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ON ADAPTIVE CONTROL OF LOW ORDER SYSTEMS

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LUND INSTITUTE OF TECHNOLOGY
DIVISION OF AUTOMATIC CONTROL

ON ADAPTIVE CONTROL OF LOW ORDER SYSTEMS †

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ABSTRACT

In this report we study the behaviour of different adaptive regulators, derived via the theory of stochastic control. The regulators are used on first and second order systems. Phenomena, due to the adaptive control, will occur. These phenomena can be eliminated by using an optimal control law, derived through Dynamic Programming. The optimal control law is derived for first order systems. When obtained the structure of the optimal control law a much simpler suboptimal control law can be derived. The suboptimal control law can be generalized to higher order systems. Results from simulations are given to illustrate the behaviour of the different regulators.

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APPENDIX : Computational aspects upon the Dynamic Programming equations

1. INTRODUCTION

There are many ways to approach the adaptive control problems. In literature there exist a great flora of suggested solutions to a wide range of processes. See e.g. [4]. Many of the solutions can be reduced to a scheme, illustrated by figure 1.1.

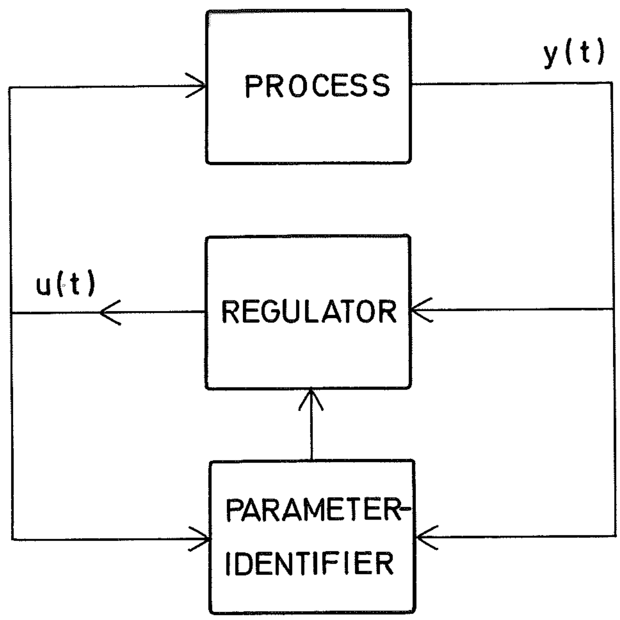


Fig. 1.1: A scheme for adaptive control system

It is assumed, that the process has time varying or unknown parameters. These parameters are calculated from input and output data and fed into the regulator. The regulator computes the input signal to the process, based upon the estimated parameter values instead of the real ones. The regulator is often designed as if the parameters were known exactly.

When using this scheme we get a control, which is nonlinear. The nonlinearity comes from the parameter identifier, which e.g. can be a Kalman filter, [3], [7].

One essential question is now, how the quality of the estimation affects upon the regulation. If the estimate is poor is it then necessary to take account to the uncertainty of the estimated parameter values? This means, can the separation of the problem, as demonstrated in figure 1.1, be justified? Further, must the regulator be redesigned because of the identification of the parameters? It will be shown in examples that the controller sometimes has to be redesigned to make a smoother control because of the uncertainty of the identified parameters.

Another important and animated discussed problem is, whether it is necessary to apply perturbation inputs to the system in order to get better identification or if the normal control signals are sufficient. For the pure identification problem there is a common opinion that perturbation signals ought to be introduced.

A phenomenon, associated with adaptive control, is the "burst phenomenon". This is well known to workers in the field, but has unfortunately not been published. It has e.g. been pointed out by Åström in unpublished notes [5]. If an adaptive system is started with poor estimates the system makes a poor control, but the estimates will improve quickly. Thus, the system starts to control good and has small errors. In that case the real parameters of the process can start drifting, and a slow parameter estimator may not detect this from the small error signal, until there is a large difference between the real and estimated parameters. Then the system will be controlled poor, and the errors begin to grow, but then it will be possible to get better estimates, and the errors become small again. The error signal of such a system will thus consist of parts with small amplitudes and parts with large oscillations.

The phenomenon can be influenced in many ways. One is to introduce a perturbation signal, which shall assure good parameter estimation. In this case an additional error will be introduced, and one have to decide between additional error and the rates of the bursts.

The mentioned problems will be discussed further in the report.

The approach to adaptive control, used in this report, goes via the theory of stochastic optimal control. By assuming, that the parameters are stochastic processes, the problem can be formulated as a stochastic optimal control problem. This has been discussed in [6].

The problem now becomes to derive recursive equations for the conditional probability distributions and to solve a deterministic control problem. The deterministic control problem becomes one of controlling a system, described by nonlinear partial differential equations. There are, however, special cases where the probability distributions can be characterized by a reasonable number of parameters and when a solution might be found. This is the case with linear systems, having a quadratic loss function. Then the conditional probability distribution is characterized by the mean and covariance functions.

The purpose with this report is to treat simple examples of adaptive control. But the examples are sufficiently complicated to demonstrate many interesting features of adaptive systems. Furthermore, when the simple case is solved and the structure of the regulator is developed, the results can be generalized to more complex systems.

The background and the statement of the problem is discussed in section 2, where also the performance index is discussed, i.e. how to measure the behaviour of the system.

In section 3 we will from the discussion around figure 1.1 derive a heuristic control law and discuss its validity. This control law will then be expanded in section 4, where the problem is discussed from a statistical point of view. This will lead to a control law, which takes account to the uncertainty of the parameter estimates. By using this, we will get a better performance, but in some cases the special features of adaptive control will turn up. The "burst phenomenon" occurs together with a related phenomenon, which we can call the "turn-off phenomenon". Briefly, this means that the regulator turns off for periods of time and after a while turns on and begins to control in a proper way again. This is discussed together with simulation results for a second order system in section 5.

In order to overcome these phenomena a more complex performance index is considered for a first order system. The derivation of the control law is done in section 6, using the technique of Dynamic Programming.

Results from simulation of first order system are discussed in section 7. The optimal control law has a two-fold action. First it takes account to the variance of the estimated parameters, as the control law derived in section 4. Secondly, which is very important, it makes control in a way to get better estimates, i.e. to get a smaller variance of the parameters. A controller of this kind is called a dual controller. The contrary controller as in sections 3 and 4 is called nondual controller.

Some computational aspects upon the derivation of the optimal control law are discussed in appendix.

Briefly, the main results of the report can be summarized into the following points:

- The nondual controller, which does not take account to the uncertainty of the estimates, has serious limitations.
- The nondual controller, which takes account to the uncertainty of the estimates, has inspite of its simplicity good performance in many cases.
- Turn off and burst phenomena can occur when using nondual controllers.
- A dual control law can be derived, which eliminates the turn off phenomena. The dual controller has a two-fold action and consists of one error correcting and one information sensing part.
- A suboptimal dual controller can be derived when the structure of the optimal control law is known. The suboptimal control law consists of two parts, first the same nondual control law, which takes account to the uncertainty of the estimates, and second a perturbation signal. Thus in order to eliminate the turn off phenomenon can the introduction of a perturbation signal be justified.

The author wishes to express his acknowledgements to Professor K.J. Åström for pointing out the problem and for the stimulating discussions during the work with this report.

2. STATEMENT OF THE PROBLEM

When interesting in input-output performance for discrete system many processes can be written in the following form:

$$y(t) + a_1y(t-1) + \ldots + a_ny(t-n) = b_0u(t) + \ldots + b_nu(t-n) + e(t) \tag{2.1}$$

where

$y(t)$	—	output
$u(t)$	—	control signal
$e(t)$	—	normal white noise

Notice the simplification, that it is assumed, that the driving noise is white.

If the parameters a_i and b_i are known, there are many ways of handling such systems, e.g. minimum variance strategies [7]. But if the parameters are time-varying in an unknown way we can get into difficulties. E.g. if the gain is varying over a wide range and especially if it changes sign.

In order to attack these problems we rewrite equation (2.1):

$$y(t) = - a_1y(t-1) - \ldots - a_ny(t-n) + b_0u(t) + \ldots + b_nu(t-n) + e(t)$$

and introduce the vectors x and θ, defined by:

$$x^T = [a_1 \ldots a_n\ b_0 \ldots b_n]$$

and

$$\theta = [- y(t-1) \ldots - y(t-n)\ u(t) \ldots u(t-n)]$$

Then (2.1) becomes:

$$y(t) = \theta(t)\ x(t) + e(t)$$

Assume that the parameters are linear stochastic processes. Then the time dependence of $x(t)$ can be written as:

$$x(t+1) = \phi\, x(t) + v(t)$$

where $\{v(t)\}$ is a sequence of gaussian random variables.

It is not obvious how to get the statistical properties of the parameters e.g. how to get the matrix ϕ and the variance of $v(t)$. But if we overlook this difficulty we now have the problem in a very attractive form:

$$\begin{cases} x(t+1) = \phi x(t) + v(t) \\ y(t) = \theta(t)\, x(t) + e(t) \end{cases} \tag{2.2}$$

where $\{v(t)\}$ and $\{e(t)\}$ are sequences of independent gaussian random variables with the properties:

$$E\ v(t) = E\ e(t) = 0$$
$$E\ v(t)\ v(t)^T = R_1$$
$$E\ e(t)\ e(t)^T = R_2$$
$$E\ v(t)\ e(t)^T = 0$$

where E denotes mathematical expectation. Further $\theta(t)$ is a linear function of known variables $y(t-1)\ \ldots\ y(t-n)$, $u(t)\ \ldots$ $\ldots\ u(t-n)$.

This structure will cover a wide range of processes, but this report will be limited to first and second order systems, where $\theta(t)$ is a function of $u(t)$.

The discussed systems are simple, but yet many interesting features of adaptive control systems can be demonstrated. The results can be transferred to the more general problems, outlined above.

In the verysimple case $n = 1$ we have:

$$\begin{cases} x(t+1) = ax(t) + v(t) \\ y(t) = u(t)\ x(t) + e(t) \end{cases}$$

This can be interpreted as a system with time-varying gain with the output corrumpted by white noise (see fig. 2.1).

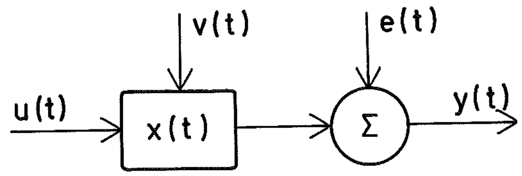


Fig. 2.1

To complete the statement of the problem we have to decide upon how the behaviour of the system shall be measured. There are many ways of specifying the performance index of a system. The complexity of the control law will depend on this choice. Therefore, we will use a couple of performance indices to be able to derive control laws of different complexity.

First in section 4 we will minimize at each time step, one stage control

$$\ell^1 = E\left(k + \theta(t)\ x(t)\right)^2 \tag{2.3}$$

where E denotes mathematical expectation.

Second in section 6 we choose to minimize

$$\ell = E \sum_{s=1}^{N} \left(k + \theta(s)\ x(s)\right)^2 \tag{2.4}$$

This is called N stage control.

These two performance indices can seem to be equal. But there is a fundamental and very important difference between them. The first case (2.3) will give a control law, which makes the best thing in a given situation, but it does not make any attempt to get a better situation, i.e. to get better estimates.

Using (2.4) will give a control law which does not look at only one time step. The difference depends on when using (2.4) a greater loss in the beginning can be accepted in order to get better identification of the parameters and then be able to make a better control.

To summarize we have the following:

Problem

Given the system:

$$\begin{cases} x(t+1) = \phi x(t) + v(t) \\ y(t) = \theta\big(u(t)\big)\ x(t) + e(t) \end{cases} \qquad (2.2)$$

where

- the order of system is equal to 1 or 2.
- $\{v(t)\}$ and $\{e(t)\}$ are independent sequences of normal random variables with mean equal to zero and the variance matrices R_1 respectively R_2.
- $\theta\big(u(t)\big)$ is a linear function of $u(t)$.

Then find a control law $u(t)$ as a function of $u(t-1)$, $u(t-2)$, ..., $y(t-1)$, $y(t-2)$, ..., which minimizes either of the performance indices

$$\ell^1 = E\Big[k + \theta\big(u(t)\big)\ x(t)\Big]^2 \qquad (2.3)$$

or

$$\ell = E\sum_{s=1}^{N}\Big[k + \theta\big(u(s)\big)\ x(s)\Big]^2 \qquad (2.4)$$

3. HEURISTIC APPROACH

In section 1 we pointed out that adaptive control system often was divided in two parts. First calculation of the parameters and second design of the controller, as if the parameters were known (see fig. 1.1). In this section we will use this argument to derive a control law for the problem, given in section 2.

In this part we will assume a special system of second order, but the arguments are also valid for other systems.

Let the system be:

$$\begin{cases} x(t+1) = \Phi x(t) + v(t) \\ y(t) = x_1(t) + u(t)\, x_2(t) \end{cases} \tag{3.1}$$

and the performance index is to minimize the square of the output signal.

If the state variables were known $y(t)$ would become equal to zero by choosing the control law:

$$u(t) = -\frac{x_1(t)}{x_2(t)} \tag{3.2}$$

But in system (3.1) it is only the output $y(t)$ which is measurable thus we have to estimate the state variables from measurements of the output signal. With the given structure of the problem the estimation can be done by using the Kalman filtering theory (see e.g. Åström [7] or Kalman [3]).

Let $\hat{x}(t+1|t)$ denote the estimated state vector based upon measurements $y(t)$, $y(t-1)$, ... and $P(t)$ the variance matrix of the estimation error, $x(t) - \hat{x}(t|t-1)$. Then we have the following recursive equations:

$$\begin{cases} \hat{x}(t+1|t) = \phi\hat{x}(t|t-1) + K(t)\left(y(t) - \theta\hat{x}(t|t-1)\right) \\ K(t) = \phi P(t)\, \theta^T \left(\theta P(t)\, \theta^T + R_2\right)^{-1} \\ P(t+1) = \phi P(t)\, \phi^T + R_1 - \phi P(t)\, \theta^T \left(\theta P(t)\, \theta^T + R_2\right)^{-1} \theta P(t)\, \phi^T \end{cases} \tag{3.3}$$

When using the estimates instead of the real values, the control law becomes:

$$u(t) = -\frac{\hat{x}_1(t|t-1)}{\hat{x}_2(t|t-1)} \tag{3.4}$$

We see that if the state variable $x_2(t)$ is small the quality of the estimate can have great influence on the regulation. But if $x_2(t)$ is large then the uncertainty of the estimate does not have the same importance. This will be examplified through simulation, and the results are discussed in section 5.

