ m} & D^2/a = 8 \text{ months} \end{array} \qquad (5.2.4)$$

The time scales for different depths (5.2.4) are of great importance for a good appreciation of the thermal processes that we discuss.

The second case to be considered is a sinusoidal surface temperature:

$$T_s(t) = T_0 + T_1 \sin\left(\frac{2\pi t}{t_0} + \varphi_o\right) \qquad (5.2.5)$$

Here t_0 is the period of the temperature variation. The temperature in the ground due to this surface temperature is given by [3B]:

$$T(z,t) = T_0 + T_1 \cdot e^{-\frac{z}{d_0}} \sin\left(\frac{2\pi t}{t_0} + \varphi_o - \frac{z}{d_0}\right) \quad (z>0) \qquad (5.2.6)$$

The penetration depth d_0 is given by

$$d_0 = \sqrt{\frac{at_0}{\pi}} \tag{5.2.7}$$

The amplitude of the temperature oscillation at the depth z is:

$$T_1 \cdot e^{-\frac{z}{d_0}} \tag{5.2.8}$$

We have the damping at $z = d_0$, $z = 2d_0$ and $z = 3d_0$:

$$e^{-1} = 0.37,\ e^{-2} = 0.14,\ e^{-3} = 0.05 \tag{5.2.9}$$

So 5% of the amplitude remains at the depth $z = 3d_0$.

Numerically we have for example:

$a = 0.75 \cdot 10^{-6}\ m^2/s$	t_0 = 1 hour	d_0 = 0.03 m
	t_0 = 1 day	d_0 = 0.14 m
	t_0 = 1 week	d_0 = 0.38 m
	t_0 = 1 month	d_0 = 0.79 m
	t_0 = 1 year	d_0 = 2.7 m

The annual variation retains 14% of the surface amplitude at the depth z = 5.4 m. A weekly variation has 5% of the surface amplitude at the depth z = 1.1 m.

The natural ground temperatures for more complex ground temperatures $T_s(t)$ may be obtained from the above basic solutions by superposition.

5.3 Ground surface influence

Let us summarize how the superposition accounts for the influence of the ground surface with its varying temperature.

The total temperature field in the ground with the boundary condition 5.1.2 or 5.1.1 and with a certain number of pipes with prescribed heat extraction rates is denoted T_{tot}. Let T_{undist} denote the temperature field in the undisturbed ground without heat extraction. The temperature field due to the heat extraction pipes T_{pipes} is defined by:

$$T_{tot} = T_{undist} + T_{pipes} \qquad (5.3.1)$$

The temperature field T_{pipes} shall account for the prescribed heat extraction rates at the pipes. The boundary condition at the ground surface becomes according to 5.1.2 and 5.3.1:

$$T_{pipes} = 0 \qquad z = 0 \qquad (5.3.2)$$

With the more general boundary condition 5.1.1 we have instead:

$$(0 - T_{pipes})\alpha_s = -\lambda \frac{\partial T_{pipes}}{\partial z} \qquad z = 0 \qquad (5.3.2')$$

The important result of this discussion is that the undisturbed heat flow process is completely decoupled from the heat extraction problem. The two processes can be dealt with separately. This simplification is due to the fact that the original problem is formulated in terms of prescribed heat extraction rates. A prescription of fluid temperatures in the pipes would have resulted in a more complicated problem without this direct decoupling.

We can in the following chapters concentrate ourselves upon the thermal process in the ground due to the heat extraction pipes with zero temperature at the ground surface according to 5.3.2.

It must always be remembered that the total temperature in the ground contains the contribution from the natural, undisturbed temperature field.

5.4 Optimal heat extraction depth

The optimal depth for heat extraction pipes is a compromise between many aspects: Digging cost, environmental impact, thermal performance and so on. We will in this section discuss an optimization for which only pure thermal aspects are taken into consideration.

Let us consider the steady-state heat extraction by one pipe at a depth z=D. The surface temperature during the annual cycle is assumed to follow 5.2.5.

The temperature field due to the steady-state heat extraction is given by 4.1.2. The undisturbed temperature field due to ground surface temperature is given by 5.2.5. The total temperature field is by superposition given by the sum of these two cases in accordance with 5.3.1. The fluid temperature in the pipe is then from 4.3.2, 4.1.9 and 5.2.6;

$$T_f(t) = -q\left(\frac{1}{2\pi\lambda}\ln\left(\frac{2D}{R}\right)+m_p\right)+T_0+T_1\cdot e^{-\frac{D}{d_0}}\cdot\sin\left(\frac{2\pi t}{t_0}+\varphi_0-\frac{D}{d_0}\right) \quad (5.4.1)$$

This expression gives the temperature $T_f(t)$ in the fluid, which is to be maintained in order to obtain the prescribed constant heat extraction rate q.

The temperature T_f varies sinusoidally during the year. The lowest fluid temperature during the cycle is from 5.4.1:

$$T_{fmin} = -q\cdot\left(\frac{1}{2\pi\lambda}\ln\left(\frac{2D}{R}\right)+m_p\right) + T_0 - T_1\, e^{-\frac{D}{d_0}} \quad (5.4.2)$$

As optimization criterion we take that the lowest extraction temperature 5.4.2 is to be as high as possible. The expression 5.4.2 is to be maximized as a function of D, D >> R > 0.

Let us first consider the following example:

$$T_0 = 10^oC \qquad T_1 = 10^oC \qquad t_0 = 1 \text{ year}$$
$$\lambda = 1.5 \text{ W/m}^oC \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \qquad (5.4.3)$$
$$q = 5 \text{ W/m} \qquad m_p = 0 \qquad R = 0.02 \text{ m}$$

The penetration depth is from 5.2.7 d_0 = 2.74 m. The resulting extraction temperature 5.4.2 as a function of D is shown in figure 5.2.

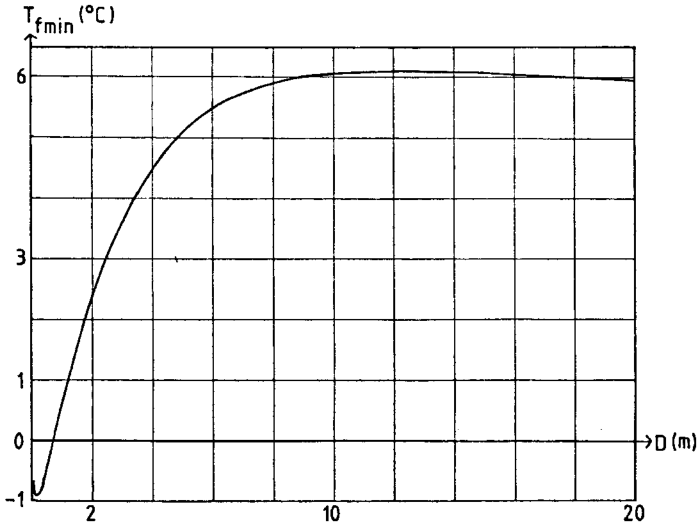


Figure 5.2. Minimum heat extraction temperature as a function of pipe depth. Data according to 5.4.3.

Values of D up to 20 m are included in order to illustrate the trend although they are not practically possible.

The curve for T_{fmin} is quite interesting. The extraction temperature T_{fmin} has a local minimum for a depth D = 0.15 m. There is a rapid increase from say D = 0.7 m to D = 4 m. In fact there is an increase all the way down to D = 12 m. From that on there is a minor decrease again. The character of the curve is due to the interaction between the logarithmic and exponential terms of 5.4.2.

It is interesting to note that the maximum lies so deep as 12 m. This is of course not a realistic depth to bury pipes. The lesson is however that the optimal depth, considering purely thermal aspects, is indeed very deep.

In order to analyse the general behaviour of $T_{fmin}(D)$ we introduce the function

$$f(s,p) = -e^{-s} - p\ \ln(s) \qquad (5.4.4)$$

The minimum extraction temperature 5.4.2 may then be written:

$$T_{fmin} = T_0 - q(\frac{1}{2\pi\lambda}\ \ln(\frac{2d_0}{R}) + m_p) + T_1 \cdot f(\frac{D}{d_0}, \frac{q}{2\pi\lambda T_1}) \qquad (5.4.5)$$

The function $f(s,p)$ is thus to be maximized as a function of s for fixed p. The derivative is

$$\frac{\partial f}{\partial s} = \frac{se^{-s}-p}{s} \qquad (5.4.6)$$

The function se^{-s} has the maximum $1/e = 0.37$ for $s=1$. The derivative is therefore always negative, if p is greater than $1/e$. The derivative has two zeros s_1 and s_2 $(0 < s_1 < 1 < s_2)$, when p is less than $1/e$. Figure 5.3 shows the character of $f(s,p)$.

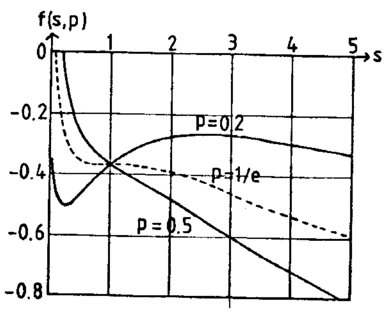


Figure 5.3. Character of the function $f(s,p)$ (5.4.4) for $p > 1/e$ and $p < 1/e$.

The minimum s_1 and the maximum s_2 are given in Table 5.1 for some values of p.

p	0.05	0.1	0.2	0.3	0.35	0.368
s_1	0.053	0.112	0.259	0.489	0.717	1
$f(s_1)$	-0.802	-0.675	-0.502	-0.399	-0.372	-0.368
s_2	4.50	3.58	2.54	1.78	1.32	1
$f(s_2)$	-0.086	-0.155	-0.265	-0.342	-0.364	-0.368

Table 5.1. Minimum and maximum of f(s,p) (1/e = 0.368).

Let us summarize. The minimum fluid temperature T_{fmin} for steady-state heat extraction by a single pipe, when the ground surface temperature varies sinusoidally during the year (5.2.5), is given by 5.4.5 and 5.4.4. The variation of this minimum temperature with the pipe depth D is as follows.

The value of T_{fmin} decreases steadily with increasing D if

$$p = \frac{q}{2\pi\lambda T_1} > \frac{1}{e} \tag{5.4.7}$$

There is a local minimum and a local maximum for T_{fmin}, when

$$0 < \frac{q}{2\pi\lambda T_1} < \frac{1}{e} \tag{5.4.8}$$

The local minimum occurs for $D/d_0 = s_1$ and the local maximum for $D/d_0 = s_2$. Here s_1 and s_2 are the two solutions of:

$$se^{-s} = p = \frac{q}{2\pi\lambda T_1} \tag{5.4.9}$$

$$(0 < s_1 < 1 < s_2)$$

Some values are given in Table 5.1. The minimum extraction temperature decreases in the interval $0 < D/d_0 < s_1$, increases in $s_1 < D/d_0 < s_2$, and decreases again in $D/d_0 > s_2$. The optimal depth occurs for $D/d_0 = s_2$.

It should be noted that the optimal depth lies quite deep.

6. HEAT EXTRACTION PULSES

The time-dependent part of the heat extraction rate may by superposition be regarded as a sum of extraction pulses of simpler character. An arbitrary extraction q(t) may in fact be regarded as a sum of elementary step changes. So the starting-point of the analysis will be the basic step line sink of 6.1.1 and figure 6.1. The temperature field of this case is discussed in section 6.1.

More complicated extraction rates are analysed in the following sections with the aid of the basic step line sink. We will not try to give a complete study of heat extraction pulses. The aim is to analyse and elucidate various dynamic effects of heat extraction.

6.1 Basic step line sink

Consider an infinite line sink with a heat extraction rate q(t) (W/m). The line sink lies along the y-axis. The temperature field is thus two-dimensional in the (x,z)-plane: T = T(x,z,t). The surrounding soil extends to infinity in all directions.

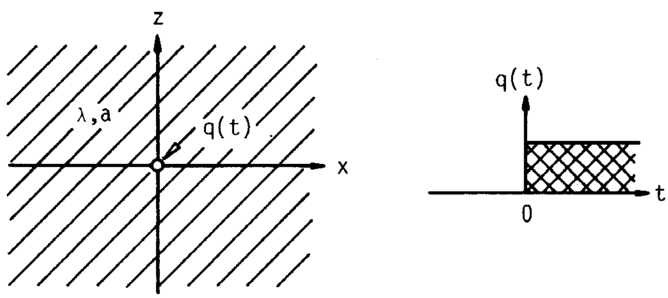


Figure 6.1. Line sink in an infinite surrounding.

The heat extraction rate is

$$q(t) = \begin{cases} 0 & t < 0 \\ q & t > 0 \end{cases} \tag{6.1.1}$$

A negative value of q means that heat is injected into the ground. The temperature in the ground at t = 0 is zero:

$$T(x,z,0) = 0 \qquad (6.1.2)$$

We will call this case the basic step line sink.

The solution for the basic step line sink is given by [3C]. The temperature is a function of the distance $r = \sqrt{x^2 + z^2}$ to the pipe and of time:

$$T(r,t) = -\frac{q}{4\pi\lambda} E_1\left(\frac{r^2}{4at}\right) \qquad (6.1.3)$$

The temperature in the ground is negative, as heat is extracted. The function E_1 is the so-called exponential integral:

$$E_1(x) = \int_x^{\infty} \frac{1}{s} e^{-s} ds \qquad (6.1.4)$$

This function is discussed and given in tables in [2A].

The expression 6.1.3 is quite simple. The temperature is a function of $r^2/(at)$ only. Let us consider the case:

$$\lambda = 1.5 \text{ W/m}^0\text{C} \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s}$$
$$q = 10 \text{ W/m} \qquad (6.1.5)$$

The resulting temperature profiles are shown in figure 6.2. The temperature development at certain distances r is shown in figure 6.3.

We are interested in the radial temperature profiles and the time development for a given radius. We therefore introduce two representations of the solution 6.1.3:

$$T(r,t) = \frac{q}{\lambda} \cdot E_r\left(\frac{r}{\sqrt{at}}\right) \qquad (6.1.6)$$

$$T(r,t) = \frac{q}{\lambda} \cdot E_t\left(\frac{at}{r^2}\right) \qquad (6.1.7)$$

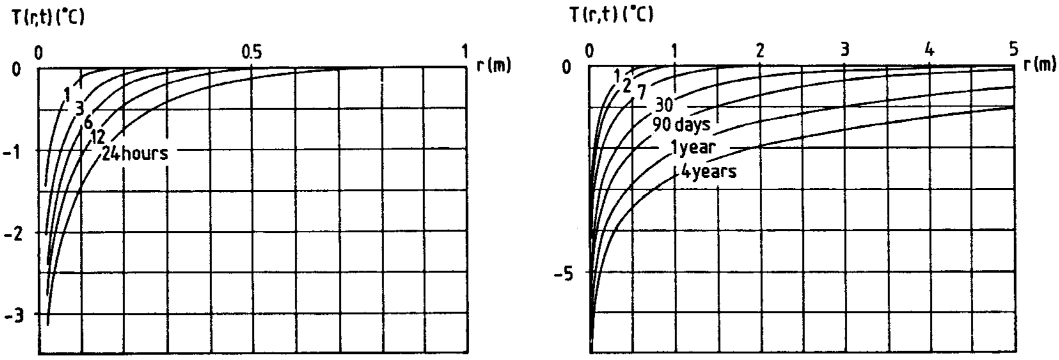


Figure 6.2. Temperature profiles at different times for the basic step line sink. Data according to 6.1.5.

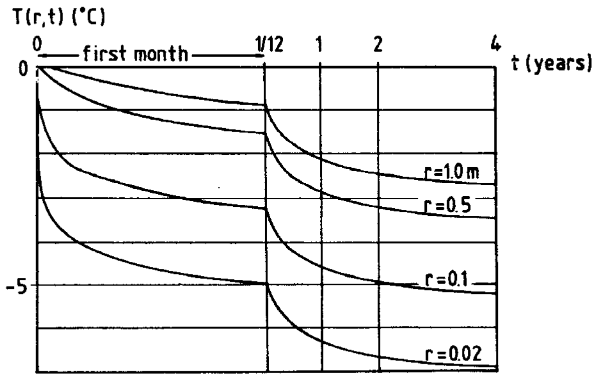


Figure 6.3 Temperature development at different distances from the basic step line sink. Data according to 6.1.5.

This means that

$$E_r(s) = -\frac{1}{4\pi} E_1\left(\frac{s^2}{4}\right) \qquad (6.1.8)$$

$$E_t(\tau) = -\frac{1}{4\pi} E_1\left(\frac{1}{4\tau}\right) \qquad (6.1.9)$$

The function $E_r(s)$, $s = r/\sqrt{at}$, gives the radial variation of the temperature at each time. It is given in table 6.1 and figure 6.4.

s	0.001	0.002	0.005	0.01	0.02	0.05	0.1
$E_r(s)$	-1.164	-1.053	-0.908	-0.797	-0.687	-0.541	-0.431

s	0.2	0.5	1	2	5	10
$E_r(s)$	-0.321	-0.180	-0.083	-0.018	$-2.2 \cdot 10^{-5}$	$-4.3 \cdot 10^{-14}$

Table 6.1. Radial temperature profile $E_r(s)$, $s = r/\sqrt{at}$, for the basic step line sink.

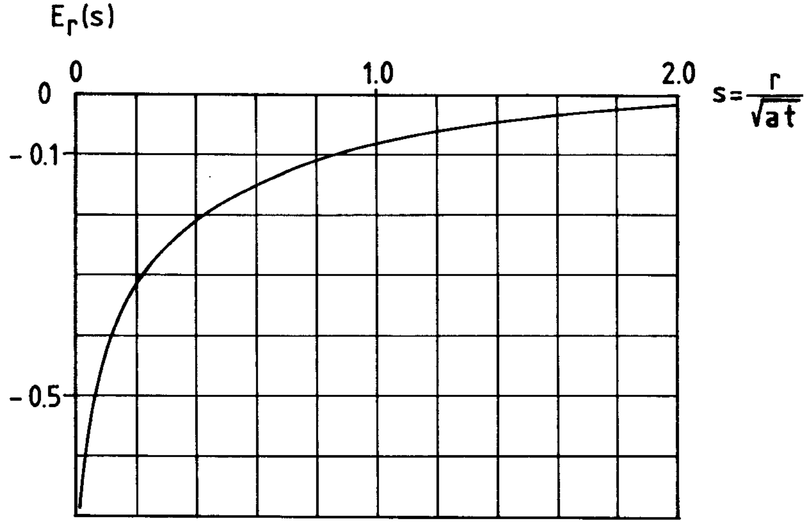


Figure 6.4. Radial temperature profile for the basic step line sink.

Asymptotically for large s we have from 6.1.8 and [2A]:

$$E_r(s) \simeq -\frac{1}{\pi s} e^{-\frac{s^2}{4}} \qquad (s = \frac{r}{\sqrt{at}} > 5) \qquad (6.1.10)$$

The temperature is extremely small for large s because of the exponential factor.

The temperature depends on $r/\sqrt{at}$. The length $\sqrt{at}$ is a measure of the influence range around the pipe. The temperature change in the region

$r/\sqrt{at} > 3$ is virtually zero. Let us consider an example:

$$a = 0.75 \cdot 10^{-6}\ m^2/s \qquad (6.1.11)$$

t	1 min	10 min	1 hour	24 hours	1 week
$\sqrt{at}$ (m)	0.007	0.02	0.05	0.25	0.67

t	1 month	3 months	1 year	3 years	10 years
$\sqrt{at}$ (m)	1.4	2.4	4.9	8.4	15

Table 6.2. Example of influence radius $\sqrt{at}$ for the basic step line sink.

The temperature development at a radius r was given by 6.1.7:

$$T = \frac{q}{\lambda} \cdot E_t(\tau) \qquad \tau = \frac{at}{r^2} \qquad (6.1.12)$$

Here τ is a dimensionless time. The function $E_t(\tau)$ is shown in figure 6.5 and given in table 6.3.

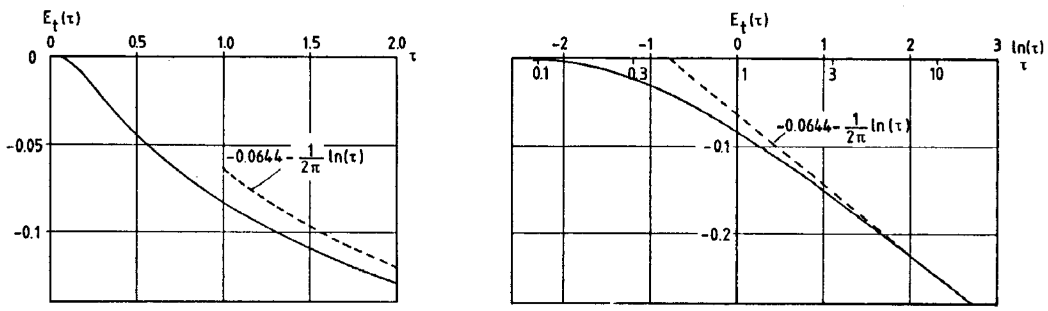


Figure 6.5. The function $E_t(\tau)$, $\tau = at/r^2$, which gives the temperature development for a given radius (6.1.12).

τ	E_t	τ	E_t	τ	E_t	τ	E_t
0.1	-0.00198	1	-0.0831	10	-0.250	100	-0.431
0.15	-0.00623	1.5	-0.109	15	-0.281	150	-0.463
0.2	-0.0117	2	-0.129	20	-0.304	200	-0.486
0.25	-0.0175	2.5	-0.145	25	-0.321	250	-0.504
0.3	-0.0233	3	-0.158	30	-0.336	300	-0.518
0.4	-0.0344	4	-0.180	40	-0.358	400	-0.541
0.5	-0.0446	5	-0.196	50	-0.376	500	-0.559
0.6	-0.0537	6	-0.210	60	-0.391	600	-0.574
0.7	-0.0621	7	-0.221	70	-0.402	700	-0.586
0.8	-0.0697	8	-0.232	80	-0.413	800	-0.596
0.9	-0.0767	9	-0.241	90	-0.423	900	-0.606

τ	E_t	τ	E_t	τ	E_t
1 000	-0.614	10 000	-0.797	100 000	-0.981
1 500	-0.646	15 000	-0.830	150 000	-1.013
2 000	-0.669	20 000	-0.852	200 000	-1.036
2 500	-0.687	25 000	-0.870	250 000	-1.053
3 000	-0.702	30 000	-0.885	300 000	-1.068
4 000	-0.724	40 000	-0.908	400 000	-1.091
5 000	-0.742	50 000	-0.925	500 000	-1.109
6 000	-0.757	60 000	-0.940	600 000	-1.123
7 000	-0.769	70 000	-0.952	700 000	-1.135
8 000	-0.780	80 000	-0.963	800 000	-1.146
9 000	-0.789	90 000	-0.972	900 000	-1.155

Table 6.3. The function $E_t(\tau)$.

For small values of τ we have from 6.1.9 and [2A]:

$$E_t(\tau) \simeq -\frac{\tau}{\pi}\, e^{-\frac{1}{4\tau}} \cdot (1 - 4\tau + 32\tau^2 \ldots) \qquad (6.1.13)$$

We see again $E_t(\tau)$ is quite small for small τ due to the exponential factor.

For large values of τ we have from 6.1.9 and [2A] the important representation:

$$E_t(\tau) \simeq -0.0644 - \frac{1}{4\pi}\ln(\tau) - \frac{1}{16\pi}\left(\frac{1}{\tau} - \frac{1}{16}\cdot\frac{1}{\tau^2} + \ldots\right) \qquad (6.1.14)$$

In particular we have the useful approximation

$$E_t(\tau) \simeq -0.0644 - \frac{1}{4\pi}\ln(\tau) \qquad (6.1.15)$$

The error in the range $\tau \geqq 5$ is less than 2%.

The denominator of dimensionless time $\tau = t/(r^2/a)$ is a time-scale for the process at the distance r from the line sink. Let us consider a numerical example:

$$a = 0.75 \cdot 10^{-6}\ m^2/s$$

r(m)	0.01	0.03	0.05	0.10	0.5
r^2/a	2 min	20 min	1 hour	4 hours	4 days

r(m)	1	2	5
r^2/a	15 days	2 months	1 year

Table 6.4. Time-scale factor for different distances.

This table is quite instructive. Consider for example the radius r = 1m. The time-scale factor is 15 days. The influence of the line sink starts to be felt for τ equal, say, 0.2. This means that it is felt after 3 days at the radius r = 1m. The temperature rise after 15 days is $-0.08 \cdot q/\lambda$. The logarithmic approximation 6.1.15 is followed reasonably after 30 days and very closely after 75 days.

Our main interest is the temperature at the radius r = R of the pipe:

$$T_R = T(R,t) = \frac{q}{\lambda} E_t\left(\frac{at}{R^2}\right) \qquad (6.1.16)$$

The function E_t shows with appropriate scaling directly the temperature development at the pipe radius. Let us consider the following case with data in accordance with reference case (3.2):

$$\lambda = 1.5\ W/m^{0}C \qquad a = 0.75 \cdot 10^{-6}\ m^2/s$$
$$q = 10\ W/m \qquad R = 0.02\ m \tag{6.1.17}$$

Then we have:

$$\frac{R^2}{a} = 533\ s = 9\ min \qquad \frac{q}{\lambda} = 6.7^{0}C \tag{6.1.18}$$

The logarathmic formula is valid with very good accuracy after 45 minutes:

$$T_R = -6.7\ (0.0644 + \frac{1}{4\pi}\ ln\left(\frac{t}{533}\right)) =$$
$$= -1.44 - 0.53\ ln\left(\frac{t}{3600}\right) \qquad \left(\frac{t}{3600} > \frac{3}{4}\right) \tag{6.1.19}$$

We get a very simple expression for T_R according to the last line of 6.1.19. We have the following values:

t	1 h	2 h	5 h	24 h	7 d	30 d	90 d	1 y
$T_R(^{0}C)$	-1.4	-1.8	-2.3	-3.1	-4.2	-4.9	-5.5	-6.3

The temperature at the pipe is to be lowered according to this suite of values in order ot obtain the prescribed extraction rate q=10 W/m. It is note-worthy that the required decrease changes rapidly in the beginning. The change becomes slower and slower as time goes. Let us compute the times that correspond to T_R equal $-1^{0}C$, $-2^{0}C$ and so on:

$$\begin{aligned}
T_R &= -1^{0}C & t &= 24\ min\\
T_R &= -2^{0}C & t &= 3\ h\\
T_R &= -3^{0}C & t &= 19\ h\\
T_R &= -4^{0}C & t &= 5\ d\\
T_R &= -5^{0}C & t &= 34\ d\\
T_R &= -6^{0}C & t &= 230\ d\\
T_R &= -7^{0}C & t &= 4\ y
\end{aligned} \tag{6.1.20}$$

We get a time-span from 24 minutes to 4 years.

The time-scale factor R^2/a at the pipe radius is quite small. The logarithmic expression 6.1.15 for E_t is therefore soon valid. We have then:

$$T_R(t) = -\frac{q}{\lambda}\left(0.0644 + \frac{1}{4\pi}\ln\left(\frac{at}{R^2}\right)\right) \quad (t > 5R^2/a) \qquad (6.1.21)$$

Formula 6.1.21 is very useful due to its simplicity.

The line heat sink extracts heat at the rate q at r = 0. The heat flux at a finite radius becomes by derivation of 6.1.3-4:

$$-\lambda 2\pi r \frac{\partial T}{\partial r} = -q \cdot e^{-\frac{r^2}{4at}} \qquad (6.1.22)$$

The heat flux is because of the exponential factor less than q for $r > 0$. In fact we have:

$\frac{at}{r^2}$	0.1	0.5	1	2	5	10	20
$e^{-\frac{r^2}{4at}}$	0.08	0.61	0.78	0.88	0.95	0.98	0.99

Our requirement that the heat flux at r = R shall be equal to q is not fulfilled for small values of at/R^2. The heat flux is within an error of 5% equal to q, when at/R^2 exceeds 5. Thus we have the following requirement.

The line sink solution is applicable after a time:

$$t > 5 \cdot \frac{R^2}{a} \qquad (6.1.23)$$

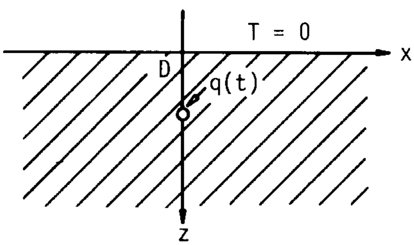
This is not any severe limitation in the present applications. In example 6.1.17 we have then from 6.1.18 the requirement that t is to be greater than 45 minutes. We discuss here time periods of months, weeks and days. Condition 6.1.23 is then of no consequence. It should however always be remembered that the present theory does not include heat extraction periods which are shorter than, say, a few hours. Shorter variations are not considered in this study.

Here and in the following we mostly deal with the temperature T_R at the pipe radius. The thermal resistance m_p between the soil at the pipe and the heat carrier fluid gives an additional temperature fall to the fluid. We have from 4.3.1:

$$T_f(t) = T_R(t) - m_p \cdot q(t) \qquad (6.1.24)$$

6.2 Step extraction pulse

The solution 6.1.3 is valid for a step extraction pulse under the assumption of an infinite surrounding ground. The corresponding step extraction pulse for a pipe at depth z = D has a semi-infinite surrounding $z > 0$, $-\infty < x < \infty$. The boundary condition at the ground surface z = 0 is that the temperature shall be zero. See figure 6.6.



$$q(t) = \begin{cases} 0 & t < 0 \\ q & t > 0 \end{cases}$$

Figure 6.6. Step extraction pulse to a pipe at a depth D.

The initial condition 6.1.2 is still valid.

The solution to this new problem is readily obtained from the previous case by introducing a mirror sink at x = 0, z = -D with the opposite strength -q(t). The temperature at the ground surface is then by anti-symmetry automatically zero. We have by superposition from 6.1.3 the temperature field due to a step extraction pulse.

$$T(x,z,t) = -\frac{q}{4\pi\lambda} \left\{ E_1\left(\frac{x^2+(z-D)^2}{4at}\right) - E_1\left(\frac{x^2+(z+D)^2}{4at}\right) \right\} \qquad (6.2.1)$$

The nominators of the arguments for E_1 are the square of the distance from the point (x,z) to the pipe and to the mirror pipe.

The thermal process in the ground has the characteristic time scale t_D which is given by (3.2):

$$t_D = \frac{2D^2}{a} \qquad (3.2)$$

Let us consider the reference example 3.2. The characteristic time t_D is then 1 month (3.3). The temperature field due to the heat extraction pulse at the times $t_D/2$, t_D, $2t_D$ and $t = \infty$ are shown in figure 3.2. The temperature field is also illustrated in figures 3.3 and 3.4. The temperature T_R at the pipe is shown in figure 3.5. We see from the figures that the thermal process around the extraction pipe is quite uninfluenced by the ground surface or the mirror pipe for $t < t_D$. The process near the pipe is on the other hand virtually a steady-state one for $t > 2t_D$.

In the limit, when t goes to infinity, we can use expression 6.1.15. The temperature 6.2.1 becomes a difference between two logarithms, where the time t cancels. The steady-state temperature distribution 4.1.2 is then obtained.

The temperature at the pipe periphery $x^2 + (z-D)^2 = R^2$ is according to 6.2.1 and 6.1.9:

$$T_R(t) = \frac{q}{\lambda}\left\{E_t\left(\frac{at}{R^2}\right) - E_t\left(\frac{at}{4D^2}\right)\right\} \qquad (6.2.2)$$

Here the approximation 4.1.5 is used for the distance from the pipe periphery to the mirror line sink.

The logarithmic expression 6.1.15 can be used for the first E_t-function of 6.2.2.

We get from 6.2.2 and 6.1.5:

$$T_R(t) = -\frac{q}{2\pi\lambda}\ln\left(\frac{2D}{R}\right) + \frac{q}{\lambda}E_p\left(\frac{at}{D^2}\right) \tag{6.2.3}$$

$$\left(\frac{at}{R^2} > 5 \quad \text{or} \quad \frac{at}{D^2} > 5\left(\frac{R}{D}\right)^2\right)$$

The function $E_p(\tau)$ is given by:

$$E_p(\tau) = -\ 0.0644 - \frac{1}{4\pi}\ln(\tfrac{\tau}{4}) - E_t(\tfrac{\tau}{4}) \tag{6.2.4}$$

The first part of 6.2.3 represents the steady-state extraction according to 4.1.7. The second part, which contains the function E_p gives the transient effect. It should be noted that this part depends only on at/D^2 with the scale factor q/λ.

The function $E_p(\tau)$ is shown in figure 6.7. For large τ we have the series expansion from 6.1.14 and 6.2.4:

$$E_p(\tau) = \frac{1}{4\pi}\left\{\frac{1}{\tau} - \frac{1}{4\tau^2} + \ldots\right\} \tag{6.2.5}$$

For small τ we have from 6.2.4 and table 6.3:

$$E_p(\tau) \simeq -\ 0.0644 - \frac{1}{4\pi}\ln\left(\tfrac{\tau}{4}\right) \qquad (\tau < 0.4) \tag{6.2.6}$$

Let us consider the reference example 3.2 again. Figure 6.8 shows the pipe temperature $T_R(t)$ from figure 3.5 with a logarithmic time-scale. The curve is closely approximated by the two straight lines 6.2.8A and 6.2.8B. The first line 6.2.8A is obtained from 6.2.3 and 6.2.6. It is in fact the temperature 6.1.21 from the pipe alone without the mirror pipe. The second line gives the steady-state heat extraction. We get a quite good approximation of T_R by using the two straight lines.

