Some Extensions of Poincaré-Bendixson Theory Applied to a Resonant Converter.

Melin, Jan

2005

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

International Journal of Hybrid Systems 3 (2003) 331-340

UPPER BOUNDS FOR THE NUMBER OF LIMIT CYCLES OF SWITCHED SYSTEMS THROUGH DISTRIBUTION THEORY.

JAN MELIN
DEPARTMENT OF CHEMISTRY AND BIOMEDICAL SCIENCES
UNIVERSITY OF KALMAR, NORRA VÄGEN 49, 39182 KALMAR, SWEDEN.

ABSTRACT. We use the theory of distributions to extend the theory of stability of limit cycles and calculating the Floquet exponents to piecewise C^1 systems possessing unique and continuous solutions. We demonstrate the use of these extensions by several examples.

Keywords: Floquet exponents, distributions, piecewise linear systems.
AMS subject classification: 34A36, 34A12, 34H05

1. INTRODUCTION

Finding upper bounds for the number of limit cycles for two-dimensional systems of ordinary differential equations has been a difficult and acknowledged mathematical problem since Hilbert published his 23 problems 1900. In many cases, upper limits for the number of limit cycles have been calculated using Floquet theory, see e.g. Zhang [10], Cherkas and Zhilevich [2], Ye et al [9] and Zhang [11]. In this paper we suggest an extension of this classical method that is suitable for systems with discontinuities in the righthand side. Our extensions requires calculation of the divergence of such systems in distributional sense. Such calculations require special mathematical knowledge and in some cases these calculations can be quite extensive. Yet, we claim that our approach yields information that otherwise would be hardly available. The organisation of our paper is as follows: We start by explaining our setting and formulating the main theorem. After this we discuss the calculation of the Floquet-exponents in distributional sense and work through examples that have appeared in the literature recently. We summarize our results in the end of the paper.

2. OUR SETTINGS AND MAIN THEOREMS

We shall work with planar autonomous systems with discontinuous righthand sides throughout this paper. We restrict the properties of the systems under consideration by four major assumptions.

We consider a planar autonomous system

(1) $$\dot{X} = f(X), X \in \Omega$$

(A1). Ω is an open domain in R^2, divided into a finite number of sub domains Ω_i (also called switching regions), such that $\cup\overline{\Omega_i} = \overline{\Omega}$.

(A2). If $\overline{\Omega_i}$ and $\overline{\Omega_j}$ are not disjoint and $i \neq j$, then $\overline{\Omega_i} \cap \overline{\Omega_j} = \Gamma_{ij}$, where Γ_{ij} (joint boundaries) are piecewise smooth.

(A3). f is in C^1 in all sub domains Ω_i and possibly discontinuous along Γ_{ij} (also called discontinuity curves).

(A4). The vector field f defines a direction in each point in Ω. In particular, at every point along Γ_{ij} the vector field specifies into which Ω_i the flow is directed.

The conditions (A3)-(A4) imply that (1) has unique, continuous and piecewise C^2-solutions in Ω. Note that (A4) gives strong restrictions on the possible discontinuities. In terms of Filippov [3] there are three kinds of sliding modes. We only allow transversal sliding mode, that is: the vector field is directed from one side to the other at the discontinuity curves. The solutions will pass the discontinuity curves in the field direction and we have uniqueness of solutions there. However we will involve repulsion sliding mode later and define it then. Now we will give an extension of a classical theorem concerning stability of limit cycles.

Theorem 1. *Consider the planar autonomous system* (1). *Let the conditions* (A1)-(A4) *be satisfied, let f be bounded in Ω and* div$f$ *(the divergence of f calculated in sense of distributions) be in $L^1(\Omega)$. Furthermore let $X(t)$ be a closed trajectory of* (1) *with period T and let $\mu = \frac{1}{T} \cdot \int_0^T \operatorname{div} f(X(t))dt$ (the Floquet-exponents). If $\mu < 0$, then $X(t)$ is asymptotically stable and if $\mu > 0$ then $X(t)$ is unstable.*