4. A STOCHASTIC CONTROL FORMULATION OF THE PROBLEM

We now consider the system:

$$\begin{cases} x(t+1) = \phi x(t) + v(t) \\ y(t) = \theta x(t) \end{cases} \tag{4.1}$$

where $\theta(t) = [1 \; u(t)]$ and $\{v(t)\}$ are a sequence of independent stochastic variables with zero mean and covariance matrix R_1.

Let the loss function be:

$$\ell^1(t) = E\left(x_1(t) + x_2(t)u(t)\right)^2 = E\left(\theta(t)x(t)\right)^2 \tag{4.2}$$

The problem will now be formulated and solved as a stochastic control problem. The solution can easily be transfered to other systems of the same class.

Introduce:

$$\mathcal{Y}_t = [y(t) \; y(t-1) \; \dots \; y(t_0)]^T$$

i.e. the sequence of output values up to time t.

Rewrite (4.2)

$$\ell^1(t) = E\big(\theta(t)x(t)\big)^2$$

$$= E\Big[E\{(\theta x)^2 | \mathcal{Y}_{t-1}\}\Big] \tag{4.2'}$$

where $E(\cdot|y)$ denotes conditional expectation given y.

To minimize (4.2') with respect to u(t) is equivalent to minimize:

$$V^1(t) = E\{(\theta x)^2 | \mathcal{Y}_{t-1}\} \tag{4.3}$$

with respect to u(t). This is true, because $\mathcal{Y}_{t-1}$ is independent of u(t).

As the system fulfils the conditions of the Kalman filtering theory (4.3) can be written as:

$$V^1(t) = \big(\theta(t)\,\hat{x}(t|t-1)\big)^2 + \theta(t)P(t)\theta(t)^T$$

$$= \big[\hat{x}_1 + u\hat{x}_2\big]^2 + p_{11} + 2up_{12} + u^2p_{22}$$

where $x(t|t-1)$ and P(t) satisfy the Kalman equations given by (3.3) in section 3.

Differentiate $V^1(t)$ with respect to u(t)

$$\frac{\partial V^1}{\partial u} = 2\hat{x}_2(\hat{x}_1 + u\hat{x}_2) + 2p_{12} + 2up_{22}$$

This will give minima of $V^1(t)$ for

$$u(t) = -\frac{\hat{x}_1(t|t-1)\,\hat{x}_2(t|t-1) + p_{12}(t)}{\hat{x}_2(t|t-1)^2 + p_{22}(t)} \tag{4.4}$$

If the estimates are exact, P(t) = 0, are (4.4) reduced to:

$$u(t) = -\frac{\hat{x}_1(t|t-1)}{\hat{x}_2(t|t-1)}$$

which is the same control law as (3.4), derived in section 3. Thus, if the estimates are good the two control laws will have about the same behaviour. But when controlling according to (4.4) when the elements in the variance matrix are large we pay regard to this and make a smoother control, and that can give a considerable reduction of loss. But yet is the complexity of (4.4) not greater than of (3.4), because the variance matrix has to be calculated in both cases, because it is the same state estimator. But there are disadvantages with (4.4) too. The control law does just handle to minimize the expected error, but does not make any attempt to get better estimates of the state variables. This type of control is called nondual.

The behaviour of the control law is discussed in the next section.

5. SIMULATION OF SECOND ORDER SYSTEM

In this section we will discuss the results from simulation of second order systems using the control laws derived in sections 3 and 4. As before the system is:

$$\begin{cases} x(t+1) = \phi x(t) + v(t) \\ y(t) = x_1(t) + u(t)\, x_2(t) \end{cases} \qquad (5.1) = (3.1)$$

where $E\, v(t)v(t)^T = R_1$.

The used control laws are

$$u(t) = -\frac{\hat{x}_1(t|t-1)}{\hat{x}_2(t|t-1)} \qquad (5.2) = (3.4)$$

and

$$u(t) = -\frac{\hat{x}_1(t|t-1)\,\hat{x}_2(t|t-1) + p_{12}(t)}{\hat{x}_2(t|t-1)^2 + p_{22}(t)} \qquad (5.3) = (4.4)$$

where $x_i(t|t-1)$ and p_{ij} are given by the Kalman equations (3.3). To evaluate the performance of the system we use the loss function:

$$V(t) = \sum_{n=1}^{t} y(n)^2 \tag{5.4}$$

If one of the state variables $x_i(t)$ is known exactly it can be shown that:

$$E\{V(t)\} = t \cdot r_{ii}$$

where r_{ii} is the ith diagonal element in the variance matrix R_1.

This can be used to compare the behaviour of the system for different control laws.

Example 5.1

Let the system be characterized by the matrixes:

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_1 = \begin{bmatrix} 0{,}01 & 0 \\ 0 & 0{,}01 \end{bmatrix}$$

ϕ equal to the unit matrix implies that the state variables are not coupled. Further the state variables are pure random walk processes.

The result of a simulation is shown in figure 5.1. After the transients in the beginning there is no major difference between the control laws.

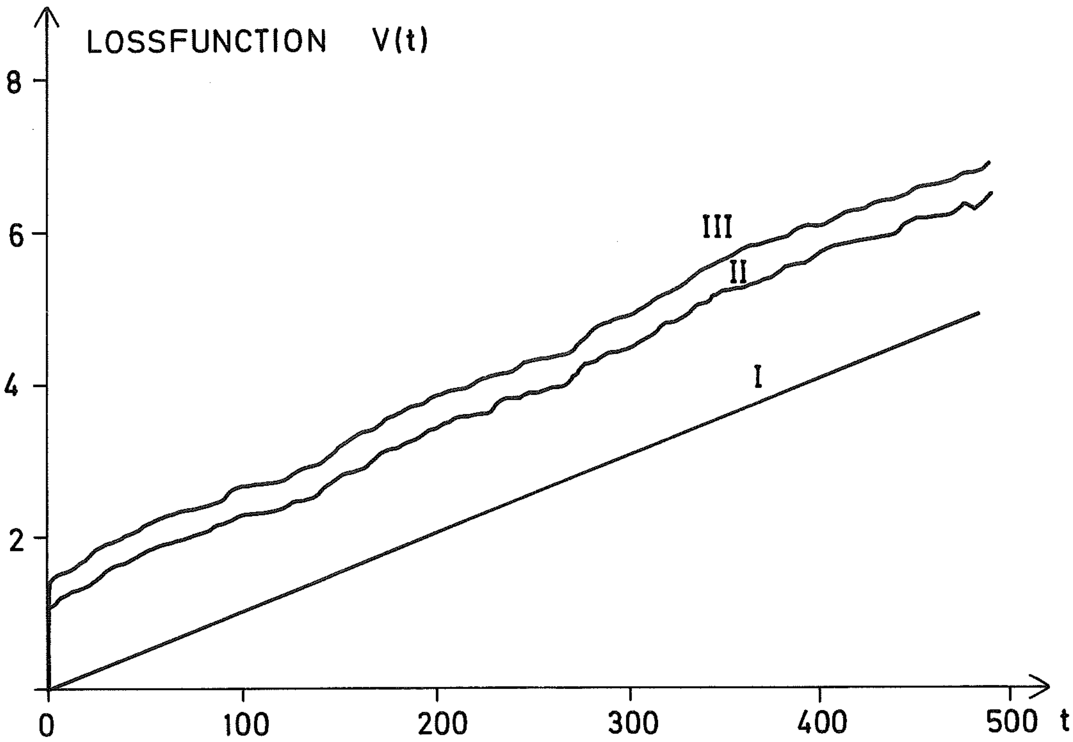


Fig. 5.1: Loss function for different control laws.

I Expected loss when one state variable is known exactly.

II Loss when using $u(t) = - \dfrac{\hat{x}_1(t|t-1)}{\hat{x}_2(t|t-1)}$

III Loss when using $u(t) = - \dfrac{\hat{x}_1(t|t-1)\ \hat{x}_2(t|t-1) + p_{12}(t)}{\hat{x}_2(t|t-1)^2 + p_{22}(t)}$

The small differences are better seen if the influence of the initial values are eliminated. This is done in table 5.1 by looking at the differences $V(t) - V(10)$ for the same cases as in figure 5.1.

t	V(t) - V(10) I One known state variable	II $u(t) = -\frac{\hat{x}_1}{\hat{x}_2}$	III $u(t) = -\frac{\hat{x}_1\hat{x}_2+p_{12}}{\hat{x}_2^2+p_{22}}$
100	0.9	1.04	1.01
200	1.9	2.24	2.20
300	2.9	3.24	3.21
400	3.9	4.44	4.40
500	4.9	5.37	5.34

Table 5.1

The equal performance of regulator (5.2) and (5.3) is due to the fact that the state variables are random walk processes. A characteristic feature for random walk processes is the long time between times when the variable changes signs. Further the elements in the R_1 matrix are small, which implies that the changes of the state variables are small between each time step and thus it is easy for the estimator to track the state variables.

Example 5.2

This example will show the sensitivity of the regulator (5.3) when the state variables are small.

Let the state variables have the deterministic values:

$$\begin{cases} x_1(t) = 1 \\ x_2(t) = \sin 0.05\ t \end{cases}$$

and put

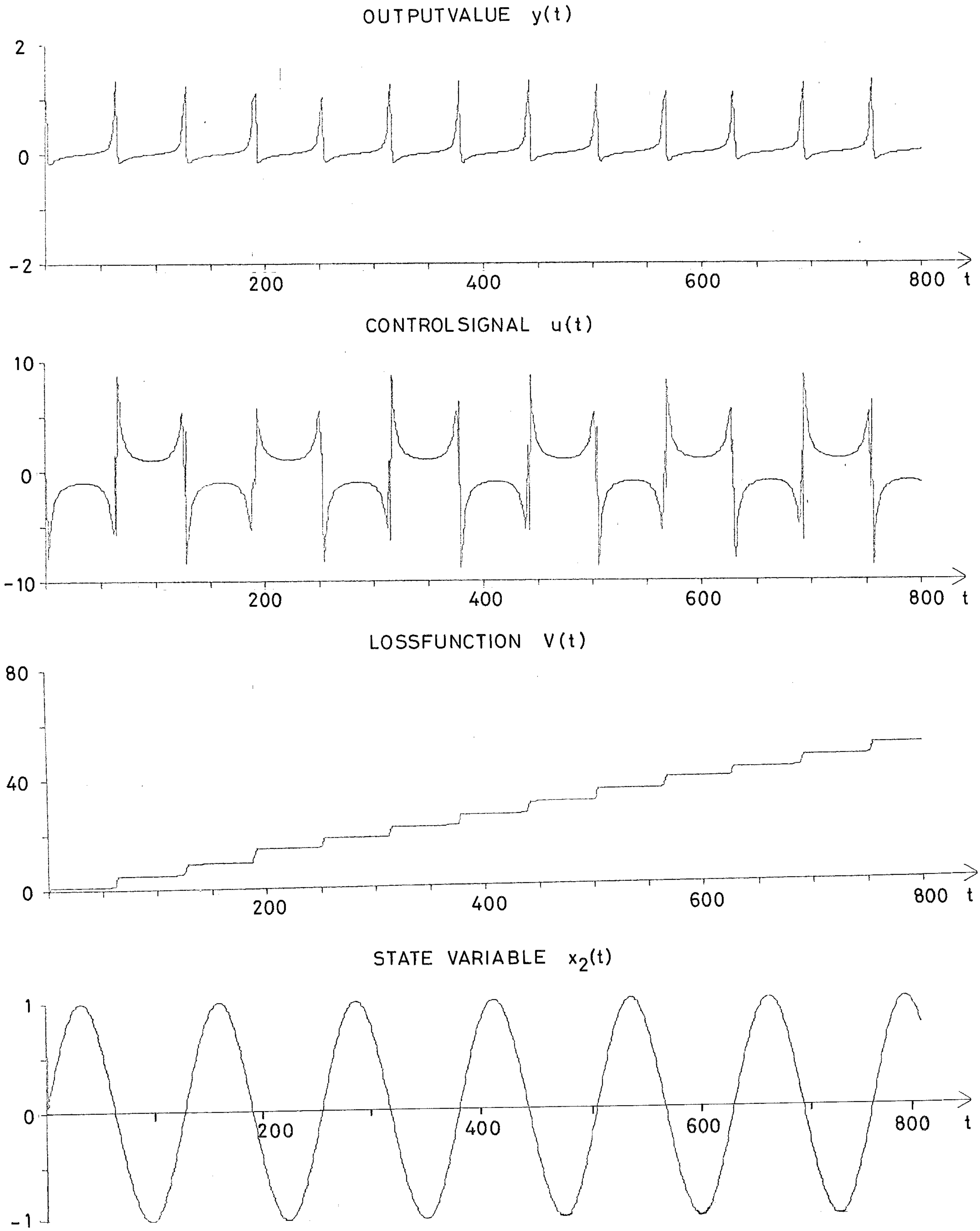
$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 10^{-3} & 0 \\ 0 & 3 \cdot 10^{-3} \end{pmatrix}$$

in the Kalman equations.

When using (5.3) the results can be seen in figure 5.2.

Even if we consider the uncertainty of the estimates we see that the performance changes drasticly when $x_2(t)$ is near zero.

Looking at the regulator we see that the system is not so sensitive for small values of $x_1(t)$ because $\hat{x}_1(t|t-1)$ only appears in the numerator.



<u>Fig. 5.2</u>

$$\begin{cases} x_1(t) = 1 \\ x_2(t) = \sin 0.05\, t \end{cases} \qquad \phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 10^{-3} & 0 \\ 0 & 3\cdot 10^{-3} \end{pmatrix}$$

$$u(t) = -\frac{\hat{x}_1 \hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}}$$

Example 5.3

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This example will show the dependence of assumed variance matrix R_1 versus actual variance matrix R_1^0 when using:

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^2 + p_{22}} \tag{5.3}$$

From table 5.2 we see that the regulator is not specially sensitive of R_1 if it is chosen too small. But if R_1 is chosen too large the estimator will become very sluggish. Further the elements in the variance matrix P will be large and thus have the major influence upon the control signal u(t).