The two straight lines coincide for

$$\tau = \frac{at}{D^2} \simeq 1.78 \tag{6.2.7}$$

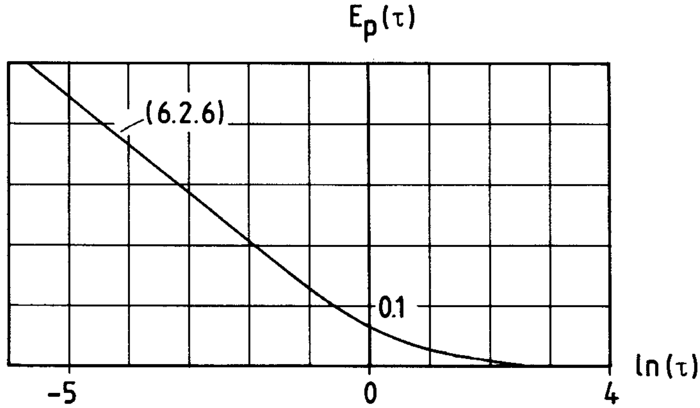


Figure 6.7. The function $E_p(\tau)$ which gives the transient part of the pipe temperature for a heat extraction pulse (6.2.3).

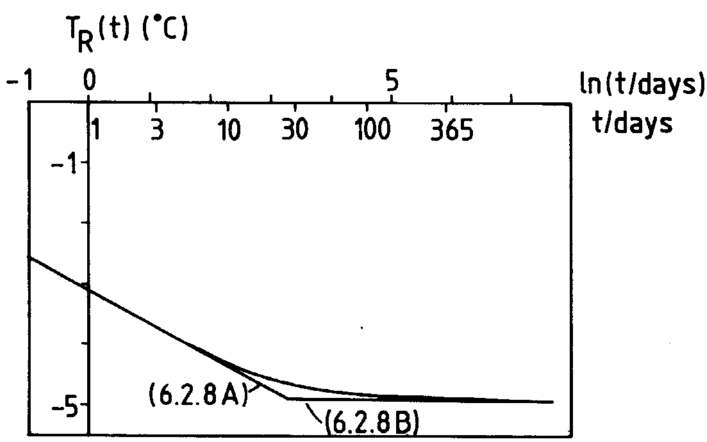


Figure 6.8 Pipe temperature T_R and the approximations 6.2.8A,B for a step extraction pulse. Data according to 3.2.

We have the following simple expressions for the pipe temperature during a step extraction pulse:

$$T_R(t) \simeq \begin{cases} -\frac{q}{\lambda}\cdot\left\{0.0644 + \frac{1}{4\pi}\ln\left(\frac{at}{R^2}\right)\right\} & \frac{at}{D^2} \leq 1.78 \quad (6.2.8A) \\ & \left(\frac{at}{R^2} > 5\right) \\ -\frac{q}{2\pi\lambda}\cdot\ln\left(\frac{2D}{R}\right) & \frac{at}{D^2} \geq 1.78 \quad (6.2.8B) \end{cases}$$

The largest error occurs at the breaking-point of the two lines. The relative error at that point is

$$\frac{T_R(\infty) - T_R(1.8D^2/a)}{T_R(\infty)} = \ldots = \frac{-2\pi\ E_t(1.78/4)}{\ln\left(\frac{2D}{R}\right)} \simeq \frac{0.25}{\ln(\frac{2D}{R})} \quad (6.2.9)$$

The logarithm in the denominator is of the order 3-5, so the maximal relative error for 6.2.8 is only around 6%.

The two expressions 6.2.8A and B illustrates the characteristic time $t_D = 2D^2/a$. For $t < t_D$ (or to be precise $t < 0.89t_D$) the single-pipe expression 6.2.8A is valid, while steady-state conditions prevail for $t > t_D$ according to 6.2.8B.

6.3 Temperature recovery after a pulse

The thermal process after a heat extraction pulse is of great interest, since during a new heat extraction period, one has to work against the residual temperature fields from previous pulses. The negative temperatures of the pipe from preceding pulses are to be added in order to get the total extraction temperature.

We consider the thermal recovery process during $t > 0$ after an extraction pulse during $-t_1 < t < 0$:

$$q(t) = \begin{cases} 0 & t < -t_1 \\ q & -t_1 < t < 0 \\ 0 & t > 0 \end{cases} \quad (6.3.0)$$

The pulse 6.3.0 may be considered as the sum of a pulse +q from $t = -t_1$ and a step pulse -q from t = 0. Then we have from 6.2.1 by superposition:

$$T(x,z,t) = -\frac{q}{4\pi\lambda}\left\{ E_1\left(\frac{x^2+(z-D)^2}{4a(t+t_1)}\right) - E_1\left(\frac{x^2+(z+D)^2}{4a(t+t_1)}\right) - E_1\left(\frac{x^2+(z-D)^2}{4at}\right) + E_1\left(\frac{x^2+(z+D)^2}{4at}\right)\right\}$$

$$(t > 0) \qquad (6.3.1)$$

The limit $t_1 \to \infty$ is also of interest. The two first terms of 6.3.1 tends to the steady-state solution 4.1.2, when t_1 tends to infinity according to 6.1.9 and 6.1.15. The temperature field 6.3.1 will in this case give the thermal recovery after a very long extraction pulse. At the time t = 0 one starts with the steady-state temperature field 4.1.2.

Let us consider the reference case 3.2. The thermal recovery for $t > 0$ after a pulse q = 10W/m during $t < 0$ is shown in figure 3.6. The temperature along the line x = 0, $z < 0$ is shown in figure 3.6A for different times. The movement of the isotherm T = -0.5°C is shown in figure 3.6B.

The temperature at the pipe is as usual the most important quantity. We have in accordance with 6.3.1, 6.2.1 and 6.2.2:

$$T_R(t) = \frac{q}{\lambda}\left\{E_t\left(\frac{a(t+t_1)}{R^2}\right) - E_t\left(\frac{a(t+t_1)}{4D^2}\right) - E_t\left(\frac{at}{R^2}\right) + E_t\left(\frac{at}{4D^2}\right)\right\} \qquad (t > 0) \qquad (6.3.2)$$

For the first and third term we can use (for greater than, say, an hour) the approximation 6.1.15:

$$T_R(t) = \frac{q}{\lambda}\left\{-\frac{1}{4\pi}\ln\left(\frac{t+t_1}{t}\right) - E_t\left(\frac{a(t+t_1)}{4D^2}\right) + E_t\left(\frac{at}{4D^2}\right)\right\} \qquad (6.3.3)$$

$$\left(\frac{at}{R^2} > 5\right)$$

If the influence of the mirror sink is negligible we get the following simple formula for the temperature recovery at the pipe after an extraction pulse:

$$T_R(t) = -\frac{q}{4\pi\lambda} \cdot \ln\left(\frac{t+t_1}{t}\right) \tag{6.3.4}$$

The temperature recovery after a pulse may be expressed with the E_p function 6.2.4. From 6.3.3 and 6.2.4 or directly by superposition from 6.2.3 we have the alternative expression:

$$T_R(t) = \frac{q}{\lambda}\left\{ E_p\left(\frac{2(t+t_1)}{t_D}\right) - E_p\left(\frac{2t}{t_D}\right)\right\} \tag{6.3.5}$$

Here the characteristic time $t_D = 2D^2/a$ is used. The dimensionless pipe temperature $\lambda T_R/q$ is a function of t/t_D with the parameter t_1/t_D. It is shown in figure 6.9.

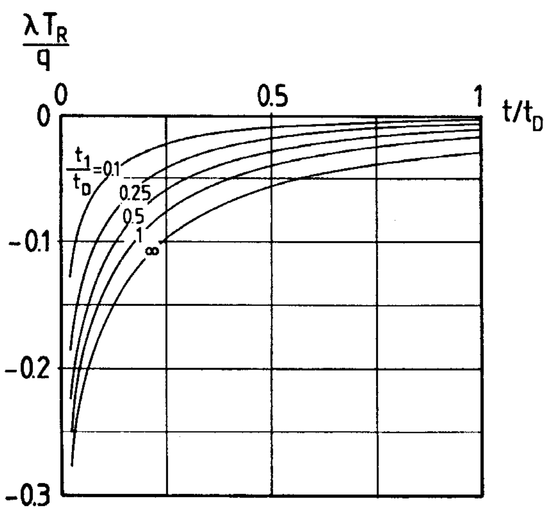


Figure 6.9. Thermal recovery after an extraction pulse during $-t_1 \leq t \leq 0$ according to 6.3.5.

The series expansion 6.2.5 gives for 6.3.5:

$$T_R(t) = -\frac{q}{8\pi\lambda} \cdot \frac{t_D t_1}{t(t+t_1)}\left(1 - \frac{t_D(t_1+2t)}{8t(t+t_1)} + \ldots\right) \tag{6.3.6}$$

So we have the following simple formula for the temperature recovery after an extraction pulse:

$$T_R(t) \simeq - \frac{q}{8\pi\lambda} \cdot \frac{t_D t_1}{t(t+t_1)} \qquad (t > 3t_D) \qquad (6.3.7)$$

The error of the approximation is obtained from the next term of 6.3.6.

Let us consider the reference example 3.2:

$$\frac{q}{\lambda} = \frac{10}{1.5}\ {}^{o}C \qquad t_D = 1 \text{ month}$$

Then we have for example with 6.3.7:

$$t = 3 \text{ months} \quad \begin{array}{ll} t_1 = 0.1 \text{ month} & T_R = -0.003^{o}C \\ t_1 = 1 \text{ month} & T_R = -0.022^{o}C \\ t_1 = 6 \text{ months} & T_R = -0.059^{o}C \\ t_1 = \infty & T_R = -0.088^{o}C \end{array} \qquad (6.3.8)$$

We see that the remaining temperature three months after the end of the pulse is quite small.

It is sometimes suggested that one should <u>recharge the ground</u> around the pipe during the summer. Let us consider a case, when the ground is recharged strongly during three months. The value of q is then negative. We take:

$$\begin{array}{ll} q = -30 \text{ W/m} & \lambda = 1.5 \text{ W/m}^{o}C \\ D = 1 \text{ m} & a = 0.75 \ \ 10^{-6} \text{ m}^2/s \\ (t_D = 1 \text{ month}) & \end{array} \qquad (6.3.9)$$

Let us compute the temperature in the mid-winter. We have from 6.3.9 and 6.3.7:

$$t_1 = 3 \text{ months} \qquad t = 3 \text{ months}$$

$$T_R = - \frac{-30}{8\pi \cdot 1.5} \cdot \frac{1 \cdot 3}{3 \cdot (3+3)} = 0.13^{\circ}C \qquad (6.3.10)$$

The ground is only $0.13^{\circ}C$ warmer than the natural case without recharge.

We can draw the important conclusion that it is completely useless to recharge the ground in order to improve the heat extraction after a period of several months. We see from the curves of figure 6.9 that the temperature decline is quite rapid. So even short term recharge is normally not of any value.

Formula 6.3.7 for the temperature recovery after a pulse is valid for times that are much greater that t_D. We will derive another asymptotic formula which is valid after a time that is much greater than the pulse length t_1. Consider a Taylor expansion of the two E_p-functions (6.3.5) around the point $\tau = (2t+t_1)/t_D$. The lowest term, which is a good approximation of $T_R(t)$ after some time, becomes:

$$T_R(t) \simeq - \frac{q}{2\pi\lambda} \cdot \frac{t_1}{2t+t_1} \cdot \left(1 - e^{-\frac{t_D}{2t+t_1}}\right) \qquad (t \geqq 2t_1) \qquad (6.3.11)$$

The next term of the Taylor expansion is:

$$- \frac{q}{2\pi\lambda} \cdot \frac{t_1}{2t+t_1} \cdot \left(\frac{t_1}{2t+t_1}\right) \cdot f \qquad 0 \leqq f \leqq 0.41 \qquad (6.3.12)$$

The factor f is a function of $t_D/(2t+t_1)$, which lies in the indicated range for positive t. The formula 6.3.11 is therefore valid for, say, t greater than $2t_1$.

Let us consider a case of a strong rechange during one day:

$$q = -30 \text{ W} \qquad \lambda = 1.5 \text{ W/m}^{\circ}\text{C}$$

$$t_1 = 1 \text{ day} \qquad t_D = 30 \text{ days} \qquad (6.3.13)$$

Then we have after the pulse from 6.3.11:

$$T_R(t) \simeq - \frac{-30}{2\pi \cdot 1.5} \cdot \frac{1}{2t+1} \cdot \left(1 - e^{-\frac{30}{2t+1}}\right) \qquad (6.3.14)$$

(t in days, t > 2 days)

t(days)	2	3	4	5	10	30	100
T_R (^oC)	0.64	0.45	0.34	0.27	0.12	0.02	0.002

We see as before that the effect of the strong recharge during one day vanishes within a few days.

6.4 Superposition of pulses

We will in this section only consider a few examples for which the process is analysed as a superposition of heat extraction pulses. We will use the data of the reference case:

$$\lambda = 1.5 \text{ W/mK} \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \quad D = 1 \text{ m} \qquad (6.4.1)$$

Example 1.

Let us first consider a case for which the heat extraction rate is constant during each month. The total amount of extracted heat is 12 0000 kWh which may represent the demand of a one-family house. The total pipe length is taken to be 200 meters. We assume that the different parts of the pipe do not influcence each other. The required extraction rate during each month is shown on top in figure 6.10. The highest extraction rate 16.7 W/m occurs during the fifth month.

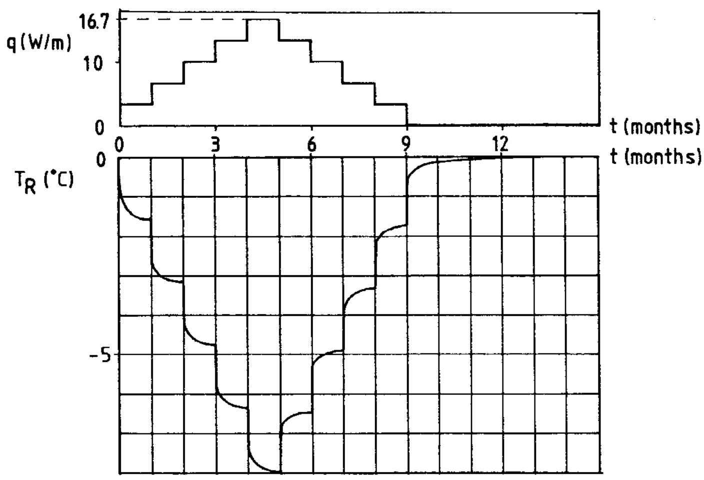


Figure 6.10. Prescribed extraction rate (top) and required pipe temperatures (bottom) of example 1.

The pipe temperature T_R is obtained as a superposition from the pulse of the present month, eq. 6.2.2, and remnants from the previous pulses, eq. 6.3.2. The procedure is quite simple.

The pipe temperature is shown in figure 6.10. The shape of the temperature curve during each month is very characteristic. There is, when the extraction rate is changed, a rapid temperature change during the first days.

The lowest extraction temperature $T_R = -7.9^oC$ is obtained at the end of the fifth month. A superimposed daily variation of the extraction rate will lower this temperature. This will be illustrated below.

There is no heat extraction during the last three months. It is noteworthy that the pipe temperature T_R has virtually recovered its initial value at the end of the year. In fact we have during the recovery period:

t(month)	9	9.1	9.5	10	11	12
$T_R(^oC)$	-1.7	-0.48	-0.21	-0.13	-0.07	-0.05

Example 2

In this example we consider an extraction period of one week and an ensuing recovery period. The case with a constant heat extraction rate q = 15 W/m is compared to a case with daily pulses as shown in figure 6.11 (top). The mean extraction rate is taken to be 15 W/m.

The pipe temperature $T_R(t)$ is obtained by superposition of the daily pulses. We see in figure 6.11 how the pipe temperature (full line) varies around the pipe temperature in the case of constant extraction (dashed line).

The minimum extraction temperature in the variable case is -10^oC, while the minimum temperature in the constant case is -6.5^oC. The temperature during the recovery is however very similar. The difference after three days is only a few tenths of a degree.

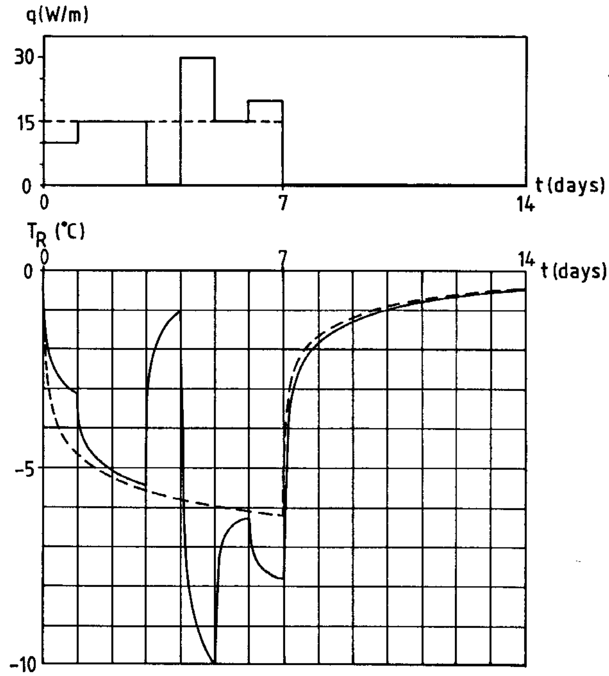


Figure 6.11. Extraction rate and pipe temperature in a comparison of a constant pulse during a week with variable daily pulses.

Example 3

Let us now consider even shorter pulses of two hours. These can still be handled by the line source solutions with reasonable accuracy. The three pulses during a period of 12 hours are shown on top in figure 6.12. The dashed line shows the corresponding case of constant extraction with the same mean effect. The ensuing pipe temperatures are shown below.

The temperature at the pipe during the thermal recovery after the extraction period is almost the same in both cases. We note that the pulsating case requires a minimum temperature of -4.3^{o}C, while the case with constant extraction requires -2.8^{o}C. The pulsation may increase considerably the required extraction temperature difference. The daily and hourly variation of extraction rate may be quite important.

Example 4

As a last example we will consider a series of diurnal pulses during 18 days. Only the pulsating part is considered, so the mean extrac-

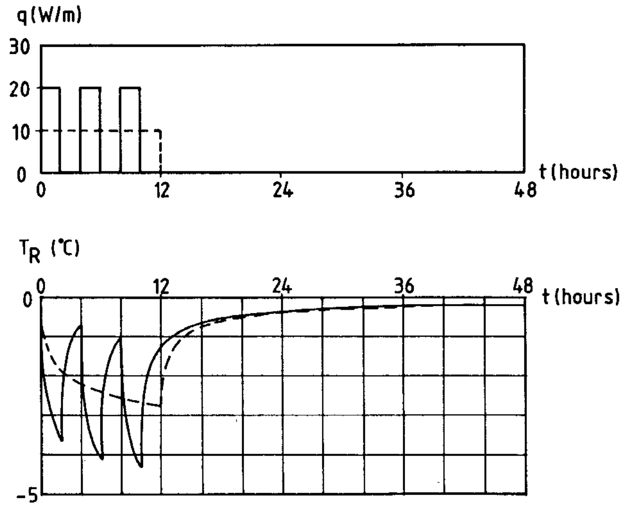


Figure 6.12. Extraction rate and pipe temperature for example 3 with two-hour pulses.

tion rate is zero. There is a unit extraction rate during the first day. During the next day there is a unit heat injection, and so on. See top of figure 6.13. The ensuing pipe temperature is shown below.

The shapes of the pulses are very similar. The process becomes rapidly periodic with the period time of 2 days. The table below shows the temperature in the middle of each extraction period:

t(days)	0.5	2.5	4.5	6.5	8.5	10.5
T_R(°C)	-0.276	-0.246	-0.240	-0.238	-0.237	-0.2367

t(days)	12.5	14.5	16.5
T_R(°C)	-0.2363	-0.2360	-0.2357

The change from the first to the second pulse is 10%, from the second to the third 2%. Thereafter the change is less than 1%. We may conclude that after a few cycles of the pulse train the process is periodic to a high degree.

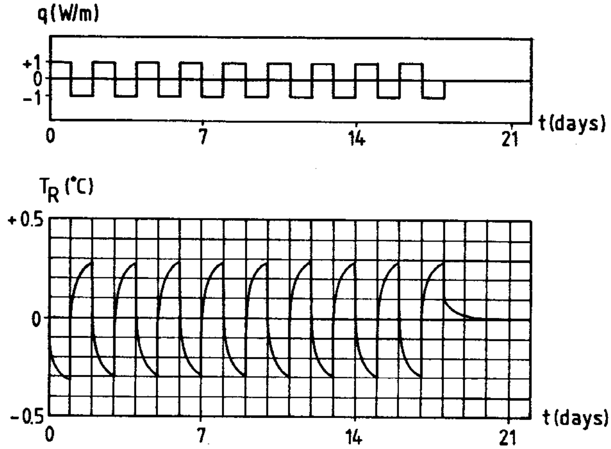


Figure 6.13. Extraction rate and pipe temperature for example 4 with a pulse train during 18 days.

6.5. Sequence of pulses

An important special case is a sequence of equal pulses. The time between the beginning of the pulses is t_0, and the length of the pulse is αt_0, $0 < \alpha < 1$. Let q_0 denote the mean extraction rate. The extraction rate of the pulse is then q_0/α. The pulse train is shown in figure 6.14.

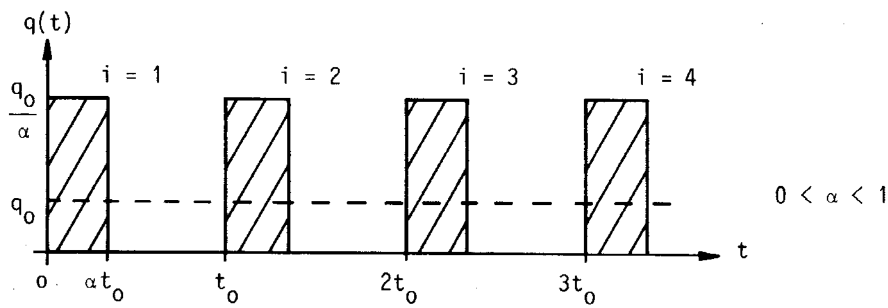


Figure 6.14. Sequence of heat extraction pulses with the mean extraction rate q_0.

The case of an infinite surrounding ground is first considered. The effect of the ground surface and a negative mirror pulse is discussed at the end of this section.

The pipe temperature is obtained by superposition of step line sinks (6.1.3). We consider first an example:

$$t_o = 1 \text{ day} \qquad q_o = 10 \text{ W/m} \qquad \alpha = 1/3$$

$$\lambda = 1.5 \text{ W/mK} \qquad a = 0.75\ 10^{-6}\ m^2/s \tag{6.5.1}$$

The obtained pipe temperature is shown in figure 6.15. The dashed line shows the extraction temperature with the constant mean value q_o.

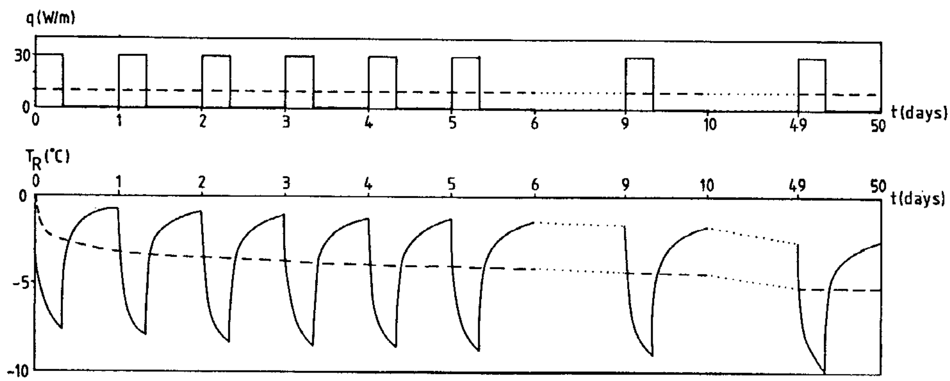


Figure 6.15. Pipe temperature for example 6.5.1. The dashed line refers to the corresponding mean extraction rate.

The pipe temperature of the pulse train oscillates around the temperature from the mean extraction rate. The transient change from day to day diminishes rapidly. The oscillation around the mean-extraction curve is very similar during the fifth, tenth and fiftieth day.

These oscillations around the mean-extraction curve represent the effect of the pulsation. This effect is to be superimposed on the effect of our previous analyses without pulsation.

The analysis does not have to be restricted to the pipe radius $r = R$. So we will consider an arbitrary radius r. Let $T^o(r,t)$ denote the temperature for the constant extractions rate q_o. From eq. 6.1.3 we have

$$T^{o}(r,t) = -\frac{q_o}{4\pi\lambda} \cdot E_1\left(\frac{r^2}{4at}\right) \qquad (6.5.2)$$

The oscillating part is denoted $T^*(r,t)$. The total temperature at a distance r from the pipe is then

$$T(r,t) = T^{o}(r,t) + T^*(r,t) \qquad (6.5.3)$$

The expression for T^* is obtained by superposition of E_1-functions with the effect factor q_o/α and the argument $r^2/4a(t-t')$. The time t' is the start of each step pulse. The two components of $T(r,t)$ are shown in figure 6.16.

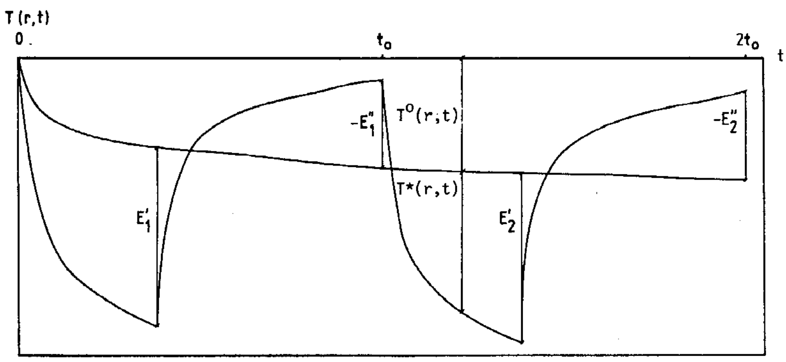


Figure 6.16. Temperature $T(r,t)$ due to a pulse train. Definition of E_i' and E_i''.

The extreme values of the oscillation T^* are our main interest. These values are obtained at the end of the pulse (E_i' for pulse i) and at the end of the period (E_i'' for period i). See figure 6.16. We introduce for period i the notations

$$T^*(r,\ t_o(i-1) + \alpha t_o) = -\frac{q_o}{\lambda} E_i'\left(\frac{r^2}{at_o}, \alpha\right)$$

$$T^*(r,\ t_o\, i) = -\frac{q_o}{\lambda} E_i''\left(\frac{r^2}{at_o}, \alpha\right) \qquad (6.5.4)$$

The values E_i' and E_i'' are functions of $r^2/(at_o)$ and the relative pulse length α. The functions E_1', E_1'', E_5' and E_5'' are shown in figures 6.17 - 6.20. The functions for other pulses are normally not necessary to have, since the variation with i is rather small.

Our first example concerns a pulse train with the period of two hours:

$$t_o = 2 \text{ hours} \qquad \alpha = 0.5$$

$$q_o = 10 \text{ w/m} \qquad R = 0.02 \text{ m} \tag{6.5.5}$$

$$\lambda = 1.5 \text{ W/mK} \qquad a = 0.75\ 10^{-6}\ \text{m}^2/\text{s}$$

The result is given in table 6.5.1.

Period	t(hours)	E_i'	E_i''	$T^o(R,t)$	$T^*(R,t)$	$T(R,t)$
i = 1	1	0.218		-1.46	-1.46	-2.92
	2		-0.166	-1.82	+1.103	-0.72
i = 2	3	0.198		-2.03	-1.316	-3.35
	4		-0.175	-2.18	+1.165	-1.02
i = 5	9	0.188		-2.61	-1.259	-3.87
	10		-0.1804	-2.67	+1.203	-1.46
i = 10	19	0.1865		-3.01	-1.244	-4.25
	20		-0.1825	-3.03	+1.126	-1.82
i = 20	39	0.1855		-3.39	-1.236	-4.62
	40		-0.1834	-3.40	+1.223	-2.18
i = 100	199	0.1846		-4.25	-1.231	-5.48
	200		-0.1842	-4.25	+1.228	-3.03
i = 500	999	0.1845		-5.106	-1.230	-6.34
	1000		-0.1844	-5.106	+1.229	-3.88

Table 6.5.1. The different contributions to the pipe temperature for a pulse train. Data according to (6.5.5)

The contribution T^o increases of course steadily. The oscillation T^* around T^o stabilizes rapidly. The change from the fifth to the hundredth pulse is only 0.03°C. The change from the first to the fifth pulse is around 0.2°C.

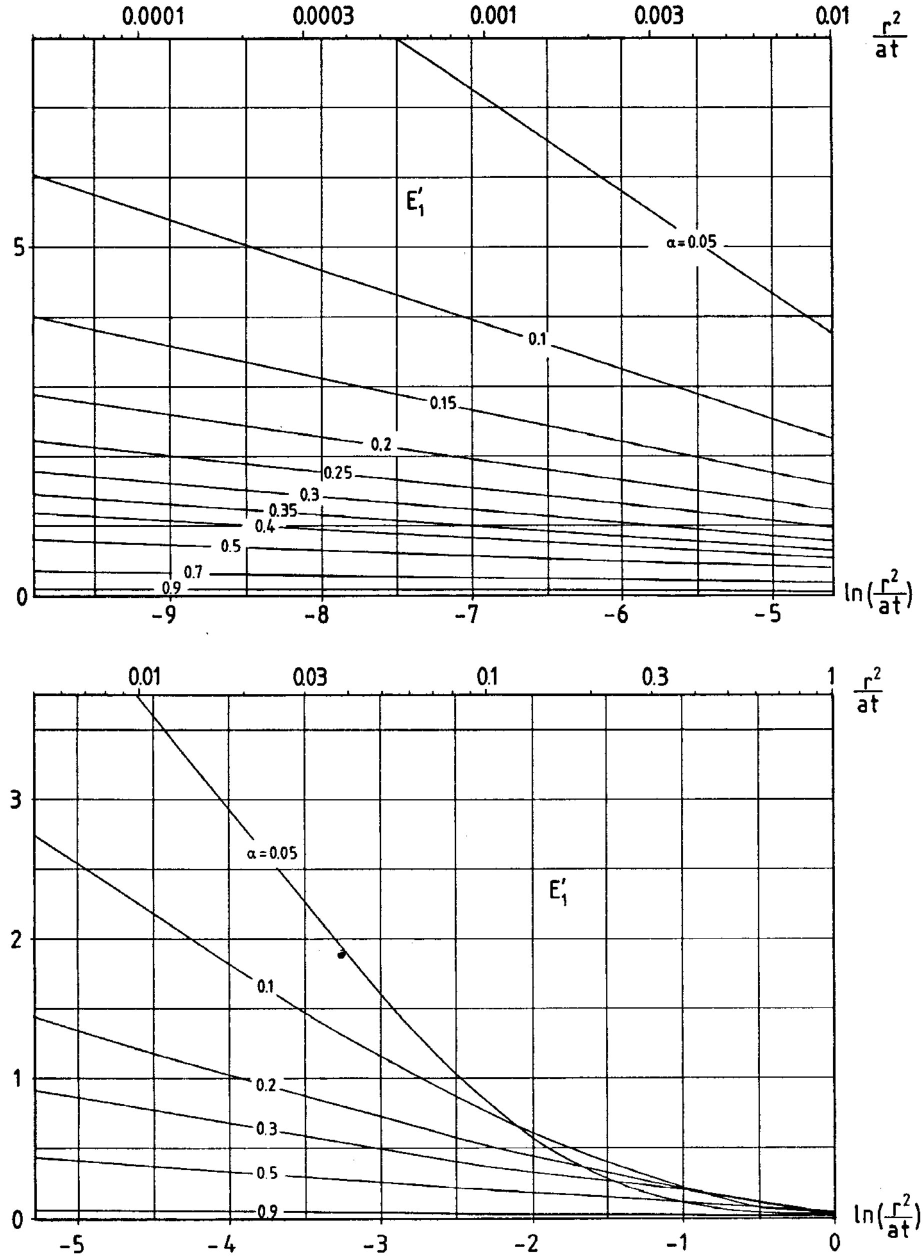


Figure 6.17. Pulse extraction function E_1' for the first pulse. See figure 6.16 and eqs. 6.5.2-4.