Proof: Suppose for simplicity that there is only one intersection point between $X(t)$ and a discontinuity curve at $t = 0$. Let $O_\varepsilon =]-\varepsilon, \varepsilon[$ be an open neighborhood of $t = 0$. Take $\varphi \in C_0^\infty(R)$ such that supp$\varphi \subseteq [-1, 1]$, $\varphi(t) \geq 0$ and $\int \varphi(t)dt = 1$ and put $\varphi_j(t) = j \cdot \varphi(j \cdot t)$, then supp$\varphi_j \subseteq [-\frac{1}{j}, \frac{1}{j}]$,$\varphi_j(t) \geq 0$ and $\int \varphi_j(t)dt = 1$. Now we have $X * \varphi_j \to X$, as $j \to \infty$ (convolution defined coordinate wise). Let $X_c(t)$ be the C^2-restriction of $X(t)$ to the interval $[-\frac{T}{2}, \frac{T}{2}] \setminus O_\varepsilon$. Suppose also that φ_j is chosen so that $(X * \varphi_j)^{(k)}(\pm\varepsilon) = X_c^{(k)}(\pm\varepsilon)$, $k = 0, 1, 2$ and define $X_j(t)$ as follows:

$$X_j(t) = \begin{cases} X_c(t) & \text{if } t \in [-\frac{T}{2}, \frac{T}{2}] \setminus O_\varepsilon \\ (X * \varphi_j)(t) & \text{if } t \in O_\varepsilon \end{cases}$$

then $X_j(t)$ is a closed C^2-curve and $X_j(t) \to X(t)$ in C^0 and:

$$\dot{X}_j(t) = \begin{cases} \dot{X}_c(t) & \text{if } t \in [-\frac{T}{2}, \frac{T}{2}] \setminus O_\varepsilon \\ (\dot{X} * \varphi_j)(t) & \text{if } t \in O_\varepsilon \end{cases}$$

and this leads to:

$$f_j(X_j(t)) = \begin{cases} f_c(X_c((t)) & \text{if } t \in [-\frac{T}{2}, \frac{T}{2}] \setminus O_\varepsilon \\ ((f \circ X) * \varphi_j)(t) & \text{if } t \in O_\varepsilon \end{cases}$$

from this we have $f_j \in C^1$ and $f_j \to f$ in D' (convergence in sense of distribution theory, and note that $\circ$ means composition) and furthermore:

$$\operatorname{div} f_j(X(t)) = \begin{cases} \operatorname{div} f_c(X_c(t)) & \text{if } t \in [-\frac{T}{2}, \frac{T}{2}] \setminus O_\varepsilon \\ (\operatorname{div}(f \circ X) * \varphi_j)(t) & \text{if } t \in O_\varepsilon \end{cases}$$

then $\mathrm{div} f_j$ is in C^0 and $\mathrm{div} f_j \to \mathrm{div} f$ in D'. Now we have:

$$\mu_j = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{div} f_j(X_j(t))dt =$$

$$= \frac{1}{T} \cdot \left(\int_{[-\frac{T}{2},\frac{T}{2}]\setminus O_\epsilon} \mathrm{div} f_c(X_c(t))dt + \int_{O_\epsilon} (\mathrm{div}(f \circ X) * \varphi_j)(t)dt \right),$$

since $\mathrm{div} f$ is in L^1 we have:

$$\int_{O_\epsilon} (\mathrm{div}(f \circ X) * \varphi_j)(t)dt = \int_{O_\epsilon} \mathrm{div} f(X(t))dt \cdot \int_{O_\epsilon} \varphi_j(t)dt = \int_{O_\epsilon} \mathrm{div} f(X(t))dt,$$

for j large enough and finally the Floquet-exponents becomes:

$$\mu_j = \frac{1}{T} \cdot \left(\int_{[-\frac{T}{2},\frac{T}{2}]\setminus O_\epsilon} \mathrm{div} f_c(X_c(t))dt + \int_{O_\epsilon} \mathrm{div} f(X(t))dt \right) = \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{div} f(X(t))dt.$$

It is obvious that f satisfies the theorem since f_j satisfies the original theorem. This can of course be generalized to a finite number of discontinuity points, and so the theorem holds.