	V(t) - V(50)			
	Assumed R_1			
t	0,01I	0,1I	I	10I
100	79	75	63	1807
200	243	239	224	11997
500	624	619	602	33947

Table 5.2: Loss function for the system

$$\begin{cases} x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + v(t) \\ y(t) = x_1(t) + u(t)\, x_2(t) \end{cases}$$

where $R_1^0 = E\, v(t)v(t)^T = I$ using the control law

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^2 + p_{22}}$$

for different values of the variance matrix R_1 in the Kalman equations (3.3).

Phenomena, occuring when chosen the variance matrix R_1 too large, will be discussed further in example 5.5.

Example 5.4

$$\phi = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix} \qquad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This is a process where $x_1(t)$ is a random walk process and $x_2(t)$ is a stochastic variable which changes signs relatively often. The rate of change in the state variables is greater than in example 5.1.

In figure 5.3 the loss functions are shown for different control laws.

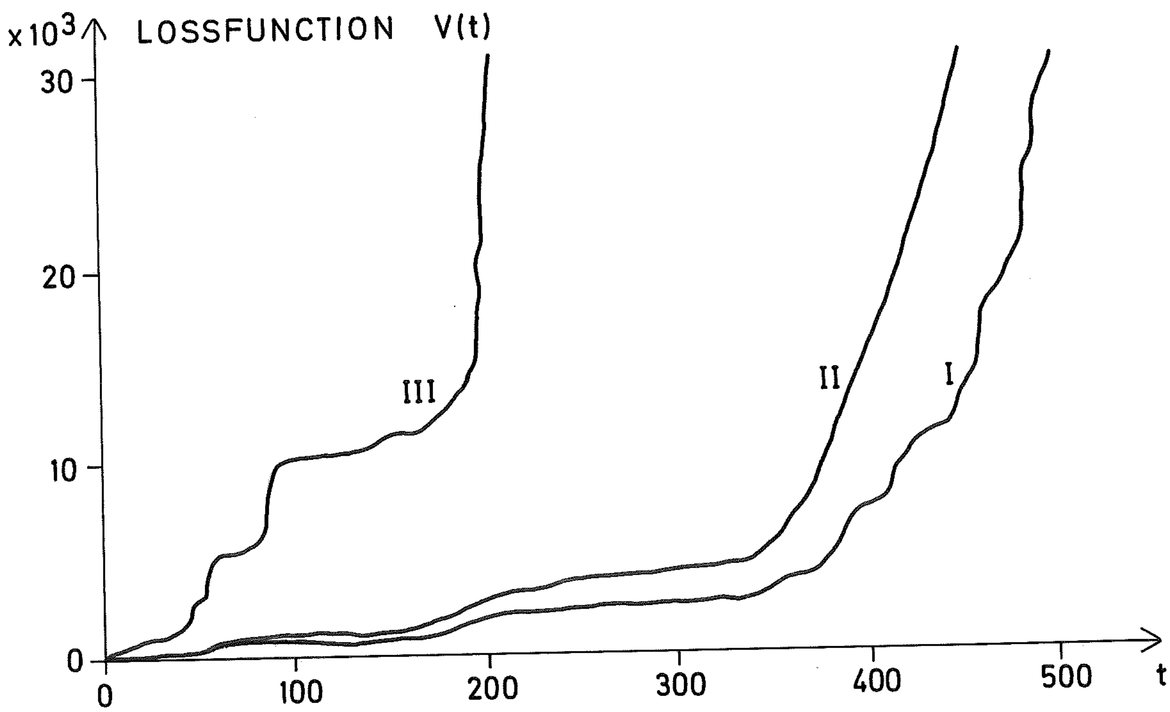


Fig. 5.3: Loss functions for the system

$$\begin{cases} x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix} x(t) + v(t) \\ y(t) = x_1(t) + u(t)\, x_2(t) \end{cases}$$

for different control laws.

Fig. 5.3 Contd.

$$\text{I} \qquad u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^2 + p_{22}}$$

$$\text{II} \qquad u(t) = 0$$

$$\text{III} \qquad u(t) = -\frac{\hat{x}_1}{\hat{x}_2}$$

As previously mentioned the control law

$$u(t) = -\frac{\hat{x}_1}{\hat{x}_2}$$

makes a poor regulation when $x_2(t)$ often is near zero. In this example it even gives a much greater loss than no control at all (curve II in fig. 5.3).

The difference between (5.2) and (5.3) is further demonstrated when looking at figures 5.4 and 5.5, where the output signals are drawn in the same scale for the two control laws.

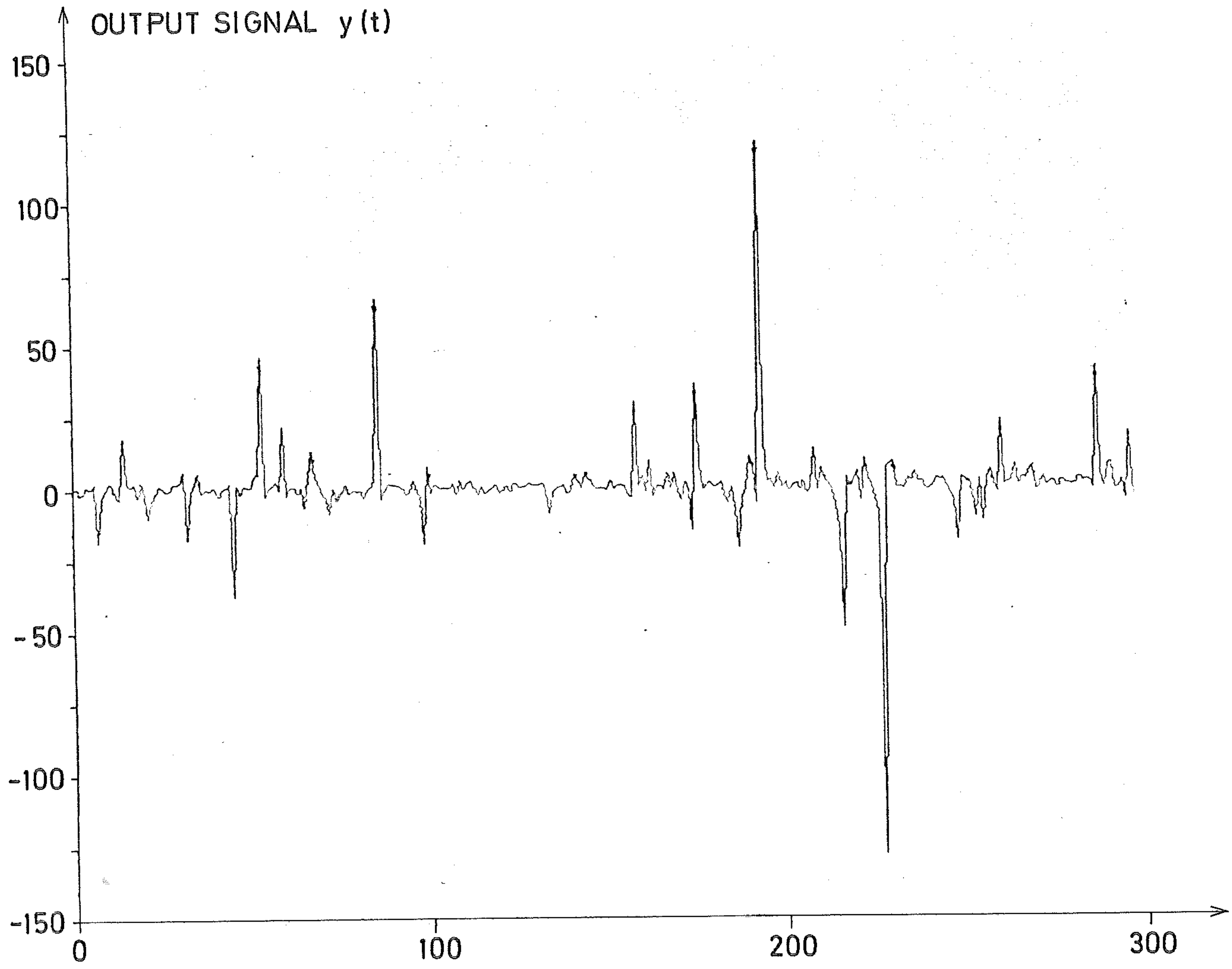


Fig. 5.4 - Output signal for exemple 5.4 using

$$u(t) = -\frac{\hat{x}_1}{\hat{x}_2}$$

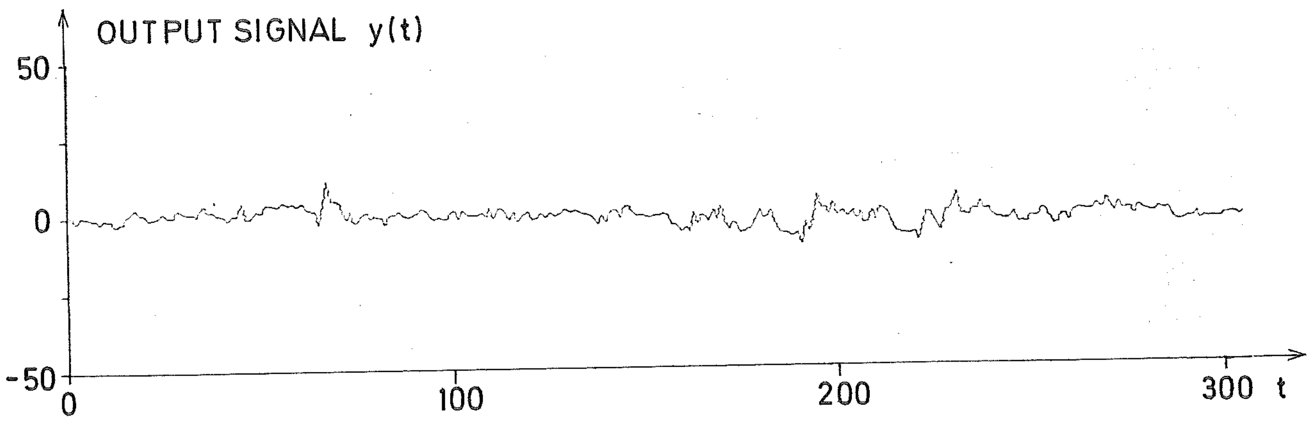


Fig. 5.5 - Output signal for exemple 5.4 using

$$u(t) = -\frac{\hat{x}_1 \hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}}$$

Example 5.5

As mentioned in example 5.3 phenomenon can occur when chosen the variance matrix R_1 too large in comparison with the actual value. This will be called the "turn-off" phenomenon.

Take the system:

$$\begin{cases} x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 0.9 \end{pmatrix} x(t) + v(t) \\ \\ y(t) = x_1(t) + u(t)x_2(t) \end{cases}$$

with $R_1^0 = I$ and use $R_1 = 10I$ in the Kalman equations and the control law

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^2 + p_{22}}$$

The result of a simulation is shown in figure 5.6.

The control signal is zero or almost zero for about 620 time steps, and this implies that the output only contains information about $x_1(t)$, and thus the best estimation of $x_2(t)$ is $\hat{x}_2(t) = 0$.

After about 620 time steps the regulator starts controlling again in a proper way. The improvement can be seen in the loss function, where the dashed curve is the loss function for u(t) equal to zero.