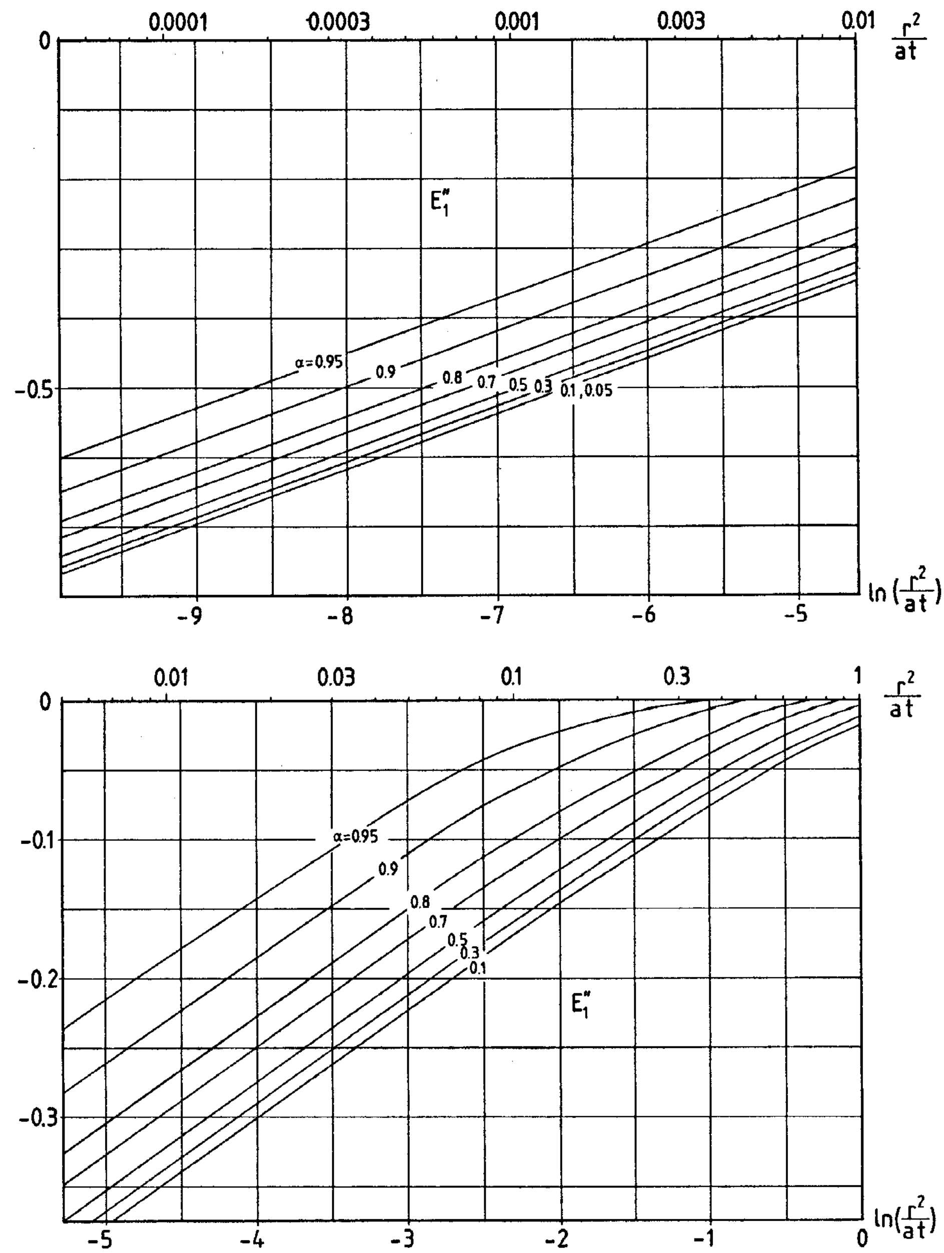


Figure 6.18. Pulse extraction function E_1'' for the first pulse. See figure 6.16 and eqs. 6.5.2-4.

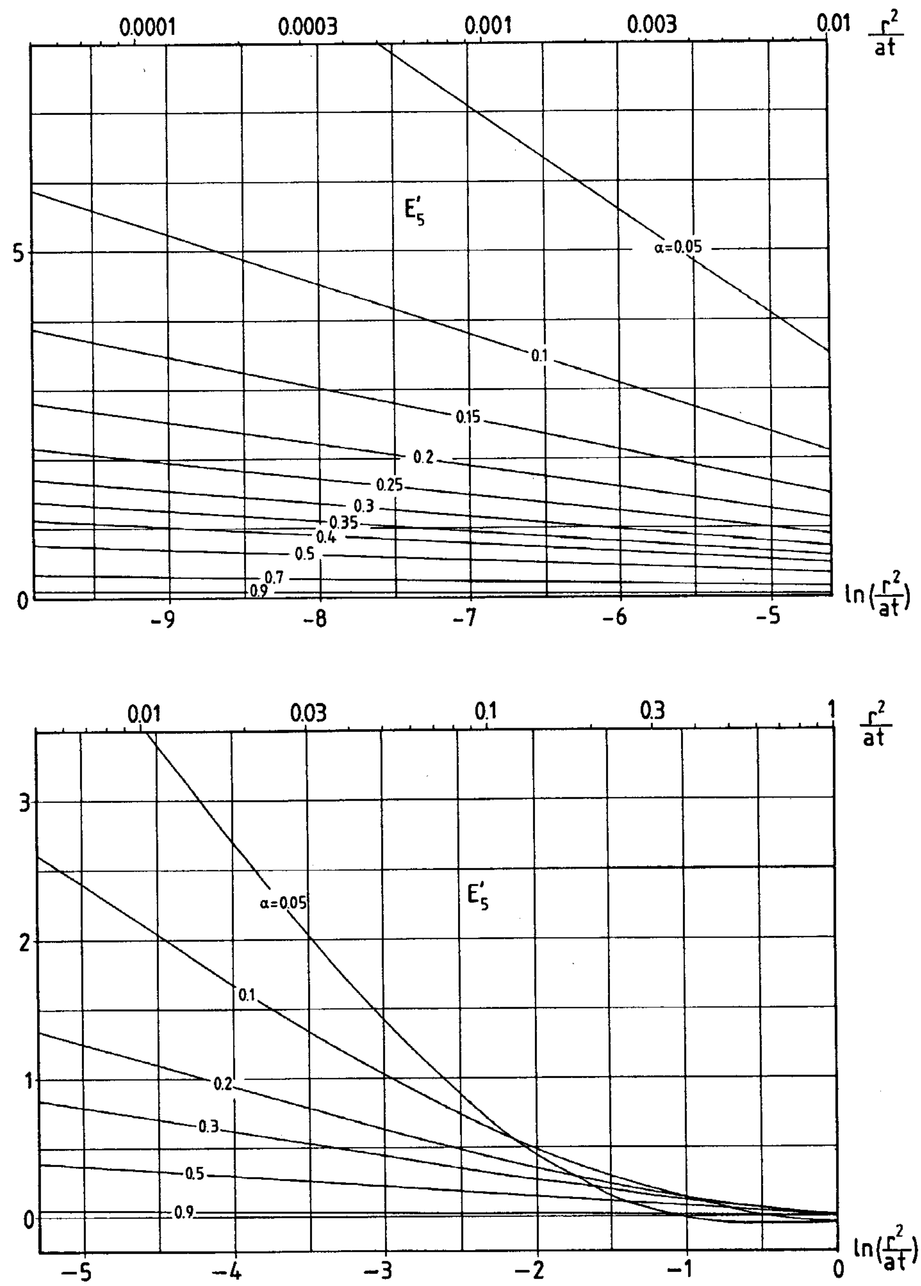


Figure 6.19. Pulse extraction function E_5' for the fifth pulse. See figure 6.16 and eqs. 6.5.2-4.

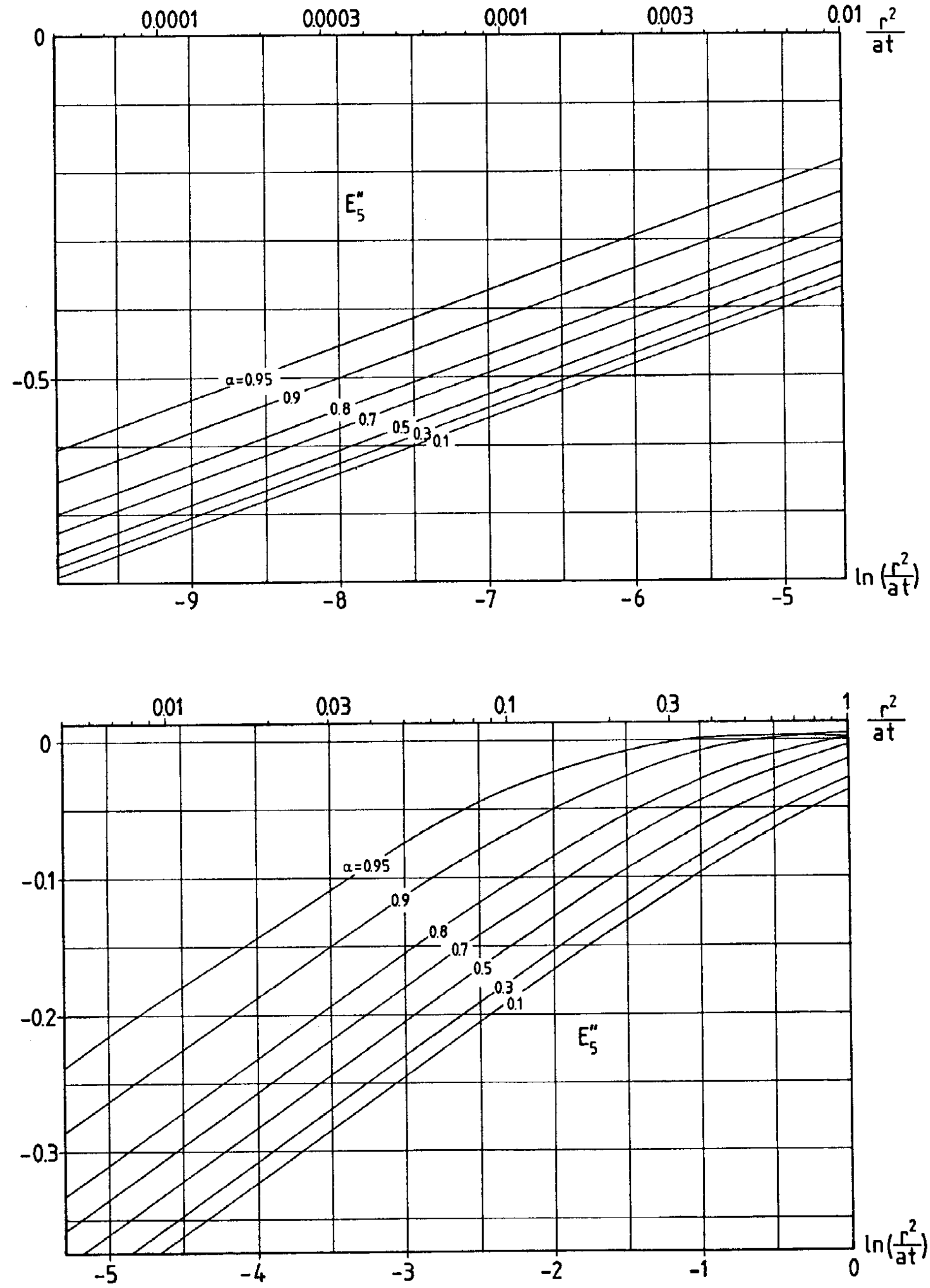


Figure 6.20. Pulse extraction function for the fifth pulse. See figure 6.16 and eqs. 6.5.2-4.

Our second example concerns a pulse train with the period of one day:

$$t_o = 1 \text{ day} \qquad \alpha = 0.3$$

$$q_o = 10 \text{ W/m} \qquad R = 0.02 \text{ m} \qquad D = 1 \text{ m} \qquad (6.5.6)$$

$$\lambda = 1.5 \text{ W/mK} \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s}$$

The result is given in table 6.5.2.

Period	t(days)	E_i'	E_i''	$T^o(R,t)$	$T^*(R,t)$	$T(R,t)$
i = 1	0.3	0.872		-2.49	-5.82	-8.31
	1		-0.375	-3.13	+2.50	-0.63
i = 5	4.3	0.812		-3.90	-5.41	-9.31
	5		-0.395	-3.98	+2.63	-1.35
i = 100	99.3	0.806		-5.57	-5.37	-10.94
	100		-0.400	-5.57	+2.67	-2.90

Table 6.5.2. The different contributions to the pipe temperature for a pulse train. Data according to (6.5.6).

We note again that the difference between the fifth and the hundredth pulse is quite small except for the contribution from T^o.

The effect of the ground surface will now be considered. The influence at the pipe is obtained with the preceding formulas putting r = 2D and changing the sign ($q_o \to -q_o$).

Let us take example 6.5.6 again (D = 1 m).The effect of the first pulse is less than 0.001°C. The effect of the hundredth pulse contains an oscillating part T* and a steady-extraction part T^o. The oscillating part is less than 0.01°C, while we have for T^o:

$$T^{o}(2D,t) = +\frac{q_o}{4\pi\lambda} E_1 \left(\frac{4D^2}{4at}\right) = 0.76 \qquad t = 100 \text{ days} \tag{6.5.7}$$

So the pipe temperature becomes:

$$\begin{aligned} t &= 99.3 \text{ days} \qquad T_R = -10.94 + 0.76 + 0 = -10.18^{o}C \\ t &= 100 \text{ days} \qquad T_R = -2.90 + 0.76 + 0 = -2.14^{o}C \end{aligned} \tag{6.5.8}$$

As a last example we consider an example with a period of one week:

$$\begin{aligned} &t_o = 1 \text{ week} \qquad \alpha = 0.5 \\ &q_o = 10 \text{ W/m} \qquad R = 0.02 \text{ m} \qquad D = 1\text{m} \\ &\lambda = 1.5 \text{ W/mK} \qquad a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \end{aligned} \tag{6.5.9}$$

The contriubtion at the pipe radius from the pipe itself is given by eqs. 6.5.2-5 with r = R. The mirror pipe has the steady-extraction contribution (6.5.7). The contribution from the pulsating part is less than 0.001^{o}C. The result is given in table 6.5.3.

Period	t(weeks)	E_i'	E_i''	$T^o(R,t)$	$T^*(R,t)$	$T^o(2D,t)$	$T(R,t)$
i = 1	0.5	0.569		-3.79	-3.97	0	-7.58
	1		-0.514	-4.16	+3.43	+0.02	-0.71
i = 10	9.5	0.535		-5.36	-3.57	+0.59	-8.34
	10		-0.531	-5.38	+3.54	+0.61	-1.23
i = 100	99.5	0.533		-6.60	-3.56	+1.73	-8.43
	100		-0.533	-6.60	+3.55	+1.73	-1.32

Table 6.5.3. The contributions to the pipe temperature for the pulse train of (6.5.9).

Let us summarize the results. The pipe temperature corresponding to a sequence of pulses is given by 6.5.2-4. The functions E_i' and E_i'' give the amplitudes of the oscillations around the mean-extraction contribution. The contribution due to the ground surface is given by the mirror source 6.5.7. The oscillating part of the mirror source is negligible except for quite long pulses ($t_0 > 1$ month). The values of E_i' and E_i'' do not change much for $i \geqq 5$.

7. PERIODIC HEAT EXTRACTION

The heat extraction rate q(t) to a pipe may be regarded as a superposition of a steady-state component and various periodic components. The temperature field due to a harmonic extraction is well-known. The solution is presented in appendix 3. We will in this chapter give some basic periodical solutions and illustrate how they can be used in the thermal analysis.

The periodic heat extraction to the pipe is given by

$$q(t) = q_1 \cdot \sin\left(\frac{2\pi t}{t_0}\right) \tag{7.1}$$

Here t_0 is the period, which may be equal to a year. We will use a complex notation:

$$q(t) = q_1 \cdot e^{2\pi i t/t_0} \tag{7.2}$$

Eq. 7.1 is the imaginary part of 7.2. The temperature fields corresponding to 7.2 will be complex-valued. The real and imaginary parts give two solutions which correspond to a cosine- and a sine-extraction respectively.

The penetration depth d_0 associated with a period t_0 is defined by

$$d_0 = \sqrt{\frac{a t_0}{\pi}} \tag{7.3}$$

The distance to the pipe is denoted r. We will use the following notations:

$$r' = \frac{r\sqrt{2}}{d_0} \qquad\qquad R' = \frac{R\sqrt{2}}{d_0} \tag{7.4}$$

7.1 Periodic sink in an infinite surrounding

Figure 7.1 shows the considered case. There is a periodic extraction q(t) according to eq. 7.2. The surrounding ground around the pipe extends sufficiently far away in all directions. The radial distance to the pipe is r.

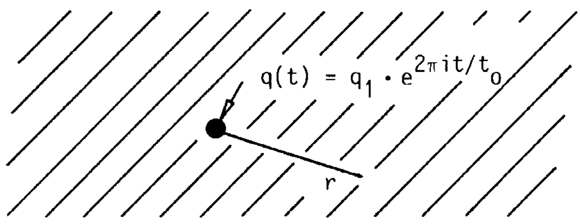


Figure 7.1. Periodic extraction to a pipe in an infinite surrounding.

We will first consider the case when the pipe radius is small compared to the penetration depth. We assume:

$$R' < 0.3 \qquad \text{or} \qquad t_0 \geq 70 \cdot \frac{R^2}{a} \tag{7.5}$$

The solution (7.6) is then valid with an error below 5%.

The periodic temperature around the pipe is from A3.9:

$$T(r,t) = -\frac{q_1}{2\pi\lambda} N_0(r')\, e^{i(2\pi t/t_0 + \phi_0(r'))} \tag{7.6}$$

Here N_0 and ϕ_0 are the amplitude and phase of a Kelvin function. They are given in table 7.2 and figure 7.2.

The following asymptotic expressions are valid with an error below 1%:

$$N_0(r') \simeq \sqrt{(\ln(2/r') - \gamma)^2 + \pi^2/16} \qquad (r' < 0.1) \tag{7.7'}$$

$$\phi_0(r') \simeq -\arctan\left(\frac{\pi/4}{\ln(2/r')-\gamma}\right) \qquad \gamma = 0.5772 \tag{7.7"}$$

$$N_0(r') \simeq \sqrt{\frac{\pi}{2r'}}\; e^{-r'/\sqrt{2}}$$

$$\phi_0(r') \simeq -\left(\frac{r'}{\sqrt{2}} + \frac{\pi}{8}\right) \qquad (r' > 7) \tag{7.8}$$

The radial variation of the amplitude of the temperature field is given by $N_0(r')$. The temperature amplitude at the pipe radius $r = R$ is represented by $N_0(R')$.

Let us consider the reference example:

$$q_1 = 10 \text{ W/m} \qquad R = 0.02 \text{ m}$$

$$\lambda = 1.5 \text{ W/mK} \qquad a \doteq 0.75 \cdot 11^{-6} \text{ m}^2/\text{s} \qquad (7.9)$$

$$\frac{q_1}{2\pi\lambda} = 1.06^{o}C$$

Condition 7.5 reads:

$$t_0 \geq 10 \text{ hours}$$

For a weekly cycle we have

$$t_0 = 7 \text{ days} \qquad d_0 = 0.38 \text{ m}$$

$$R' = 0.074 \qquad r' = \frac{r}{0.27}$$

The temperature amplitude and phase is then for different r:

r(m)	0.02	0.10	0.5	1	2
$-1.06 \cdot N_0(r/0.27)^{o}C$	-2.97	-1.41	-0.23	-0.046	-0.003
$\phi_0(r/0.27)$	-0.275	-0.54	-1.66	-3.0	-5.6
$t_0\phi_0/2\pi$ (days)	-0.31	-0.60	-1.8	-3.3	-6.2

The last line gives the time lag of the temperature compared to the effect. So the temperature amplitude at the pipe is $-3^{o}C$. The maximum temperature occurs +0.3 days after the maximum effect. The temperature amplitude is halved at $r = 0.10$ m. Only 10% of the amplitude remains at $r = 0.4$ m. At $r = 2$ m only 0.1% of the amplitude remains.

The temperature amplitude as a function of r for different periods t_0 is given in table 7.1.

t_0	1 day	1 week	1 month	1 year
r = 0.02 m	-2.06	-3.08	-3.71	-5.08
r = 0.1 m	-0.60	-1.40	-2.12	-3.40
r = 0.2 m	-0.22	-0.83	-1.44	-2.70
r = 0.5 m	-0.02	-0.23	-0.68	-1.75
r = 1 m	-0.0004	-0.046	-0.23	-1.11
r = 2 m	0	-0.003	-0.05	-0.57
r = 5 m	0	0	-0.0008	-0.13
r = 10 m	0	0	0	-0.01

Table 7.1. Temperature amplitude at different distances for different periods t_0. Data according to (7.9).

So 10% of the pipe amplitude occurs at the distance 0.2 m for t_0 = 1 day, 0.4 m for t_0 = 1 week, 0.8 m for t_0 = 1 month and 2 m for t_0 = 1 year. Only 1% of the amplitude remains at the distance 0.5 m for t_0 = 1 day, 1.2 m for t_0 = 1 week, 2 m for t_0 = 1 month and 6 m for t_0 = 1 year.

The exact solution without the restriction (7.5) is obtained by correcting the heat flux of the line source at r = R. We have from appendix 3 :

$$T(r,t) = -\frac{q_1}{2\pi\lambda} \cdot \frac{N_0(r')}{F(R')} e^{i(2\pi t/t_0 + \phi_0(r') + G(R'))} \qquad (7.10)$$

The temperature at the pipe radius r = R is of particular interest. We have:

$$T(R,t) = T_R(t) = -\frac{q_1}{2\pi\lambda} \cdot A(R') \cdot e^{i(2\pi t/t_0 - B(R'))} \qquad (7.11)$$

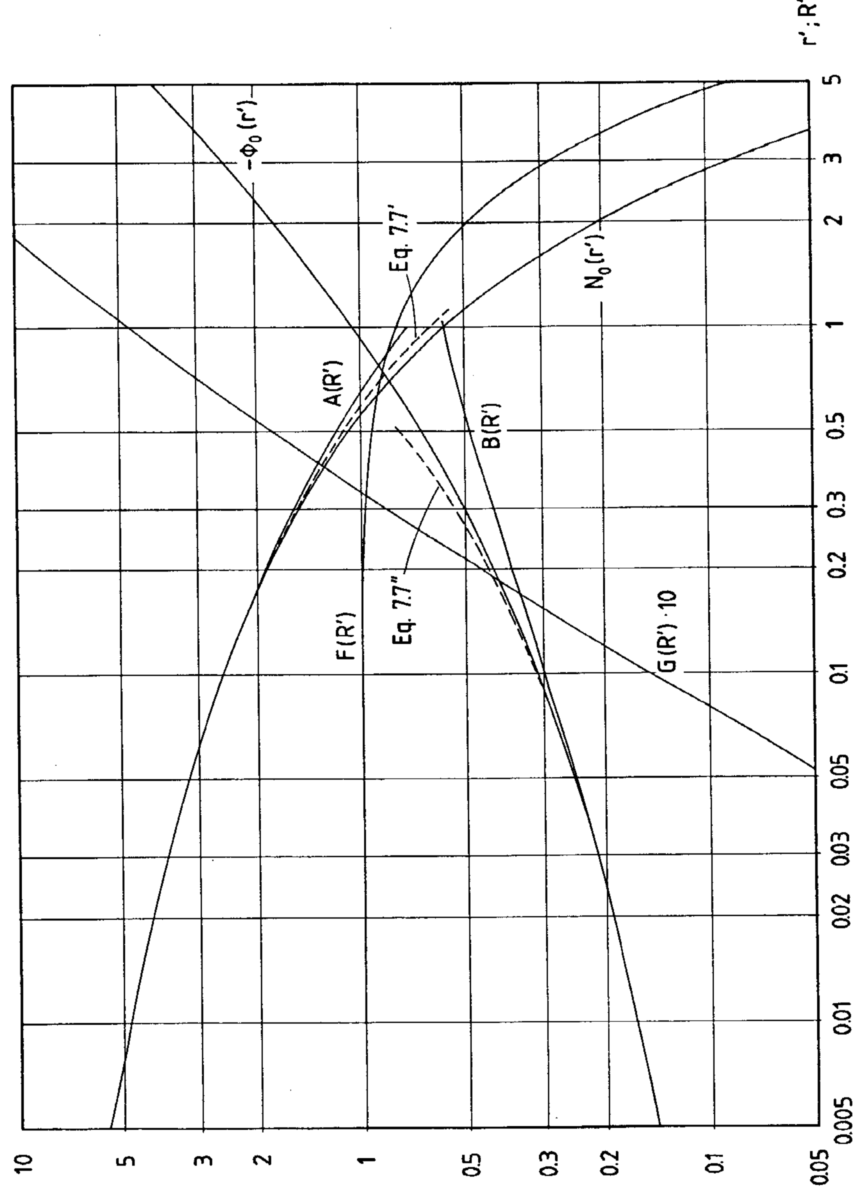


Figure 7.2. Amplitude and phase functions for a periodic heat sink according to eqs. 7.6, 7.10 and 7.11.

The functions F, G, A and B are given in table 7.2 and figure 7.2.

r';R'	$N_0(r')$	$-\phi_0(r')$	F(R')	G(R')	A(R')	B(R')	1/A(R')
0.001	7.067	0.111	1	0	7.067	0.111	0.142
0.002	6.379	0.123	1	0	6.379	0.123	0.157
0.003	5.977	0.132	1	0	5.977	0.132	0.167
0.004	5.692	0.138	1	0	5.692	0.138	0.176
0.005	5.471	0.144	1	0	5.471	0.144	0.183
0.006	5.291	0.149	1	0	5.291	0.149	0.189
0.007	5.138	0.153	1	0	5.138	0.153	0.195
0.008	5.006	0.158	1	0	5.006	0.158	0.200
0.009	4.890	0.161	1	0	4.890	0.161	0.204
0.01	4.786	0.165	1	0.000	4.786	0.165	0.209
0.02	4.104	0.193	1	0.001	4.104	0.192	0.244
0.03	3.707	0.214	1.000	0.002	3.707	0.212	0.270
0.04	3.426	0.231	0.999	0.003	3.429	0.228	0.292
0.05	3.209	0.247	0.999	0.005	3.212	0.242	0.311
0.06	3.033	0.261	0.999	0.006	3.036	0.255	0.329
0.07	2.884	0.276	0.998	0.008	2.890	0.268	0.346
0.08	2.756	0.289	0.998	0.010	2.762	0.279	0.362
0.09	2.643	0.302	0.997	0.012	2.651	0.290	0.377
0.1	2.542	0.311	0.996	0.015	2.552	0.296	0.392
0.2	1.892	0.412	0.986	0.045	1.919	0.367	0.521
0.3	1.525	0.501	0.971	0.086	1.571	0.415	0.637
0.4	1.275	0.585	0.949	0.131	1.344	0.454	0.744
0.5	1.088	0.665	0.925	0.181	1.176	0.484	0.850
0.6	0.942	0.744	0.899	0.235	1.048	0.509	0.954
0.7	0.823	0.820	0.870	0.291	0.946	0.529	1.057
0.8	0.725	0.896	0.840	0.349	0.863	0.547	1.16
0.9	0.643	0.971	0.810	0.408	0.794	0.563	1.26
1	0.572	1.046	0.779	0.469	0.734	0.577	1.36
2	0.207	1.774	0.489	1.119			
3	0.084	2.490	0.284	1.800			
4	0.036	3.202	0.158	2.492			
5	0.0161	3.913	0.086	3.189			

Table 7.2. Amplitude and phase functions for a periodic heat sink according to eqs. 7.6, 7.10 and 7.11.

7.2 Correction for the ground surface

Let us now consider the periodic extraction from a pipe at the depth z = D below the ground surface. The temperature at the ground surface shall be equal to zero. This is obtained by adding a negative mirror sink at $z = -D$ above the real pipe, see fig. 7.3.

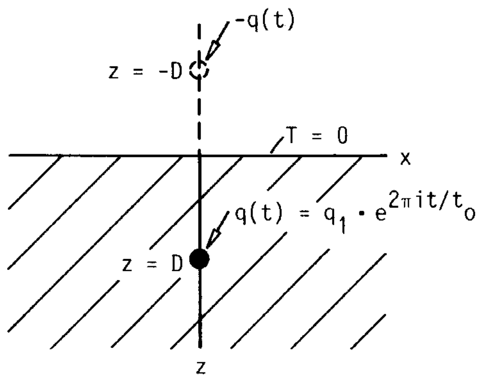


Figure 7.3. Periodic heat sink and a mirror sink.

The total temperature in the ground is given by the two sinks $q(t)$ at $(x,z) = (0,D)$ and $-q(t)$ at $(x,z) = (0, -D)$. We get two terms of type 7.6 or 7.11. For the mirror pipe we can always use the simpler expression 7.6. The reason is that the period is quite short if eq. 7.5 is not fulfilled. The effect of the mirror source is then negligible in the ground since it lies at the distance $2D$ from the sink.

The pipe temperature is now:

$$T_R(t) = -\frac{q_1}{2\pi\lambda} A(R')\, e^{i(2\pi t/t_o - B(R'))} + \\ + \frac{q_1}{2\pi\lambda} N_o(D')\, e^{i(2\pi t/t_o + \phi_o(D'))} \tag{7.12}$$

Here we have used

$$D' = \frac{\sqrt{2}\cdot 2D}{d_o} \tag{7.13}$$

Let us consider case (7.9) again. We have for $D = 1$ m:

t_o = 1 week

$$T_R(t) = -3.1\ e^{i(2\pi t/t_o - 0.37)} + 3 \cdot 10^{-7} \cdot e^{i(2\pi t/t_o - 14.3)}$$

t_o = 1 month

$$T_R(t) = -3.7\ e^{i(2\pi t/t_o - 0.22)} + 0.046\ e^{i(2\pi t/t_o - 2.9)}$$

t_o = 1 year

$$T_R(t) = -5.1\ e^{i(2\pi t/t_o - 0.17)} + 0.56\ e^{i(2\pi t/t_o - 1.1)}$$

(7.14)

We see that the effect of the mirror pulse is negligible for the monthly period and certainly for the weekly one. The case t_o = 1 year is shown in figure 7.4. The total pipe temperature $T_R(t)$ (full line) is shown during the year together with its two components (dashed lines). It is interesting to compare $T_R(t)$ with a steady-state temperature using the actual q(t) at each time. This should be reasonable since the time-scale to attain steady-state conditions is t_D = 1 month while the period is one year. We have the approximation:

$$T_R(t) \simeq -\frac{q(t)}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) = -4.89 \cdot \sin\left(\frac{2\pi t}{t_o}\right) \qquad (7.14')$$

(steady-state approximation)

This approximation is shown by the dotted line of fig. 7.4. The approximation gives a rather good fit. The difference between 7.14 and 7.14' is characteristically 0.4°C. There is a time lag of around 3 days. The difference of temperature amplitude is only 0.15°C.

7.3 Two pipes

The steady-state analysis for two or more pipes may be repeated in an analogous way for the periodic extraction. In order to illustrate this we will consider two pipes as shown in figure 7.5. We assume that there is symmetry with the same temperature at the two pipes.

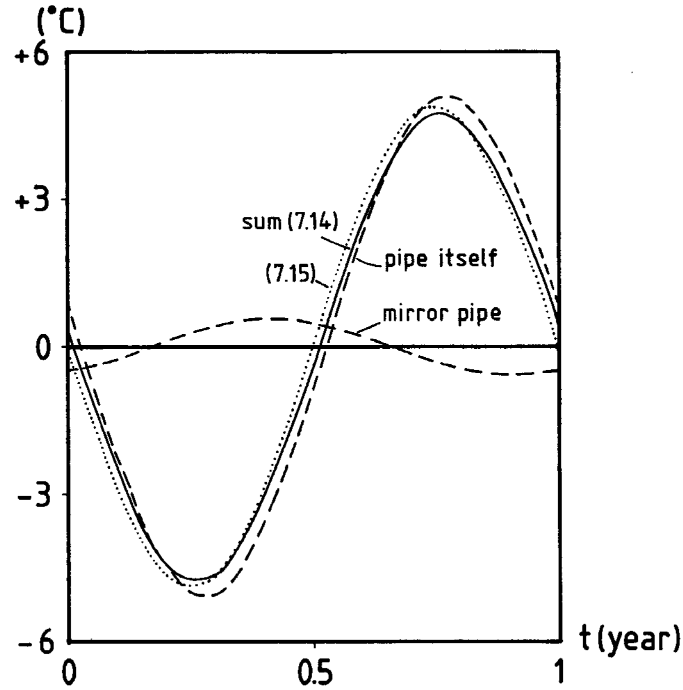


Figure 7.4. Annual periodic temperature (7.14) for example 7.9. The dashed lines show the two components from the pipe and its mirror. The dotted line shows the steady-state approximation at each time (7.14')

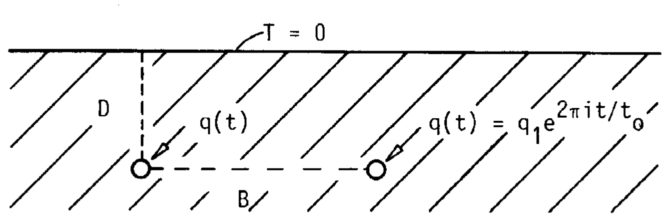


Figure 7.5. Periodic heat extraction from two pipes.

The pipe temperature T_R is now obtained as a superposition from the pipe itself, from the other pipe, and from the two mirror pipes. This gives:

$$T_R(t) = -\frac{q_1}{2\pi\lambda}\Big\{A(R')^{i(2\pi t/t_0 - B(R'))} +$$

$$+ N_0(B')\, e^{i(2\pi t/t_0 + \phi_0(B'))} -$$

$$- N_0(D')\, e^{i(2\pi t/t_0 + \phi_0(D'))} -$$

$$- N_0(B'')\, e^{i(2\pi t/t_0 + \phi_0(B''))}\Big\} \tag{7.15}$$

Here we have

$$D' = \frac{\sqrt{2}\cdot 2D}{d_0} \qquad B' = \frac{\sqrt{2}\cdot B}{d_0} \qquad B'' = \frac{\sqrt{2}\cdot\sqrt{4D^2 + B^2}}{d_0} \tag{7.15}$$

The quantity B" represents the distance between one of the pipes and the mirror of the other pipe. We assume here that B is much larger than R.