Now we will give a theorem which involves "repulsion sliding mode". According to Filippov [3] that is: the vector field is directed away from the discontinuity curve on both sides. This means that the motion along the discontinuity curve is unstable and the direction of the vector field is not unique.

Theorem 2. *If the conditions* (A1)-(A4) *are satisfied, then a closed trajectory of* (1) *must enclose at least one fixed point or at least one interval of repulsion sliding mode.*

Proof: Choose the approximations X_j and f_j of X respectively f from theorem 1. The original theorem (see for example Grimshaw [8]) holds for the system $\dot{X}_j = f_j(X)$, since $f_j \in C^1$ and $X_j \in C^2$. The original theorem does not involve repulsion sliding mode, but it is obvious that the theorem requires uniqueness of the vector field. Since X_j and f_j are arbitrary close to X respectively f for j large enough, this theorem holds.

3. CALCULATION OF THE FLOQUET-EXPONENTS

Put $X = \begin{pmatrix} x \\ y \end{pmatrix}$ in (1) and suppose that we only have one discontinuity curve Γ represented by the equation $\psi(x,y) = 0$, where ψ is smooth.
Let $\Omega_+ = \{(x,y) \in \Omega; \psi(x,y) > 0\}$, $\Omega_- = \{(x,y) \in \Omega; \psi(x,y) < 0\}$ and
$f = \begin{cases} f^+, & \text{if } (x,y) \in \Omega_+ \\ f^-, & \text{if } (x,y) \in \Omega_- \end{cases}$ where f^+ and f^- is in C^1 and suppose that the solutions passes Γ from Ω_- into Ω_+. Let χ_+ and χ_- be the characteristic functions of Ω_+ respectively Ω_-, then $\chi_+ = H(\psi(x,y))$ and $\chi_- = 1 - H(\psi(x,y))$ where H is the Heavyside function.
We have $f = f^+ \cdot \chi_+ + f^- \cdot \chi_-$ and this could be written as
$f = f^- + (f^+ - f^-) \cdot H(\psi(x,y))$. The divergence of f now becomes:

$$\mathrm{div} f = \mathrm{div} f^- + (\mathrm{div} f^+ - \mathrm{div} f^-) \cdot H(\psi(x,y)) + \langle f^+ - f^-, \mathrm{grad}\psi\rangle \cdot \delta(\psi(x,y)),$$

where δ is the Dirac impulse and $\langle\cdot,\cdot\rangle$ is the inner product. The divergence of f is in L^1 so the integral

$$\mu = \frac{1}{T}\cdot\int_0^T \operatorname{div} f(x(t),y(t))dt$$

is well defined and finite, and $(x(t),y(t))$ is a closed trajectory of (1) with period T. Now consider the infinite part $\operatorname{div} f_d$ of $\operatorname{div} f$, that is

$$\operatorname{div} f_d(x,y) = c(x,y)\cdot\delta(\psi(x,y)) \text{ where } c = \langle f^+ - f^-, \operatorname{grad}\psi\rangle \text{ is in } C^1.$$

We will now calculate the integral I of $\operatorname{div} f_d$ over the interval $[t_0 - \frac{\Delta t}{2}, t_0 + \frac{\Delta t}{2}]$, where Δt is the infinitely small interval length and $(x_0,y_0) = (x(t_0),y(t_0))$ is the intersection between the closed trajectory and the discontinuity curve. We have

$$I = \int_{t_0-\frac{\Delta t}{2}}^{t_0+\frac{\Delta t}{2}} \operatorname{div} f_d(x(t),y(t))dt = \int_{t_0-\frac{\Delta t}{2}}^{t_0+\frac{\Delta t}{2}} c(x(t),y(t))\cdot\delta(\psi(x(t),y(t)))dt.$$

A change of variables $s = \psi(x(t),y(t))$ [5] gives us

$$I = \int_{s_1}^{s_2} \frac{c(x(t),y(t))}{|\frac{ds}{dt}|}\cdot\delta(s)ds, \text{ where}$$