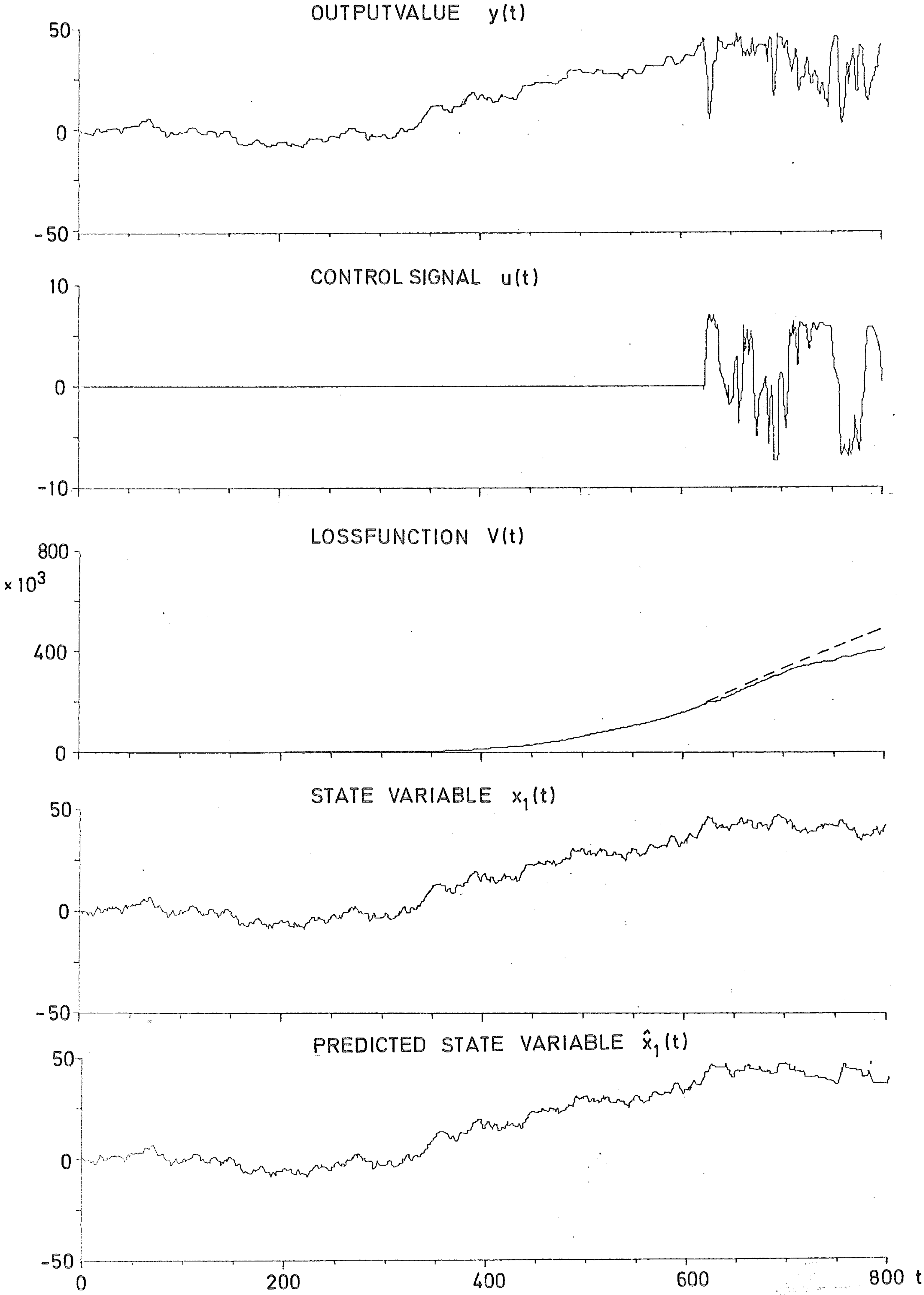


Fig. 5.6 - Simulation results from example 5.5 showing the turn-off phenomena

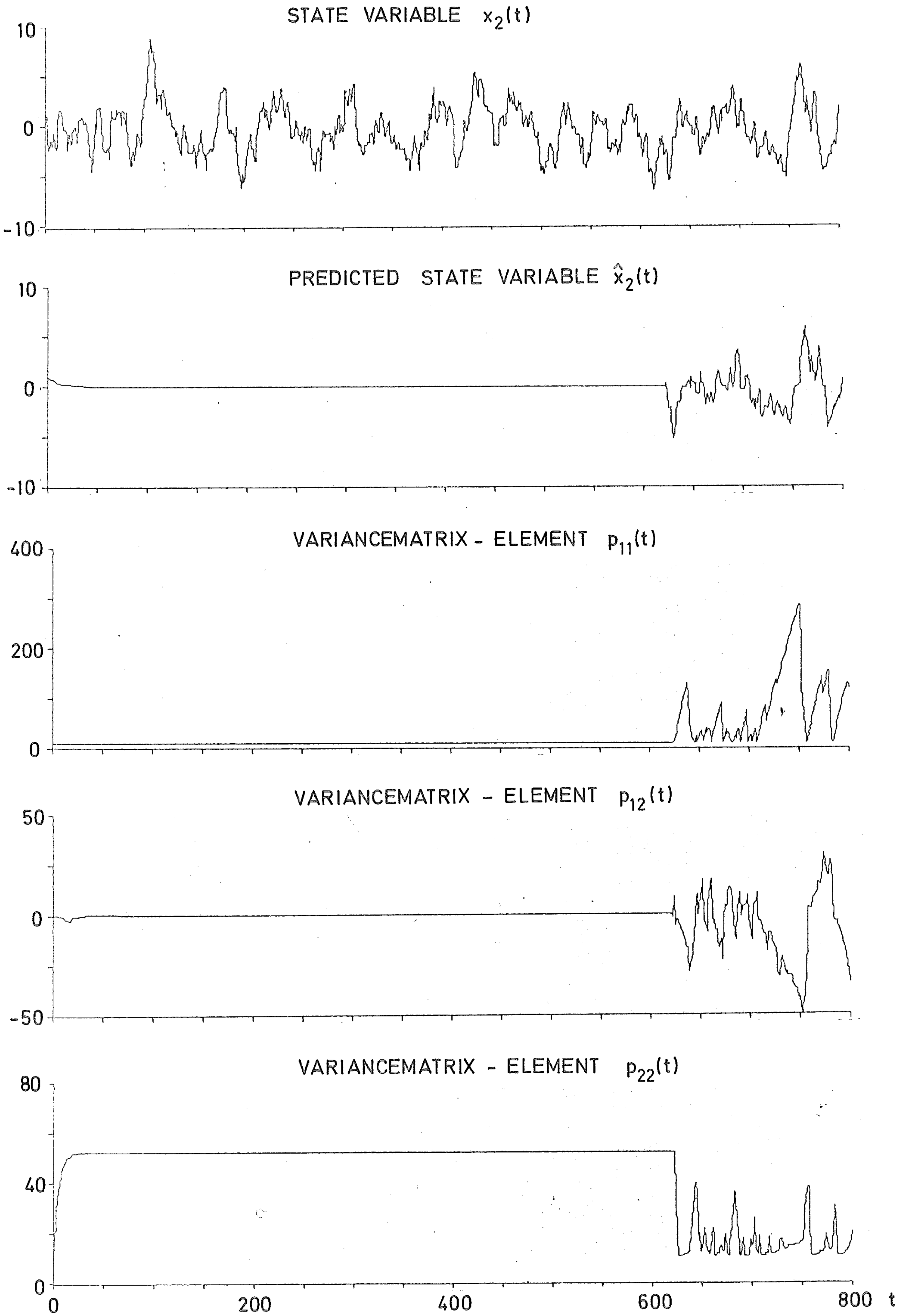

STATE VARIABLE $x_2(t)$
10
0
-10
PREDICTED STATE VARIABLE $\hat{x}_2(t)$
10
0
-10
VARIANCEMATRIX - ELEMENT $p_{11}(t)$
400
200
0
VARIANCEMATRIX - ELEMENT $p_{12}(t)$
50
0
-50
VARIANCEMATRIX - ELEMENT $p_{22}(t)$
80
40
0
0
200
400
600
800
t

Fig. 5.6 - Contd.

Example 5.6

In example 5.5 it was shown, that turn-off phenomena can occur when R_1 was chosen too large. But turn-off can emerge even if the parameter values are correct.

Take the system:

$$\begin{cases} x(t+1) = \begin{pmatrix} 0.95 & 0 \\ 0 & 0.9 \end{pmatrix} x(t) + v(t) & R_1 = I \\ y(t) = \begin{pmatrix} 1 & u(t) \end{pmatrix} x(t) \end{cases}$$

we can get curves as in figure 5.7 when using

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}}$$

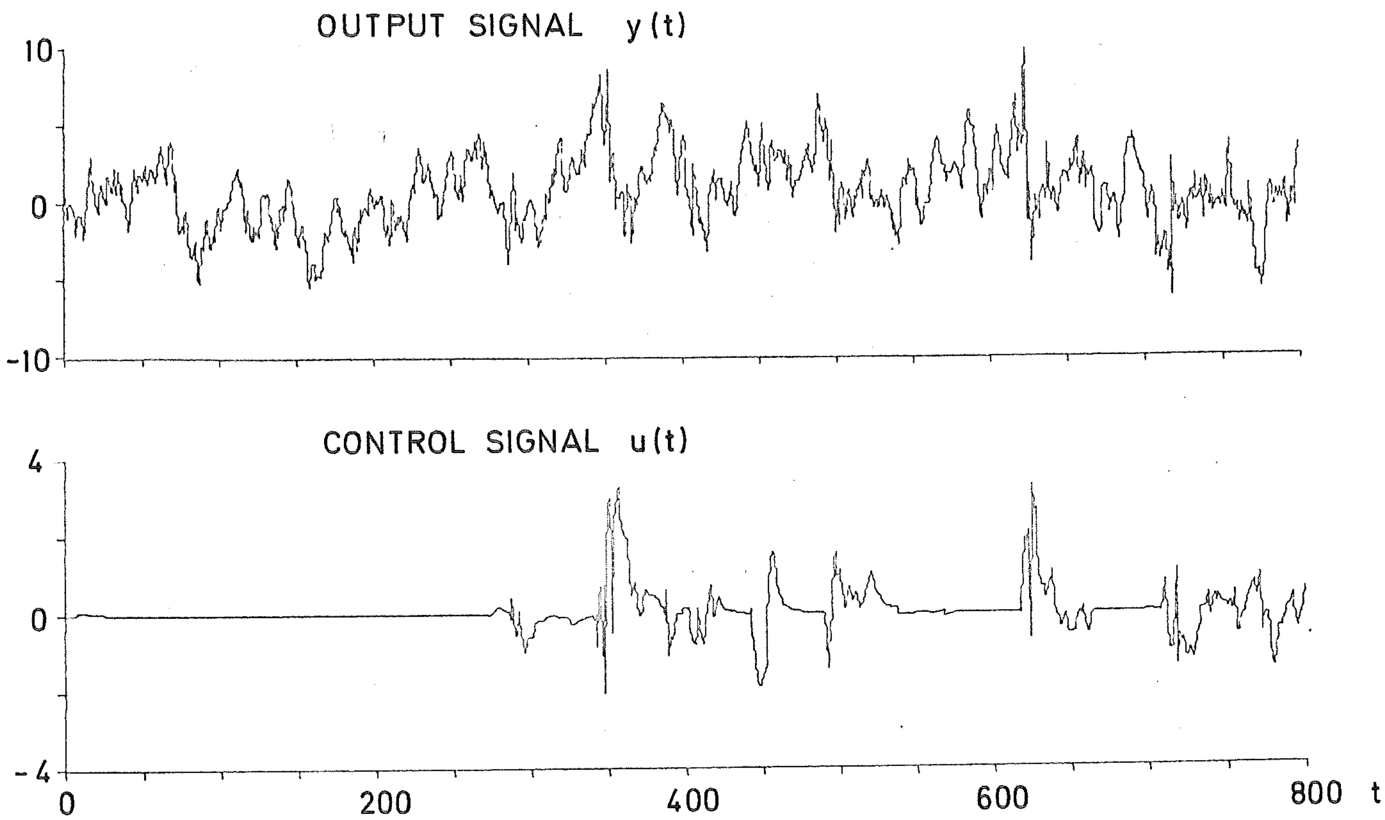


Fig. 5.7: Output and control signals for example 5.6 when using

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}}$$

To overcome the turn-off of the control we can introduce a perturbation signal in order to excite all the modes of the system. One way of doing this is to use the following control law:

$$u(t) = -\frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}} + (-1)^t \cdot \delta \tag{5.5}$$

i.e. a superposition of a square wave upon the control law (5.3), used previously.

The effect of this can be studied in figure 5.8, where the loss functions are mapped for the control laws (5.3), (5.5) and $u(t) = 0$.

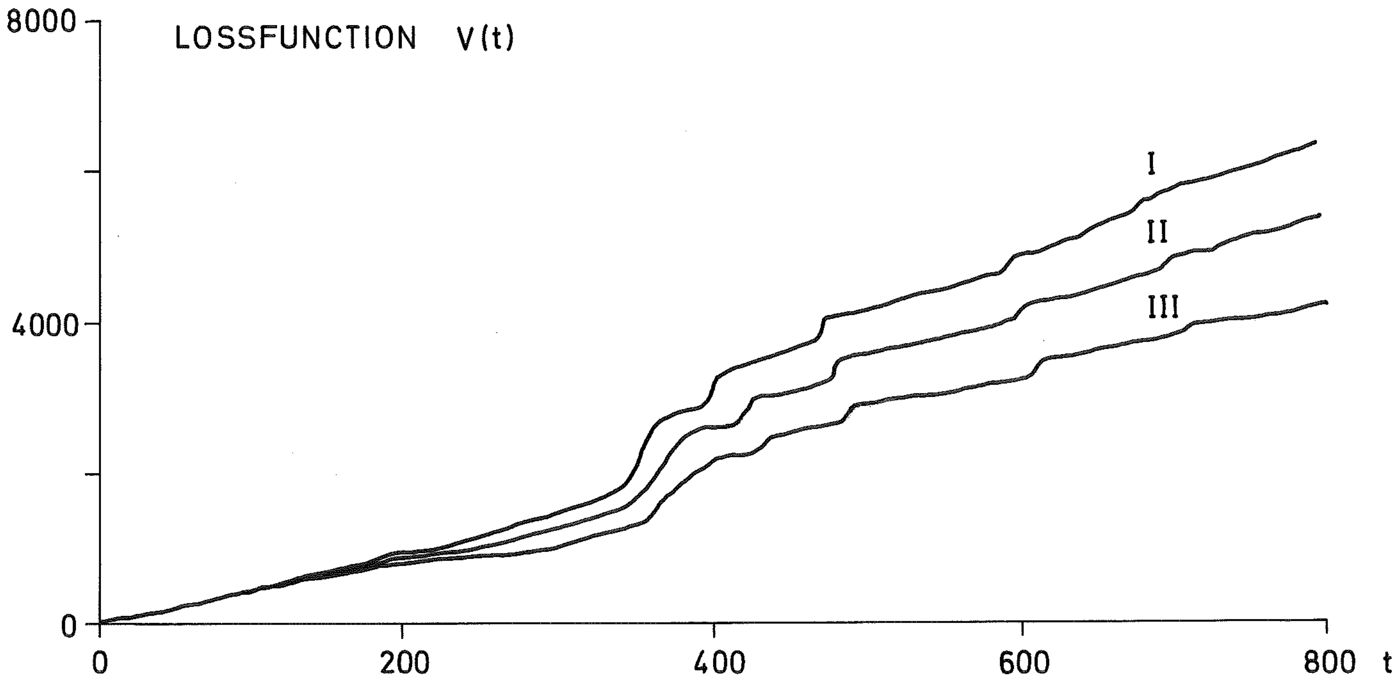


Fig. 5.8: Loss function for the system in example 5.6 using the control laws:

I $\quad u(t) = 0$

II $\quad u(t) = -\dfrac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}}$

III $\quad u(t) = -\dfrac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^{\,2} + p_{22}} + (-1)^t\,\delta;\ \delta = 0.15$

The effect of the perturbation signal is that the control signal does not turn off and thus we all the time get a proper estimation of both state variables, i.e. the elements in the variance matrix P(t) become smaller.

Example 5.7

The burst phenomenon was mentioned in section 1, and we will now give an example of this.

Let the system be:

$$\begin{cases} x(t+1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + v(t) \\ \\ y(t) = \begin{pmatrix} 1 & u(t) \end{pmatrix} x(t) \end{cases}$$

and the control law

$$u(t) = - \frac{\hat{x}_1\hat{x}_2 + p_{12}}{\hat{x}_2^2 + p_{22}}$$

Then we can get a result as in fig. 5.9.

The burst occurs around $t = 500$ and when looking at the state variable $x_2(t)$ around $t = 500$ we see that it is near zero, and thus the control law is sensitive for the variance of the estimates (compare example 5.2). But for $t \approx 250$ is $x_2(t)$ also near zero, and there is no burst. The reason for this is, that the variance matrix element $p_{22}(t)$ is smaller for $t \approx 500$ than for $t \approx 250$.