7.4 Steady-state and periodic heat extraction

Let us as another illustration consider a case with a steady-state and a periodic part for the heat extraction q(t):

$$q(t) = q_0 + q_1\, e^{2\pi i t/t_0} \tag{7.16}$$

There is also a periodically varying ground surface temperature. The situation is shown in fig. 7.6. A phase factor φ_0 is included in the surface temperature.

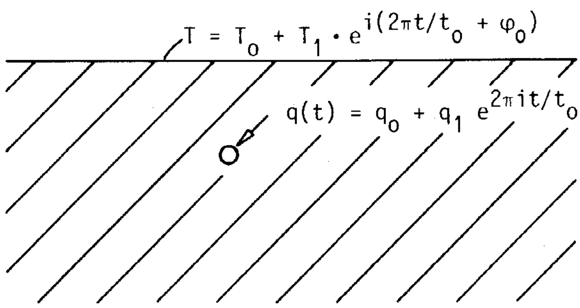


Figure 7.6. Steady-state and periodic heat extraction. The ground surface temperature varies periodically.

The contribution to the total temperature field from the steady-state extraction q_o and from the ground surface temperatures is given by an expression of type 5.4.1. The contribution from the periodic extraction is given by 7.12. The total pipe temperature is then:

$$T_R(t) = -\frac{q_o}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) + T_o + T_1\, e^{-D/d_o} \cdot e^{i(2\pi t/t_o + \varphi_o - D/d_o)} -$$

$$-\frac{q_1}{2\pi\lambda}\left\{A(R')\, e^{i(2\pi t/t_o - B(R'))} - N_o(D')\, e^{i(2\pi t/t_o + \phi_o(D'))}\right\} \tag{7.17}$$

Here we have used the notation (7.13).

Let us as an example consider a heat pump application with a heat demand which is proportional to the difference between the room temperature T_2 and the air temperature:

$$q_o + q_1\, e^{2\pi i t/t_o} = \alpha\left\{T_2 - T_o - T_1\, e^{i(2\pi t/t_o + \varphi_o)}\right\}$$

or

$$q_o = \alpha\,(T_2 - T_o)$$

$$q_1 = -\alpha T_1 \cdot e^{i\varphi_o} \tag{7.18}$$

As an example we consider the following case with data for T_o and T_1 from Grenoble:

$T_o = 13^oC$	$T_1 = 7^oC$	$T_2 = 20^oC$
$e^{i\varphi_o} = -1$	t_o = 1 year	$m_p = 0$
$\lambda = 1.5$ W/mK	$a = 0.75 \cdot 10^{-6} m^2/s$	$R = 0.02$ m
$D = 1$ m	$\alpha = 1.0$ W/mK	

(7.19)

The mean effect and the effect amplitude become:

$$q_o = 7\ \text{W/m} \qquad q_1 = 7\ \text{W/m} \tag{7.19'}$$

Taking the imaginary part of the complex contributions we get:

$$T_R(t) = -\frac{7}{2\pi\cdot 1.5}\ \ln\left(\frac{2}{0.02}\right) + 13 - 7\cdot e^{-1/2.74}\ \sin\left(\frac{2\pi t}{t_o} - \frac{1}{2.74}\right)$$

$$-\frac{7}{2\pi\cdot 1.5}\left\{4.79\cdot \sin\left(\frac{2\pi t}{t_o} - 0.165\right) - 0.56\cdot \sin\left(\frac{2\pi t}{t_o} - 1.07\right)\right\}$$

or

$$T_R(t) = 13 - 4.86\cdot \sin\left(\frac{2\pi t}{t_o} - 0.36\right) - 3.42 -$$

$$- 3.56\cdot \sin\left(\frac{2\pi t}{t_o} - 0.165\right) + 0.42\cdot \sin\left(\frac{2\pi t}{t_o} - 1.07\right) \tag{7.20}$$

The pipe temperature 7.20 and the different contributions are shown in figure 7.7. The lowest extraction temperature is 1.5^oC.

The variation of the extraction temperature with the depth D is of great interest. Let us consider case 7.19 for

$$D = 0.4,\ \ 0.7,\ \ 1.0,\ \ 1.5,\ \ 2.0\ \text{m} \tag{7.21}$$

The resulting extraction temperatures during the annual cycle are shown in figure 7.8.

The minimum extraction temperature lies around 1.5^oC or higher.

It is interesting to compare the obtained minimum extraction temperature with the steady-state analysis of chapter 5. We use the maximum extraction rate

$$q = q_o + q_1 = 14\ \text{W/m} \tag{7.22}$$

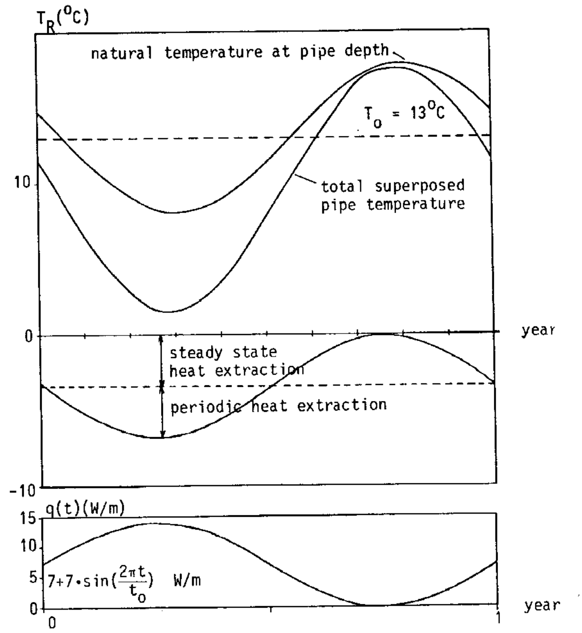


Figure 7.7. Pipe temperature of example (7.19) together with the different contributions according to (7.20). The heat extraction rate is shown below.

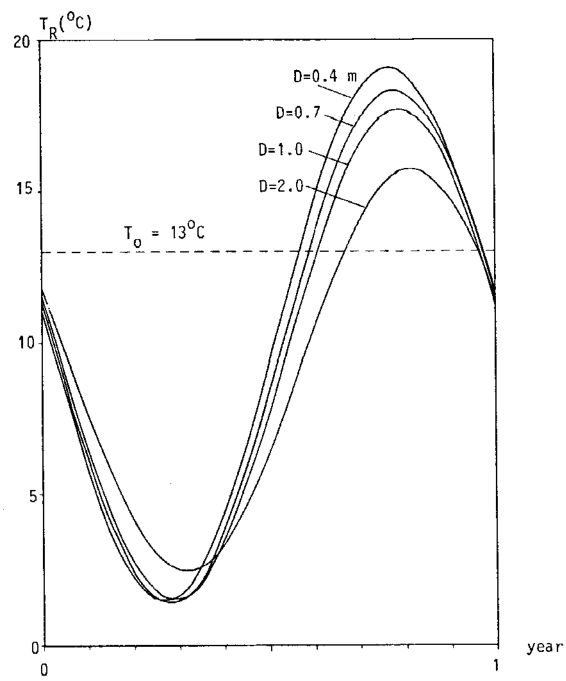


Figure 7.8. Pipe temperature of example 7.19 for different values of the depth D to the pipe.

The minimum temperature is then given by eq. 5.4.2 ($m_p = 0$) or eq. 5.4.5. We have the following result:

D(m)	0.4	0.7	1.0	1.5	2.0
T_R,min, fig. 7.8	1.5	1.4	1.5	1.9	2.5
T_R, min, eq. 5.4.5.	1.5	1.3	1.3	1.5	1.8

(7.23)

We see that the simpler steady-state approximation using the maximum extraction rate gives quite good results. The error increases of course with D, since the steady-state assumption becomes less valid with increasing D. The variation of T_{fmin} with the depth D is in the steady-state approximation given by the function f(s,p)(5.4.4). The steady-state minimum as a function of D is shown in figure 7.9. There is a minimum extraction temperature for D = 0.8 m. The exact values are also indicated.

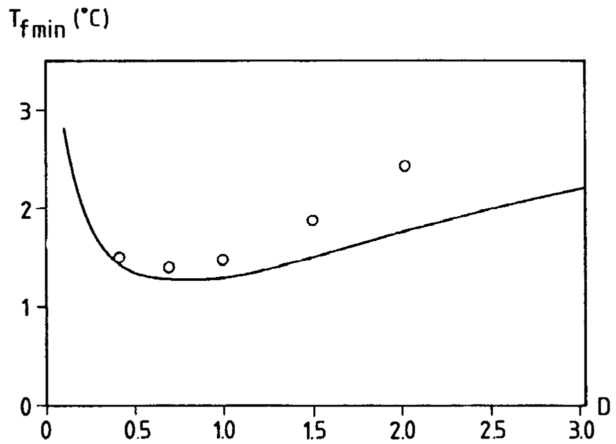


Figure 7.9. Minimum extraction temperature of case (7.19) as a function of D in the steady-state approximation with maximum extraction rate (7.22). The rings show exact values.

The factor α determines the magnitude of the extraction. Let us consider case 7.19 for different α:

$$\alpha = 0.5, \quad 0.75, \quad 1.0, \quad 1.25, \quad 1.5 \text{ W/mK} \qquad (7.24)$$

The result is shown in figure 7.10. The pipe temperature falls just below zero for $\alpha = 1.25$.

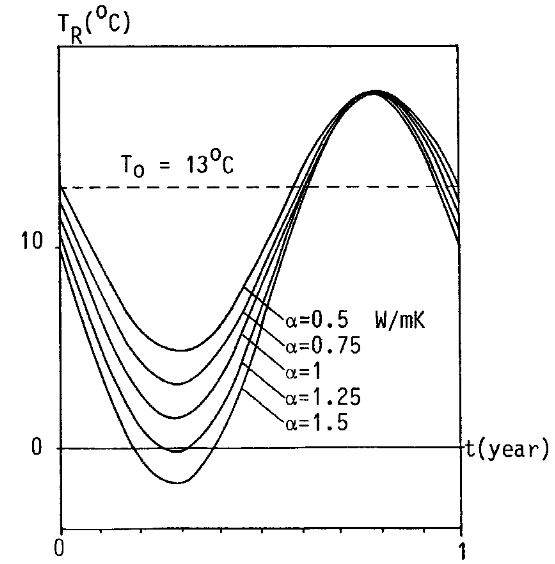


Figure 7.10. Pipe temperature of example 7.19 for different values of the extraction factor α.

The steady-state approximation of chapter 5 using (7.22) gives the following result:

α (W/mK)	0.5	0.75	1.0	1.25	1.5
T_R, min, fig. 7.10	4.8	3.2	1.6	-0.2	-1.8
T_R, min, eq. 5.4.5	4.7	3.0	1.3	-0.4	-2.1

We see that the steady-state analysis gives quite good results.

7.5 Infinite array of pipes

The extreme case of an infinite array of pipes is shown in figure 4.15. The distance between the pipes is B. The steady-state heat extraction was studied in section 4.9. The thermal resistance between one of the pipes and the ground was given by eq. 4.9.3. The corresponding periodic part of the heat extraction process will be studied in this section.

The infinite array of pipes is shown in figure 7.11. The two closest pipes have index 1, while the next two pipes to the left and to the right have index 2, and so on. The mirror pipes above the ground surface are also shown in the figure. We assume as usual that B is much greater than the pipe radius R.

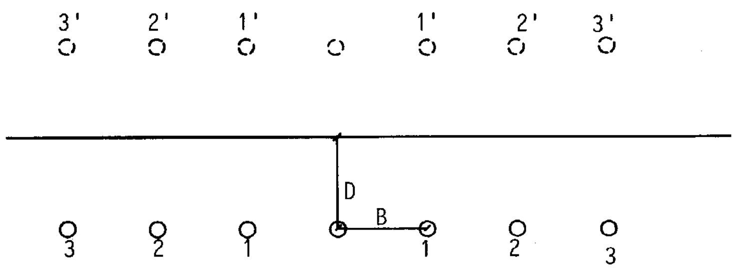


Figure 7.11. Infinite array of pipes and the mirror pipes above the ground.

The periodic heat extraction rate at the pipes is given by eq. 7.1 or 7.2. The heat extraction rate at the mirror pipes is given by the corresponding negative value. The contribution to the pipe temperature from the considered pipe itself and its mirror pipe is given by eq. 7.12. The other contributions from the pipes 1, 1', 2, 2' and so on are obtained as an infinite sum of expressions of type (7.6). We then have

$$T_R(t) = -\frac{q_1}{2\pi\lambda}\, e^{2\pi i t/t_0} \Big\{A(R')e^{-iB(R')} - N_0(D')e^{i\phi_0(D')} + M_0(D/d_0,B/D)e^{i\Psi_0(D/d_0,B/D)}\Big\} \qquad (7.25)$$

Here R' is given by eq. 7.4 and D' by 7.13.

$$M_o \cdot e^{i\Psi_o} = 2\sum_{j=1}^{\infty} \left\{N_o(B_j)e^{i\phi_o(B_j)} - N_o(B_j')e^{i\phi_o(B_j')}\right\} \quad (7.26)$$

The arguments B_j and B_j' are obtained from eq. 7.4 with r equal to the distance between the considered pipe and pipe j and j' respectively:

$$B_j = \frac{jB\sqrt{2}}{d_o} \qquad B_j' = \frac{\sqrt{(jB)^2 + 4D^2}\cdot\sqrt{2}}{d_o} \quad (7.27)$$

The amplitude M_o and the phase Ψ_o of the contribution from the surrounding pipes depend according to 7.27 on D/d_o and B/D. These functions are given in figures 7.12 and 7.13.

As an illustration we consider the following case:

$$D = 1 \text{ m} \qquad B = 0.5 \text{ m} \qquad t_o = 1 \text{ year}$$

$$a = 0.75 \cdot 10^{-6} \text{ m}^2/\text{s} \qquad R = 0.02 \text{ m} \quad (7.28)$$

Then we have:

$$d_o = 2.7438 \text{ m}$$

$$R' = 0.0103 \qquad D' = 1.0308 \quad (7.29)$$

$$D/d_o = 0.3645 \qquad B/D = 0.5$$

The amplitudes and phases of the terms of 7.25 are:

$$A(R') = 4.76 \qquad B(R') = 0.17$$

$$N_o(D') = 0.561 \qquad \phi_o(D') = -1.07 \quad (7.30)$$

$$M_o(0.364,\ 0.5) = 5.724 \qquad \Psi_o(0.364,\ 0.5) = -0.414$$

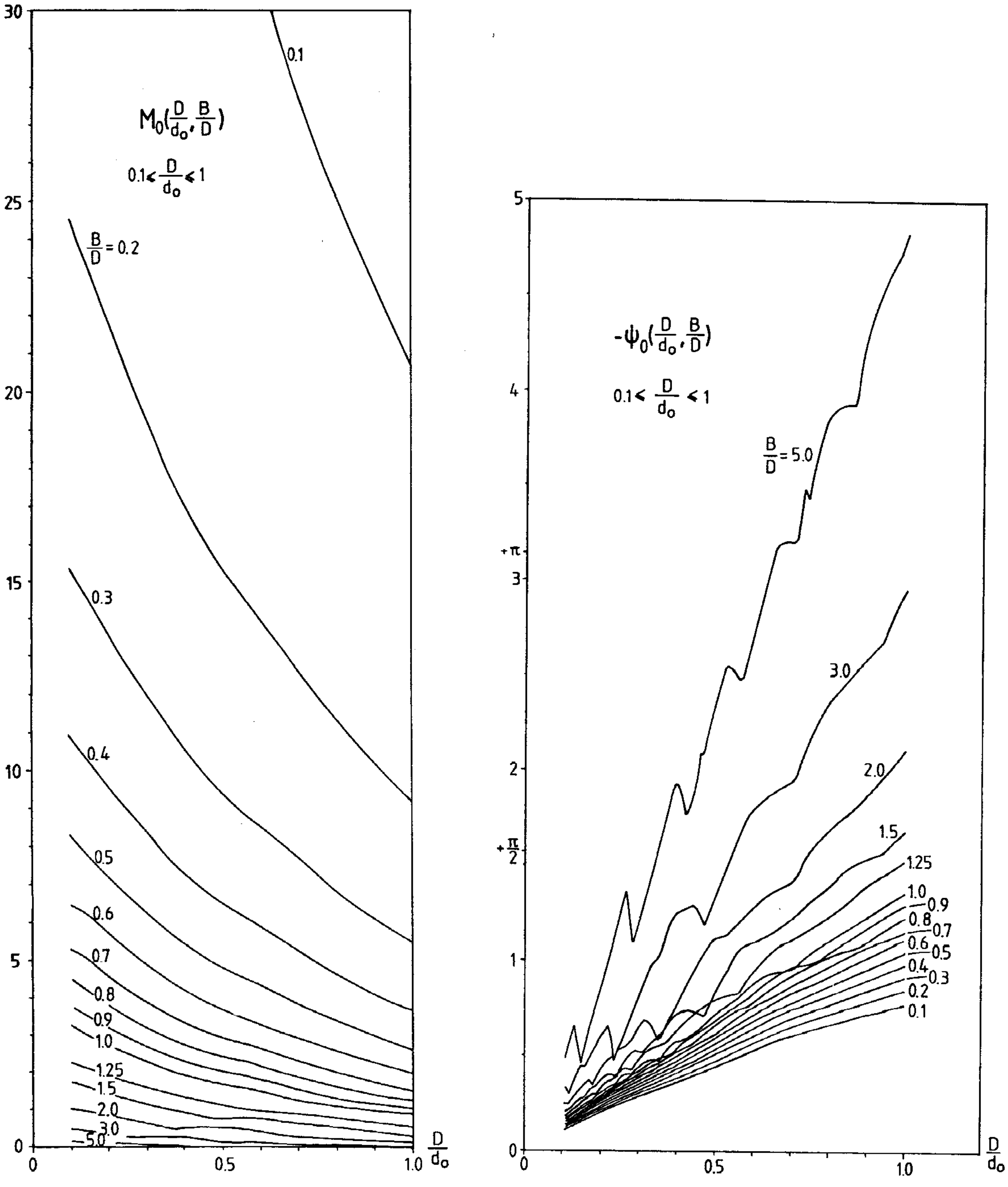
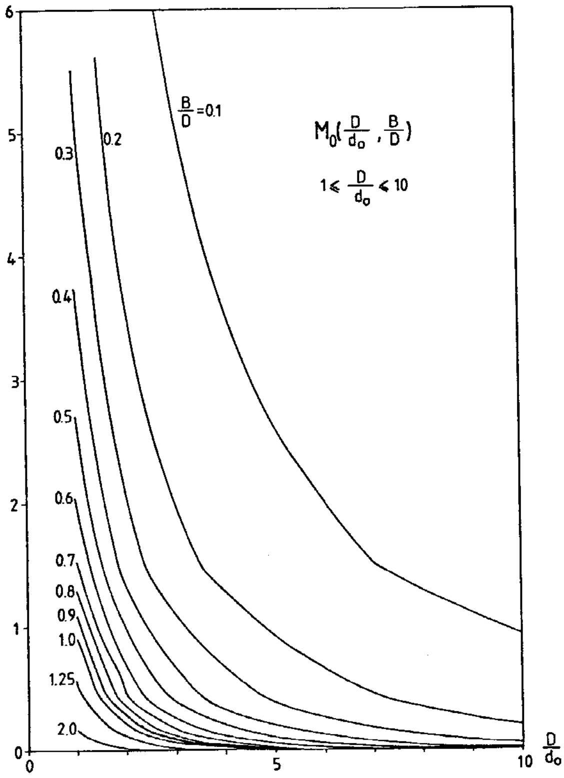


Figure 7.12. The functions M_0 and Ψ_0, which give the contribution from an infinite array of pipes, eq. 7.25.



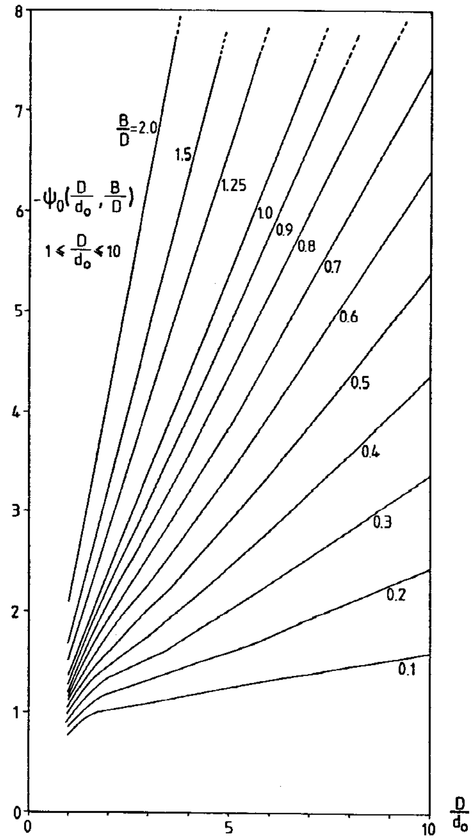


Figure 7.13. The functions M_0 and Ψ_0, which give the contribution from an infinite array of pipes, eq. 7.25.

The pipe temperature becomes

$$T_R(t) = -\frac{q_1}{2\pi\lambda} \cdot e^{2\pi i t/t_o} \left\{4.76e^{-i0.17} - 0.561 \cdot e^{-i1.07} + 5.724e^{-i0.414}\right\} \tag{7.31}$$

The three complex contributions of 7.31 are shown in figure 7.15. Adding these we may rewrite 7.31:

$$T_R(t) = -\frac{q_1}{2\pi\lambda} \cdot 10.0\, e^{\,i(2\pi t/t_o - 0.263)} \tag{7.32}$$

The total temperature amplitude is thus $q_1/(2\pi\lambda) \cdot 10.0\ ^oC$. The phase lag is

$$0.263 \cdot \frac{t_o}{2\pi} = 15.3 \text{ days} \tag{7.33}$$

So the minimum temperature occurs 15 days after the maximum extraction. As a comparison we have for the single pipe the amplitude $q_1/(2\pi\lambda) \cdot 4.4^oC$ and a time-lag of 4 days.

The contributions from the surrounding pipes, eq. 7.26, are for the present example shown in figure 7.14. The first terms for j = 1, i.e. the two pipes 1 and their mirrors 1' of figure 7.11, give a complex vector denoted 1 + 1'. The next term j = 2 gives the vector 2 + 2' and so on.

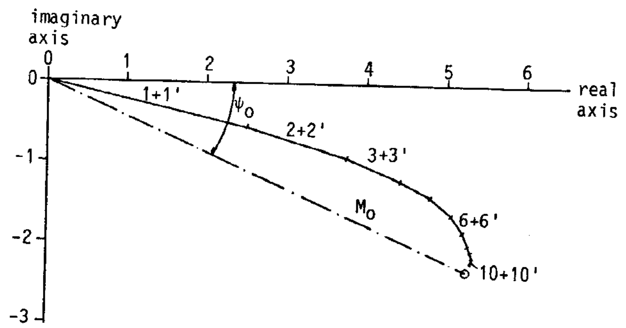


Figure 7.14. Contribution from surrounding pipes to the sum $M_o \cdot \exp(i\psi_o)$. See figure 7.11 and eq. 7.26.

As a second illustration we consider case 7.28 for three spacings B:

$$B = 0.5,\ 0.8,\ 1.0\ \mathrm{m} \qquad (7.34)$$

We get:

$$\begin{aligned} B &= 0.8\ \mathrm{m} \qquad M_o = 2.92 \qquad \Psi_o = -0.476 \\ B &= 1.0\ \mathrm{m} \qquad M_o = 2.03 \qquad \Psi_o = -0.502 \end{aligned} \qquad (7.35)$$

The three cases 7.34 are shown in figure 7.15.

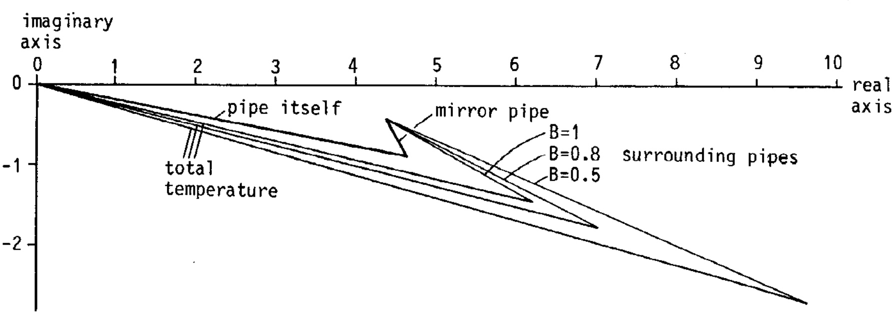


Figure 7.15. The three complex contributions of eq. 7.25. Data according to 7.28 and 7.34.

The total complex amplitudes and phase lags are summarized below:

B(m)	0.5	0.8	1.0	∞
Amplitude (^{o}C)	10.0	7.2	6.3	4.4
Phase lag (days)	15.3	13.3	11.7	3.8

This example shows the strong influence between the pipes for small values of B compared to D. We also note that the influence is small for larger spacing B.

The curves for Ψ_o have certain maxima and minima. There is for example for B/D = 2 a maximum near D/d_o = 0.32. We have:

$$B/D = 2 \quad \begin{array}{lll} D/d_o = 0.30 & M_o = 0.634 & \Psi_o = -0.641 \\ D/d_o = 0.32 & M_o = 0.584 & \Psi_o = -0.785 \\ D/d_o = 0.35 & M_o = 0.505 & \Psi_o = -0.592 \end{array} \tag{7.36}$$

The occurence of local maxima and minima is due to the fact that the influence from different pipes are added with different phase lags.

Let us now consider the outermost pipe in a large array of pipes. The situation is shown in figure 7.16.

o o_1 o_2 o_3 ......

Figure 7.16. The end pipe in a large array of pipes.

The end pipe has an infinite array of pipes 1,2,... to the right.

We assume a periodic heat extraction rate 7.2 to each pipe. The heat extraction rate is assumed to be the same for all pipes. The required temperature will then be different from pipe to pipe. For a pipe far away from the end we can use the preceding expressions.

The end pipe sees an infinite array to the right. In the previous case there was also an infinite array to the left. See figure 7.11. The influence will due to symmetry be halved. The pipe temperature for the end pipe is thus from eq. 7.25 given by

$$T_R(t) = -\frac{q_1}{2\pi\lambda} e^{2\pi it/t_o} \Big\{ A(R')e^{-iB(R')} - N_o(D')e^{i\phi_o(D')} + \tfrac{1}{2} M_o(D/d_o, B/D)e^{i\Psi_o(D/d_o, B/D)} \Big\} \tag{7.37}$$

8. EFFECT OF GROUND WATER FLOW AND INFILTRATION

The previous analyses were made under the assumption of pure heat conduction. Convective heat transfer in the ground due to moving ground water or infiltration from rain was not considered. In this chapter these effects will be analysed.

The analysis is based on analytical solutions. Only steady-state cases are dealt with. The heat extraction rate is constant, and the convective flow does not change in time.

8.1 Steady-state line sink in moving ground water

We will first consider the case with one pipe in an infinite surrounding ground with moving ground water. The pipe lies at the center $(x,z) = (0,0)$. The heat extraction rate q is constant. The ground water flows in the $\hat{x}$-direction. The volumetric water flow is q_w (m_w^3/m^2s or m/s). The flow $q_w\hat{x}$ does not vary in time. The considered case is shown in figure 8.1.

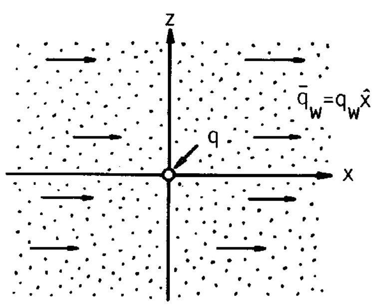


Figure 8.1. Constant heat extraction q to a pipe in moving ground water.

The thermal conductivity in the ground is λ, and the volumetric heat capacity of the ground with its water is C. The volumetric capacity of the water is C_w:

$$C_w \simeq 4.18 \quad MJ/m_w^3K \tag{8.1}$$

The temperature far away from the pipe is zero, since we are only considering the contribution from the sink.

The water flow q_w will displace a temperature field in the flow direction with a certain velocity v_T. This thermal velocity depends on the capacity ratio C_w/C and the water flow q_w. In fact we have

$$v_T = \frac{C_w}{C} q_w \tag{8.2}$$

The temperatuer field from a sink in a moving stream is given in ref. 3E ($U = v_T$). We have

$$T(x,z) = -\frac{q}{2\pi\lambda} e^{x/\ell} K_0\left(\frac{\sqrt{x^2+z^2}}{\ell}\right) \tag{8.3}$$

Here $K_0(s)$ is a modified Bessel function. It is shown in figure 8.2.

The parameter ℓ has the dimension of a length:

$$\ell = \frac{2\lambda}{Cv_T} = \frac{2\lambda}{C_w q_w} \tag{8.4}$$

The length ℓ has the following physical interpretation. Consider a temperature difference ΔT over a length $\ell/2$. The steady-state heat flow is $\Delta T \cdot \lambda/(\ell/2) = \Delta T\ C_w q_w$. The right-hand side is the convective heat transfer in the corresponding case. So ℓ is the length scale for which convective and diffusive heat transfer are of the same order.

Let us look a bit closer on the temperature field (8.3) around the pipe. We have from the pipe downstream, upstream and perpendicular to the flow the following temperature profiles:

$$\begin{aligned} &x > 0,\ z = 0 \qquad && T \sim e^{s} K_0(s) \qquad && s = x/\ell \\ &x < 0,\ z = 0 && T \sim e^{-s} K_0(s) && s = -x/\ell \\ &x = 0 && T \sim K_0(s) && s = |z|/\ell \end{aligned} \tag{8.5}$$

These three curves are shown in figure 8.2. The influence region around the pipes is scaled with the basic length ℓ.

Near the pipe, i.e. for small s, we have the approximation:

$$K_0(s) \simeq - \ln\left(\frac{s}{2}\right) - \gamma \qquad (s < 0.2)$$

$$\gamma = 0.5772 \tag{8.6}$$

The temperature downstream approaches asymptotically the following expression:

$$T(x,0) = - \frac{q}{2\pi\lambda} \cdot \sqrt{\frac{\pi\ell}{2x}} \qquad (x/\ell > 3) \tag{8.7}$$

The temperature near the pipe is given by

$$T(x,z) \simeq - \frac{q}{2\pi\lambda} \left\{ - \ln\left(\frac{\sqrt{x^2+z^2}}{2\ell}\right) - \gamma \right\} \tag{8.8}$$

$$\left(\frac{\sqrt{x^2+z^2}}{\ell} < 0.05 \right)$$

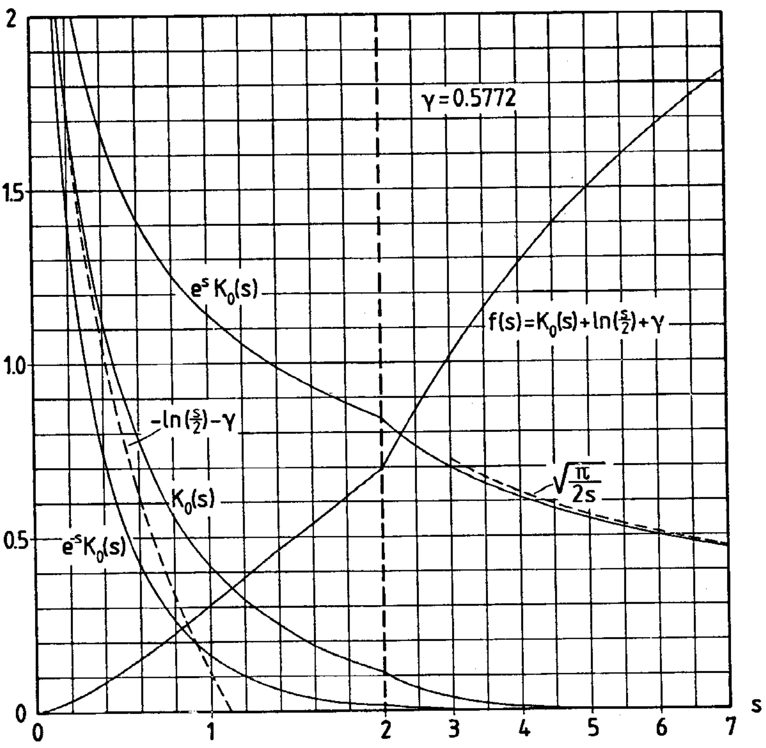


Figure 8.2. Functions associated with the pipe in moving ground water.

Let us consider the case:

$$\lambda = 1.5 \text{ W/mK} \qquad q_w = n \text{ m/year} = \frac{n}{3600 \cdot 24 \cdot 365} \text{ m/s}$$

$$R = 0.02 \text{ m} \qquad D = 1 \text{ m} \tag{8.9}$$

Then we have:

$$\ell = \frac{2 \cdot 1.5}{4.18 \cdot 10^6} \cdot \frac{3600 \cdot 24 \cdot 365}{n} = \frac{22.6}{n} \text{ m}$$

So we have:

$q_w = 0.1$ m/year $\quad \ell = 226$ m

$q_w = 1$ m/year $\quad \ell = 22.6$ m

$q_w = 10$ m/year $\quad \ell = 2.26$ m

$q_w = 100$ m/year $\quad \ell = 0.23$ m

The temperature at the pipe radius is from eq. 8.8:

$$T_R = -\frac{q}{2\pi\lambda} \left\{ \ln\left(\frac{2\ell}{R}\right) - \gamma \right\} \tag{8.10}$$

We assume here

$$\ell >> R \tag{8.11}$$

This assumption is made throughout this chapter.