$s_1 = \psi(x(t_0 - \frac{\Delta t}{2}), y(t_0 - \frac{\Delta t}{2})) < 0$ and $s_2 = \psi(x(t_0 + \frac{\Delta t}{2}), y(t_0 + \frac{\Delta t}{2})) > 0$. The chain rule implies $\frac{ds}{dt} = \psi_x\cdot\dot{x} + \psi_y\cdot\dot{y} = \langle f, \operatorname{grad}\psi\rangle$. On the boundary we put, according to Filippov [3] $f = f^0$, where $f^0 = f^- + \alpha\cdot(f^+ - f^-)$ and α is a parameter in the interval $[0,1]$. Then the integral becomes

$$I = \int_{s_1}^{s_2} \frac{c(x(t),y(t))}{|\langle f^0(x(t),y(t)), \operatorname{grad}\psi(x(t),y(t))\rangle|}\cdot\delta(s)ds.$$

We have $s = 0 \Leftrightarrow t = t_0 \Leftrightarrow (x(t),y(t)) = (x_0,y_0)$, this implies

$$I = \frac{c(x_0,y_0)}{|\langle f^0(x_0,y_0), \operatorname{grad}\psi(x_0,y_0)\rangle|}, \tag{2}$$

where $f^0(x_0,y_0)$ is set valued, so I belongs to an interval as α varies in the interval $[0,1]$. Suppose now that we have a finite number of intersection points between the closed trajectory and the discontinuity curves. The Floquet-exponents of (1) can therefore be written as

$$\mu = \frac{1}{T}\cdot\int_0^T v(x(t),y(t))dt + \frac{1}{T}\cdot\sum_k I_k,$$

where v is bounded and L^1 in Ω and $\sum_k I_k$ is a finite sum of integrals (v is obtained from $\operatorname{div} f$). Each integral I_k is calculated as in formula (2) above.

Note that μ is set valued and belongs to an interval $[\mu_{min}, \mu_{max}]$. If we like to determine the type of stability for the closed trajectory $(x(t),y(t))$ of (1), then we have to consider the following two conditions:

1) If $\mu_{max} < 0$ then $(x(t),y(t))$ is asymptotically stable.
2) If $\mu_{min} > 0$ then $(x(t),y(t))$ is unstable.

We now introduce a number of examples, in which we show how to calculate the

Floquet-exponents, if possible determine the type of stability and the uniqueness of closed orbits.

4. EXAMPLES

Exampel 1 (Branicky [1]). Consider the system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + (100 - 90\lambda)y + 90(2\lambda - 1) \cdot y \cdot (H(x) + H(y) - 2H(x)H(y)) \\ -(90\lambda + 10)x - y + 90(2\lambda - 1) \cdot x \cdot (H(x) + H(y) - 2H(x)H(y)) \end{pmatrix}$$

The parameter λ is in the interval $[0, 1]$. According to J. Melin [6] the divergence of the righthand side f is

$$\text{div} f(x, y) = -2 + 90 \cdot (1 - 2\lambda) \cdot (|y| \cdot \delta(x) + |x| \cdot \delta(y)).$$

A necessary condition for a closed orbit is if div$f$ changes signs, so choose $0 \leq \lambda < \frac{1}{2}$. Due to symmetry a closed orbit passes through the discontinuity points: $(r, 0)$, $(0, -r)$, $(-r, 0)$ and $(0, r)$ where $r > 0$, see figure 1. The switching regions are $\Omega_1 : x > 0, y < 0$, $\Omega_2 : x < 0, y < 0$, $\Omega_3 : x < 0, y > 0$ and $\Omega_4 : x > 0, y > 0$ (the four quadrants). We will now calculate the integrals I_k point by point.
We start with the discontinuity point 1 with the coordinates (x_0, y_0)=$(r, 0)$, and obtain