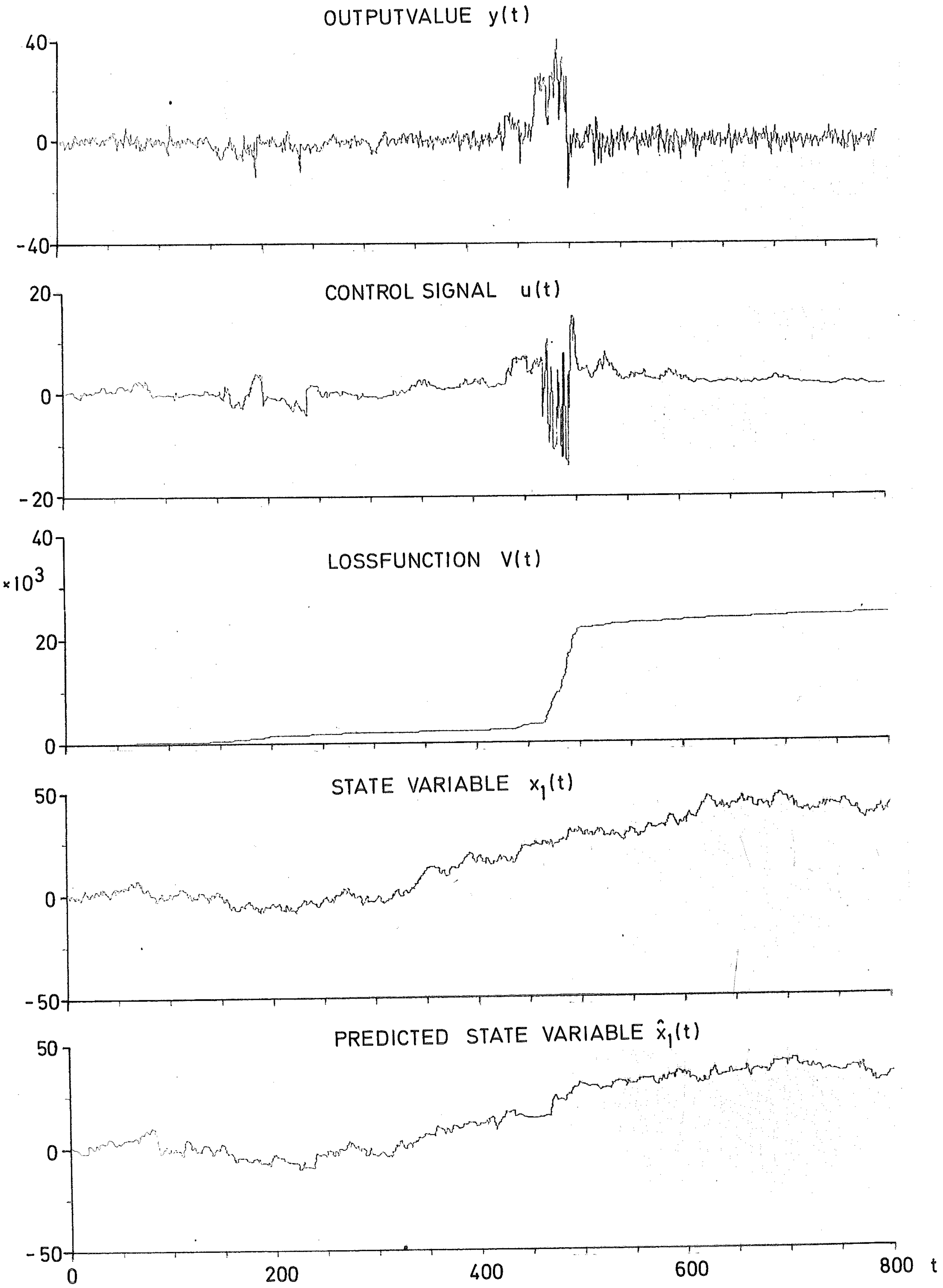


Fig. 5.9 - Simulation results from example 5.7 showing the burst-phenomena

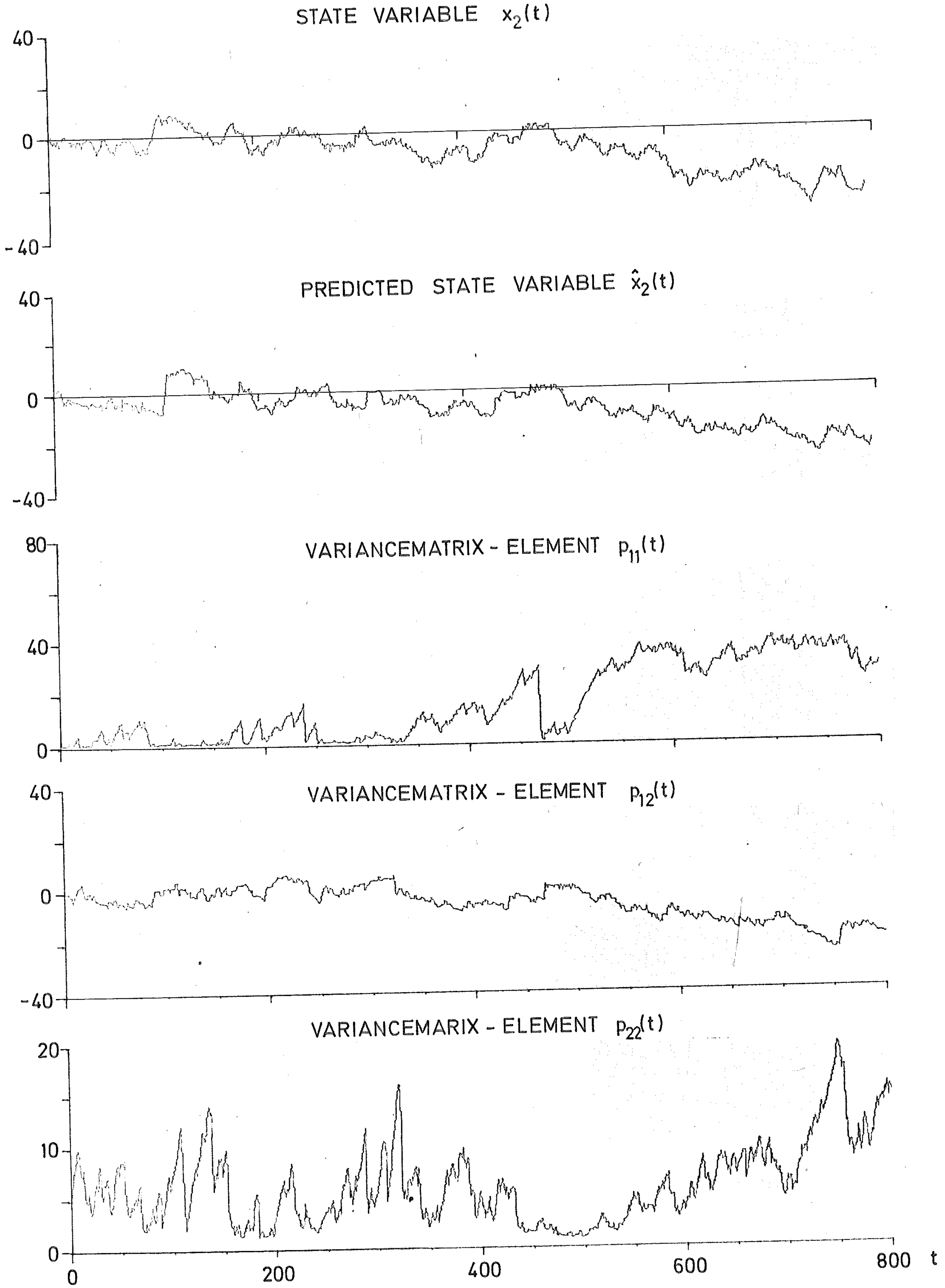

STATE VARIABLE $x_2(t)$
40
0
-40
PREDICTED STATE VARIABLE $\hat{x}_2(t)$
40
0
-40
VARIANCEMATRIX - ELEMENT $p_{11}(t)$
80
40
0
VARIANCEMATRIX - ELEMENT $p_{12}(t)$
40
0
-40
VARIANCEMARIX - ELEMENT $p_{22}(t)$
20
10
0
0
200
400
600
800
t

Fig. 5.9 - Contd.

6. OPTIMAL CONTROL LAW FOR FIRST ORDER SYSTEM

In this section an optimal control law for first order system will be derived. The solution will lead to Dynamic Programming equations. Because of the computational limitations for high order systems when using Dynamic Programming we will only treat first order system. But when the structure of the optimal control law is obtained we can choose a suboptimal control law, which will give good performance, and this can be transfered to higher order systems.

Let the system be:

$$\begin{cases} x(t+1) = ax(t) + v(t) \\ y(t) = x(t)u(t) + e(t) \end{cases} \qquad (6.1)$$

where $\{v(t)\}$ and $\{e(t)\}$ are sequences of independent normal random variables with

$$E\, v(t) = E\, e(t) = 0$$

$$E\, v(t)^2 = 1$$

$$E\, e(t)^2 = \lambda^2$$

The problem is to find an admissible control law which minimizes the loss function:

$$\ell = E \sum_{s=1}^{N} \left(1 + u(s)x(s)\right)^2 \qquad (2.4)$$

with respect to the control sequence $u(1) \ldots u(N)$.

A control law is admissible if $u(t)$ is a function of $u(t-1)$, $u(t-2)$, ..., $y(t-1)$, $y(t-2)$,...

Introduce the notation as in section 4

$$\mathcal{Y}_t = [y(t)\ y(t-1)\ \ldots\ y(1)]^T$$

the sequence of old output values.

Now consider the situation at time t. The output values y(1) ... ... y(t-1) have been observed and the control signal u(t) shall be determined. The loss function can be written as a sum of two terms.

$$\ell = E \sum_{s=1}^{t-1} \left(1 + u(s)x(s)\right)^2 + E \sum_{s=t}^{N} \left(1 + u(s)x(s)\right)^2$$

The first term does not depend on u(t) u(t+1) ... u(N), and to minimize the loss function with respect to these control signals is equivalent to minimize the second term in the loss function. This term can be rewritten as:

$$E \sum_{s=t}^{N} \left(1 + u(s)x(s)\right)^2 = E\{E\left[\Sigma\left(1 + u(s)x(s)\right)^2 \middle| \mathcal{Y}_{t-1}\right]\}$$

To minimize ℓ is now equivalent to minimize:

$$E\{ \sum_{s=t}^{N} \left(1 + u(s)x(s)\right)^2 \big| \mathcal{Y}_{t-1}\} \qquad (6.2)$$

with respect to the control sequence u(t) ... u(N).

From the Kalman filter theory it follows, that the conditional distribution of x(t) given $\mathcal{Y}_{t-1}$ is normal with mean value and variance, given by the equations:

$$\begin{cases} \hat{x}(t+1|t) = a\hat{x}(t|t-1) + K(t)\left[y(t) - u(t)\hat{x}(t|t-1)\right] \\ K(t) = \dfrac{ap(t)u(t)}{p(t)u(t)^2 + \lambda^2} \\ p(t+1) = \dfrac{\lambda^2 a^2 p(t)}{p(t)u(t)^2 + \lambda^2} + 1 \end{cases} \qquad (6.3)$$

Now consider the situation for $t = N$. Then it only remains to choose $u(N)$ to minimize (6.2) and this is equivalent to minimize:

$$E\{(1 + u(N)x(N))^2 | \mathcal{Y}_{N-1}\} =$$

$$= (1 + u(N)\hat{x}(N|N-1))^2 + u(N)^2 p(N) =$$

$$= (\hat{x}(N|N-1)^2 + p(N))\left(u(N) + \frac{\hat{x}(N|N-1)}{\hat{x}(N|N-1)^2 + p(N)}\right)^2 + \frac{p(N)}{\hat{x}(N|N-1)^2 + p(N)}$$

Minimum is given for:

$$u(N) = -\frac{\hat{x}(N|N-1)}{\hat{x}(N|N-1)^2 + p(N)} \tag{6.4}$$

This is the first order system correspondence to the control law derived in section 4.

Define:

$$V(\hat{x}(t|t-1), p(t), t) = \min_{u(t)\ \ldots\ u(N)} E\{\sum_{s=t}^{N} (1 + u(s)x(s))^2 | \mathcal{Y}_{t-1}\} \tag{6.5}$$

For $t = N$ we have:

$$V(\hat{x}(N|N-1), p(N), N) = \frac{p(N)}{\hat{x}(N|N-1)^2 + p(N)} \tag{6.6}$$

Rewrite (6.5):

$$V(\hat{x}(t|t-1), p(t), t) = \min_{u(t)\ldots u(N)} E\{\sum_{s=t}^{N} (1 + u(s)x(s))^2 | \mathcal{Y}_{t-1}\}$$

$$= \min_{u(t)\ldots u(N)} E\{(1 + u(t)x(t))^2 + \sum_{s=t+1}^{N} (1 + u(s)x(s))^2 | \mathcal{Y}_{t-1}\}$$

$$= \min_{u(t)} \Big[\big(1 + u(t)\hat{x}(t|t-1)\big)^2 + u(t)^2 p(t) +$$

$$+ \min_{u(t+1)\dots u(N)} E\{ \sum_{s=t+1}^{N} \big(1 + u(s)x(s)\big)^2 | \mathcal{Y}_{t-1} \} \Big]$$

$$= \min_{u(t)} \Big[\big(1 + u(t)\hat{x}(t|t-1)\big)^2 + u(t)^2 p(t) + E\{ V\big(\hat{x}(t+1|t), p(t+1), t+1\big) | \mathcal{Y}_{t-1} \} \Big] \tag{6.7}$$

The equation (6.7) is a recuration formula from which an optimal control signal u(t) can be chosen.