From 8.10 we have the thermal resistance between the pipe and the surrounding ground:

$$m = \frac{1}{2\pi\lambda} \left\{ \ln\left(\frac{2\ell}{R}\right) - \gamma \right\} \tag{8.12}$$

This resistance may be represented by an equivalent circle around the pipe with a radius R_w:

$$m = \frac{1}{2\pi\lambda} \ln\left(\frac{R_w}{R}\right) \qquad R_w = 2 \cdot e^{-\gamma} \ell \simeq 1.1 \cdot \ell \tag{8.13}$$

Considering again example (8.9) we have:

$$m \cdot 2\pi\lambda = \ln\left(\frac{1.1 \cdot 22.6}{0.02 \cdot n}\right) = \ln\left(\frac{1243}{n}\right) = \begin{cases} 7.1 & n = 1 \\ 4.8 & n = 10 \end{cases} \qquad (8.14)$$

These values may be compared to our previous reference case 4.1.12 and the value 4.6 (eq. 4.1.15).

The effect of the ground surface can only be neglected if the depth D to the pipe is large compared to ℓ. Figure 8.3 shows a case which is easily solved analytically with the mirror sink technique. The pipe lies at the depth z = D. We assume that there is a constant horizontal ground water flow q_w in the $\hat{x}$-direction. The ground water level is assumed to lie very close to the ground surface. This is not a very realistic case but the result is anyhow of interest for an understanding of the effects of the ground water movement. The temperature at the ground surface is zero. The effect of variable surface conditions is as usual accounted for by superposition.

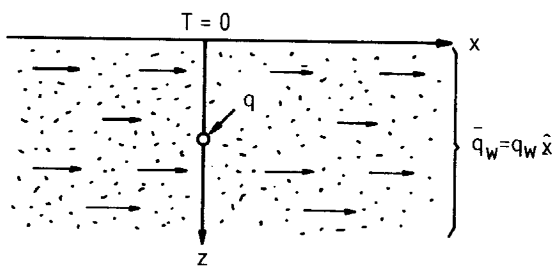


Figure 8.3. Heat extraction in moving ground water with the ground water level virtually at the ground surface.

The boundary condition at z = 0 is fulfilled if we imagine a negative mirror sink at (x,z) = (0,-D). We get two contributions of type (8.3): The temperature becomes:

$$T(x,z) = -\frac{q}{2\pi\lambda} e^{x/\ell} \cdot \left\{ K_o\left(\frac{\sqrt{x^2+(z-D)^2}}{\ell}\right) - K_o\left(\frac{\sqrt{x^2+(z+D)^2}}{\ell}\right) \right\} \qquad (8.15)$$

The temperature at the pipe radius is with the usual approximation for the distance to the mirror pipe:

$$T_R = -\frac{q}{2\pi\lambda}\left\{ K_0(R/\ell) - K_0(2D/\ell)\right\} \tag{8.16}$$

From eqs. 8.6 and 8.11 we have

$$\begin{aligned} K_0(R/\ell) - K_0(2D/\ell) &= -\ln\left(\frac{R}{2\ell}\right) - \gamma - K_0(2D/\ell) = \\ &= \ln\left(\frac{2D}{R}\right) - \left(K_0\left(\frac{2D}{\ell}\right) + \ln\left(\frac{D}{\ell}\right) + \gamma\right) \end{aligned} \tag{8.17}$$

We introduce the function

$$f(s) = K_0(s) + \ln(s/2) + \gamma \tag{8.18}$$

Then we have

$$T_R = -\frac{q}{2\pi\lambda}\left\{ \ln\left(\frac{2D}{R}\right) - f\left(\frac{2D}{\ell}\right) \right\} \tag{8.19}$$

So the thermal resistance for a pipe with moving ground water according to figure 8.3 is

$$m = \frac{1}{2\pi\lambda}\left\{ \ln\left(\frac{2D}{R}\right) - f\left(\frac{2D}{\ell}\right) \right\} \tag{8.20}$$

The first logarithm is the usual resistance. The function f represents the decrease due to the ground water flow. The function f is shown in figure 8.2.

Let us consider example 8.9. We have:

$$\begin{aligned} &\ln\left(\frac{2D}{R}\right) = 4.61 \qquad \frac{2D}{\ell} = \frac{n}{11.3} \\ &n = 1 \qquad f\left(\frac{2D}{\ell}\right) = f(0.09) = 0.001 \\ &n = 10 \qquad f\left(\frac{2D}{\ell}\right) = f(0.9) = 0.25 \\ &n = 100 \quad f\left(\frac{2D}{\ell}\right) = f(8.85) = 2.06 \end{aligned} \tag{8.21}$$

We see that the effect of the ground water flow is completely negligible in the first case ($q_w \sim$ 1m/year). There is a reduction of the thermal resistance with 5% in the second case ($q_w \sim$ 10m/year). In the third case ($q_w \sim$ 100m/year) the resistance is reduced with 45%.

8.2 Vertical infiltration

The case with a steady-state vertical infiltration may be treated with the same solution as above. The situation is shown in figure 8.4.

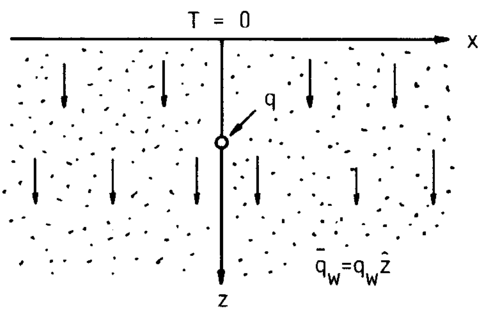


Figure 8.4. Steady-state vertical infiltration around a heat extraction pipe.

The assumption of steady-state vertical infiltration is not a realistic one, since the infiltration often varies with rain and snow melting. But the results obtained will still be of use if a suitable average infiltration rate is considered.

The temperature at the ground surface is to be zero. So we imagine a negative mirror sink at (x,z) = (0, -D). There are two contributions of type (8.3). But the temperature fields are not symmetrical in the z-direction. The boundary condition at z = 0 turns out to be fulfilled if the strength of the mirror sink is $-q\ e^{-2D/\ell}$. The temperature field is then:

$$T(x,z) = -\frac{q}{2\pi\lambda}\left\{e^{(z-D)/\ell}\cdot K_0\left(\frac{\sqrt{x^2+(z-D)^2}}{\ell}\right) - \right.$$
$$\left. -e^{-2D/\ell}\cdot e^{(z+D)/\ell}\cdot K_0\left(\frac{\sqrt{x^2+(z+D)^2}}{\ell}\right)\right\} \qquad (8.22)$$

The pipe temperature becomes:

$$T_R = -\frac{q}{2\pi\lambda}\left\{K_0(R/\ell) - K_0(2D/\ell)\right\} \qquad (8.23)$$

The expression is identical with (8.16). The thermal resistance of the case shown in figure 8.4 is then:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - f\left(\frac{2D}{\ell}\right)\right\} \qquad (8.24)$$

Here f is given by (8.18).

Let us again consider example 8.9. A reasonable value of q_w is

$$q_w = 0.5 \text{ m/year} \qquad (8.25)$$

Then we have

$$\ell = \frac{22.6}{0.5} = 45.2 \text{ m} \qquad f\left(\frac{2\cdot 1}{45.2}\right) = 0.002 \qquad (8.26)$$

$$2\pi\lambda\cdot m = 4.605 - 0.002$$

The infiltration gives a 0.04% decrease of thermal resistance. A very high value of q_w is

$$q_w = 5 \text{ m/year}$$

Then we have

$$\ell = \frac{22.6}{5} = 4.52 \text{ m} \qquad f\left(\frac{2\cdot 1}{4.52}\right) = 0.10 \qquad (8.27)$$

$$2\pi\lambda\ m = 4.61 - 0.10$$

The very strong infiltration reduces the thermal resistance with 2%. Let us finally consider the extreme case

$$q_w = 50 \text{ m/year}$$

Then we have

$$\ell = \frac{22.6}{50} = 0.452 \text{ m} \qquad f\left(\frac{2 \cdot 1}{0.452}\right) = f(4.4) = 1.38$$

$$2\pi\lambda\, m = 4.61 - 1.38 = 3.23$$

This extreme infiltration reduces the thermal resistance with 30%.

We may from this example conclude that the infiltration is in normal cases unimportant for the heat extraction rates. It must be remembered that we are talking about the convective effect. The strong infiltration may change the water content in the ground and the thermal conductivity. This will of course directly change the heat extraction potential. The periodic boundary condition at the ground surface causes a damped periodic variation at the pipe. This is discussed in section 5.2. The infiltration will increase the amplitude of the periodic variation at the pipe. This effect of the infiltration is however negligible, if ℓ is much bigger than the penetration depth d_o.

8.3 Ground water flow below a pipe

The ground water level lies in most cases below the heat extraction pipes. The steady-state heat extraction from a single pipe with a horizontal ground water flow in a region below the pipe will be discussed in this section.

The case is shown in figure 8.5. The ground water level lies at the depth $z = H$, $H > D$. There is a constant horizontal ground water flow in the region $z > H$. The thermal conductivity in the ground above the ground water level is λ. The value in the ground water region is λ_1.

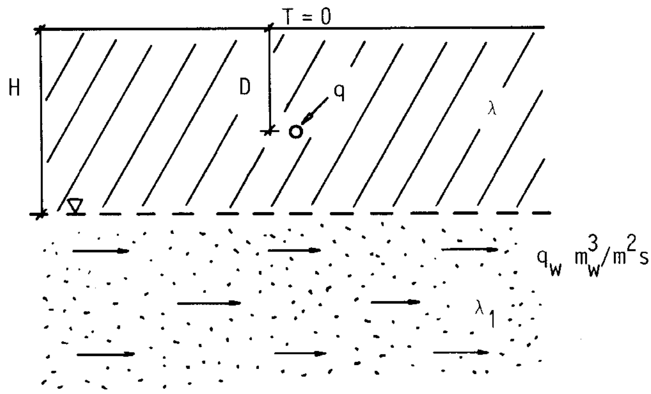


Figure 8.5. Steady-state heat extraction from a pipe with an underlying ground water flow.

The temperature field of this case is derived in appendix 4. The solution is rather complicated. The thermal resistance m between the pipe radius and the ground surface is given by eq. A4.15. We have:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - P_w\right\} \tag{8.28}$$

The logarithm is our standard thermal resistance of a pipe. The function P_w represents the effect of the flowing ground water and the other thermal conductivity λ_1. The function P_w depends on three dimensionless parameters:

$$P_w = P_w\left(\frac{D}{H}, \ \frac{\lambda_1}{\lambda}, \ \frac{HC_w q_w}{\lambda_1}\right) \tag{8.29}$$

The exact expression is given by eq. A4.17.

The limit with a very strong ground water flow gives the maximum effect of the ground water. The limit $q_w \to \infty$ implies that the temperature at the ground water level z = H is zero ; T(x,H) = 0. The solution in this case is given by A4.19. We have from A4.21:

$$P_w = \ln\left(\frac{\pi\ D/H}{\sin(\pi D/H)}\right) \qquad q_w = \infty \quad (T = 0,\ z = H) \tag{8.30}$$

This function is given below:

D/H	0	0.1	0.2	0.3	0.4	0.5
$P_w(D/H,\lambda_1/\lambda,\infty)$	0	0.02	0.07	0.15	0.28	0.45

(8.31)

D/H	0.6	0.7	0.8	0.9	0.95
$P_w(D/H,\lambda_1/\lambda,\infty)$	0.68	1.00	1.45	2.21	2.95

These values are to be compared with $\ln(2D/R)$, which is equal to 4.61 in reference case (4.1.12).

We have the following general conclusion. The maximal increase of the heat extraction with a very strong ground water flow is less than around 10% for $D/H \leq 0.5$. It is less than about 1% for $D/H \leq 0.2$.

In order to illustrate the effect of the ground water flow for different flows q_w we will take a few examples. The function P_w, which is given by the integral A4. 17 , has been computed numerically. We take:

$$D = 1 \text{ m} \qquad R = 0.02 \text{ m} \quad : \quad \ln(2D/R) = 4.61$$

$$\lambda = 1 \text{ W/mK} \qquad \lambda_1 = 2 \text{ W/mK} \quad : \quad \lambda_1/\lambda = 2 \tag{8.32}$$

The table below shows the computed values of P_w for different flow rates q_w for three values of the depth H to the ground surface:

q_w(m/year)	0	1.5	7.5	37.5	187.5	∞
H = 1.5 m	0.21	0.22	0.30	0.50	0.69	0.88
H = 2 m	0.10	0.11	0.17	0.28	0.37	0.45
H = 4 m	0.02	0.03	0.05	0.08	0.09	0.11

(8.33)

These values are to be subtracted from 4.61.

We see that the ground water flow must be quite strong and the ground water level must lie rather close to the pipe in order to affect significantly the heat extraction.

The flowing ground water may cause an increase of the apparent thermal conductivity. This so-called macrodispersion is due to inhomogeneities in the ground and the ground water flow pattern. Let us therefore also consider a case with a very high thermal conductivity λ_1. We take the case:

$$D = 1 \text{ m} \qquad H = 2 \text{ m}$$
$$\lambda = 1 \text{ W/mK} \qquad \lambda_1 = 10 \text{ W/mK} \tag{8.34}$$

Then we get:

q_w(m/year)	0	1.5	7.5	37.5	187.5	∞
P_w	0.32	0.32	0.33	0.36	0.41	0.45

We note again that the effect of the ground water is rather small even in this extreme case.

9. TEMPERATURE VARIATION ALONG THE PIPE. PIPE ARRANGEMENT.

This chapter is devoted to a study of the temperature variation along the pipes and the effect of this on the heat extraction rate. Only steady-state is considered.

New factors that affect the heat extraction is the fluid flow rate, the length of the pipes and the fluid flow arrangements through the different pipes.

The three-dimensional temperature field around the end of a heat extraction pipe is discussed in appendix 5. It is shown that the particular end effects only concern one or two meters of pipe. So these end effects are not considered in this chapter.

9.1 Temperature variation along a single pipe

Figure 9.1 shows the considered case. There is a single heat extraction pipe along $(x,y,z) = (0,y,D)$, $0 \leq y \leq L$. The length of the pipe is L. The inlet fluid temperature is T_{in}. The temperature at the outlet at y = L is denoted T_{out}. The fluid flow rate is Q_f (m^3/s), and the volumetric fluid heat capacity is C_f (J/m^3K).

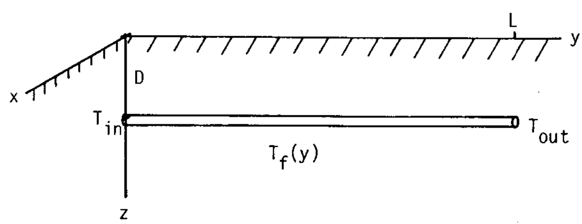


Figure 9.1. Steady-state heat extraction with a varying temperature along the pipe.

The temperature at the ground surface is zero. Let q(y) (W/m) denote the heat extraction rate and $T_f(y)$ the fluid temperature. The total thermal resistance between pipe and ground surface is $m+m_p$. Then we have from eq. 4.3.2:

$$-T_f(y) = (m + m_p)\, q(y) \tag{9.1.1}$$

The heat extraction increases the fluid temperature:

$$Q_f C_f \frac{dT_f}{dy} = q(y) \tag{9.1.2}$$

We have for T_f the differential equation

$$\frac{dT_f}{dy} = -\frac{1}{y_f} \cdot T_f(y) \qquad T_f(0) = T_{in} \tag{9.1.3}$$

where we have introduced a characteristic thermal length:

$$y_f = (m + m_p)\, C_f Q_f \tag{9.1.4}$$

The solution of (9.1.3) is simple:

$$T_f(y) = T_{in} \cdot e^{-y/y_f} \tag{9.1.5}$$

In particular we have the outlet temperature:

$$T_{out} = T_{in} \cdot e^{-L/y_f} \tag{9.1.6}$$

The mean fluid velocity v_f (m/s) is

$$v_f = \frac{Q_f}{\pi R_-^2} \tag{9.1.7}$$

Let us consider the reference example:

$$\begin{array}{lll} \lambda = 1.5 \text{ W/mK} & a = 0.75 \cdot 10^{-6} \text{m}^2/\text{s} & \\ D = 1\text{m} & R_- \simeq R = 0.02\text{m} & L = 100\text{m} \\ m_p = 0 & m = \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) = 0.489 \text{ mK/W} & \\ C_f = 4.18 \text{ MJ/m}^3\text{K} & & \\ v_f = 0.38 \text{ m/s} & & \end{array} \tag{9.1.8}$$

Then we have

$$y_f = 0.489 \cdot 4.18 \cdot 10^6 \cdot 0.38 \cdot \pi \cdot 0.02^2 = 976 \text{ m} \tag{9.1.9}$$

Let $\bar{q}$ (W/m) denote the mean heat extraction over the pipe length:

$$\bar{q} = \frac{1}{L} \int_0^L q(y)dy \tag{9.1.10}$$

Then we have, using eq. 9.1.6

$$L\bar{q} = C_f Q_f (T_{out} - T_{in}) = C_f Q_f (1 - e^{-L/y_f}) \cdot (-T_{in}) \tag{9.1.11}$$

We can define a total mean thermal resistance $\bar{m}_t$ per unit length:

$$0 - T_{in} = \bar{m}_t \cdot \bar{q} \tag{9.1.12}$$

$$\bar{m}_t = \frac{L}{(1-e^{-L/y_f})Q_f C_f} \tag{9.1.13}$$

The temperature along the pipe declines exponentially according to eq. 9.1.5. The length y_f is often, as example 9.1.8 shows, much larger than the pipe length L. A linear approximation is then valid with good accuracy:

$$T_f(y) = T_{in} \cdot e^{-y/y_f} \simeq T_{in} \cdot (1 - \frac{y}{y_f}) \quad (y \ll y_f) \tag{9.1.14}$$

The error is less than 2% when

$$\frac{y}{y_f} < 0.2 \text{ or } \frac{L}{y_f} < 0.2 \tag{9.1.15}$$

This linear approximation will be used in the following.

Let us introduce the quantity μ:

$$\mu = \frac{L}{C_f Q_f} \tag{9.1.16}$$

The dimension of μ is that of a thermal resistance per unit length (Km/W). The length y_f becomes

$$y_f = \frac{m+m_p}{\mu} L \qquad (9.1.17)$$

The mean thermal resistance $\bar{m}_t$ from eq. 9.1.13 may be written:

$$\bar{m}_t = \frac{\mu}{1-e^{-\mu/(m+m_p)}} \qquad (9.1.18)$$

A series expansion in $\mu/(m+m_p)$ of 9.1.18 gives:

$$\bar{m}_t = (m+m_p+\frac{\mu}{2})\left(1 - \frac{\mu^2}{24(m+m_p)(m+m_p+\mu)} + \ldots\right) \qquad (9.1.19)$$

The first factor of 9.1.19 is a linear approximation:

$$\bar{m}_t \simeq m+m_p+\frac{\mu}{2} \qquad (9.1.20)$$

We see from 9.1.19 that the relative error is

$$\frac{1}{24}\cdot\left(\frac{\mu}{m+m_p}\right)^2 \qquad (9.1.21)$$

For the example 9.1.8 we have:

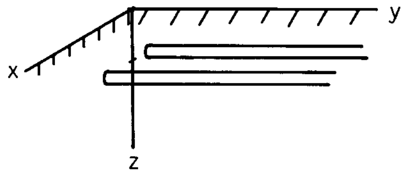
$$\mu = \frac{100}{4.18^6 \cdot 0.38 \cdot \pi \cdot 0.02^2} = 0.05 \text{ mK/W}$$

$$m+m_p = 0.489 \text{ mK/W}$$

The relative error 9.1.21 is then only 0.0004. We may conclude that the linear approximation may be used with very good accuracy in normal situations.

Formula 9.1.20 gives a very simple correction for the changing temperature along the pipe. The mean thermal resistance is equal to the earlier cross-section resistance $m+m_p$ plus an additional term $\mu/2$.

Figure 9.2 shows two parallel pipes at the depth z = D. This case is easily solved with the above solution.



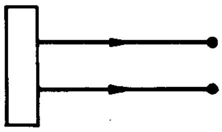


Figure 9.2. Two parallel pipes at the depth z = D.

The fluid temperature $T_f(y)$ along the two pipes is the same due to symmetry. A relation of type (9.1.1) is still valid. We have from eq. 4.5.10:

$$-T_f(y) = (m_{t1} + m_{12})\cdot q(y) \qquad (9.1.22)$$

We get the same solution 9.1.6 as before with a new thermal resistance in the formula for y_f:

$$y_f = (m_{t1} + m_{12})\cdot C_f Q_f \qquad (9.1.23)$$

9.2 Linear temperature approximation for two pipes

In the following we assume that the steady-state temperature varies linearly along each pipe. The case with two pipes is studied in this section, while the case with an array of N pipes is dealt with in section 9.4.

Figure 9.3 shows a cross-section through the two pipes for any y, $0 \leq y \leq L$.

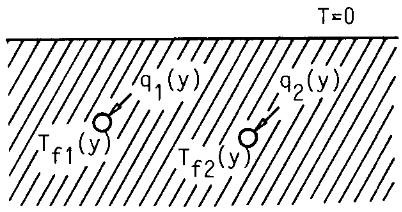


Figure 9.3. Steady-state heat extraction by two pipes, $0 \leq y \leq L$.

The two-dimensional analysis of section 4.5 may be used with good accuracy for each y. We have from eqs. 4.5.7 the general relations:

$$-T_{f1}(y) = q_1(y) \cdot m_{t1} + q_2(y) \cdot m_{12}$$
$$-T_{f2}(y) = q_2(y) \cdot m_{t2} + q_1(y) \cdot m_{12} \qquad (9.2.1)$$

Integration over $0 \leq y \leq L$ gives

$$-\bar{T}_{f1} = \bar{q}_1 m_{t1} + \bar{q}_2 m_{12}$$
$$-\bar{T}_{f2} = \bar{q}_2 m_{t2} + \bar{q}_1 m_{12} \qquad (9.2.2)$$

Here $\bar{q}_1$ and $\bar{q}_2$ denotes the mean extraction rate for pipe 1 and pipe 2. The left-hand side gives the mean fluid temperature along the pipe. In our linear approximation we have:

$$\bar{T}_{fi} = \tfrac{1}{2}(T_{fi}(0) + T_{fi}(L)) \qquad (9.2.3)$$

There are two possibilities for the fluid flow in the two pipes. The flow may be in parallel as shown in figure 9.4A. The pipes are connected in series in the other case as shown in figure 9.4B.

In the parallel case we have from eq. 9.2.3:

$$\bar{T}_{fi} = T_{in} + \tfrac{1}{2}(T_{fi}(L) - T_{in}) = T_{in} + \tfrac{1}{2} \cdot \frac{L\,\bar{q}_i}{C_f Q_f} \quad (i = 1,2) \qquad (9.2.4)$$

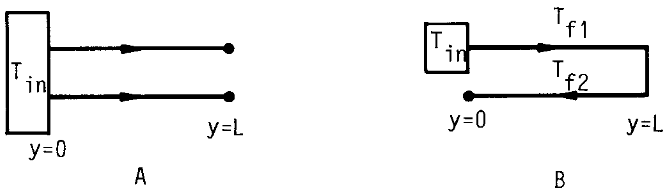


Figure 9.4. Parallel (A) or series (B) connection of two pipes.

Eq. 9.2.4 may, using notation 9.1.16, be written:

$$\bar{T}_{f1} = T_{in} + \frac{\mu}{2}\bar{q}_1$$
$$\bar{T}_{f2} = T_{in} + \frac{\mu}{2}\bar{q}_2 \qquad (9.2.5)$$

From eqs. 9.2.2 and 9.2.6 we have in the parallel case:

$$-T_{in} = (m_{t1} + \frac{\mu}{2})\bar{q}_1 + m_{12}\bar{q}_2$$
$$-T_{in} = (m_{t2} + \frac{\mu}{2})\bar{q}_2 + m_{12}\bar{q}_1 \qquad (9.2.6)$$

In the series case of figure 9.4B we have for pipe 1 as above:

$$\bar{T}_{f1} = T_{in} + \frac{\mu}{2}\bar{q}_1 \qquad (9.2.7)$$

For the second pipe we get:

$$\bar{T}_{f2} = \tfrac{1}{2}(T_{f2}(L) + T_{f2}(0)) = T_{f2}(L) + \tfrac{1}{2}(T_{f2}(0) - T_{f2}(L)) \qquad (9.2.8)$$

But we also have:

$$T_{f2}(L) = T_{f1}(L) = T_{in} + \mu\bar{q}_1$$
$$T_{f2}(0) - T_{f2}(L) = \mu\bar{q}_2 \qquad (9.2.9)$$

Finally we get for the second pipe:

$$\bar{T}_{f2} = T_{in} + \mu\,\bar{q}_1 + \frac{\mu}{2}\,\bar{q}_2 \qquad (9.2.10)$$

For the series case we get from eqs. 9.2.2, 7 and 10:

$$\begin{aligned} -T_{in} &= (m_{t1} + \tfrac{\mu}{2})\,\bar{q}_1 + m_{12}\,\bar{q}_2 \\ -T_{in} &= (m_{t2} + \tfrac{\mu}{2})\,\bar{q}_2 + (m_{12} + \mu)\,\bar{q}_1 \end{aligned} \qquad (9.2.11)$$

The mean thermal resistance per unit length of the two pipes is defined by:

$$-T_{in} = \bar{m}_{1+2} \cdot (\bar{q}_1 + \bar{q}_2) \qquad (9.2.12)$$

We get from eq. 9.2.6 in the case with parallel flow

$$\bar{m}_{1+2} = \frac{(m_{t1} + \frac{\mu}{2})(m_{t2} + \frac{\mu}{2}) - m_{12}^2}{m_{t1} + m_{t2} + \mu - 2m_{12}} \qquad (9.2.13)$$

In the series case we have from eq. 9.2.11:

$$\bar{m}_{1+2} = \frac{(m_{t1} + \frac{\mu}{2})(m_{t2} + \frac{\mu}{2}) - m_{12}(m_{12}+\mu)}{m_{t1} + m_{t2} - 2m_{12}} \qquad (9.2.14)$$

The quantities of 9.2.13-14 are defined by eqs. 4.5.8, 4.5.4 and 9.1.16.

9.3 Three and four pipes

The analysis of sections 4.6 and 4.7 for three and four pipes is readily extended to account for the linear variation along the pipes.

Let us first consider the case with three pipes at the depth D. The case is shown in figure 4.10. We assume that the three pipes are in parallel. See figure 9.5.

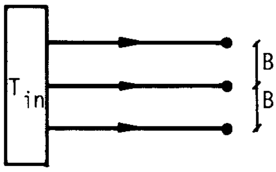


Figure 9.5. Three pipes in parallel.

We have directly with the notations of section 4.6 from eqs. 4.6.1:

$$-T_{in} = (m+m_p + \frac{\mu}{2})\ \bar{q}_1 + 2m_{12}\ \bar{q}_2$$
$$-T_{in} = (m+m_p + \frac{\mu}{2} + m_{22})\ \bar{q}_2 + m_{12}\ \bar{q}_1 \qquad (9.3.1)$$

The thermal resistance of the three pipes together is:

$$-T_{in} = \bar{m}_{2+1+2} \cdot (\bar{q}_1 + 2\bar{q}_2) \qquad (9.3.2)$$

$$\bar{m}_{2+1+2} = \frac{(m+m_p + \frac{\mu}{2})(m+m_p + \frac{\mu}{2} + m_{22}) - 2m_{12}^2}{3(m+m_p + \frac{\mu}{2}) + m_{22} - 4m_{12}} \qquad (9.3.3)$$

The considered case with four pipes is shown in figure 4.11. The notations of section 4.7 are used. Two fluid flow arrangements will be discussed. See figure 9.6.

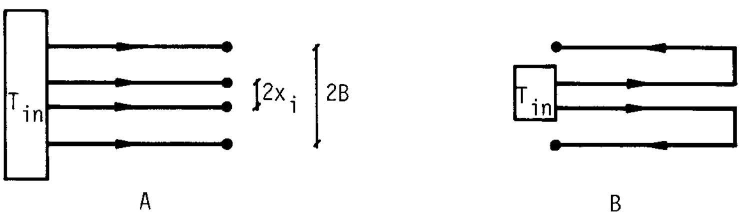


Figure 9.6. Four pipes in parallel (A) or in a bifilar arrangement (B).

In the parallel case we have in accordance with eqs. 4.7.1:

$$-T_{in} = (m+m_p + m_{11} + \tfrac{\mu}{2})\,\bar{q}_1 + (m_{12} + m'_{12})\,\bar{q}_2$$

$$-T_{in} = (m+m_p + m_{22} + \tfrac{\mu}{2})\,\bar{q}_2 + (m_{12} + m'_{12})\,\bar{q}_1 \qquad (9.3.4)$$

The mean thermal resistance per unit length of the four pipes becomes:

$$\bar{m}_{2+1+1+2} = \tfrac{1}{2}\cdot\frac{(m+m_p + m_{11} + \frac{\mu}{2})(m+m_p + m_{22} + \frac{\mu}{2}) - (m_{12} + m'_{12})^2}{2m + 2m_p + m_{11} + m_{22} + \mu - 2m_{12} - 2m'_{12}} \qquad (9.3.5)$$

In the bifilar case of figure 9.6B we have instead:

$$-T_{in} = (m+m_p + m_{11} + \tfrac{\mu}{2})\,\bar{q}_1 + (m_{12} + m'_{12})\,\bar{q}_2$$

$$-T_{in} = (m+m_p + m_{22} + \tfrac{\mu}{2})\,\bar{q}_2 + (m_{12} + m'_{12} + \mu)\,\bar{q}_1 \qquad (9.3.6)$$

The thermal resistance becomes:

$$\bar{m}_{2+1+1+2} = \tfrac{1}{2}\cdot\frac{(m+m_p+m_{11}+ \frac{\mu}{2})(m+m_p+m_{22}+ \frac{\mu}{2}) - (m_{12}+m'_{12})(m_{12}+m'_{12}+\mu)}{2m + 2m_p + m_{11} + m_{22} - 2m_{12} - 2m'_{12}} \qquad (9.3.7)$$

The above expression is not changed if the fluid flow direction is reversed so that the inlet is into the outer pipes.

9.4 General formulas for N pipes

Let us now consider the general case with N pipes that lie in parallel. We use the notations of section 4.8. The mean relation for pipe number i is from eq. 4.8.6:

$$-\bar{T}_{fi} = (m_i + m_{pi})\,\bar{q}_i + \sum_{\substack{j=1\\ j\neq i}}^{N} m_{ij}\,\bar{q}_j \qquad (9.4.1)$$

The mean temperature $\bar{T}_{fi}$ along pipe i is determined by the inlet temperature T_{in} and the fluid flow arrangement in the pipes. Let S_i denote the set of indices of the pipes through which the fluid flows before it reaches the considered pipe i. The set is empty if the pipe is directly connected to the inlet with the temperature T_{in}.

As an example we can take a case when pipes 1,6,7 and 10 lies in series. Then we have:

$$S_1 = [-] \qquad S_6 = [1]$$
$$S_7 = [1,6] \qquad S_{10} = [1,6,7] \tag{9.4.2}$$

The increase of temperature along a pipe k is $\mu\bar{q}_k$. So we have

$$\bar{T}_{fi} = T_{in} + \frac{\mu}{2}\bar{q}_i + \sum_{k \in S_i} \mu\,\bar{q}_k \tag{9.4.3}$$

From 9.4.1 and 9.4.3 we have the following equation system for the N pipes:

$$-T_{in} = (m_i + m_{pi} + \frac{\mu}{2})\,\bar{q}_i + \sum_{\substack{j=1 \\ j \neq i}}^{N} m_{ij}\,\bar{q}_j + \sum_{k \in S_i} \mu\,\bar{q}_k \tag{9.4.4}$$

The equation system is solved by inversion, when T_{in} is given. We have in particular a relation between T_{in} and the total extraction rate $\bar{q}_1 + \bar{q}_2 + \ldots + \bar{q}_N$ per unit length of pipes.

The mean thermal resistance of the N pipes is defined by:

$$-T_{in} = \bar{m}_{1+2+..+N} \cdot (\bar{q}_1 + \bar{q}_2 + \ldots + \bar{q}_N) \tag{9.4.5}$$

9.5 Comparison of pipe arrangements

In this section the heat extraction rates for a few simple pipe arrangements will be compared. The data of 9.1.8 will be used. So we use:

$$\lambda = 1.5 \text{ W/mK} \qquad \frac{R}{D} = 0.02$$
$$m_p = 0 \qquad \mu = 0.05 \text{ mK/W} \qquad (9.5.1)$$

Our first comparison concerns the effect when a second pipe is added to a single pipe. The heat extraction rate of the single pipe is given by eq. 9.1.20. The extraction rate of two pipes in parallel or in series are given by eqs. 9.2.13 and 9.2.14. The relative increase of the steady-state heat extraction rate as a function of the spacing B/D is shown in figure 9.7 for four cases. The first and second case concern two pipes at the same depth D with parallel and series flow respectively. The third and fourth cases concern two pipes in the same ditch with parallel and series flow respectively. The added pipe lies below the first one.