$$c_1(x, y) = 90(1 - 2\lambda) \cdot |x| \Rightarrow c_1(x_0, y_0) = 90(1 - 2\lambda) \cdot r$$
$$\psi(x, y) = y \Rightarrow \text{grad}\psi(x_0, y_0) = (0, 1)$$
$$f^+(x, y) = \begin{pmatrix} -x + (100 - 90\lambda)y \\ (90\lambda - 100)x - y \end{pmatrix} \Rightarrow f^+(x_0, y_0) = \begin{pmatrix} -r \\ (90\lambda - 100)r \end{pmatrix}$$
$$f^-(x, y) = \begin{pmatrix} -x + (100 - 90\lambda)y \\ -(90\lambda + 10)x - y \end{pmatrix} \Rightarrow f^-(x_0, y_0) = \begin{pmatrix} -r \\ -(90\lambda + 10)r \end{pmatrix}$$
$$\Rightarrow f^0(x_0, y_0) = \begin{pmatrix} -r \\ -(90\lambda + 10)r \end{pmatrix} + \alpha_1 \cdot \begin{pmatrix} 0 \\ (180\lambda - 90)r \end{pmatrix}, 0 \leq \alpha_1 \leq 1$$

and simplify this to $f^0(x_0, y_0) = \begin{pmatrix} -r \\ -(90\lambda + 10)r - 90\alpha_1(1 - 2\lambda)r \end{pmatrix}$

this gives $|\langle f^0(x_0, y_0), \text{grad}\psi(x_0, y_0)\rangle| = 10r(9\lambda + 1 + 9\alpha_1(1 - 2\lambda))$

and finally the integral becomes $I_1 = \dfrac{9(1 - 2\lambda)}{9\lambda + 1 + 9\alpha_1(1 - 2\lambda)}$.

We can in a similar way calculate the other three integrals and obtain

$$I_k = \frac{9(1 - 2\lambda)}{9\lambda + 1 + 9\alpha_k(1 - 2\lambda)}, \text{ where } k = 1, 2, 3, 4 \text{ and } 0 \leq \alpha_k \leq 1.$$

Then the Floquet-exponents becomes

$$\mu = -2 + \frac{1}{T} \cdot \sum_{k=1}^{4} \frac{9(1 - 2\lambda)}{9\lambda + 1 + 9\alpha_k(1 - 2\lambda)}$$

and the maximum respectively minimum of μ are

$$\mu_{max} = -2 + \frac{36(1-2\lambda)}{T(9\lambda+1)} \text{ when } \alpha_k = 0$$

$$\mu_{min} = -2 + \frac{36(1-2\lambda)}{T(10-9\lambda)} \text{ when } \alpha_k = 1$$

According to J. Melin [6] we have for $\lambda = 0.482544...$ (this value of λ implies the period $T = 0.114286...$) infinitely many closed orbits. The chosen values of λ and T gives $\mu \in [-0.056, 0.058]$ so the theorem does not tell us anything, which of course is expected.

Exampel 2 (Giannakopoulos-Pliete [4]). Consider the system introduced by Giannakopoulos-Pliete in 2001.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + y + b_1 \cdot \text{sgn}(x) \\ -p \cdot x + b_2 \cdot \text{sgn}(x) \end{pmatrix}, \text{ where } p > \frac{1}{4}, b_1 > 0 \text{ and } 0 < b_2 < b_{20}.$$

In the paper of J.Melin [6] we obtain $\text{div} f(x,y) = -1 + b_1 \cdot \delta(x)$ and the fact that it exists a limit cycle, of this system, for this parameter range. According to symmetry, a closed orbit passes through the following two discontinuity points: $(0,r)$ and $(0,-r)$, where $r > b_1$, see figure 2. The switching regions are $\Omega_1 : x > 0$ and $\Omega_2 : x < 0$. The discontinuity curve is represented by $\psi(x,y) = x$. Now we calculate the integrals I_1 and I_2 as we did in example 1 and the calculation gives:

$$I_k = \frac{2b_1}{r - b_1 + 2\alpha_k \cdot b_1}, \text{ where } k = 1,2 \text{ and } 0 \le \alpha_k \le 1,$$

and the Floquet-exponents are

$$\mu = -1 + \frac{2b_1}{T} \cdot \left(\frac{1}{r - b_1 + 2\alpha_1 b_1} + \frac{1}{r - b_1 + 2\alpha_2 b_1} \right)$$

and this gives

$$\mu_{max} = -1 + \frac{4b_1}{T(r-b_1)}, \text{ when } \alpha_1 = \alpha_2 = 0$$

$$\mu_{min} = -1 + \frac{4b_1}{T(r+b_1)}, \text{ when } \alpha_1 = \alpha_2 = 1.$$