Now introduce (6.3) into (6.7):

$$V\big(\hat{x}(t|t-1), p(t), t\big) = \min_{u(t)} \Big[\big(1 + u(t)\hat{x}(t|t-1)\big)^2 + u(t)^2 p(t) +$$

$$+ E\{ V \Big(a\hat{x}(t|t-1) + K(t)\big(y(t) - u(t)\hat{x}(t|t-1)\big), \frac{\lambda^2 a^2 p(t)}{u(t)^2 p(t) + \lambda^2} + 1, t+1 \Big) | \mathcal{Y}_{t-1} \} \Big]$$

Furthermore, the distribution for y(t) given $\mathcal{Y}_{t-1}$ is normal with mean value $\hat{x}(t|t-1)u(t)$ and variance $u(t)^2 p(t) + \lambda^2$. Given the distribution of y(t) we can evaluate the last term in the equation above and we get:

$$V\big(\hat{x}(t|t-1), p(t), t\big) = \min_{u(t)} \Big[\big(1 + u(t)\hat{x}(t|t-1)\big)^2 = u(t)^2 p(t) +$$

$$+ \frac{1}{\sqrt{2\Pi\big(u(t)^2 p(t) + \lambda^2\big)}} \int_{-\infty}^{\infty} V\big(\hat{x}(t+1|t), p(t+1), t+1\big) \cdot$$

$$\cdot \exp - \frac{\big(s' - u(t)\hat{x}(t|t-1)\big)^2}{2\big(u(t)^2 p(t) + \lambda^2\big)} \cdot ds' \Big]$$

Introduce the new variables:

$$\begin{cases} s = \dfrac{s' - u(t)\hat{x}(t|t-1)}{\sqrt{u(t)^2 p(t) + \lambda^2}} \\ \omega(t) = \dfrac{p(t)}{\hat{x}(t|t-1)^2} \\ z(t) = -\,u(t)\hat{x}(t|t-1) \end{cases}$$

Here $\omega(t)$ can be interpreted as a measure of the relative accuracy, small $\omega(t)$ is equivalent to good relative accuracy. $z(t)$ is a normalized control signal.

Then

$$V_1\left(\hat{x}(t|t-1),\ \omega(t),\ t\right) = V\left(\hat{x}(t|t-1),\ p(t),\ t\right) =$$

$$= \min_{z(t)} \left[\left(1 - z(t)\right)^2 + z(t)^2\omega(t) + \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} V_1\left(\hat{x}(t+1|t),\ \omega(t+1),\ t+1\right) e^{-\frac{s^2}{2}}\, ds \right] \qquad (6.8)$$

where

$$\begin{cases} \hat{x}(t+1|t) = a\hat{x}(t|t-1)\left[1 - \dfrac{\omega(t)z(t)s}{\sqrt{z(t)^2\omega(t) + \lambda^2}}\right] \\ \omega(t+1) = \dfrac{1}{\hat{x}(t+1|t)^2}\left[1 + \dfrac{\lambda^2 a^2 \omega(t)\hat{x}(t|t-1)^2}{z(t)^2\omega(t) + \lambda^2}\right] \end{cases} \qquad (6.9)$$

The loss function V_1 is a function of the variables $\hat{x}(t|t-1)$ and $\omega(t)$, and the minimization is done with respect to the normalized control signal $z(t) = -\,u(t)\hat{x}(t|t-1)$. The equations (6.6), (6.8) and (6.9) now defines a Dynamic Programming solution to the given problem. The initial value is given by (6.6) and V_1 can be evaluated backwards using (6.8) and (6.9). The computational aspects are discussed in appendix.

The solution is not given in analytical form, but as a loss table and a control table, in which values for the loss function V_1 and the control signal z are given for discrete values of the two variables $\hat{x}(t|t-1)$ and $\omega(t)$. The backstepping is done until the changes in the control table are small. The steady state control table is then used in the simulations.

The actual control signal in the simulations is obtained by looking in the control table for the actual values of $\hat{x}(t|t-1)$ and $\omega(t)$. This will give $z(t)$ and then:

$$u(t) = - \frac{1}{\hat{x}(t|t-1)} \cdot z(t) \tag{6.10}$$

The variable $z(t)$ can be interpreted as a weighing parameter modifying the heuristic control law $u(t) = - 1/\hat{x}(t|t-1)$ (compare section 3) with respect to the uncertainty in the estimate of the state variable. To examplify this we look at two special cases.

First, assume that the estimate is exact e.g. $p(t) = 0$. Then (6.9) reduces to:

$$\begin{cases} \hat{x}(t+1|t) = a\hat{x}(t|t-1) \\ \omega(t+1) = \dfrac{1}{\hat{x}(t+1|t)^2} \end{cases} \tag{6.9'}$$

This inserted into (6.8) gives for $t = N - 1$:

$$V_1(\hat{x}, \omega, N-1) = \min_z \left[(1 - z)^2 + \frac{1}{(a\hat{x})^2 + 1} \right] = \frac{1}{(a\hat{x})^2 + 1}$$

where the minimum is obtained for $z = 1$. As expected the control law will be

$$u(t) = - \frac{1}{\hat{x}(t|t-1)} \qquad \forall t$$

Second , assume p(t) is larger than for t = N-1:

$$\omega(t+1) \approx \frac{k}{\omega(t)}$$

$$V_1 = \min_z \left[(1-z)^2 + z^2\omega + \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \frac{k}{\omega + k} e^{-\frac{s^2}{2}} ds \right]$$

$$\approx \min_z \left[(1-z)^2 + z^2\omega \right] \qquad \text{for large } \omega$$

and minimum is obtained for z equal to zero. From these two special cases it seems likely that minimum is obtained for z(t) in the interval (0, 1), and it is verified through the numerical calculations.

Using $\omega = p/\hat{x}^2$ in (6.4) will give:

$$u(t) = -\frac{1}{\hat{x}(t|t-1)} \cdot \frac{1}{1 + \omega(t)} \tag{6.4'}$$

Comparing (6.4') and (6.10) we see that the factor $\frac{1}{1+\omega}$ in (6.4') also can be interpreted as a weighing factor. This resemblance between the two control laws (6.4') and (6.10) will be further discussed in section 7 in connection with results from simulations.

A very important question is:How large will the expected loss be, when using the optimal control law? This question can be answered, using the loss table. Take the difference between the loss tables in two consecutive points of time at stationarity. The difference table will give the loss per time step. When starting the system we do not know $\hat{x}$ and p for every time step ahead, but we can get the statistics of $\hat{x}$ in the following way:

First, as the Kalman estimate is unbiased the mean value of $\hat{x}$ will be the same as for x(t), i.e. $E\,\hat{x} = 0$. Further the control will be best if the variance of the estimation error is as small as possible. The smallest possible value of p(t) is 1, which is the variance of the measurement error. As the estimation error $\tilde{x}$ and the estimate $\hat{x}$ are independent and we have

$$\hat{x} + \tilde{x} = x$$

then

$$\operatorname{var} \hat{x} + \operatorname{var} \tilde{x} = \operatorname{var} x = \frac{1}{1 - a^2}$$

This will give

$$\sigma_{\hat{x}}^2 = \operatorname{var} \hat{x} = \frac{a^2}{1 - a^2}$$

Thus $\hat{x}$ is a normal stochastic variable with zero mean and the variance

$$\frac{a^2}{1 - a^2}$$

This can now be used to compute the expected loss per time step.

$$E\Delta V = E\Delta V_1(\hat{x}, \omega, t) =$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{\hat{x}}} \cdot \int_{-\infty}^{\infty} \Delta V_1(s, \frac{1}{s^2}, t)\, e^{-\frac{s^2}{2\sigma_{\hat{x}}^2}} ds$$

Through numerical integration the expected loss can be computed, and this value can then be used to evaluate the results from the simulations.

7. SIMULATION OF FIRST ORDER SYSTEM

In this section we will investigate the behaviour of a first order system using the optimal control law, derived in section 6.

The system used is:

$$\begin{cases} x(t+1) = 0.9\ x(t) + v(t) \\ y(t) = u(t)x(t) + e(t) \end{cases} \qquad (7.1)$$

where

$$E\ v(t) = E\ e(t) = 0$$

$$E\ v(t)^2 = 1$$

$$E\ e(t)^2 = 0.25$$

With these numerical values the loss and control tables have been calculated. To get the steady state control table it was necessary to go backwards and compute twenty tables. As mentioned in section 6, it is possible to calculate the expected loss per step. When using the optimal control law it was found that the expected loss would be:

$$E\ \ell = E \sum_{s=1}^{N} \big(1 + u(s)x(s)\big)^2 = 0.58 \cdot N$$

This can be compared with the case when $u(t) \equiv 0$, then

$$E\ \ell = 1 \cdot N$$

First we will investigate the behaviour of the system when using the nondual controller:

$$u(t) = - \frac{1}{\hat{x}(t|t-1)} \cdot \frac{1}{1 + \omega(t)} \qquad (6.4')$$

To get a statistical sample the results from 75 different simulations have been compared. The mean value of the loss and the standard deviation have been estimated. The result from the "most average" simulation is seen in figure 7.1. As for the second order system in section 5 the "turn-off" phenomena occur. The regulator turns off the control for long periods of time.

When using the optimal control law, derived in section 6, the result from a typical simulation will be as in figure 7.2. With this control law the "turn-off" phenomena do not occur. Another major difference between the two control laws is seen in the variance of the estimation error $p(t)$. When using the nondual controller (6.4') the variance is much greater than when using the optimal control law. This depends on the fact that the optimal control law acts in such a way to get better estimates. This control law is with the terminology introduced in section 1 a dual controller.

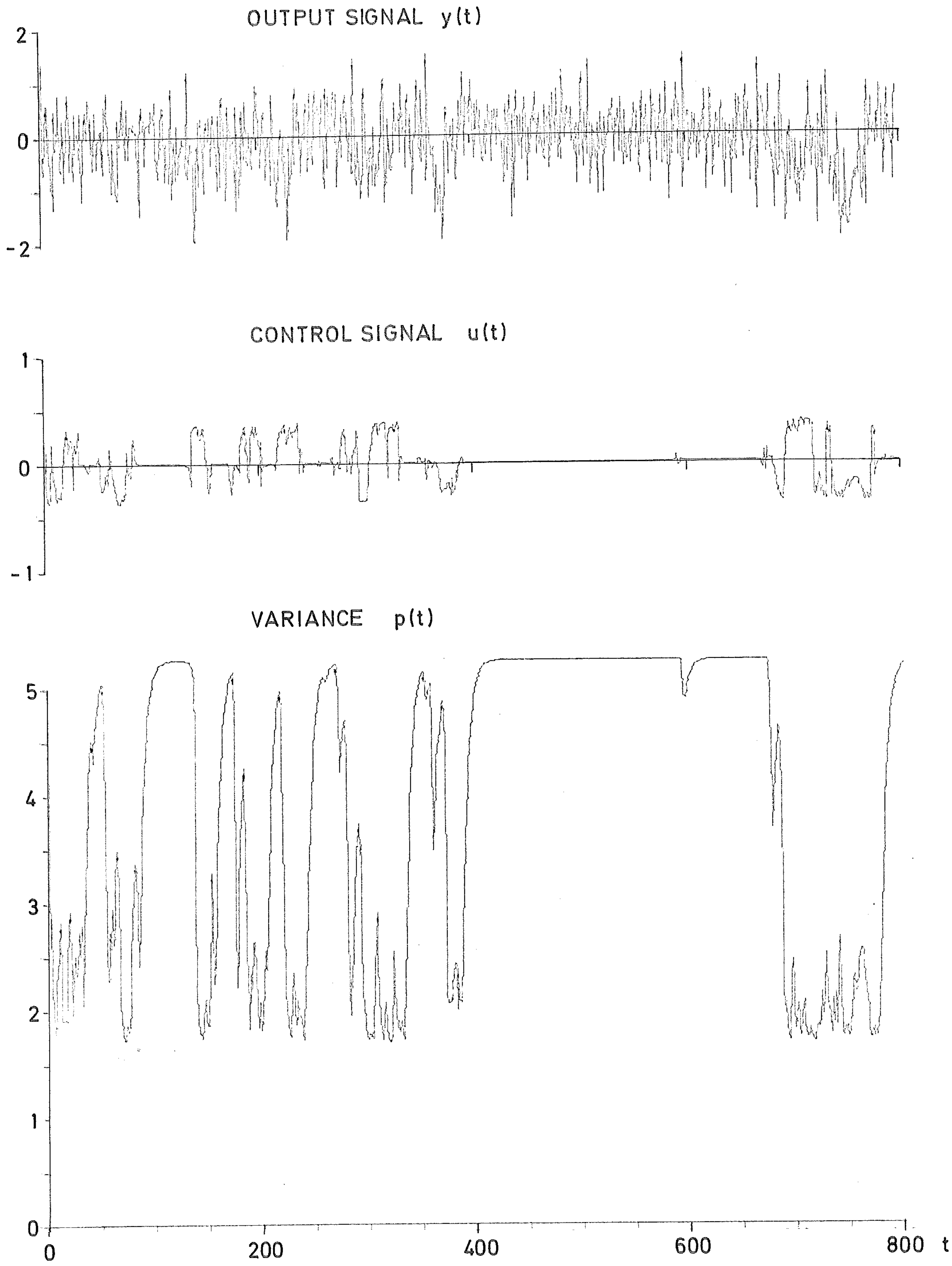


Fig. 7.1: Result from simulation of system, defined by equation (7.1) using the control law