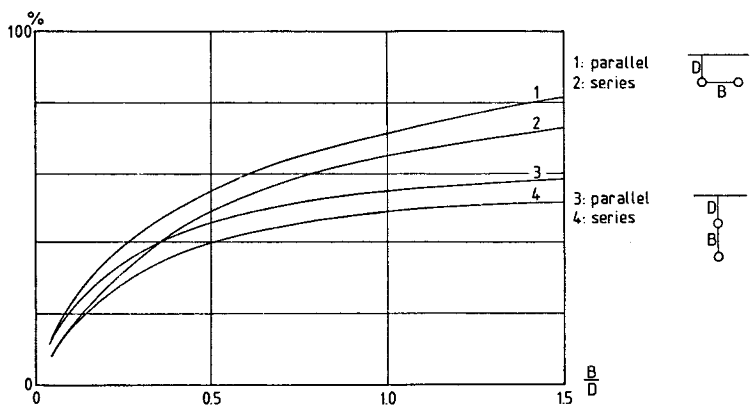


Figure 9.7. Relative increase of steady-state heat extraction rate when the second pipe is added. Data according to eq. 9.5.1 or 9.1.8.

We see from figure 9.7 that the case with parallel flow is better, but the difference between parallel and series flow is rather small.

Our other comparison concerns N pipes that all lie at the depth D with a spacing B. The heat extraction is compared to that of N independent pipes. The relative extraction rate for N equal 2 and 4 is shown in figure 9.8 as a function of the spacing B/D. Parallel (a), bifilar (b) and series (c) flow are considered.

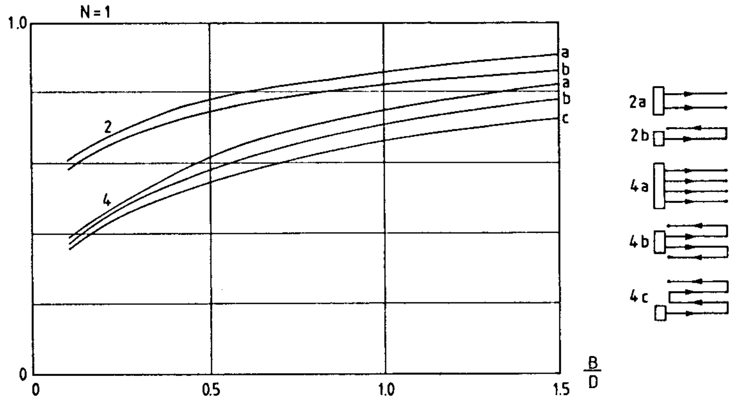


Figure 9.8. Relative extraction efficiency of N pipes at the depth D with a spacing B for different pipe arrangements. Data according to eq. 9.5.1 or 9.1.8.

We see that the extraction decreases when the water flows in series for two or more pipes. The decrease between parallel and series flow for four pipes (4a and 4c) is around 10% in this particular example. The arrangement of pipes in series may decrease the extraction rate relatively strongly.

9.6 A pipe through two regions

A heat extraction pipe may pass through different regions. The ground material or the depth to the ground may change. Such cases can also be analysed. We will consider a single example.

Figure 9.9 shows the considered case. The pipe passes two different regions with different thermal conductivities and height to the ground surface.

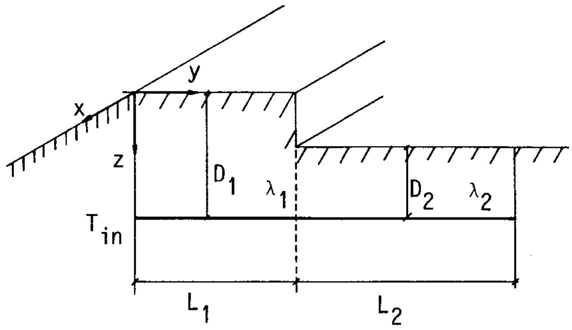


Figure 9.9. Steady-state heat extraction to a pipe that passes two different regions.

The analysis of section 9.1 is applicable for each region. We introduce:

$$m_1 = \frac{1}{2\pi\lambda_1} \ln\left(\frac{2D_1}{R}\right) \qquad m_2 = \frac{1}{2\pi\lambda_2} \ln\left(\frac{2D_2}{R}\right)$$
$$\mu_1 = \frac{L_1}{C_f Q_f} \qquad \mu_2 = \frac{L_2}{C_f Q_f} \tag{9.6.1}$$

Then we have for the first region:

$$-T_{in} = (m_1 + m_p + \frac{\mu_1}{2}) \cdot \bar{q}_1 \tag{9.6.2}$$

For the second region we get

$$-T_{in} = \mu_1 \bar{q}_1 + (m_2 + m_p + \frac{\mu_2}{2}) \bar{q}_2 \tag{9.6.3}$$

The mean thermal resistance per unit length is defined by

$$-T_{in} = \bar{m}_{1+2} \cdot \bar{q} \qquad \bar{q} = \frac{L_1\bar{q}_1 + L_2\bar{q}_2}{L_1 + L_2} \tag{9.6.4}$$

From eqs. 9.6.2-4 we have

$$\bar{m}_{1+2} = (L_1 + L_2) \frac{(m_1+m_p + \frac{\mu_1}{2})(m_2+m_p + \frac{\mu_2}{2})}{L_1(m_2+m_p + \frac{\mu_2}{2}) + L_2(m_1+m_p - \frac{\mu_1}{2})} \qquad (9.6.5)$$

As an example we consider the case:

$$L_1 = L_2 = 100\text{m} \qquad D_1 = 1\text{m} \qquad D_2 = 0.75\text{m}$$

$$\lambda_1 = 1.0 \text{ W/mK} \qquad \lambda_2 = 2.0 \text{ W/mK} \qquad m_p = 0 \qquad (9.6.6)$$

$$\mu_1 = \mu_2 = 0.05 \text{ mK/W} \qquad R = 0.02\text{m}$$

The mean thermal resistance 9.6.5 becomes:

$$\bar{m}_{1+2} = 0.519 \text{ mK/W}$$

Suppose that we want to extract 10 W/m in mean. The inlet temperature is then

$$T_{in} = -0.519 \cdot 10 = -5.19^{\circ}\text{C}$$

So the inlet temperature must lie 5.2°C below the undisturbed ground temperature in order to obtain the prescribed heat extraction.

10. THERMAL IMPACT ON SURROUNDING GROUND

We will in this chapter discuss the disturbance of the ground temperature due to the heat extraction pipes. We will deal with the temperature field at some distance from the pipes.

The two-dimensional temperature field around a pipe is discussed in chapter 3. Figure 3.2 a-d and 3.3-5 show the increasing thermal disturbance around a heat extraction pipe. The steady-state part gives the largest disturbance except for the region near the pipe. We will therefore now restrict ourselves to the steady-state situation.

The thermal influence region for steady-state heat extraction of a single pipe is discussed in section 4.2. The temperature field is given by the simple expression (4.2.3) at a distance greater than 3D from the pipe. The expression is a dipole approximation of the line source with its mirror source above the ground surface. The corresponding three-dimensional analysis will be made here.

10.1 Single pipe of finite length

Our first case concerns a single straight heat extraction pipe with the length 2L. The pipe lies at the depth $z = D$ along the y-axis. The center-line of the pipe is defined by

$$(x,y,z) = (o,y,D) \qquad -L \leq y \leq L \qquad (10.1.1)$$

The heat extraction rate from the pipe will vary along the pipe. But the variation along the pipe is normally rather small except for the very ends of the pipe. We assume as a necessary simplification that the heat extraction rate q (W/m) is constant along the pipe. So we have a finite line sink along the pipe. There is a negative mirror sink above the ground surface.

The steady-state temperature field of this case may be obtained explicitly. The expression is greatly simplified, if a dipole approximation along the pipe is used. The solution is derived in appendix 6. The dipole approximation is only valid for points that lie more than 3D from the pipe.

We have according to eq. A6.6 the following approximate expression for the temperature field around a heat extraction pipe of finite length:

$$T(x,y,z) \simeq -\frac{q}{2\pi\lambda} \cdot \frac{Dz}{x^2 + z^2} \cdot \left\{ \frac{L - y}{\sqrt{x^2 + (L-y)^2 + z^2}} + \right.$$

$$\left. + \frac{L + y}{\sqrt{x^2 + (L+y)^2 + z^2}} \right\} \qquad (10.1.2)$$

(valid at distances greater than 3D from the pipe)

The first two factors are the two-dimensional dipole expression (4.2.3), which is valid for $L = \infty$. The last factor accounts for the finite length.

Let us consider the following case

$q = 10$ W/m $\qquad \lambda = 1.5$ W/mK

$D = 1$ m $\qquad 2L = 20$ m $\qquad R = 0.02$ m $\qquad$ (10.1.3)

$-\frac{q}{2\pi\lambda} = 1.06^\circ C \qquad -\frac{q}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) = -4.9^\circ C$

The temperature at the pipe radius is $-4.9^\circ C$. We consider the temperature along three lines at the pipe depth $z = 1$ m. The first line lies perpendicular to the pipe mid-point ($y = 0$). The second and third ones start from the end of the pipe ($y = L = 10$ m). They lie perpendicular and parallel to the pipe. We have in these three cases from eq. 10.1.2:

$$T(d,0,1) = -1.06 \cdot \frac{1}{1 + d^2} \cdot \frac{20}{\sqrt{d^2 + 101}}$$

$$T(d,10,1) = -1.06 \cdot \frac{1}{1 + d^2} \cdot \frac{20}{\sqrt{d^2 + 401}} \qquad (d \geq 3)$$

$$T(0,10 + d,1) = -1.06 \cdot \left(\frac{-d}{\sqrt{d^2 + 1}} + \frac{20 + d}{\sqrt{(20+d)^2 + 1}}\right) \qquad (10.1.4)$$

We get the following temperatures:

d(m)	3	5	10	25	50	100
$-T(d,0,1)$ (^oC)	0.202	0.073	0.015	0.0013	0.00017	0.00002
$-T(d,10,1)$	0.105	0.039	0.010	0.0011	0.00016	0.00002
$-T(0,10+d,1)$	0.053	0.020	0.004	0.0006	0.00011	0.00002

These values are to be compared with the pipe temperature -4.9^oC.

We can conclude from this example that the thermal disturbance from a single heat extraction pipe is quite small. The temperature change relative to undisturbed conditions at a distance of 5 m or more from the pipe at the pipe depth will never exceed 0.07^oC in this case.

10.2 Rectangular heat extraction area

The heat extraction pipes often cover a certain rectangular area. We will consider the case, when a rectangular area $-L < y < L$, $-M < x < M$ is used. The pipes lie at the depth $z = D$. The steady-state heat extraction per unit area of the rectangular field is denoted q_a (W/m^2). If the spacing between the pipes is B and the steady-state heat extraction per unit pipe length is q, then for a single layer of pipes

$$q_a = \frac{q}{B} \tag{10.2.1}$$

The heat extraction q_a will vary over the field. But normally this variation is small except for the outer pipes. We assume here however that q_a is constant over the rectangular area:

$$q_a \ \mathrm{W/m^2} \qquad \begin{matrix} -L < y < L \\ -M < x < M \end{matrix} \tag{10.2.2}$$

The heat extraction is in reality located to the pipes. The temperature field at a distance larger than 3D from all pipes may be approximated by dipole fields. The exact position of the extraction sources is not important. In this approximation the rectangular heat extraction field may be replaced by a rectangular dipole field. The solution in this case is given in appendix 6. We have from A6.8:

$$T(x,y,z) = -\frac{q_a D}{\lambda} \Big\{ a(M-x,\ L-y,z) + a(M+x,\ L-y,z)$$

$$+ a(M-x,\ L+y,z) + a(M+x,\ L+y,z) \Big\} \tag{10.2.3}$$

(valid at distances larger than 3D from the pipes)

Here we have used the notation:

$$a(x',y',z) = \frac{1}{2\pi} \arctan \left(\frac{x'\,y'}{z\sqrt{(x')^2 + (y')^2 + z^2}} \right) \tag{10.2.4}$$

The temperature field (10.2.3) has another interpretation which is illustrated in figure 10.1. There is steady-state heat conduction in the ground region $z > 0$. The temperature at the ground surface is $T = -q_a D/\lambda$ over the rectangular region $-L < y < L$, $-M < z < M$. The temperature is zero outside the rectangle at the ground surface. The temperature field in this case is exactly given by (10.2.3).

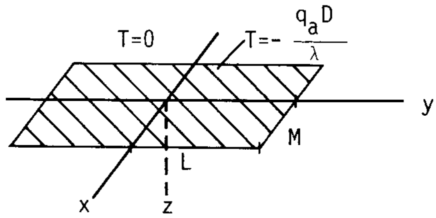


Figure 10.1. Equivalent interpretation of the dipole field (10.2.3)

So the temperature disturbance due to a rectangular heat extraction field at distances larger than 3D from the pipes is the same as if there were a constant temperature T_a on the rectangular area at the ground surface. This equivalent surface temperature is

$$T_a = - \frac{Dq_a}{\lambda} = - \frac{Dq}{\lambda B} \qquad (10.2.5)$$

We have for example:

$q = 10$ W/m $\quad D = 1$ m $\quad B = 1$ m

$\lambda = 1.5$ W/mK

$$T_a = - \frac{1 \cdot 10}{1.5 \cdot 1} = -6.7^oC \qquad (10.2.6)$$

As an example of the thermal effect of the rectangular heat extraction area we take

$$2\,L = 10 \text{ m} \qquad 2\,M = 20 \text{ m} \qquad (10.2.7)$$

Let us first consider the z-axis downwards:

1. $\quad x = 0 \quad y = 0 \quad z = d$

d(m)	0	1	2	3	4	5	6	7	8	9
T/T_a	(1)	(0.86)	(0.73)	(0.61)	0.52	0.44	0.37	0.32	0.27	0.24

d(m)	10	12	15	20	30	40	50	100
T/T_a	0.20	0.16	0.11	0.07	0.033	0.019	0.012	0.003

We see that the temperature disturbance extends far downwards. The values for $d < 4$ are not valid, since we are too close to the pipes.

2. $x = 0$, $y = 5+d$, $z = 1$ m

d(m)	0	3	5	15
T/T_a	(0.46)	0.06	0.03	0.004

3. $x = 10+d$, $y = 0$, $z = 1$ m

d(m)	0	3	5	15
T/T_a	(0.44)	0.05	0.02	0.003

We see from these values that the thermal disturbance is quite small at the pipe depth $z = 1$m outside the pipe field. The value 3 meters outwards is about $-6.7 \cdot 0.06 = -0.4^{\circ}C$. Five meters away at the depth of one meter the disturbance is $-0.2^{\circ}C$.

We can in general conclude that the disturbance at the pipe level (and upwards) is negligible at a distance of 5D from the pipes.

11. SUMMARY OF FORMULAS

Steady-state heat extraction

The thermal resistance m is defined by the equation

$$\Delta T = q \cdot m$$

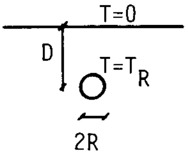
Here ΔT is the driving temperature difference for example between the fluid in the pipe and the ground surface. The heat extraction rate per unit length of the pipe is q. The dimension of m is K/(W/m) = Km/W. The inverse 1/m gives the heat flux per unit temperature difference:

$$q = \Delta T \cdot \frac{1}{m}$$

It must be kept in mind that m refers to a unit length of the pipe. It must also be kept in mind that the thermal resistance only concerns the steady-state component of the heat extraction.

Single pipe

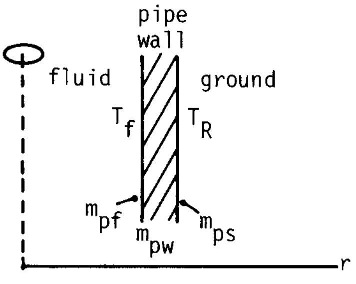
Soil resistance:



$$m = \frac{1}{2\pi\lambda} \ln\left(\frac{2D}{R}\right)$$

$$0 - T_R = q \cdot m$$

Fluid-soil or pipe resistance:



$$m_p = m_{pf} + m_{pw} + m_{ps}$$

$$T_R - T_f = q \cdot m_p$$

m_{pf} : eqs. 4.3.9-11

m_{pw} : eq. 4.3.6

m_{ps} : from measurement

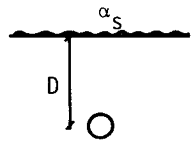
Total resistance between fluid and ground surface:

$$m_t = m + m_p$$

$$0 - T_f = q \cdot m_t$$

Soil resistance including ground surface resistance:

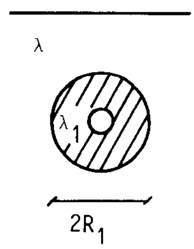
$$m = \frac{1}{2\pi\lambda} \cdot \left[\ln\left(\frac{2D}{R}\right) + g_s(D\alpha_s/\lambda) \right]$$



g_s: eq. 4.4.4, table 4.3

Soil resistance including a different region around the pipe:

$$m = \frac{1}{2\pi\lambda} \cdot \ln\left(\frac{2D}{R_1}\right) + \frac{1}{2\pi\lambda_1} \cdot \ln\left(\frac{R_1}{R}\right)$$



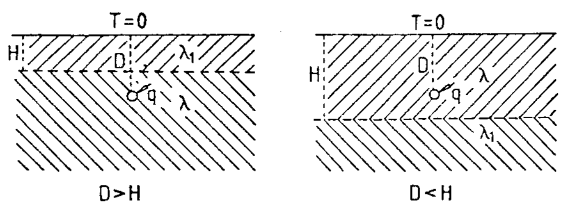
More general $\lambda_1 = \lambda_1(r)$:

m: eq. 4.3.13

Total resistance between ground surface and pipe fluid:

$$m_t = m + m_p \qquad\qquad 0 - T_f = q \cdot m_t$$

Two layers of soil:

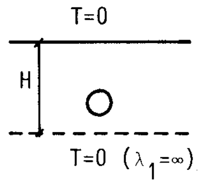


$$m = \frac{1}{2\pi\lambda} \left\{ \ln\left(\frac{2D}{R}\right) + p(H/D,\sigma) \right\} \qquad\qquad \sigma = \frac{\lambda - \lambda_1}{\lambda + \lambda_1}$$

p: fig. 4.23

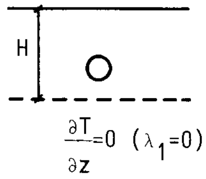
In particular $\lambda_1 = \infty$:

m: eq. 4.12.3



In particular $\lambda_1 = 0$:

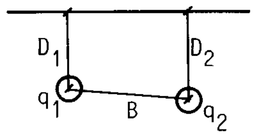
m: eq. 4.12.4



<u>Two pipes</u>

$$0 - T_{f1} = q_1 \cdot m_{t1} + q_2 \cdot m_{12}$$

$$0 - T_{f2} = q_2 \cdot m_{t2} + q_1 \cdot m_{12}$$



m_{t1}, m_{t2} m_{12}: eq. 4.5.8, 4.5.4

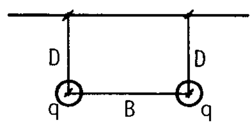
In particular $T_{f1} = T_{f2} = T_f$:

$$0 - T_f = (q_1 + q_2) \cdot m_{1+2}$$

m_{1+2}: eq. 4.5.12

Two pipes at the same depth:

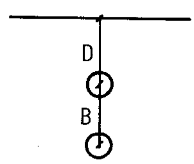
$$0 - T_f = q \cdot (m_p + m)$$



m: eqs. 4.5.15-16, table 4.4

Two pipes in the same ditch:

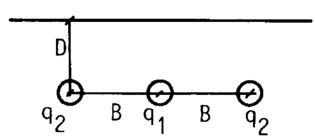
m: 4.5.21



<u>Three pipes</u>

$$0 - T_{f1} = q_1 \cdot (m_p + m) + q_2 \cdot (m_{12} + m_{12})$$

$$0 - T_{f2} = q_2 (m_p + m + m_{22}) + q_1 \cdot m_{12}$$

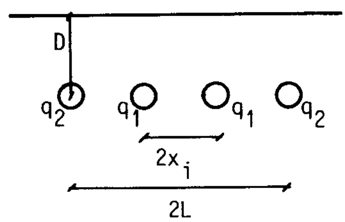


m_{12}, m_{22}: eq. 4.6.2

In particular $T_{f1} = T_{f2} = T_f$:

$$0 - T_f = (q_1+2q_2) \cdot m_{2+1+2} \qquad m_{2+1+2}\text{: eq. 4.6.4}$$

Four pipes



$$0 - T_{f1} = q_1 \cdot (m_p+m+m_{11}) + q_2 \cdot (m_{12}+m'_{12})$$

$$0 - T_{f2} = q_2 \cdot (m_p+m+m_{22}) + q_1 \cdot (m_{12}+m'_{12})$$

m_{11} etc: eq. 4.7.2

In particular $T_{f1} = T_{f2} = T_f$:

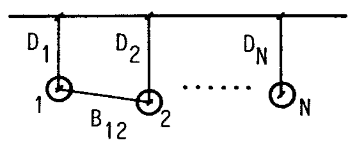
$$0 - T_f = (2q_1+2q_2) \cdot m_{2+1+1+2} \qquad m_{2+1+1+2}\text{: eq. 4.7.4}$$

N pipes

$$0 - T_{fi} = q_i \cdot (m_{pi}+m_i) + \sum_{\substack{j=1 \\ j \neq i}}^{N} q_j \cdot m_{ij} \qquad i = 1,2,..,N$$

m_i: eq. 4.8.2

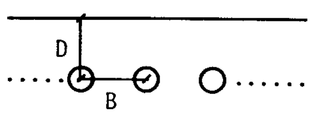
m_{ij}: eq. 4.8.5



Infinite array of pipes

$$m = \frac{1}{2\pi\lambda} \left\{ \ln\left(\frac{2D}{R}\right) + f\left(\frac{B}{D}\right) \right\}$$

f: eq. 4.9.4, table 4.10

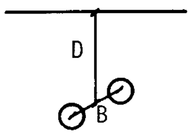


A few pipes at moderate distances

$$R \ll B \ll D$$

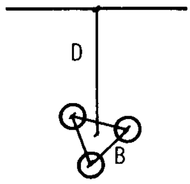
$$0 - T_f = q \cdot \left\{ m_p + \frac{1}{2\pi\lambda} \left(\ln\left(\frac{2D}{R}\right) + g' \right) \right\}$$

Two pipes:



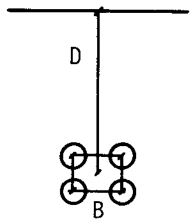
$$g' = \ln\left(\frac{2D}{B}\right)$$

Three pipes in an equilateral triangle:



$$g' = 2 \cdot \ln\left(\frac{2D}{B}\right)$$

Four pipes in a square:

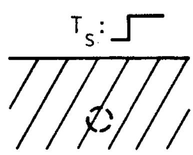


$$g' = 2 \cdot \ln\left(\frac{2D}{B}\right) + \ln\left(\frac{2D}{B\sqrt{2}}\right)$$

Effect of ground surface temperatures

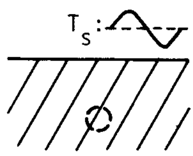
The undisturbed temperature field in the ground is determined by the ground surface temperatures. This temperature is to be added to the effect of the heat extraction pipes.

Step change at the ground surface:



Eq. 5.2.2

Periodic ground surface temperature:



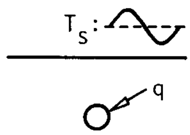
Eq. 5.2.6

Penetration depth:

$$d_o = \sqrt{\frac{at_o}{\pi}} \qquad (\text{amplitude} \sim e^{-z/d_o})$$

Steady-state heat extraction by one pipe + periodic ground surface temperature:

$T_f(t)$: eq. 5.4.1



Minimum extraction temperature: eq. 5.4.2 or eq. 5.4.5 and figure 5.3.

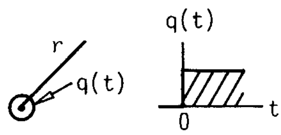
Heat extraction pulses

The temperature due to heat extraction pulses is by superposition to be added to the undisturbed ground temperature (and a steady-state contribution).

Basic step line sink

$$T(r,t) = -\frac{q}{4\pi\lambda}\, E_1\left(\frac{r^2}{4at}\right)$$

E_1: eq. 6.1.4



Radial profile function

$$T(r,t) = \frac{q}{\lambda} \cdot E_r\left(\frac{r}{\sqrt{at}}\right)$$

E_r: eq. 6.1.8, table 6.1, figure 6.4

Temperature development at a given radius:

$$T(r,t) = \frac{q}{\lambda}\, E_t\left(\frac{at}{r^2}\right)$$

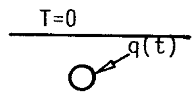
E_t: eq. 6.1.9, figure 6.5, table 6.3

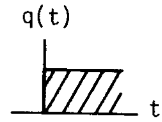
Pipe temperature approximation:

$$T_R(t) = T(R,t) = -\frac{q}{\lambda} \cdot \left(0.0644 + \frac{1}{4\pi} \cdot \ln\left(\frac{at}{R^2}\right)\right) \quad (t > 5R^2/a)$$

Step extraction pulse below the ground surface

$$T_R(t) = -\frac{q}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) + \frac{q}{\lambda} E_p\left(\frac{at}{D^2}\right)$$





$E_p(\tau)$: eq. 6.2.4, figure 6.7

Characteristic time-scale t_D:

$$t_D = \frac{2D^2}{a}$$

The process is essentially steady-state for $t > t_D$. The influence from the ground surface is small for $t < t_D$. We have the approximations

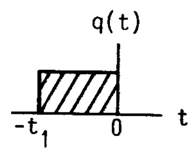
$$T_R(t) = \begin{cases} -\frac{q}{\lambda}\left(0.0644 + \frac{1}{4\pi} \ln\left(\frac{at}{R^2}\right)\right) & t < t_D \\ -\frac{q}{2\pi\lambda} \ln\left(\frac{2D}{R}\right) & t > t_D \end{cases}$$

The maximum relative error is 0.25/ln(2D/R)

Temperature recovery after a pulse

Without mirror pulse:

$$T_R(t) = -\frac{q}{4\pi\lambda} \cdot \ln\left(\frac{t+t_1}{t}\right) \qquad t > 0$$



With mirror pulse, i.e. the ground surface is accounted for:

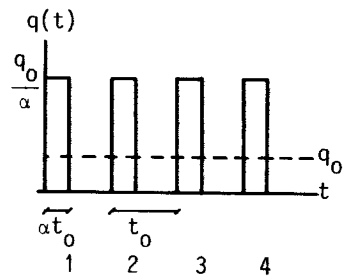
$T_R(t)$: eq. 6.3.3

Approximations:

$$T_R(t) \simeq - \frac{q}{8\pi\lambda} \cdot \frac{t_D t_1}{t(t+t_1)} \qquad t > 3t_D$$

$$T_R(t) \simeq - \frac{q}{2\pi\lambda} \cdot \frac{t_1}{2t+t_1} \left(1 - e^{-t_D/(2t+t_1)}\right) \qquad t > 2t_1$$

Sequence of pulses (pulse train)



q_o mean extraction rate

α pulse fraction

$$T = T^o + T^*$$

T^o is the contribution from $q(t) = q_o$. It is given by the basic step line sink below the ground surface.

T^* is the pulsating part. The largest values are obtained at the end of a pulse i and at the end of a period i.

$$T^*(R,t_o(i-1) + \alpha t_o) = - \frac{q}{\lambda} E_i' \left(\frac{R^2}{at_o}, \alpha\right)$$

$$T^*(R,t_o i) = - \frac{q}{\lambda} E_i'' \left(\frac{R^2}{at_o}, \alpha\right)$$

E_i', E_i'' : figures 6.17-20

Periodic heat extraction

$$q(t) = q_1 \cdot e^{2\pi i t/t_o}$$

Periodic sink in infinite surrounding

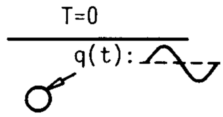
$$T(r,t) : \begin{cases} \text{eq. 7.6, } N_o \text{ and } \phi_o \text{ from figure 7.2} \\ \text{eq. 7.10, F and G from figure 7.2} \end{cases}$$

$$T_R(t) = -\frac{q_1}{2\pi\lambda} A(R') \; e^{i(2\pi t/t_o - B(R'))} \qquad R' = \frac{R\sqrt{2}}{d_o}$$

A, B: figure 7.2 and table 7.2

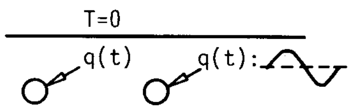
Correction for ground surface

$T_R(t)$: eq. 7.12



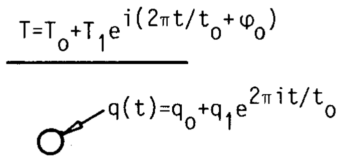
Two pipes

$T_R(t)$: eq. 7.15



Steady-state and periodic extraction, and periodic ground surface temperature

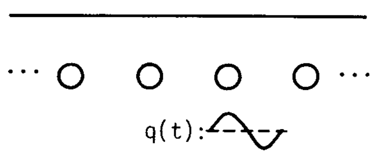
$T_R(t)$: eq. 7.17



Infinite array of pipes

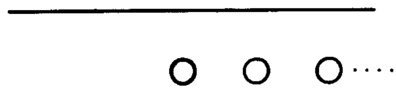
$T_R(t)$: eq. 7.25

M_o, Ψ_o: figures 7.12-13



End pipe (in semi-infinite array):

$T_R(t)$: eq. 7.37

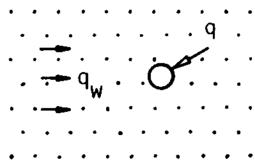


Effect of ground water flow and infiltration

We only consider steady-state cases with a constant volumetric ground water flow q_w

Steady-state line sink in moving ground water

Infinite surrounding:

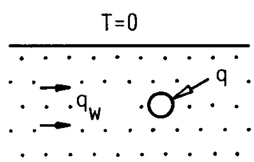


T: eq. 8.3

$$0 - T_R = q \cdot m$$

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2\ell}{R}\right) - \gamma\right\} \qquad \ell = \frac{2\lambda}{q_w C_w} \qquad \gamma = 0.5772$$

Ground surface at the water table:

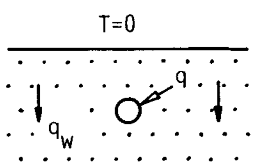


T: eq. 8.15

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - f\left(\frac{2D}{\ell}\right)\right\}$$

f: eq. 8.18, figure 8.2

Vertical infiltration



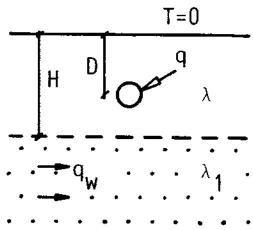
T: eq. 8.22

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - f\left(\frac{2D}{\ell}\right)\right\}$$

f: eq.8.18, figure 8.2

Ground water flow below a pipe

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - P_w\right\}$$



P_w: eq. A4.17, tables 8.31-34

In particular for a very strong ground water flow ($q_w = \infty$):

P_w: eq. 8.30

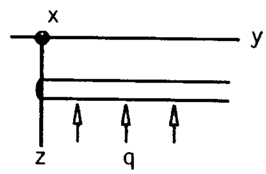
Temperature variation along the pipe

The formulas below concern only the steady-state case. The previous formulas are extended to the three-dimensional case with a changing fluid temperature along the pipe.

Temperature field around the end of a pipe

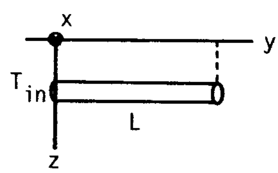
$T(x,y,z)$: eq. A5.2

$T_R(y)$: eq. A5.3



Temperature variation along a single pipe

$T_f(y)$: eq. 9.1.4-5



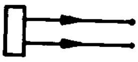
Mean thermal resistance:

$0 - T_{in} = \bar{m}_t \cdot \bar{q}$ $\qquad$ $\bar{m}_t$: eq. 9.1.13 or 9.1.18

Approximation: $\bar{m}_t = m + m_p + \mu/2$ $\qquad$ $\mu = \frac{L}{C_f Q_f}$

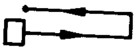
Two pipes

Parallel flow



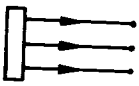
Eq. 9.2.6 $\bar{m}_{1+2}$: eq. 9.2.12-13

Series flow

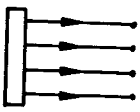


Eq. 9.2.11 $\bar{m}_{1+2}$: eq. 9.2.12,14

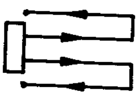
Three and four pipes



Eq. 9.3.1 $\bar{m}_{2+1+2}$: eq. 9.3.3



Eq. 9.3.4 $\bar{m}_{2+1+1+2}$: eq. 9.3.5



Eq. 9.3.6 $\bar{m}_{2+1+1+2}$: eq. 9.3.7

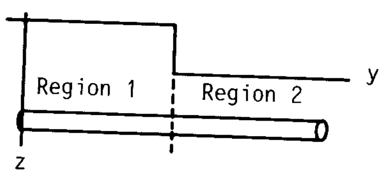
General formulas for N pipes

Eq. 9.4.4 $\bar{m}_{1+2+..+N}$: eq. 9.4.5

A pipe through two regions

Eqs. 9.6.2-3

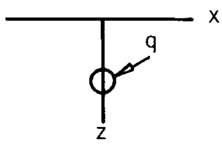
$\bar{m}_{1+2}$: eq. 9.6.4-5



Thermal impact on surrounding ground

Steady-state heat extraction to a single pipe

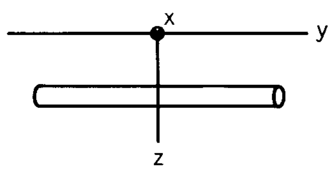
T(x,z): eq. 4.1.2



Dipole approximation: eq. 4.2.3

Heat flux at the ground surface: eq. 4.13.2

Single pipe of finite length



T(x,y,z): eq. 10.1.2

Rectangular heat extraction area

T(x,y,z): eqs. 10.2.3-4 See figure 10.1

12. CONCLUSIONS AND SUMMARY OF RESULTS

We will in this chapter make a summary of the results and present various conclusions that can be made from this mathematical analysis of the thermal processes.