It is not a trivial problem to estimate the parameters r and T, but we can always do this numerically. From a simulation we get: $r = 30000$, $T = 2$ and $b_1 = 8000$. This gives $\mu \in [-0.579, -0.273]$ and this means that the limit cycle is asymptotically stable and unique. The calculations can of course be repeated for other parameter values. This result of course coincides with the result of Giannakopoulos-Pliete [4].

Exampel 3 (J.Melin-A.Hultgren [7]). A resonant converter can, in the two dimensional case, be modelled as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{y}{C} \\ -\dfrac{x + Ry - u_i}{L} \end{pmatrix}, \text{ where } R > 0, L > 0, C > 0 \text{ and } \frac{L}{C} > \frac{R^2}{4}.$$

The switching regions are

$\Omega_1 : x^2 + y^2 < i_r^2, y > 0$
$\Omega_2 : x^2 + y^2 < i_r^2, y < 0$
$\Omega_3 : x^2 + y^2 > i_r^2, y > 0$
$\Omega_4 : x^2 + y^2 > i_r^2, y < 0$

and the discontinuity curves are represented by $\psi_1(x, y) = y$ and $\psi_2(x, y) = x^2+y^2-i_r^2$. We have six discontinuity points symmetrically located at the x-axis and the circle, see figure 3. The bifurcation parameters U_0 and $E - U_0$ satisfies in this example $U_0 < i_r$ and $E - U_0 > i_r$. The control parameters u_i takes the values of $u_1 = E - U_0$, $u_2 = U_0 - E$, $u_3 = -U_0$ and $u_4 = U_0$ in their respectively switching regions Ω_i. It is shown in the paper of Melin-Hultgren [7] that this system has an unique limit cycle in the special case $R = 0$. In fact there exists a limit cycle for $R > 0$, but we will not show it in this paper. Instead we will show type of stability and uniqueness of the limit cycle. The divergence of the righthand side f in sense of distribution theory is

$$\text{div} f(x, y) = -\frac{R}{L} + \frac{2}{L} \cdot (E - U_0 - E \cdot H(x^2 - i_r^2)) \cdot \delta(y) - \frac{2E}{L} \cdot |y| \cdot \delta(x^2 + y^2 - i_r^2).$$

The divergence changes signs and therefore there is a necessary condition for existence of limit cycles. Now just consider the two discontinuity points at the x-axis, then $\text{div} f(x, y) = -\frac{R}{L} - \frac{2U_0}{L} \cdot \delta(y)$ so the coefficients $c_i(x, y)$ are negative. If we just consider the four discontinuity points at the circle we have $\text{div} f(x, y) = -\frac{R}{L} - \frac{2E}{L} \cdot |y| \cdot \delta(x^2 + y^2 - i_r^2)$, so even in this case the coefficients $c_i(x, y)$ are negative. That means that the integrals I_k are negative and therefore the Floquet-exponents are negative and a limit cycle of this type is asymptotically stable and unique.

5. SUMMARY

In this paper we have suggested an extension of the Floquet-exponent theory that can be applied to many systems of ordinary differential equations with discontinuous righthand sides. Our method yields possibilities for estimating the maximum number of limit cycles in such systems and involves calculation with distributions. We have excluded some difficult discontinuities from our framework but claim that most systems appearing in the applications can be analyzed by the method presented. We analyzed examples that have appeared in the literature in the light of our results. Our first example was due to Branicky [1] and has strictly negative divergence in classical sense. Yet, this system possess infinitely many closed orbits for a specified parameter value. The divergence in distribution sense integrated over a closed orbit in this system is an closed interval containing zero, which explains the possibility for infinitely many closed orbits. We continue by the example of Giannakopoulos-Pliete [4] and this example shows how extensive explicit calculations can be reduced considerably using our new method. We end up with an example by Melin-Hultgren [7] where explicit calculations of the trajectories is difficult to do and our method yields information of the presence of a unique limit cycle.