$$u(t) = -\frac{1}{\hat{x}(t|t-1)} \cdot \frac{1}{1+\omega(t)} \qquad (6.4')$$

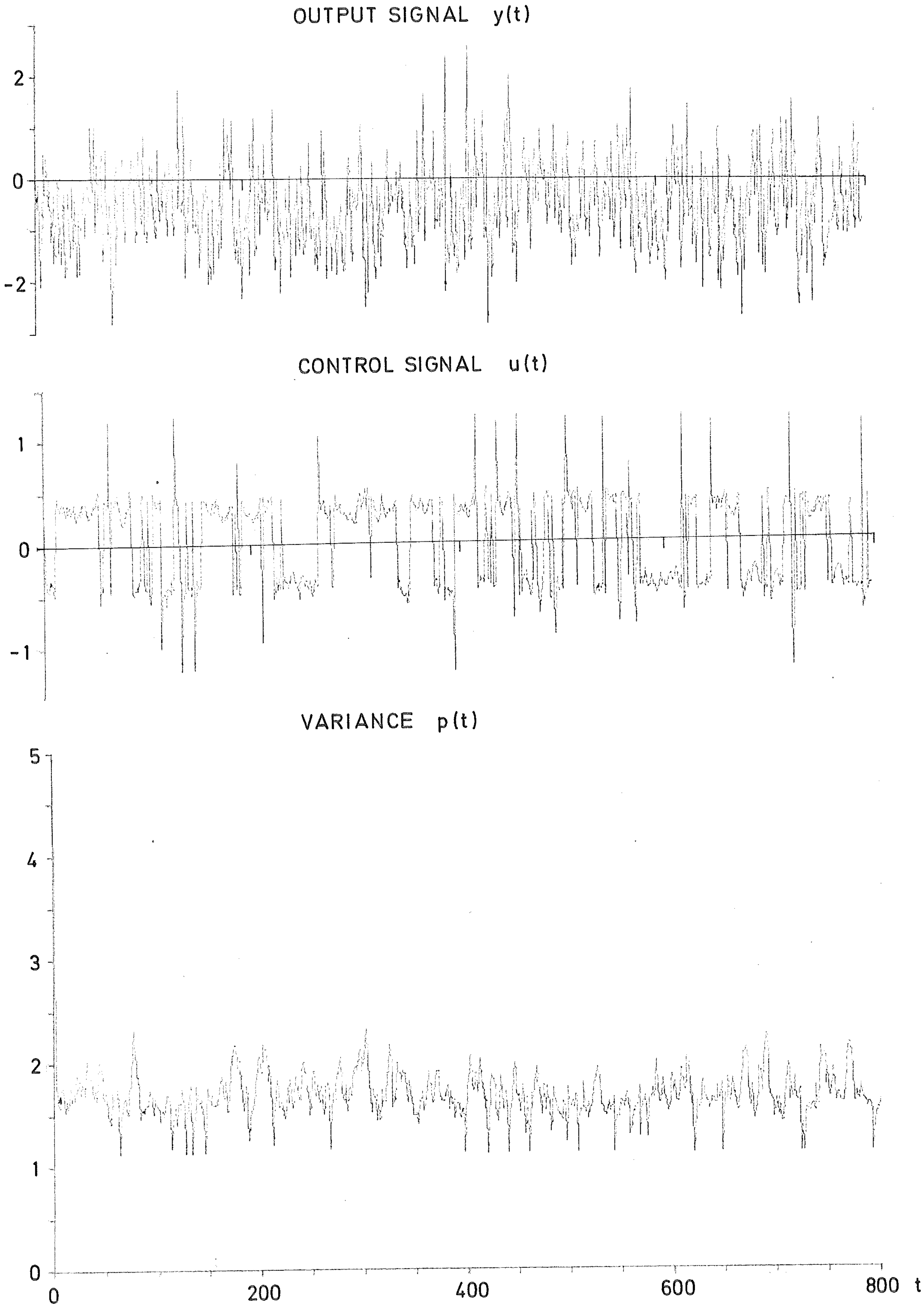


Fig. 7.2 - Result from simulation of system, defined by equation (7.1) using the optimal control law

As the variance of the estimation error is almost constant when using the optimal control law the influence of the variance in the control table can be eliminated. The most suitable way is to express the normalized control signal z as a function of $\omega = p/\hat{x}^2$. This has been done in figure 7.3, where $z(\omega)$ is plotted for the optimal control law and for $z(\omega) = \frac{1}{1+\omega}$.

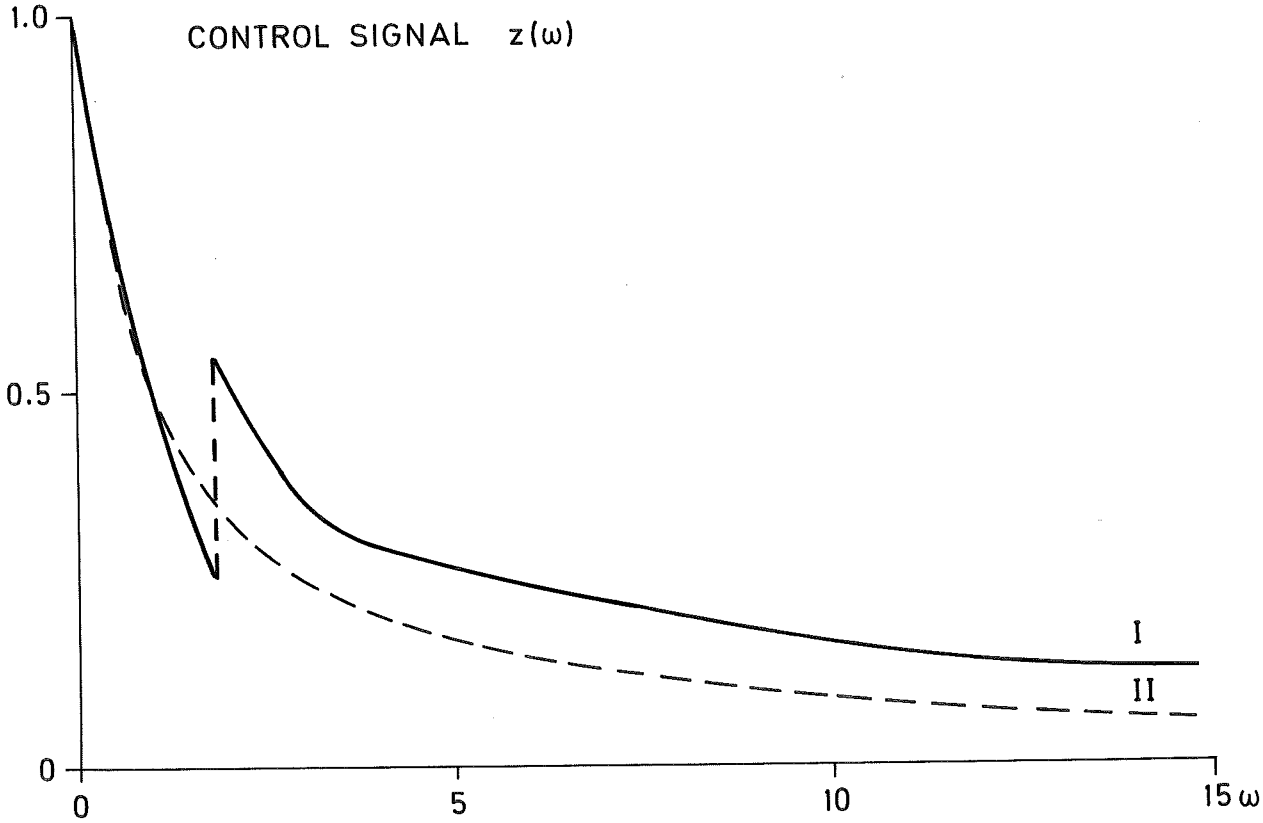


Fig. 7.3: The normalized control signal as function of $\omega = p/\hat{x}^2$ for

I Optimal control

II $z(\omega) = \frac{1}{1+\omega}$

For ω less than about 2 the two control laws are almost the same. This seems natural, because for small ω the relative accuracy of the estimate is good, and nothing has to be done to improve the estimate. In this case it is sufficient to use the regulator, defined by (6.4'). But when the relative accuracy decreases, i.e. ω increases, it becomes more and more necessary to improve the estimate. For ω greater than 2 it is the improvement of the estimate which predominates. This change in action of the optimal control law will give the discontinuity in control function $z(\omega)$. Similar results when using dual control laws have recently been given by Jacobs and Langdon [2].

The jump in the control function is further explained when looking at the loss as function of z and ω. As seen in figure 7.4 there are two valleys in the loss function. For ω less than about 2 it is one valley which gives the smallest values of the loss. But when ω exceeds 2 it is the other which gives the absolute minimum of the loss function.

The optimal controller thus contains of two parts, one error-correcting and one information-sensing part. When noticing this structure and observing that for small values of ω the control law (6.4') gives good performance it seems as if a good suboptimal control law would be:

$$u(t) = -\frac{1}{\hat{x}(t|t-1)} \cdot \frac{1}{1+\omega(t)} + (-1)^t \cdot \delta \qquad (7.2)$$

It can be discussed if the second term is the best one for the information-sensing part. But (7.2) has been chosen here to show that inspite of its simplicity it will be a good suboptimal control law.

Other functions for the information-sensing could be:

$$\begin{cases} (-1)^t \cdot \delta & \omega \geq \omega_0 \\ 0 & \omega < \omega_0 \end{cases}$$

$$\begin{cases} -\delta/\hat{x}(t/t-1) & \omega \geq \omega_0 \\ 0 & \omega < \omega_0 \end{cases}$$

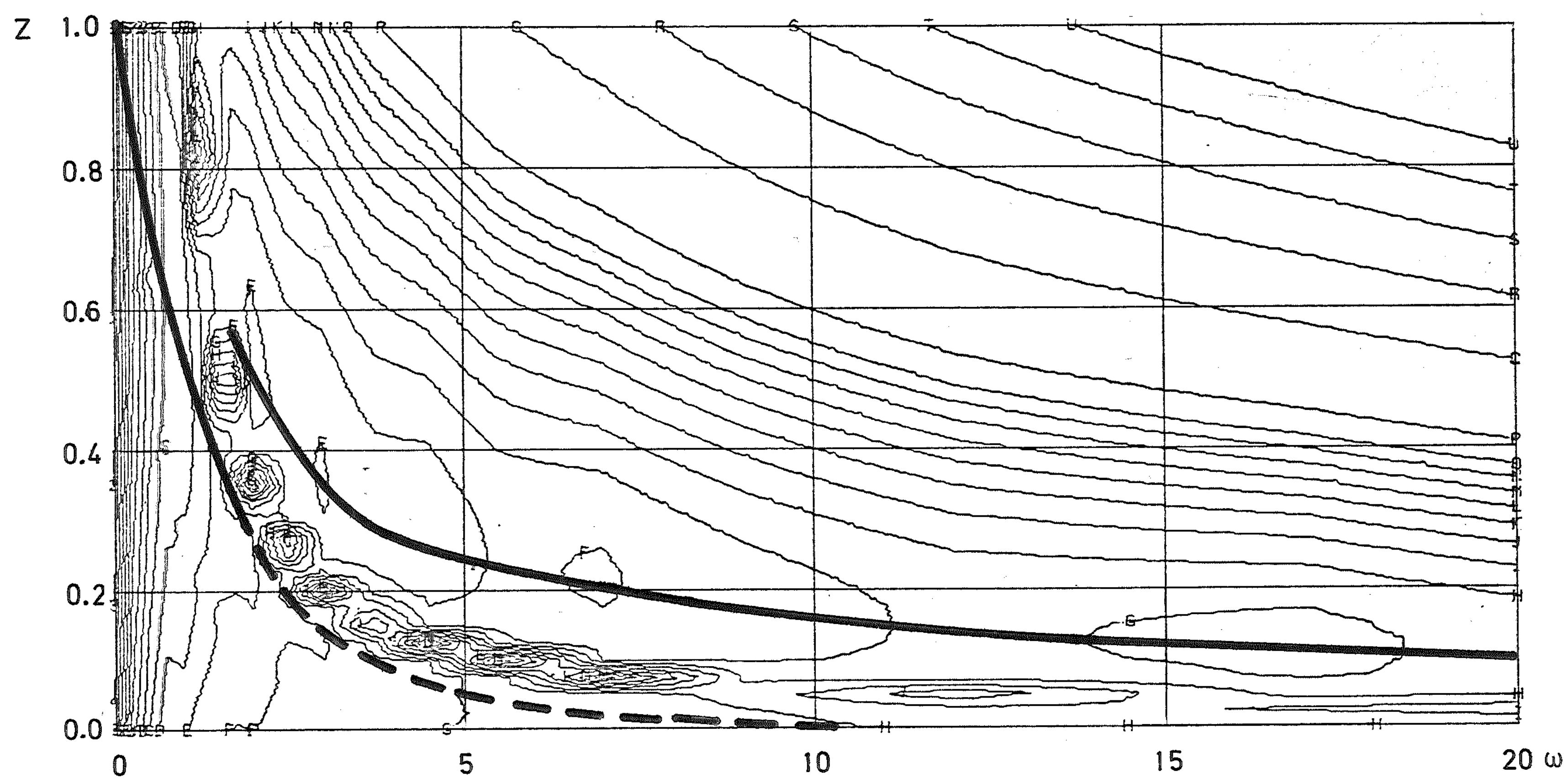


Fig. 7.4 - Contourlevels of the loss as function of z and ω. The thick line shows the absolute minimum and the dashed line indicates the second valley of the loss function

Results from an average simulation when using (7.2) with $\delta = 0.125$ can be seen in figure 7.5.

The loss functions of the average simulations in figures 7.1, 7.2 and 7.5 are mapped in figure 7.6. As seen the optimal control law gives an average slope of the loss function which for long periods of time is very close to the expected value derived as shown in the end of section 6. The suboptimal control law gives a loss which lies considerably under the loss of the nondual controller (6.4'). The numerical values of the average loss and the standard deviation for the different control law are shown in table 7.1.

Control law	Average slope of the loss	
	Mean value	Standard deviation
I	0.86	0.07
II	0.68	0.04
III	0.61	0.05
Expected slope	0.58	

Table 7.1: Mean value and standard deviation for the average slope of the loss obtained from 75 simulations with different control laws.

$$\text{I} \qquad u(t) = -\frac{1}{\hat{x}} \cdot \frac{1}{1+\omega} \tag{6.4'}$$

$$\text{II} \qquad u(t) = -\frac{1}{\hat{x}} \cdot \frac{1}{1+\omega} + (-1)^{t}\delta \tag{7.2}$$

III Optimal control law

The simple suboptimal control law (7.2) gives in percentage a reduction of 68% of the maximally obtainable reduction. For the optimal control law the reduction is 93%.