The basic assumption throughout this study is that the superposition principle for the heat transfer process is valid. The complete thermal process or temperature field is obtained as a superposition of basic, simpler ones.

This requirement of superposition or linearity entails the assumptions:

1. No freezing in the ground.
2. Linear boundary condition at the ground surface.
3. No time-varying moisture migration or ground water flow.

The soil is in most cases assumed to be homogeneous.

The use of superposition makes it possible to isolate various processes, influences and effects from each other. It provides the tools for a better and deeper understanding of the heat extraction process.

The thermal process has one component due to natural conditions. It is governed by the temperature and other conditions at the ground surface. This is the process with zero heat extraction to the pipes. Superimposed on this there is the temperature field due to the heat extraction to the pipes. This part has zero ground surface temperature.

The basis of the analysis is given or prescribed heat extraction rates to the pipes. The ensuing temperature fields and in particular the required fluid temperature in the pipes are obtained. The analysis becomes much simpler, if one starts with given extraction rates and then calculates the fluid temperature.

The heat extraction from several pipes is obtained by superposition from the analysis of a single pipe. A basic problem is the heat extraction by a single pipe with zero temperature at the ground surface. The heat extraction rate to the pipe is any function of time. This extraction rate may be considered as a superposition of more fundamental ones. Firstly, there is an important steady-state component which accounts for the mean extraction rate. Secondly, there is a time-varying component, which may be regarded as a sum of step-extraction pulses. Another important component is a harmonically varying pulse. Any periodic extraction is obtained by a Fourier expansion. These basic thermal processes are analysed in this study.

There is a characteristic time-scale for the heat extraction with pipes at the depth D: $t_D = 2D^2/a$. The effect of the ground surface on the pipe temperature is virtually negligible during the first time $o < t < t_D$ of an extraction pulse that starts at $t = o$. The pipe temperature is more or less equal to the steady-state limit for $t > t_D$.

The characteristic time t_D is one month in reference case (3.2).

The thermal recovery after the termination of a heat extraction pulse is quite rapid. One third of the heat extraction temperature remains after the time $t_D/30$. Only 10% remains after the time $t_D/3$. See eq. 3.5.

The steady-state component of the heat extraction accounts for the base load for mean values over times that are longer than t_D. The steady-state heat extraction temperature at the pipe is proportional to the prescribed heat extraction rate. The proportionality constant is the thermal resistance (per unit pipe length).

The temperature gradients become very steep near the pipes. Characteristically around 50% of the extraction temperature lies in a circle with a radius D/5 around a pipe. The value of the thermal conductivity near the pipe is therefore much more important than the values further away.

The thermally influenced region around a heat extraction pipe is rather limited. A simple dipole approximation is given by eq. 4.2.3. The temperature at the pipe depth D decreases rapidly outwards. Less than 10% of the pipe temperature remains at the distance 2D. At the distance 5D around 1-2% remains.

The steady-state component gives the maximum thermal impact on the surrounding ground except for the vicinity of the pipe where dynamical effects also are of importance.

The thermal resistance of the soil for a single pipe is given by the simple expression (4.1.9). The resistance depends logarithmically on pipe depth over pipe radius (D/R). So the steady-state heat extraction is not too sensitive to moderate changes of the depth or pipe radius.

The thermal resistance between the fluid in the pipes and the soil at the outer pipe radius may be quite important. Materials with low thermal conductivity should be avoided for the pipe wall. The fluid to pipe thermal resistance is negligible for turbulent flow. An air gap between the pipe and the adjacent soil can increase the thermal resistance considerably.

The thermal resistance at the ground surface contributes characteristically with about 4% to the total thermal resistance. So it is essentially negligible, and it is henceforth neglected in this study.

The steady-state heat extraction for two parallel pipes is readily obtained by superposition. There are two linear equations that relate the extraction temperatures to the heat extraction rates. The single-pipe thermal resistance and a coupling resistance occur as coefficients. The special case with a single fluid temperature gives a relation between this temperature and the total heat extraction from the two pipes. This may be expressed as a simple formula for the thermal resistance of the two pipes.

A few cases with three and four pipes with two independent temperatures or heat fluxes are dealt with in the same way. A simple system of two equations is obtained, and formulas for the total thermal resistance of the pipes with a common fluid temperature are given.

A study of four pipes at the same depth shows that the heat extraction potential is rather insensitive to the exact position of the pipes. The maximum extraction is virtually obtained for equal spacing between the pipes. These maxima are quite flat as a function of the lateral position.

The general formulas for N pipes are readily obtained by superposition. The relation between the fluid temperatures and the heat extraction rates is given by N linear equations. The coefficients are the single-pipe thermal resistances and coupling resistances. The solution of the equation system on a computer is immediate.

The heat extraction potential for a pipe in an infinite array of pipes is also given. The formula for the thermal resistance is quite simple.

The influence between N pipes at the same depth with equal spacing is illustrated in figure 4.14 for a particular case. The relative heat extraction compared to N independent pipes, η, is given as a function of relative spacing B/D. We have for example for $N = 2,3,4,10$ and ∞ and $B = D$ the values $\eta = 0.85$, 0.78, 0.74, 0.64 and 0.55. We see that the influence between pipes and the decrease of heat extractions increase rather strongly with the number of pipes. The value of η lies between 0.93 ($N = 2$) and 0.80 ($N = \infty$) for $B = 2D$. For $B = 0.5D$ η lies between 0.77 ($N = 2$) and 0.33 ($N = \infty$). The influence of surrounding pipes is quite small for $B > 2D$, and it is considerable for $B < 0.5D$.

A few pipes may be put together in a bundle. The heat extraction potential is however increased when the pipes are separated from each other as much as possible. Formulas for the improvement, when the pipes are separated a moderate distance, are derived. From table 4.11 we see that there is a gain around 10-20% when two, three or four pipes are separated a distance $B = 0.1D$. The gain is around 20-40% for $B = 0.2D$. Pipes that are put in the same ditch should from this point of view be separated as much as possible from each other.

The thermal resistance for a pipe in the case of a soil with two layers is given. It may be advantageous to put the pipe deeper, if the lower region has a higher thermal conductivity. See figure 4.24 A.

The extracted heat is obtained through the ground surface. Half the heat to a single pipe is provided through the strip from $-D$ to $+D$, while 87% is provided from $-5D$ to $+5D$.

The natural undisturbed temperatures at the pipes are needed in the superposition in order to get the total temperature at the pipes. They are determined by the ground surface temperatures. The time-scale for a step-change at the ground surface to be felt to 50% at the pipe depth D is $t = D^2/a$. This time is in a reference case two weeks. Daily fluctuations at the ground surface are certainly not felt at the pipe depth. A periodic temperature variation at the ground surface is associated with a penetration depth d_0, which depends on the length of the period, t_0. In a reference example we had: $d_0 = 0.14$ m for $t_0 = 1$ day and $d_0 = 0.8$ m for $t_0 = 1$ month.

The optimal depth for steady-state extraction to a single pipe with an annual periodical ground surface temperature is discussed in section 5.4. There may be an optimal depth that lies quite deep. In this optimization only purely thermal aspects are considered.

The time-dependent part of the heat extraction process may be analyzed by superposition of step extraction pulses, which have a constant extraction rate from the starting-time. The temperature process around a pipe in an infinite surrounding for a step extraction pulse is discussed in detail. The radial temperature profile and the temperature response at a certain distance from the pipe are given by simple functions and diagrams. The temperature at the pipe radius is given by the simple expression 6.1.21.

The pipe temperature for a step extraction pulse for a pipe at a depth D is given by simple formulas and diagrams. An approximation that uses the steady-state temperature for $t > t_D$ and the single-pipe analysis for $t < t_D$ turns out to give very precise results. The pipe temperature for the step pulse is given by the simple approximations 6.2.8.

The thermal recovery after a pulse is analysed. Simple formulas are provided. The results may be used to analyse the effect of a thermal recharge of the ground during the summer. Let us quote example 6.3.9-10. There is a strong recharge during three months. The temperature at the pipe after another three months is only increased 0.1 ^{o}C. The conclusion is that it is futile to recharge during the summer in order to improve the heat extraction during the winter. The recharge for shorter periods is also of small or no value due to the rapid recovery after an injection or extraction pulse.

The superposition of step pulses in order to analyse various time-dependent heat extraction cases are illustrated by several examples in section 6.4. The technique is quite simple.

A pulsating extraction is of particular interest. The mean extraction provides a base load. It is analysed with the simple step pulse. The superimposed oscillating part is added. It is often quite important. The particular case with a sequence of equal pulses is dealt with in detail. Extensive diagrams to assess the extreme values of the temperature are provided. The oscillating part of the pipe temperature is essentially periodic after five pulses. The thermal recovery after a sequence of pulses is not very different from the corresponding simplified case with a constant mean extraction rate.

A periodic heat extraction is an important basic case. The period t_0 may be the whole year or a shorter one. The temperature solutions are given in a complex-valued form. The real temperatures are obtained from the real and the imaginary parts.

The periodic process is associated with a penetration depth d_0. The temperature amplitude is dampened outwards with the length-scale d_0. It is negligible at the distance $3d_0$.

The basic solution is the periodic sink in an infinite surrounding. The pipe temperature is given by diagrams and simple asymptotic expressions. The dampening of the temperature amplitude may be illustrated by the example of table 7.1. Only 10% of the amplitude at the pipe remains at the distance 0.2 m for t_0 = 1 day, 0.4 m for t_0 = 1 week, 0.8 m for t_0 = 1 month, and 2 m for t_0 = 1 year.

The periodic sink below the ground surface is as usual obtained by the introduction of a mirror sink above the ground surface. The contribution from the mirror to the pipe temperature is however negligible for periods below one week. It is normally also insignificant even for a monthly period. The annual periodic case is on the other hand closely approximated by a steady-state analysis in which the actual heat extraction rate is used.

The periodic case for several pipes is readily obtained by superposition. The formulas for two pipes are given.

The complete process for a pipe with steady-state and periodic heat extraction, and with a periodic ground surface temperature is dealt with in section 7.4. A critical point is the lowest extraction temperature during the year, when freezing is to be avoided. The considered examples again show that a steady-state analysis for the slowly varying extraction rate gives quite accurate results concerning the minimum extraction temperature.

The periodic case for an infinite array of pipes is dealt with. The contribution from surrounding pipes is given by extensive diagrams. The distance between the pipes is of course quite important.

Effects of ground water flow and infiltration are analysed in chapter 8. A few solutions for steady-state water flow and steady-state heat extraction are given. The basis is a steady-state sink in moving ground water.

There is a characteristic length scale ℓ (eq. 8.4) for the convective-diffusive thermal process. The length ℓ is inversely proportional to the volumetric ground water flow q_w.

The thermal resistance for an extraction pipe in an infinite surrounding is given by a simple formula. The case with horizontal ground water flow and a ground water level close to the ground surface is also solved. The effect of the ground water movement is negligible for slow water movement (q_w = 1 m/year). It is modest (5% change) for q_w = 10 m/year. The thermal resistance is halved for q_w = 100 m/year.

The thermal resistance in the case of steady-state vertical infiltration is given by a simple formula. The effect of the infiltration is surprisingly small for normal infiltration rates.

It should here be remembered that we are only considering the convective effect. The change of thermal conductivity with the water content is not dealt with here. Let us quote the results of example 8.9. The change of thermal resistance for the infiltration rate $q_w = 0.5$ m/year is only 0.04%. The very strong infiltration $q_w = 5$ m/year gives a change of only 2%. The conclusion is that normal vertical infiltration is not important for the heat extraction (except for the influence of water on the thermal conductivity).

The case with horizontal ground water flow in a region $z > H$ below the pipe at $z = D$ is also solved. The numerical examples show that the effect of the ground water is quite small except for the case of strong ground water movement with the ground water table close to the pipe.

The limit with a very strong ground water flow gives an upper limit on the influence of the ground water. The formula for the thermal resistance is simple. Let us quote the results of (8.31). The maximal increase of heat extraction due to ground water flow is less than 10% for $H \geqq 2D$. It is less than 1% for $H > 5D$.

The effects of the temperature variation along the pipes are discussed in chapter 9. Only steady-state cases are considered. The three-dimensional effects around the end of a pipe affect only one to two meters so they are essentially negligible.

The average thermal resistance along a single pipe is given by a simple formula. The quantity $\mu/2$ is to be added to the previous thermal resistance. The resistance μ, eq. 9.1.16, is inversely proportional to the fluid flow rate in the pipe.

The average thermal resistance for two parallel pipes are given. One must distinguish between the case of parallel fluid flow in the pipes and the case of series flow.

The previous thermal resistance formulas for three and four pipes are extended to account for the variable temperature. A few flow arrangements through the pipes are considered.

The general relations for N pipes are given. The only change is the addition of $\mu/2$ and μ in the thermal resistance factors of the equation system. Any flow arrangement is dealt with in a rather simple way.

The relative heat extraction of two and four pipes as a function of the spacing for different flow arrangements is discussed. Let us quote the results of figure 9.8. The parallel and series cases for two pipes are studied. The parallel case is about 4% better in this particular example with a pipe length of 100 m. Parallel, bifilar and series flow are compared for four pipes. The parallel case is about 8% better than the series case. The bifilar case lies in between.

Formulas for a pipe that passes two different regions are given.

The thermal impact of the pipes on the surrounding ground is readily obtained from dipole approximations of the temperature field. These are valid for distances larger than 3D from the pipes. Here only steady-state need to be considered.

The temperature field for a single pipe is given. Let us quote example 10.1.3-4. The temperatures three meters from the pipe lie in the region 0.2 - 0.05^{o}C. Ten meters away they are around 0.01^{o}C. We can conclude that the thermal impact a few meters or more from the pipes is quite small.

The case with a rectangular area with heat extraction pipes are also dealt with. The dipole approximation corresponds to the case when there is a certain temperature T_a, eq. 10.2.5, on the rectangular surface. See figure 10.1. Let us quote example 10.2.6-7. The equivalent temperature is -7^{o}C. The temperature change due to the pipes is around 0.4^{o}C at three meters distance from the pipes on the pipe level z = 1. Fifteen meters away the temperature disturbance is 0.03^{o}C. We can in general conclude that the disturbance at the pipe level and upwards is negligible at a distance of 5D from the pipes.

The multifarious formulas are summarized in chapter 11.

NOTATIONS

Symbol	Defining equation	Definition, (dimension)
a	λ/C	thermal diffusivity of the ground (m^2/s)
$A(R')$	7.11, App.3	temperature amplitude at the pipe for a periodic heat sink (-)
B	Fig. 4.6	distance between two adjacent pipes (m)
B_-	4.5.3	distance between one pipe and the mirror of the other (m)
B_{ij}	4.8.3	distance between pipe i and pipe j (m)
$B(R')$	7.11, App.3	phase lag of the temperature at the pipe for a periodic heat sink (-)
C		volumetric heat capacity of the ground (J/m^3K)
C_w		volumetric heat capacity of water ($\simeq 4.18\ MJ/m^3K$)
C_f		volumetric heat capacity of the fluid in the pipes (J/m^3K)
d_o	5.2.7	penetration depth of a periodic surface temperature (m)
D		depth to the heat extraction pipe (m)
D_i		depth to pipe i (m)
D'	7.13	dimensionless depth to the pipe (-)

Symbol	Defining equation	Definition, (dimension)	
erfc	ref. 2B	complementary error function	(-)
E_1	6.1.4	exponential integral	(-)
E_r	6.1.8	function that gives the radial profile for a step pulse	(-)
$E_p(\tau)$	6.2.5	transient part of pipe temperature for a step pulse below the ground surface	(-)
$E_t(\tau)$	6.1.9	function that gives temperature response at a fixed distance for a step pulse	(-)
E'_i	6.5.4	temperature amplitude factor at the end of pulse i in a pulse train, figure 6.16	(-)
E''_i	6.5.4	temperature amplitude factor at the end of period i in a pulse train, figure 6.16	(-)
f(s,p)	5.4.4, fig. 5.3		(-)
f(B/D)	4.9.3-4	contribution from surrounding pipes in an infinite array to the thermal resistance factor	(-)
$f(2D/\ell)$	8.18-20	contribution from moving ground water in the cases of figures 8.3 and 8.4 to the thermal resistance factor	(-)
F(R')	7.10, App.3	amplitude function for a periodic heat sink, figure 7.2.	(-)

Symbol	Defining equation	Definition, (dimension)
$F_z(x)$	4.13.1	heat flux downwards at the ground surface due to steady-state heat extraction (W/m^2)
g_s	4.4.4	contribution from the ground surface resistance to the thermal resistance factor (-)
g'	4.5.16	contribution from a second pipe at the same depth to the thermal resistance factor (-)
$G(R')$	7.10, App.3	phase function for the periodic heat sink, figure 7.2 (-)
$h_N(B/D)$	4.10.3	contribution to thermal resistance factor from N adjacent pipes (-)
H	fig. 4.22, 8.5	thickness of top soil layer or depth to the ground water level (m)
i		pipe number or pulse number (-)
i	$\sqrt{-1}$	imaginary unit (-)
j		pipe number (-)
k		summation index
k_2, k_3, k_4	4.11.6,10,18	relative heat extraction rate of 2, 3 or 4 pipes at a moderate distance B when compared to the corresponding bundle of pipes (-)

Symbol	Defining equation	Definition, (dimension)
$K_o(s)$	fig. 8.2	modified Bessel function of zeroth order (-)
ℓ	8.4	length associated with the convective-diffusive process in ground water (m)
L		length or half the length of the pipe (m)
L	fig. 4.11	half the distance between the outer pipes for four pipes at the depth D (m)
L_1, L_2	fig. 9.9	length of pipe in region 1 and region 2, respectively (m)
m	4.1.8	thermal resistance per unit length of the pipe between the pipe periphery and the ground surface for a single pipe (Km/W) (Km/W)
m_i	4.8.2	thermal resistance for pipe i (Km/W)
m_p	4.3.1	thermal resistance per unit length of the pipe between the fluid in the pipe and the outer pipe periphery in the ground (Km/W)
m_{pf}		thermal resistance between fluid and inner pipe wall (Km/W)
m_{pw}	4.3.6	thermal resistance of pipe wall (Km/W)

Symbol	Defining equation	Definition, (dimension)
m_{ps}		thermal resistance between outer pipe wall and the adjacent ground (Km/W)
m_{12}	4.3.4	thermal resistance over an annulus (Km/W)
m_{12}	4.5.4	coupling thermal resistance between pipe 1 and pipe 2 (Km/W)
m_{ij}	4.8.5	coupling thermal resistance between pipes i and j (Km/W)
m_N	4.11.21	thermal resistance of N pipes near each other (Km/W)
m_{pi}		fluid-ground thermal resistance m_p for pipe i (Km/W)
m_{ti}	4.5.8	total thermal resistance between fluid and ground surface for pipei (Km/W)
m_{22}	fig.4.10, 4.6.2	coupling thermal resistance for the case of figure 4.10 (Km/W)
m_{11}, m_{22} m_{12}, m'_{12}	fig. 4.11, 4.7.2	coupling thermal resistance for the case of figure 4.11 (Km/W)
m_{1+2}	4.5.11	thermal resistance for two pipes (Km/W)
m_{2+1+2}	4.6.4	thermal resistance for the three pipes of figure 4.10 (Km/W)
$m_{2+1+1+2}$	4.7.4	thermal resistance for the four pipes of figure 4.11 (Km/W)

Symbol	Defining equation	Definition, (dimension)
$\bar{m}_t$	9.1.13,20	mean total thermal resistance along a pipe (Km/W)
$\bar{m}_{1+2}$	9.2.12-14	mean thermal resistance for two pipes (Km/W)
$\bar{m}_{1+2}$	9.6.4,5	mean thermal resistance for a pipe that passes through two regions (Km/W)
$\bar{m}_{2+1+2}$	9.3.3	mean thermal resistance for the three pipes of figures 4.10 and 9.5 (Km/W)
$\bar{m}_{2+1+1+2}$	9.3.7	mean thermal resistance for the four pipes of figures 4.11 and 9.6 (Km/W)
M	fig. 10.1	half the width of rectangular heat extraction area (m)
$M_o(D/d_o,B/D)$	7.26	amplitude of the contribution from an infinite array of pipes (-)
N		number of pipes (-)
$N_o(r')$	7.6	amplitude of the zeroth Kelvin function, figure 7.2 (-)
p	5.4.7	(-)
$p(H/D,\sigma)$	4.12.1, fig.4.23	contribution to the thermal resistance factor when the soil consists of two layers (-)
P_w	8.28-29, app.4	contribution to the thermal resistance factor from moving ground water in a layer below the pipe, figure 8.5 (-)

Symbol	Defining equation	Definition, (dimension)
q		constant heat extraction rate to a pipe (W/m)
$q(t)$		time-dependent heat extraction rate to a pipe (W/m)
q_a	10.2.1	heat extraction rate per unit area (W/m^2)
q_b	E.g. 4.11.1	heat extraction rate of a bundle of pipes (W/m)
q_i		constant heat extraction rate to pipe i (W/m)
$q_i(t)$		time-dependent heat extraction rate to pipe i (W/m)
q_w	fig. 8.1,3,4,5	volumetric ground water flow (m_w^3/m^2s or m/s)
q_o	fig. 6.14	mean extraction rate of a pulse train (W/m)
q_o, q_1	7.16	mean value and amplitude of periodic heat extraction (W/m)
$\bar{q}$	9.1.10	mean extraction rate along the pipe (W/m)
$\bar{q}$	9.6.4	mean extraction rate for a pipe through two regions (W/m)
$\bar{q}_i$		mean extraction rate along pipe i (W/m)

Symbol	Defining equation	Definition, (dimension)
Q_f		volumetric fluid flow in the pipe (m^3/s)
r		radial distance to the pipe (m)
r'	7.4	dimensionless distance to the pipe in the periodic case (-)
R		outer radius of the pipe (m)
R_1, R_2	fig. 4.5	inner and outer radius of an annulus around the pipe (m)
R_-		innner pipe radius (m)
R_{eq}		equivalent radius of a bundle of pipes (m)
R'	7.4	dimensionless pipe radius (-)
S_i	section 9.4	set of indices of pipes through which the water flows before it reaches pipe i (-)
t		time (s)
t_D	3.1	characteristic time scale of the extraction process of figure 3.1 (s)
t_1		duration of heat extraction pulse (s)
t_o		period time (s)
T		temperature (^oC)

Symbol	Defining equation	Definition, (dimension)
$T(x,z)$ $T(x,z,t)$		temperature in a vertical cross-section of the ground (^oC)
T_a	10.2.5	equivalent surface temperature of a heat extraction area, figure 10.1 (^oC)
T_f		temperature of the fluid in the pipe (^oC)
$T_f(y)$		fluid temperature along the pipe (^oC)
T_{fi}		fluid temperature for pipe i (^oC)
T_{fmin}		minimum fluid temperature during a cycle (^oC)
T_{in}		inlet fluid temperature (^oC)
T_{out}		outlet fluid temperature (^oC)
T_R, $T_R(t)$		heat extraction temperature at the pipe radius R in the ground (^oC)
T_{Ri}		heat extraction temperature of pipe i (^oC)
$T_s(t)$		temperature at the ground surface (^oC)
T_o		mean annual temperature at the ground surface (^oC)
T_1	5.2.5	amplitude of periodic ground surface temperature (^oC)

Symbol	Defining equation	Definition, (dimension)
T_2-T_1	fig. 4.5	temperature difference over an annulus around the pipe (^oC)
T_2	7.18	room temperature (^oC)
$\bar{T}_{fi}$		mean fluid temperature along pipe i (^oC)
T^o	6.5.3	contribution to the temperature from the mean extraction rate of a pulse train (^oC)
T^*	6.5.3	contribution to the temperature from the pulsating part of a pulse train (^oC)
v_f	9.1.7	mean fluid velocity in the pipe (m/s)
v_T	8.2	thermal displacement velocity in moving ground water (m/s)
x	fig. 4.1, 9.1	horizontal coordinate in the plane perpendicular to the pipe (m)
x_i	fig. 4.6	x-coordinate of the center of pipe i (m)
x_i	fig. 4.11	half the distance between the inner pipes of figure 4.11 (m)
y	fig. 9.1	horizontal coordinate along the pipe (m)
y_f	9.1.4	characteristic thermal length along the pipe (m)
z		vertical downward coordinate (m)

Symbol	Defining equation	Definition, (dimension)
α	fig. 6.14	relative pulse length in a pulse train (-)
α	7.18	heat extraction proportionality factor (W/mK)
α_s	5.1.1	thermal contact resistance at the ground surface (W/m^2K)
γ	$\gamma \simeq 0.5772$	Euler's constant (-)
η	4.5.18, 4.7.5 4.6.6	relative heat extraction rate for N pipes compared to N independent pipes (-)
λ		thermal conductivity of the ground (W/mK)
λ_p		thermal conductivity of pipe wall (W/mK)
λ_1, $\lambda_1(r)$	section 4.3	thermal conductivity of an annulus $R \leq r \leq R_1$ around the pipe (W/mK)
λ_1	fig. 4.22	thermal conductivity of the second soil layer (W/mK)
λ_1, λ_2	fig. 9.9	thermal conductivities of two ground regions along the pipe (W/mK)
μ	9.1.16	thermal resistance associated with the temperature change along the pipe. (Km/W)
μ_i	9.6.1	value of μ in region i (Km/W)

Symbol	Defining equation	Definition, (dimension)	
σ	4.12.2		(-)
τ	5.2.3, 6.1.12 6.2.7	dimensionless time	(-)
$\phi_0(r')$	7.6	phase of the zeroth Kelvins function, figure 7.2	(-)
φ_0	5.2.5, fig. 7.6	phase of ground surface temperature	(-)
Ψ_0	7.26	phase of the contribution from an infinite array of pipes	(-)

REFERENCES

1 Rohsenow, Hartnett. Handbook of Heat Transfer. (McGraw-Hill, 1973) Section 7.B.2.

2 Abramowitz, Stegun. Handbook of Mathematical Functions (AMS 55, National Bureau of Standards, 1970), 2A: Chapter 5, 2B: Chapter 7, 2C: Chapter 9.

3 Carslaw, Jaeger. Conduction of Heat in Solids. (Oxford, 1959). 3A: Page 60, 3B: Page 67, 3C: Page 261, 3D: Page 263, 3E: Page 267.

4 Gradshteyn, Ryzhik. Table of integrals, series and products (Academic Press, 1965). Page 493.

APPENDIX 1. EFFECT OF GROUND SURFACE RESISTANCE.

We have the steady-state, two-dimensional heat extraction problem for a single pipe as described in section 4.1 and Figure 4.1 except for the boundary condition at z=0. The temperature T(x,z) shall at the ground surface satisfy 4.4.1:

$$T - \frac{\lambda}{\alpha_s}\frac{\partial T}{\partial z} = 0 \qquad z=0 \tag{A1.1}$$

Let us define u(x,z) by

$$T(x,z) = \frac{q}{2\pi\lambda}\cdot\left(\ln\left(\frac{\sqrt{x^2+(z-D)^2}}{\sqrt{x^2+(z+D)^2}}\right) + u(x,z)\right) \tag{A1.2}$$

The first part represents the line sink solution for $\alpha_s = +\infty$ according to section 4.1. It gives the prescribed heat sink at (0,D). This means that u(x,z) does not have any heat sink.

The function u(x,z) shall satisfy the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad z > 0,\ -\infty < x < \infty \tag{A1.3}$$

When A1.2 is inserted in A1.1 we get the following condition for u:

$$u - \frac{\lambda}{\alpha_s}\frac{\partial u}{\partial z} = -\frac{2\lambda}{\alpha_s}\cdot\frac{D}{x^2+D^2} \qquad z=0 \qquad -\infty < x < \infty \tag{A1.4}$$

A relatively general solution of A1.3 is

$$u(x,z) = \int_0^\infty f(s)\, e^{-sz}\cdot\cos(sx)ds \tag{A1.5}$$

Insertion of A1.5 in A1.4 gives:

$$\int_0^\infty f(s)\left[1 + \frac{\lambda s}{\alpha_s}\right]\cos(sx)ds = -\frac{2\lambda}{\alpha_s}\cdot\frac{D}{x^2+D^2} \tag{A1.6}$$

This is a Fourier integral. We have:

$$\int_0^\infty e^{-Ds}\cos(sx)ds = \frac{D}{x^2+D^2} \tag{A1.7}$$

A comparison of A1.6 with A1.7 gives:

$$f(s) = -\frac{2\lambda}{\alpha_s}\frac{1}{1+\frac{\lambda s}{\alpha_s}}e^{-Ds} \tag{A1.8}$$

Formulas A1.5 and A1.8 give the solution u(x,z).

We need the complex notation:

$$\cos(sx) = \mathrm{Re}(e^{-isx}) \tag{A1.9}$$

Here Re signifies 'the real part of'.

From A1.5, 8 and 9 we have:

$$u(x,z) = -\mathrm{Re}\left\{\frac{2\lambda}{\alpha_s}\int_0^\infty \frac{1}{1+\frac{\lambda s}{\alpha_s}} e^{-Ds}\cdot e^{-sz}\cdot e^{-ixs}\, ds\right\} \tag{A1.10}$$

This may with elementary substitutions be rewritten in the following way:

$$u(x,z) = -2\cdot\mathrm{Re}\left\{e^{v}\int_v^\infty \frac{1}{s}e^{-s}\, ds\right\} \tag{A1.11}$$

$$v = \frac{\alpha_s(D+z+ix)}{\lambda}$$

The integral in A1.11 is the exponential integral $E_1(v)$. This function is given in [2A] in the form $ze^zE_1(z)$ for complex arguments.

The solution T(x,z) is then:

$$T(x,z) = \frac{q}{2\tau\lambda}\left\{\ln\left(\frac{\sqrt{x^2+(z-D)^2}}{\sqrt{x^2+(z+D)^2}}\right) - 2\mathrm{Re}(e^{v}\,E_1(v))\right\}$$

$$E_1(v) = \int_v^\infty \frac{1}{s}e^{-s}ds \tag{A1.12}$$

$$v = \frac{\alpha_s(D+z+ix)}{\lambda}$$

(Re = real part of)

The temperature at the pipe radius becomes:

$$T_R = \frac{q}{2\pi\lambda}\cdot[\ln(\frac{R}{2D}) - 2\, e^{\frac{2D\alpha_s}{\lambda}}\, E_1(\frac{2D\alpha_s}{\lambda})] \quad (A1.13)$$

APPENDIX 2. STEADY-STATE HEAT EXTRACTION IN A TWO-LAYERED SOIL

The steady-state heat extraction by a single pipe in a two-layered soil is discussed in section 4.12. The solutions will be derived in this appendix.

The soil consists of two horizontal layers. The top layer has the thickness H. The pipe lies at the depth z=D. Figure A2.1 shows the two possible cases, when D is greater or less than H. The thermal conductivity in the soil around the pipe is λ. It is λ_1 in the other layer.

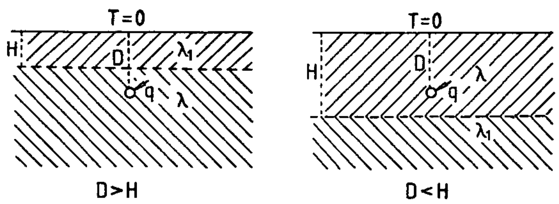


Figure A2.1. Steady-state heat extraction by a single pipe in a two-layered soil.

We consider first the case D > H. The temperature in the ground may be written:

$$T(x,z) = \frac{q}{2\pi\lambda}\cdot\left[\ln\left(\frac{\sqrt{x^2+(D-z)^2}}{\sqrt{x^2+(D+z)^2}}\right) + u(x,z)\right] \qquad \text{(A2.1)}$$

The coordinates of Figure 4.1 is used. The logarithmic term takes care of the heat extraction requirement q (W/m). The temperature correction u(x,z) shall be regular at the pipe (0,D).

The function u(x,z) shall satisfy the Laplace equation $\Delta u=0$ in the two layers $0 < z < H$ and $z > H$. We start with the following expressions:

$$u(x,z) = \begin{cases} \int_0^\infty f(s)\,\frac{\sinh(zs)}{\sinh(Hs)}\cos(xs)ds & 0 < z < H \\ \int_0^\infty f(s)\,e^{-(z-H)s}\cos(xs)\,ds & z > H \end{cases} \qquad \text{(A2.2)}$$

The function f(s) is to be determined.

According to A2.2 u(x,z) satisfies:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad 0 < z < H,\ z > H \qquad \text{(A2.3)}$$

$$u = 0 \qquad z = 0 \qquad \text{(A2.4)}$$

$$u = \int_0^\infty f(s)\cos(xs) \qquad z = H \qquad \text{(A2.5)}$$

The boundary condition T(x,0) = 0 is satisfied according to A2.1 and A2.4.