6. ACKNOWLEDGEMENTS

I had the benefit of many helpful suggestions in writing this text and I like to thank Associate professor Torsten Lindström and Senior Lecturer Anders Hultgren of the University of Kalmar for their help.

REFERENCES

[1] M. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE transactions on automatic control*, 43(4):475–482, 1998.

[2] L. Cherkas and L. Zhilevich. Some criteria for the absence of limit cycles and for the existence of a single cycle. *Differential Equations*, 6:891–897, 1970.

[3] A. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publisher, 1988.

[4] F. Giannakopoulos and K. Pliete. Planar systems of piecewise linear differential equations with a line of discontinuity. *Nonlinearity*, (14):1611–1632, 2001.

[5] L. Hörmander. *The analysis of Linear partial differential operators I*. Springer, 1990.

[6] J. Melin. Does distribution theory contain means for extending poincaré-bendixson theory?, 2003.

[7] J. Melin and A. Hultgren. A limit cycle of a resonant converter. In H. G. S. Engell and J. Zaytoon, editors, *Analysis and design of hybrid systems 2003*, pages 169–174. IFAC, Elsevier, 2003.

[8] R.Grimshaw. *Nonlinear Ordinary Differential Equations*. CRC Press, 1993.

[9] Ye Y.-Q. et al. *Theory of Limit Cycles*. American Mathematical Society, second edition, 1986.

[10] Zhang Z.-F. Proof of the uniqueness theorem of limit cycles of generelized Liénard equations. *Applicable Analysis*, 23:63–76, 1986.

[11] Zhang Z.-F. *Qualitative theory of differential equations*. Providence American Mathematical Society, 1992.

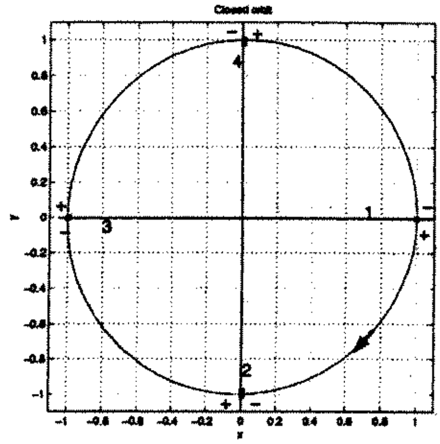

FIGURE 1. closed orbit ex.1

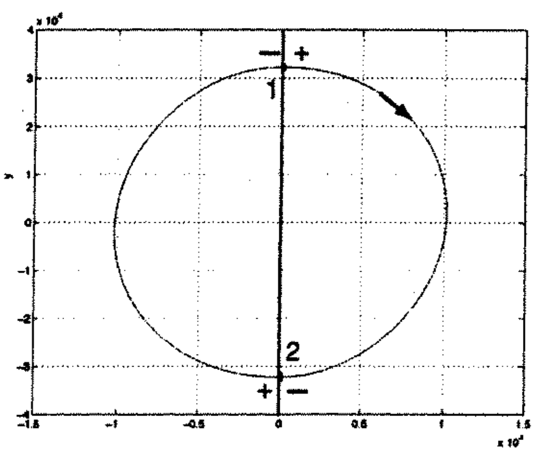

FIGURE 2. limit cycle ex.2

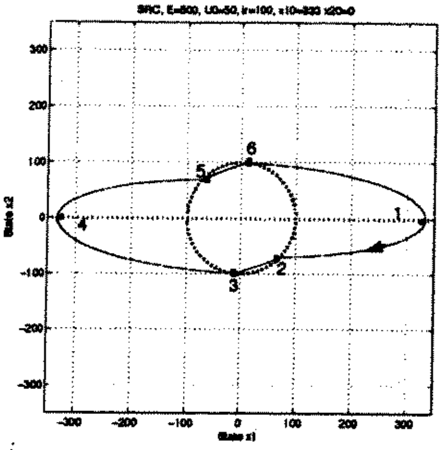

FIGURE 3. limit cycle ex.3