To summarize it is possible to derive an optimal control law by using the Dynamic Programming technique. As always there is the course of dimensionality, limiting the order of system. But as

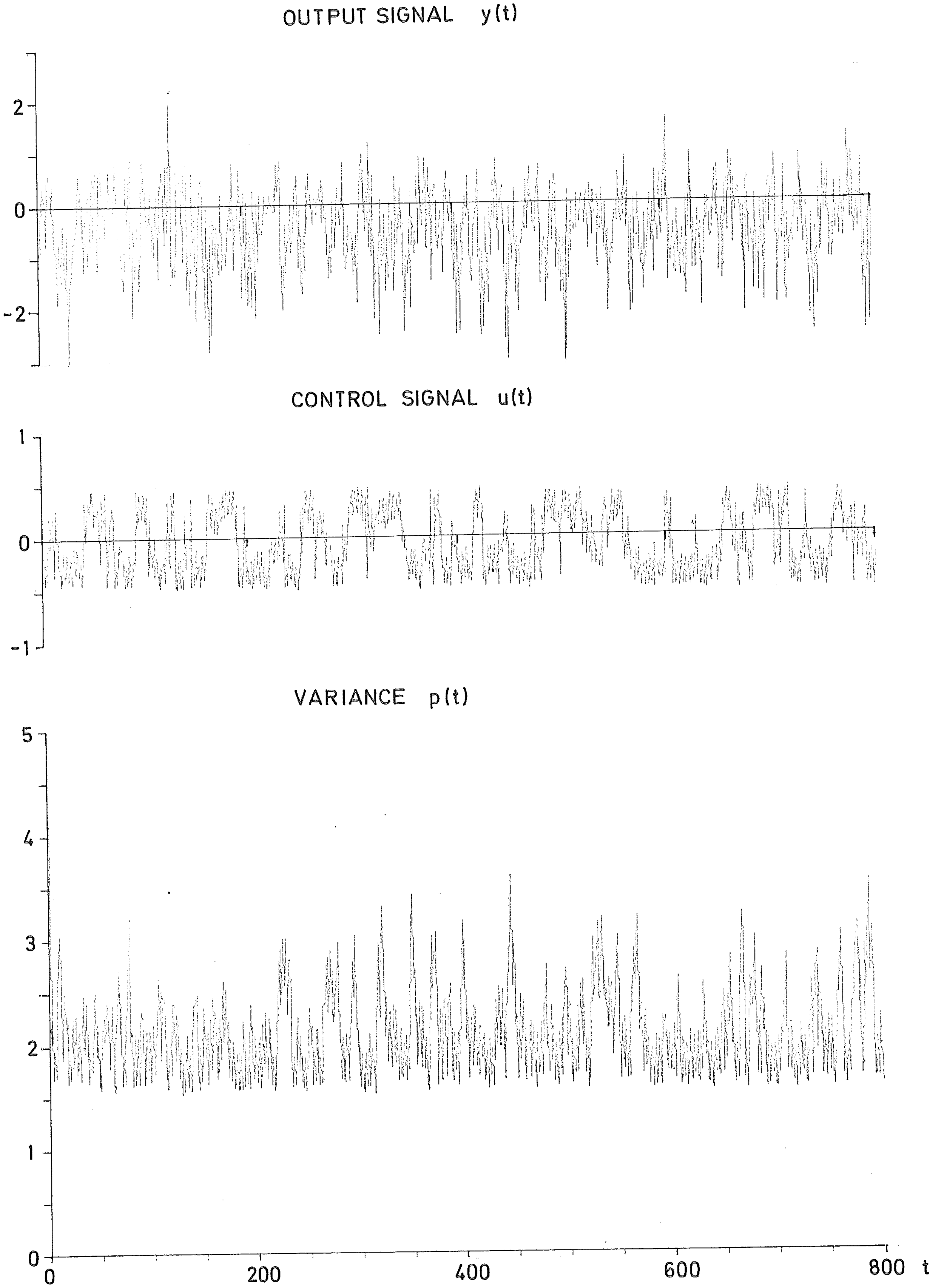


Fig. 7.5 - Results from simulation using the control law

$$u(t) = -\frac{1}{\hat{x}} \cdot \frac{1}{1+\omega} + (-1)^t \cdot \delta \; ; \quad \delta = 0.125$$

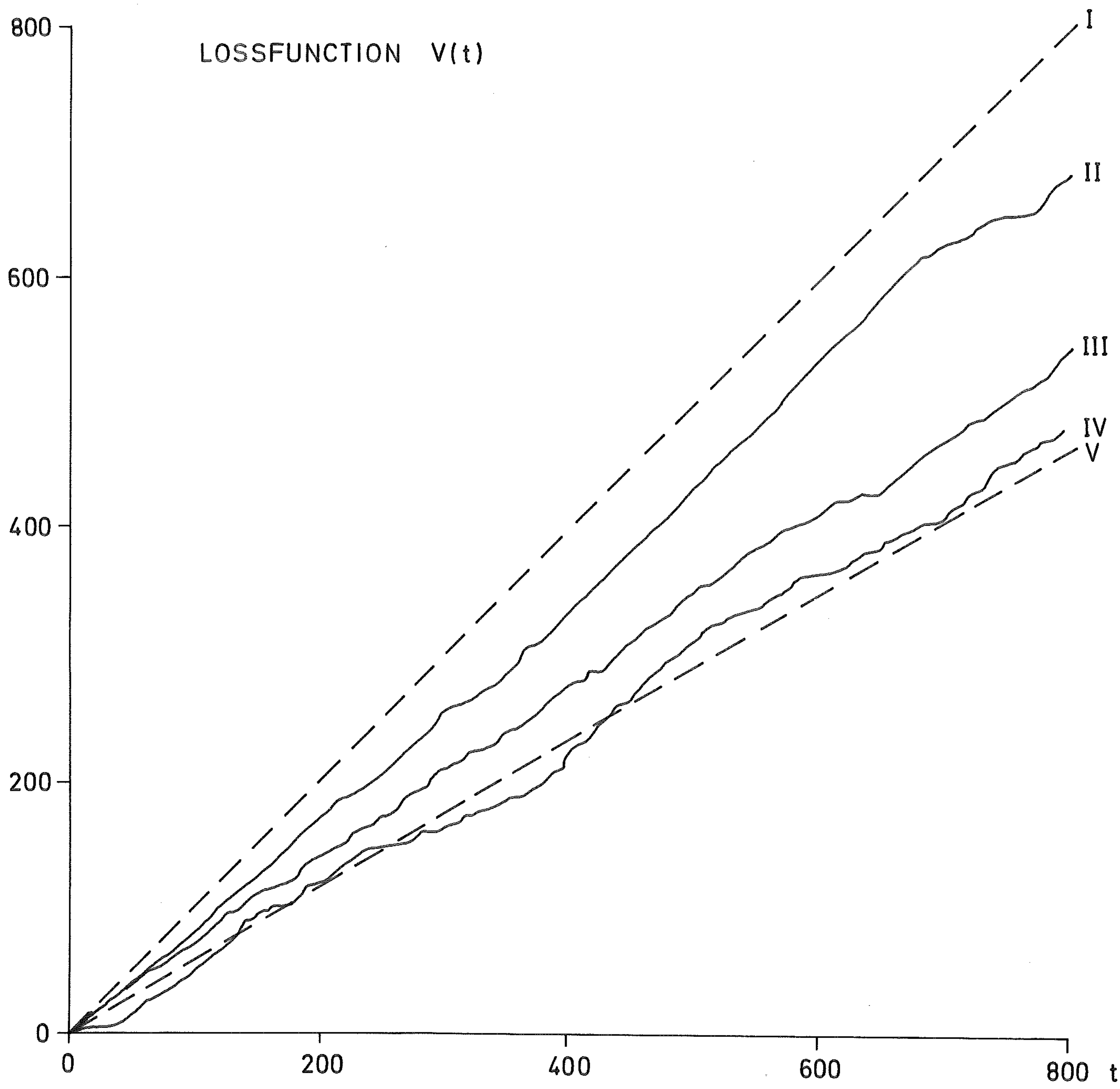


Fig. 7.6. Lossfunction for the system defined by equation (7.1) when using different controllers.

I No control $u(t) = 0$

II $u(t) = -\dfrac{\hat{x}}{\hat{x}^2+p}$

III $u(t) = -\dfrac{\hat{x}}{\hat{x}^2+p} + (-1)^t \cdot 0.125$

IV Optimal control derived through Dynamic Programming

V Expected loss derived through integration in losstable obtained by Dynamic Programming seen in this section it is possible to construct a suboptimal control law which is very easy to transfer to higher order system. E.g. it is used for a second order system in example 5.6. The suboptimal control law consists of two parts, one which eliminates the expected error and one information-sensing part. The second part can for instance be a perturbation signal as in (7.2).

The "turn-off" phenomena thus justify an introduction of a perturbation signal in order to persist good estimation of the state variables.

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APPENDIX

In appendix we will point out some special features in the solution of the Dynamic Programming problem, stated by the equations:

$$V_1\left(\hat{x}(t|t-1),\ \omega(t),\ t\right) = \min_{z(t)}\left[\left(1 - z(t)\right)^2 + z(t)^2\omega(t) + \right.$$

$$\left. + \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} V_1\left(\hat{x}(t+1|t),\ \omega(t+1),\ t+1\right) e^{-\frac{s^2}{2}} ds\right] \qquad (6.8)$$

where

$$\begin{cases} \hat{x}(t+1|t) = a\hat{x}(t|t-1)\left[1 - \dfrac{\omega(t)z(t)s}{\sqrt{z(t)^2\omega(t) + \lambda^2}}\right] \\ \omega(t+1) = \dfrac{1}{\hat{x}(t+1|t)^2}\left[1 + \dfrac{\lambda^2 a^2 \omega(t)\hat{x}(t|t-1)^2}{z(t)^2\omega(t) + \lambda^2}\right] \end{cases} \qquad (6.9)$$

and the initial value

$$V_1\left(\hat{x}(N|N-1),\ \omega(N),\ N\right) = \frac{\omega(N)}{1 + \omega(N)} \qquad (6.6)$$

The equations have to be solved backwards from $t = N$. Furthermore, the solution is given in table form, because it is impossible to find analytical expression for V_1 when $t \leq N-1$. Therefore the values of the loss and normalized control signal are given in discrete points. The discretized parameters are $\hat{x}(t|t-1)$, $\omega(t)$ and $z(t)$. Thus the solution is given as twodimensional matrices, one for the loss V_1 and one for the control signal z.

To solve the problem we first compute the loss table from (6.6), then for each value of $\hat{x}(t|t-1)$ and $\omega(t)$ search for minimum of the expression inside brackets of (6.8). The z-value, which gives minimum, now defines the optimal control signal and is stored in the control table. The minimal value of the loss is stored in a new loss table. We thus get control and loss tables for another time step backwards. By repeating this we go backwards until there are only small changes in the control table e.g. until we get the steady state control. This final control table is then used in the simulation of the system.

There are some details in the Dynamic Programming solution which shall be discussed further.

Discretation and normalization

The discretized parameters are $\hat{x}(t|t-1)$, $\omega(t)$ and $z(t)$. The problem is now how to choose the grid points to get a suitable net of grids all over the interesting intervals.

We make two observations: first the loss is symmetric in $\hat{x}(t|t-1)$, thus we only have to store tables for positive $\hat{x}(t|t-1)$. This will reduce the required storage for the tables and the computation time by a factor two. Secondly, $\omega(t) = p(t)/\hat{x}(t|t-1)^2$ can only have positive values.

For $\hat{x}$ and ω we now introduce a normalization which transforms the positive semiaxis into a finite interval:

$$X_1 = \frac{\hat{x}}{1 + \frac{\hat{x}}{a_1}}$$

$$X_2 = \frac{\omega}{1 + \frac{\omega}{a_2}}$$

This will give functions which are almost linear for $\hat{x}$ and ω less than a_1 resp. a_2. As an example of these functions see figure A.1.

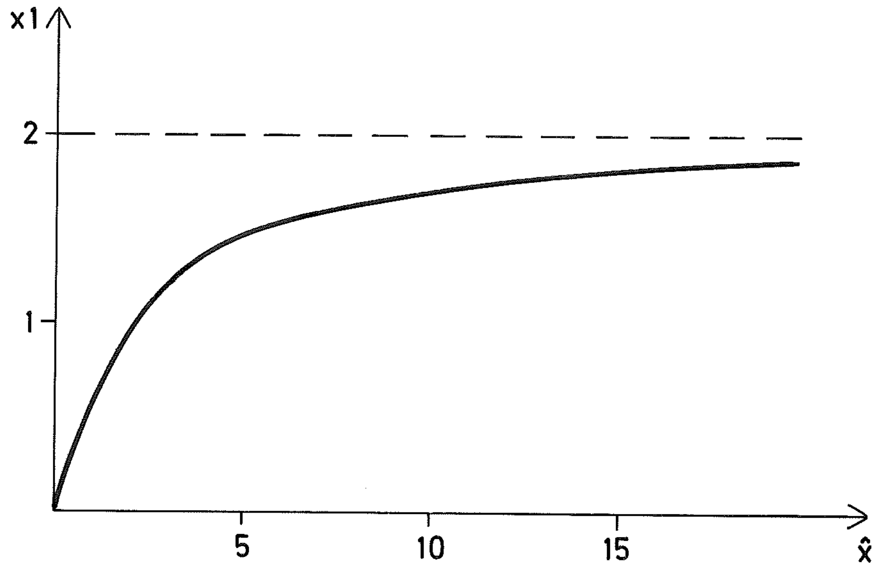

Fig. A.1: The function $X_1 = \dfrac{\hat{x}}{1 + \dfrac{\hat{x}}{a_1}}$ for $a_1 = 2$.

By choosing a_1 and a_2 and dividing the intervals $(0, a_1)$ and $(0, a_2)$ into equidistance points we can cover all interesting values for $\hat{x}$ and ω.

In the first computations the same normalization was done for z(t) but as discussed in section 6 it was found that minimum occured for z(t) in the interval(0, 1). This interval was thus divided in equidistance points.

Integration

When evaluating equation (6.8) a numerical integration has to be done. The method used is Simpson's formula:

$$\int_a^b f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n)$$

where $h = \dfrac{b - a}{n}$

The integral in (6.8) contains an exponential function multiplied by a smooth and limited function. As the exponential function decreases quickly it is sufficient to integrate over the interval (-4, 4) and use n = 8.

To see the magnitude of the introduced error we can evaluate:

$$I = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} ds = \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} f(s)ds = 1$$

Using Simpson's formula we get:

$$I^1 = \frac{1}{\sqrt{2\Pi}} \cdot \frac{1}{3}\big(f(-4) + 4f(-3) + \ldots + 2f(2) + 4f(3) + f(4)\big)$$

$$= 0.995112$$

The error in this case is 0.49%.

Interpolation

When evaluating the integral we must have function values of $V_1\big(\hat{x}(t+1|t),\ \omega(t+1),\ t+1\big)$ and the wanted value does not need to be a grid point. Thus we must have an interpolation routine. The one chosen is to do a linear interpolation in the normalized parameters X_1 and X_2 and not in the actual parameters $\hat{x}(t+1|t)$ and $\omega(t+1)$. This is just a matter of taste when using the interior of the loss table.