There are two internal boundary conditions at z=H. The temperature and the vertical heat flux are to be continuous. The temperature is continuous according to A2.5. The heat flux condition is:

$$\lambda_1 \left.\frac{\partial T}{\partial z}\right|_{z=H-0} = \lambda \left.\frac{\partial T}{\partial z}\right|_{z=H+0} \qquad \text{(A2.6)}$$

In the second case, when D is less than H, the thermal conductivities are to change places.

Inserting A2.2 and A2.1 in A2.6 gives the following equation for f(s):

$$\int_0^\infty f(s)s\cdot\{\lambda+\lambda_1 \coth(Hs)\}\cos(xs)ds = $$
$$= (\lambda_1-\lambda)\left[\frac{H+D}{x^2+(H+D)^2} + \frac{D-H}{x^2+(D-H)^2}\right] \qquad \text{(A2.7)}$$

We need the integral:

$$\int_0^\infty e^{-ps}\cos(xs)ds = \frac{p}{p^2+x^2} \qquad p > 0 \qquad \text{(A2.8)}$$

We introduce the conductivity parameter σ:

$$\sigma = \frac{\lambda-\lambda_1}{\lambda+\lambda_1} \qquad \text{(A2.9)}$$

By identification we have from A2.7-8:

$$f(s) = -\frac{1}{s}\cdot\frac{\sigma}{1-\sigma e^{-2Hs}}\cdot\left(e^{-(D-H)s} - e^{-(D+3H)s}\right) \quad D > H \quad (A2.10)$$

The second factor of A2.10 may be rewritten:

$$\frac{\sigma}{1-\sigma e^{-2Hs}} = \sum_{m=0}^{\infty} \sigma^{m+1}\cdot e^{-2mHs} \quad (A2.11)$$

The temperature u(x,z) is obtained from A2.2 and A2.10. The last factor of A2.10 contains the factor sinh(Hs) of A2.2, which thus cancels from the upper integral of A2.2. We then get, when A2.10 and 11 are inserted in A2.2, integrals of the following type:

$$\int_0^{\infty} \frac{e^{-ps}-e^{-qs}}{s}\cdot\cos(xs)ds = \ln\left(\frac{\sqrt{x^2+q^2}}{\sqrt{x^2+p^2}}\right) \quad p > 0,\ q > 0 \quad (A2.12)$$

The validity of A2.12 are easily checked by derivation with respect to p. It then reduces to A2.8.

From A2.1, 2, 10, 11 and 12 we get after some manipulations the following expressions in the case D > H.

0 < z < H:

$$T(x,z) = \frac{q}{2\pi\lambda}(1+\sigma)\sum_{m=0}^{\infty}\sigma^m \ln\left(\frac{\sqrt{x^2+(D-z+2mH)^2}}{\sqrt{x^2+(D+z+2mH)^2}}\right) \quad (D>H) \quad (A2.13)$$

z > H:

$$T(x,z) = \frac{q}{2\pi\lambda}\left[\ln\left(\frac{\sqrt{x^2+(D-z)^2}}{\sqrt{x^2+(D+z)^2}}\right) - \sum_{m=0}^{\infty}\sigma^{m+1}\ln\left(\frac{\sqrt{x^2+(D+z+2H+2mH)^2}}{\sqrt{x^2+(D+z-2H+2mH)^2}}\right)\right] \quad (D>H) \quad (A2.14)$$

We are in particular interested in the pipe temperature T_R at $x^2+(D-z)^2 = R^2$. We have from A2.14 with x=0, z=D in the infinite sum:

$$T_R = \frac{q}{2\pi\lambda}\left[\ln\left(\frac{R}{2D}\right) - \sum_{m=0}^{\infty}\sigma^{m+1}\cdot\ln\left(\frac{D+H+mH}{D-H+mH}\right)\right] \quad (D>H) \quad (A2.15)$$

The thermal resistance between the pipe and the ground surface is with definition 4.1.8:

$$m = \frac{1}{2\pi\lambda}\{\ln(\frac{2D}{R}) + p(\frac{H}{D}, \sigma)\}$$

$$p(\frac{H}{D}, \sigma) = \sum_{m=0}^{\infty} \sigma^{m+1}\cdot\ln\left(\frac{1+\frac{H}{D}(m+1)}{1+\frac{H}{D}(m-1)}\right) \qquad (D>H) \qquad (A2.16)$$

The function $p(h,\sigma)$ is here defined for $0 \leq h < 1$, $-1 \leq \sigma < 1$. In particular we have:

$$p(h,0) = 0$$

$$p(h,1) = \infty \qquad (\sigma = 1 \Leftrightarrow \lambda_1 = 0)$$

$$p(h,-1) = \ln(1-h) \qquad (\sigma = -1 \Leftrightarrow \lambda_1 = +\infty) \qquad (A2.17)$$

$$p(0,\sigma) = 0$$

The function $p(h,\sigma)$ is given in Figure 4.23.

The pipe lies in the upper soil layer in the second case $D \leq H$. See Figure A2.1. Formulas A2.1-5 are still valid. The conductivities λ and λ_1 are to change places in A2:6 and 7. The factor D-H in A2.7 is now negative. This must be observed, when A2.8 is used. We get:

$$f(s) = \frac{1}{s}\cdot\frac{-\sigma}{1+\sigma e^{-2Hs}}(1-e^{-2Hs})(e^{-(H-D)s} - e^{-(H+D)s}) \quad D<H \qquad (A2.18)$$

The second factor of A2.18 may be expanded in a series like A2.11 with σ replaced by $-\sigma$.

The temperature field is given by A2.1, 2 and 18. With the use of a series like A2.11 we get integrals of type A2.12. The temperature becomes:

$0 < z < H$:

$$T(x,z) = \frac{q}{2\pi\lambda}\cdot\left[\ln\left(\frac{\sqrt{x^2+(D-z)^2}}{\sqrt{x^2+(D+z)^2}}\right) + \right.$$
$$\left. + \sum_{m=1}^{\infty}(-\sigma)^m \ln\left[\frac{\sqrt{x^2+(2Hm+D-z)^2}}{\sqrt{x^2+(2Hm+D+z)^2}}\cdot\frac{\sqrt{x^2+(2Hm-D+z)^2}}{\sqrt{x^2+(2Hm-D-z)^2}}\right]\right] \quad (D<H) \qquad (A2.19)$$

z > H:

$$T(x,z) = \frac{q}{2\pi\lambda}(1+\sigma)\cdot\sum_{m=0}^{\infty}(-\sigma)^m \ln\left(\frac{\sqrt{x^2+(2Hm-D+z)^2}}{\sqrt{x^2+(2Hm+D+z)^2}}\right) \quad (D<H) \quad (A2.20)$$

The pipe temperature becomes from A2.19:

$$T_R = \frac{q}{2\pi\lambda}\left[\ln\left(\frac{R}{2D}\right) + \sum_{m=1}^{\infty}(-\sigma)^m \ln\left(\frac{H^2m^2}{H^2m^2-D^2}\right)\right] \quad (D<H) \qquad (A2.21)$$

The thermal resistance between the pipe and the ground surface is with definition 4.1.8:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) + p\left(\frac{H}{D},\sigma\right)\right\}$$

$$p\left(\frac{H}{D},\sigma\right) = -\sum_{m=1}^{\infty}(-\sigma)^m \ln\left(\frac{\left(\frac{H}{D}\right)^2 m^2}{\left(\frac{H}{D}\right)^2 m^2-1}\right) \qquad (D<H) \qquad (A2.22)$$

The function $p(h,\sigma)$ is here defined for $h > 1$, $-1 \leq \sigma \leq 1$. In particular we have:

$$p(h,0) = 0$$

$$p(h,-1) = \ln\left(\frac{h}{\pi}\sin\left(\frac{\pi}{h}\right)\right) \qquad (\sigma = -1 \Leftrightarrow \lambda_1 = +\infty)$$

$$p(h,1) = \ln\left(\frac{2h}{\pi}\tan\left(\frac{\pi}{2h}\right)\right) \qquad (\sigma = +1 \Leftrightarrow \lambda_1 = 0) \qquad (A2.23)$$

$$p(+\infty,\sigma) = 0$$

APPENDIX 3. PERIODIC HEAT EXTRACTION

Figure 7.1 shows a periodic line sink in an infinite surrounding. The amplitude of the heat extraction rate is q_1 (W/m), and the period is t_o. The sink lies at the center r = o.

A complex notation is used according to eqs. 7.1-2. The temperature becomes complex-valued. The actual temperature is given by the real or the imaginary part of the complex solution. They correspond to a cosine and a sine heat extraction respectively.

We will first consider the line sink for which a periodic heat extraction occurs at r = o. The case with a finite pipe radius R is then considered.

The complex-valued temperature is given in ref. 3D.

$$T(r,t) = -\frac{q_1}{2\pi\lambda} K_o\left((1+i)r/d_o\right) e^{2\pi it/t_o} \qquad \text{(A3.1)}$$

Here d_o is the penetration depth, eq. 7.3.

The total radial heat flux (W/m) towards the pipe at a distance r becomes:

$$2\pi r\lambda \frac{\partial T}{\partial r} = q_1 \frac{(1+i)r}{d_o} K_1\left((1+i)r/d_o\right)e^{2\pi it/t_o} \qquad \text{(A3.2)}$$

The functions $K_o(z)$ and $K_1(z)$ are modified Bessel functions. See ref. 2C. In this appendix z denotes the complex argument. In the derivation of A3.2 we have used

$$\frac{dK_o}{dz} = -K_1(z) \qquad \text{(A3.3)}$$

The following series expansions are valid:

$$K_o(z) = -(\ln(z/2)+\gamma)(1+\frac{z^2}{4}+\frac{z^4}{64}+\,..) + \frac{z^2}{4}+\frac{3z^4}{128}+\,.. \qquad \text{(A3.4)}$$

$$z\,K_1(z) = 1 + \ln(z/2)\left(\frac{z^2}{2} + \frac{z^4}{16} + ..\right) + \frac{z^2}{2}\left(\gamma - \frac{1}{2}\right) + \frac{z^4}{16}\left(\gamma - \frac{5}{4}\right) + ... \qquad \text{(A3.5)}$$

Here γ is Euler's constant:

$$\gamma \simeq 0.5772 \qquad \text{(A3.6)}$$

We see from eq. A3.5 that $zK_1(z)$ tends to 1, when z tends to zero. So we have from A3.2 that the heat flux tends to the prescribed value of eq. 7.2, when r tends to zero.

The modified Bessel functions are needed for the argument

$$z = \frac{(1+i)r}{d_o} = e^{i\pi/4} \cdot \frac{r\sqrt{2}}{d_o} = e^{i\pi/4} \cdot r' \qquad \text{(A3.7)}$$

We have used notation 7.4. We have the so-called Kelvin functions:

$$K_o(e^{i\pi/4} \cdot r') = \mathrm{ker}_o(r') + i \cdot \mathrm{kei}_o(r') = N_o(r')e^{i\phi_o(r')} \qquad \text{(A3.8')}$$

$$K_1(e^{i\pi/4} \cdot r') = i(\mathrm{ker}_1(r') + i \cdot \mathrm{kei}_1(r')) = i\, N_1(r')e^{i\phi_1(r')} \qquad \text{(A3.8")}$$

The functions N_j and ϕ_j give the modulus and phase of the complex-valued Kelvin functions.

The temperature in the ground outside the periodic line sink of figure 7.1 is now from eqs. A3.1 and A3.8':

$$T(r,z) = -\frac{q_1}{2\pi\lambda} N_o(r')e^{i(2\pi t/t_o + \phi_o(r'))}$$

$$r' = \frac{r\sqrt{2}}{d_o} \qquad d_o = \sqrt{\frac{at_o}{\pi}} \qquad \text{(A3.9)}$$

The real-valued temperature is obtained in the following way. Let the periodic heat extraction be

$$q(t) = q_1 \cdot \sin(2\pi t/t_0 + \varphi_0) = \mathrm{Im}\left\{q_1\, e^{i(2\pi t/t_0 + \varphi_0)}\right\} \qquad (A3.10)$$

A phase φ_0 has been included. Then we have for the imaginary part of A3.9 (including an added phase φ_0):

$$T(r,t) = -\frac{q_1}{2\pi\lambda} N_0(r') \cdot \sin\left(\frac{2\pi t}{t_0} + \phi_0(r') + \varphi_0\right) \qquad (A3.11)$$

From eqs. A3.4-5 we have the following series expansions for the Kelvin functions:

$$K_0(e^{i\pi/4} \cdot r') = \ln(2/r') - \gamma - \frac{i\pi}{4} +$$

$$+ \frac{(r')^2}{4} \left(\frac{\pi}{4} + i(1 + \ln(2/r') - \gamma)\right) +.. \qquad (A3.12')$$

$$e^{i\pi/4}\, r' \cdot K_1(e^{i\pi/4}\, r') = 1 - \frac{(r')^2}{2} \left(\frac{\pi}{4} + i(\ln(2/r') + 0.5 - \gamma)\right) +.. \qquad (A3.12'')$$

For large r' there is the following asymptotic expansion:

$$K_0(e^{i\pi/4}\, r') \simeq \sqrt{\frac{\pi}{2r'}}\, e^{-r'/\sqrt{2} - i(\pi/8 + r'/\sqrt{2})} \cdot \left\{1 - \frac{e^{-i\pi/4}}{8r'} +..\right\} \qquad (A3.13)$$

From eqs. A3.12' and A3.8' we have for small r':

$$N_0(r') \simeq \sqrt{(\ln(2/r') - \gamma)^2 + \pi^2/16}$$

$$\phi_0(r') \simeq -\arctan\left(\frac{\pi/4}{\ln(2/r') - \gamma}\right) \qquad (r' < 0.1) \qquad (A3.14)$$

The error of the approximations is less than 1% for $r' < 0.1$.

For large r' we have from eqs. A3.13 and A3.8':

$$N_o(r') \simeq \sqrt{\frac{\pi}{2r'}}\, e^{-r'/\sqrt{2}}$$
$$\phi_o(r') \simeq -\frac{r'}{\sqrt{2}} - \frac{\pi}{8} \qquad (r' > 7) \tag{A3.15}$$

The error is less than 1% for $r' > 7$. The functions $N_o(r')$ and $\phi_o(r')$ are given in figure 7.2 and table 7.2.

The heat flux of the line sink at the distance r is given by eq. A3.2. It may be written in the following way

$$2\pi r\lambda \frac{\partial T}{\partial r} = q_1\, e^{2\pi it/t_o} \cdot F(r')\, e^{-iG(r')} \tag{A3.16}$$

Here we have introduced

$$F(r')\, e^{-iG(r')} = e^{i\pi/4}\, r' K_1(e^{i\pi/4}\, r') \tag{A3.17}$$

The functions F and G represent the amplitude and phase of the heat flux at the distance r. From eq. A3.8" we have

$$F(r') = r' N_1(r')$$
$$G(r') = -\phi_1(r') - 3\pi/4 \tag{A3.18}$$

For small r' we have from A3.12":

$$F(r') \simeq 1 \qquad G(r') \simeq 0 \qquad (r' < 0.1) \tag{A3.19}$$

The functions F and G are given in table 7.2 and figure 7.2.

Let us now consider the general case with the prescribed periodic heat flux at the pipe radius $r = R$. We have the condition:

$$q(t) = q_1\, e^{2\pi it/t_o} \qquad r = R \tag{A3.20}$$

The line sink solution A3.1 can still be used. But we have to divide by the factor A3.17 in order to obtain the prescribed heat flux at $r = R$.

The temperature outside the pipe, $r \geqq R$, is then from eqs. A3.9 and A3.16 given by:

$$T(r,t) = -\frac{q_1}{2\pi\lambda} \cdot \frac{N_o(r')}{F(R')} e^{i(2\pi t/t_o + \phi_o(r') + G(R'))}$$

$$R' = \frac{R\sqrt{2}}{d_o} \qquad \text{(A3.21)}$$

Our main interest is the temperature at the pipe radius $r = R$. We have from A3.21:

$$T_R(t) = T(R,t) = -\frac{q_1}{2\pi\lambda} A(R')e^{i(2\pi t/t_o - B(R'))} \qquad \text{(A3.22)}$$

Here we have used the notations:

$$A(R') = \frac{N_o(R')}{F(R')} \qquad B(R') = -\phi_o(R') - G(R') \qquad \text{(A3.23)}$$

The functions A and B are given in figure 7.2. and table 7.2.

The distinction between the line sink and the pipe with a radius R is negligible for $R' < 0.1$. The error in the temperature is less than 5%, if $R' < 0.3$. So the line sink approximation is rather good if

$$R' = \frac{R\sqrt{2}}{d_o} < 0.3 \qquad \text{or } t_o < 70\,\frac{R^2}{a} \qquad \text{(A3.24)}$$

APPENDIX 4. GROUND WATER FILTRATION BELOW A PIPE

The steady-state heat extraction to a pipe, when there is a horizontal ground water filtration or flow in a region below the pipe, is discussed in section 8.3. The thermal resistance of the pipe contained an additional part P_w according to eq. 8.29. The explicit expression for P_w will be derived in this appendix.

The considered case is shown in figure 8.5. The pipe lies at the depth $z = D$. The ground water level lies at a lower depth $z = H$ $(H > D)$. The thermal conductivity in the upper part $0 < z < H$ is λ, while it is λ_1 in the ground water region $z > H$. There is a constant horizontal ground water flow q_w (m_w^3/m^2s or m/s) for $z > H$. The steady-state heat extraction rate is q (W/m). The temperature at the ground surface is zero.

The ordinary heat conduction equation is valid in the upper region outside the pipe:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} = 0 \qquad \begin{array}{l} 0 < z < H \\ x^2 + (z-D)^2 > R^2 \end{array} \qquad \text{(A4.1)}$$

In the lower region there is also a convective heat transfer. The heat balance equation is:

$$\lambda_1 \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) - q_w C_w \cdot \frac{\partial T}{\partial x} = 0 \qquad z > H \qquad \text{(A4.2)}$$

At the internal boundary $z = H$ the temperature and the vertical heat flux must be continuous:

$$T(x,H-0) = T(x,H+0) \qquad \text{(A4.3)}$$

$$\lambda \left.\frac{\partial T}{\partial z}\right|_{z = H-0} = \lambda_1 \left.\frac{\partial T}{\partial z}\right|_{z = H+0} \qquad \text{(A4.4)}$$

The temperature T(x,z) shall satisfy the conditions of figure A4.1.

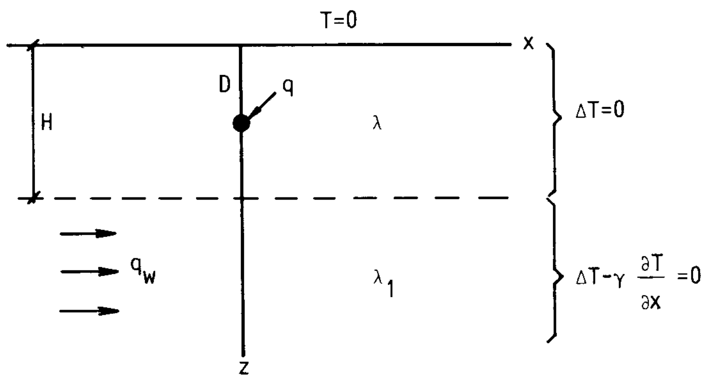


Figure A4.1. Steady-state heat extraction with ground water filtration below.

The following notation is used:

$$\gamma = \frac{C_w q_w}{\lambda_1} \tag{A4.5}$$

The temperature field T(x,z) will in the upper region consist of the previous solution 4.1.2 and an additional part. In order to obtain that part we note that the following function satisfies eq. A4.1 for any s:

$$e^{ixs} \cdot \sinh(zs) \tag{A4.6}$$

This function is also zero for z = 0.

We therefore start with the following expression for the temperature in the upper region $0 < z < H$:

$$T(x,z) = -\frac{q}{4\pi\lambda}\left\{\ln\left(\frac{x^2+(z+D)^2}{x^2+(z-D)^2}\right) - \int_{-\infty}^{\infty} f(s)e^{ixs}\frac{\sinh(zs)}{\sinh(Hs)}\,ds\right\} \tag{A4.7}$$

The function f(s) is to be determined.

A sufficiently general solution to A4.2 in the lower region is:

$$e^{ixs} \cdot e^{-\sqrt{s^2+i\gamma s} \cdot z} \tag{A4.8}$$

The square root in the exponent is complex-valued. In the lower region, z > H, we start with the following expression

$$T(x,z) = -\frac{q}{4\pi\lambda} \int_{-\infty}^{\infty} g(s) e^{ixs} e^{-\sqrt{s^2+i\gamma s} \cdot (z-H)} ds \tag{A4.9}$$

The function g(s) is to be determined below.

The two expressions A4.7 and A4.9 satisfies all the conditions of the problem for any f(s) and g(s) except the two conditions A4.3 and A4.4. These two determine the functions f and g.

We need a Fourier expansion of the logarithm in A4.7. From ref. 4 we have

$$\ln\left(\frac{x^2+(z+D)^2}{x^2+(z-D)^2}\right) = \int_{-\infty}^{\infty} f_1(s,z) e^{isx} ds$$

$$f_1(s,z) = \frac{e^{-|z-D|\cdot|s|} - e^{-|z+D|\cdot|s|}}{|s|} \tag{A4.10}$$

The boundary condition A4.3 is now fulfilled, if

$$f_1(s,H) - f(s) = g(s) \tag{A4.11}$$

Here A4.7, 9 and 10 were used.

The second boundary condition A4.4 involves the derivatives with respect to z at z = H. We have from A4.7, 9 and 10 the condition:

$$\lambda \left\{ \frac{-|s| e^{-(H-D)\cdot|s|} - (-|s|) e^{-(H+D)|s|}}{|s|} - f(s)\ s \coth(Hs) \right\} =$$

$$= \lambda_1 \cdot g(s) \cdot (-\sqrt{s^2+i\gamma s}) \tag{A4.12}$$

In the absolute values we used that $H > D$. The two functions $f(s)$ and $g(s)$ are determined from A4.11-12 with the use of expression A4.10 for f_1.

The following expression is obtained for $f(s)$:

$$f(s) = e^{-H\cdot|s|} \frac{e^{D|s|} - e^{-D|s|}}{|s|} \cdot f_1(s)$$
$$f_1(s) = \frac{\lambda_1 \sqrt{s^2+i\gamma s} - \lambda|s|}{\lambda_1 \sqrt{s^2+i\gamma s} + \lambda s \coth(Hs)} \qquad \text{(A4.13)}$$

Our main interest is as usual the pipe temperature at the pipe periphery, eq. 4.1.3. The usual approximation 4.1.4-5 is used. In the integrand of A4.7 we put $x = 0$ and $z = D$. Then we have:

$$T_R = -\frac{q}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - \frac{1}{2}\int_{-\infty}^{\infty} f(s)\ \frac{\sinh(DS)}{\sinh(Hs)}\ ds\right\} \qquad \text{(A4.14)}$$

The logarithm represents our usual thermal resistance, and the integral the effect of the ground water flow.

The thermal resistance is from A4.14:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - P_w\right\} \qquad \text{(A4.15)}$$

The function P_w is from A4.13-15 given by:

$$P_w = \frac{1}{2}\int_0^{\infty} \frac{\sinh(Ds)}{\sinh(Hs)}\,(f(s) + f(-s))\ ds \qquad \text{(A4.16)}$$

The integrand of A4.16 is changed with the substitution $t = Hs$. The factor $f_1(s) + f_1(-s)$ is rewritten so that the complex root is eliminated.

We finally have the following expression for P_w:

$$P_w = \int_0^{\infty} e^{-2(1-D')t}\ \frac{(1-e^{-2D't})^2}{t(1-e^{-2t})} \cdot B(t)\ dt \qquad \text{(A4.17)}$$

$$B(t) = \frac{(\lambda')^2 t\sqrt{t^2+p^2} + \lambda' t(\coth(t)-1)A - t^2\coth(t)}{(\lambda')^2 t\sqrt{t^2+p^2} + 2\lambda' t\coth(t)A + t^2\coth^2(t)}$$

$$A = \sqrt[4]{t^4+p^2t^2}\cdot\cos\left(\frac{1}{2}\arctan\left(\frac{p}{t}\right)\right) \qquad \text{(A4.17)}$$

Here we have used the dimensionless parameters:

$$D' = \frac{D}{H} \qquad \lambda' = \frac{\lambda_1}{\lambda} \qquad p = \gamma H = \frac{HC_w q_w}{\lambda_1} \qquad \text{(A4.18)}$$

We note that P_w is a function of D', λ' and p.

The limit with a very strong ground water flow is of particular interest since it gives the maximum effect of the ground water. The limit $q_w \to \infty$ implies that the temperature in the ground water region is zero. We have for the upper region the boundary condition:

$$T(x,H) = 0 \qquad \text{(A4.19)}$$

The considered limit is shown in figure A4.2.

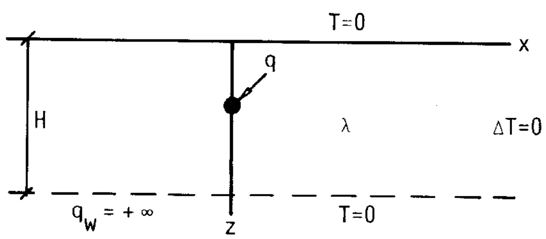


Figure A4.2. Steady-state heat extraction in the limit of very strong ground water flow.

The solution to the problem of figure A4.2 is well-known. We have

$$T(x,z) = -\frac{q}{4\pi\lambda}\ln\left[\frac{\cosh\left(\frac{\pi x}{H}\right) - \cos\left(\frac{\pi(z+D)}{H}\right)}{\cosh\left(\frac{\pi x}{H}\right) - \cos\left(\frac{\pi(z-D)}{H}\right)}\right] \qquad (A4.20)$$

The temperature at the pipe radius, eq. 4.1.3, is obtained from A4.20 in the following way. In the nominator we take $x = 0$ and $z = D$. The denominator is expanded in Taylor series. The lowest term contains the factor 4.1.3. We get after some rearrangements:

$$-T_R = \frac{q}{2\pi\lambda}\ln\left(\frac{2H\sin(\pi D/H)}{\pi R}\right) \qquad (A4.21)$$

The thermal resistance is then:

$$m = \frac{1}{2\pi\lambda}\left\{\ln\left(\frac{2D}{R}\right) - \ln\left(\frac{\pi D/H}{\sin(\pi D/H)}\right)\right\} \qquad (A4.22)$$

APPENDIX 5. TEMPERATURE FIELD AROUND THE END OF A PIPE

The three-dimensional temperature field around the very end of a heat extraction pipe will be studied here. Consider a semi-infinite pipe that lies along $(x,y,z) = (0,y,D)$, $0 \leq y \leq \infty$. See figure 9.1. The temperature at the ground surface is zero. Let q (W/m) be the steady-state heat extraction rate per unit length.

We have a semi-infinite line sink along the pipe and a negative mirror sink above the ground. The temperature field in the ground is obtained by the integral:

$$T(x,y,z) = -\int_0^\infty \frac{q}{4\pi\lambda} \cdot \left\{ \frac{1}{\sqrt{x^2+(y-s)^2 + (z-D)^2}} - \frac{1}{\sqrt{x^2+(y-s)^2 + (z+D)^2}} \right\} ds \qquad (A5.1)$$

The fluid temperature along the pipe will be more or less constant. The heat extraction rate q is then more or less constant along the pipe, except for the end region, where three-dimensional effects occur.

The integral A5.1 may be evaluated, when $q(s)$ is known. We will now assume that q is constant along the pipe including the end region. The integral is then rather simple to evaluate. We get the following temperature field in the ground:

$$T(x,y,z) = -\frac{q}{4\pi\lambda} \left\{ \ln\left(\frac{x^2+(z+D)^2}{x^2+(z-D)^2}\right) - \ln\left(\frac{\sqrt{x^2+y^2+(z+D)^2}+y}{\sqrt{x^2+y^2+(z-D)^2}+y}\right) \right\} \qquad (A5.2)$$

The temperature at the pipe radius, eq. 4.1.3, is of particular interest. We have with the usual approximations from eqs. 4.1.4-5:

$$T_R(y) = -\frac{q}{2\pi\lambda} \left\{ \ln\left(\frac{2D}{R}\right) - \frac{1}{2} \ln\left(\frac{\sqrt{y^2+4D^2}+y}{\sqrt{y^2+R^2}+y}\right) \right\} \quad (y \geq 0) \qquad (A5.3)$$

The last term, which depends on y, represents the end effect of the considered case. The variation with y is given by

$$T'_R(y) = \frac{T_R(y)}{T_R(\infty)} = 1 - \frac{1}{2} \cdot \frac{\ln\left[\dfrac{\sqrt{y^2+4D^2}+y}{\sqrt{y^2+4R^2}+y}\right]}{\ln\left(\dfrac{2D}{R}\right)} \qquad (y \geq 0) \qquad \text{(A5.4)}$$

We note that the temperature at the end of the pipe is halved:

$$T'_R(0) = \frac{1}{2} \qquad T'_R(\infty) = 1 \qquad \text{(A5.5)}$$

This is due to symmetry.

Let us consider the case:

$$\frac{R}{D} = 0.02 \qquad \text{(A5.6)}$$

Then we get the following values:

y/D	0	0.01	0.05	0.1	0.2	0.3	0.4
T'_R	0.50	0.55	0.68	0.75	0.81	0.85	0.88

y/D	0.5	0.75	1	2	3	5	10
T'_R	0.90	0.93	0.95	0.98	0.990	0.996	0.999

We see that the end region is quite small. There is only a 5% decrease for y = D.

From this we can conclude that three-dimensional effects only influence a length, say, D of the end region of the pipe. This will be true also in the case of constant fluid temperature along the pipe instead of constant heat extraction rate.

APPENDIX 6. DIPOLE APPROXIMATIONS OF THE TEMPERATURE FIELD

The thermal impact on the surrounding ground from the heat extraction pipes is discussed in chapter 10. We are there interested in the temperature field further away from the pipes. Simple, so-called dipole approximations were used. These formulas will be derived here.

We start by considering a point heat extraction q (W) at (x,y,z) = = (0,0,D). The temperature at the ground surface z = 0 becomes as usual zero by the introduction of a negative mirror sink above the ground surface. The steady-state temperature due to the point sink is then:

$$T(x,y,z) = -\frac{q}{4\pi\lambda}\left\{\frac{1}{\sqrt{x^2+y^2+(z-D)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+D)^2}}\right\} \qquad (A6.1)$$

We are interested in the temperature further away from the point sink; i.e. for

$$r = \sqrt{x^2+y^2+z^2} >> D \qquad (A6.2)$$

A Taylor expansion in 1/r is needed. We get after som manipulations from A6.1:

$$T(x,y,z) = -\frac{qD}{2\pi\lambda}\cdot\frac{z}{r^3}\left\{1-\frac{3}{2}\cdot\frac{D^2}{r^2}+\frac{15}{8}\cdot\frac{D^2}{r^2}\cdot\frac{D^2+2z^2}{r^2}+..\right\} \qquad (A6.3)$$

The dipole approximation of a point sink q at a depth D is obtained by the first term in the expansion A6.3.

$$T(x,y,z) \simeq -\frac{qD}{2\pi\lambda}\cdot\frac{z}{r^3} \qquad (r >> D) \qquad (A6.4)$$

A single pipe of finite length is discussed in section 10.1. The position of the pipe is defined by eq. 10.1.1. The heat extraction rate q (W/m) is assumed to be constant along the pipe. The temperature is in this case given as an integral of contributions of the type A6.1 along the pipe length. The corresponding dipole approximation is obtained by an integral of A6.4. We have:

$$T(x,y,z) \simeq -\frac{qD}{2\pi\lambda}\int_{-L}^{L}\frac{L}{\sqrt{x^2+(y-s)^2+z^2}^{\,3}}\,ds \qquad (A6.5)$$

The integral is not difficult to evaluate. The temperature may after some rearrangements be written in the following way:

$$T(x,y,z) = -\frac{q}{2\pi\lambda}\cdot\frac{Dz}{x^2+z^2}\cdot\left\{\frac{L-y}{\sqrt{x^2+(L-y)^2+z^2}}+\frac{L+y}{\sqrt{x^2+(L+y)^2+z^2}}\right\} \qquad (A6.6)$$

The case with a rectangular heat extraction area is dealt with in section 10.2. The pipes lie at the depth $z = D$. They cover an area $-L < y < L$, $-M < x < M$. The heat extraction rate per unit area is denoted q_a (W/m^2). It is determined by eq. 10.2.1. We assume that q_a is constant over the rectangular area. The dipole approximation is again considered. The rectangular area may be considered as a superposition of line sinks for $-L < y < L$. The contributions from these are given by A6.6. The total temperature is obtained as an integral in the x-direction from -M to +M. We have then:

$$T(x,y,z) = -\frac{q_a}{2\pi\lambda}\int_{-M}^{M}\frac{Dz}{(x-s)^2+z^2}\cdot\left\{\frac{L-y}{\sqrt{(x-s)^2+(L-y)^2+z^2}}+\right.$$

$$\left.+\frac{L+y}{\sqrt{(x-s)^2+(L+y)^2+z^2}}\right\}ds \qquad (A6.7)$$

The evaluation of the integral gives the following result:

$$T(x,y,z) = -\frac{q_aD}{\lambda}\cdot\left\{a(M-x,\ L-y),z) + a(M+x,L-y,z)\right.$$

$$\left.+\ a(M-x,L+y,z) + a(M+x,L+y),z)\right\} \qquad (A6.8)$$

$$a(x',y',z) = \frac{1}{2\pi}\ \arctan\left\{\frac{x'y'}{z\cdot\sqrt{(x')^2+(y')^2+z^2}}\right\}$$