Characterization of the Concept of 'Persistently Exciting' in the Frequency Domain

Ljung, Lennart

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LENNART LJUNG

REPORT 7119 NOVEMBER 1971
LUND INSTITUTE OF TECHNOLOGY
DIVISION OF AUTOMATIC CONTROL

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ABSTRACT

The concept of persistently exciting signals is treated in the frequency domain. Necessary and sufficient conditions for a signal to be persistently exciting are given. The effect of filtering a persistently exciting signal is discussed.

This work has been supported by the Swedish Board for Technical Development under Contract 71-50/U33.

1. INTRODUCTION.

In connection with identification problems it is necessary to have some conditions on the input signals which will ensure consistent estimates. For the determination of transfer functions by correlation techniques it is e.g. necessary to have an input, satisfying a condition such as $\phi_u(\omega) > 0$, where ϕ is the spectrum of the input. In parametric identification it has been shown that the notion of persistent excitation is useful, see [1] and [2]. In this paper necessary and sufficient conditions for a signal to be persistently exciting of finite order are given. The notion of persistent excitation of infinite order is briefly discussed.

It is also shown how the property of persistent excitation of a given order is transformed when the signal is filtered. Relations between the notion of persistent excitation and the possibility to predict the signal ahead are also given.

2. PRELIMINARIES AND NOTATION.

Let $\{u(t), t = 0, 1, 2, \ldots\}$ denote a discrete time signal, i.e. a sequence of real numbers. Assume that

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} u(t) \tag{1}$$

and

$$r(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N} \left(u(t) - \bar{u}\right)\left(u(t+k) - \bar{u}\right) \tag{2}$$

exist.

The function $r(k)$ is non-negative definite, which means that the $N \times N$ covariance matrix

$$R_N = \begin{pmatrix} r(0) & r(1) & r(2) & \ldots & r(N-1) \\ r(1) & r(0) & r(1) & \ldots & r(N-2) \\ \vdots & & & & \vdots \\ r(N-1) & . & . & . & . \quad r(0) \end{pmatrix} \tag{3}$$

is non-negative definite.

DEFINITION:

If R_N is positive definite the signal u is said to be persistently exciting of order N.

If $r(k)$ is a non-negative definite function the trigonometric moment problem

$$r(k) = \int_{-\pi}^{\pi} e^{ikx}\, dF(x) \qquad k = 0, \pm 1, \ldots \qquad (4)$$

has a unique solution F(x). F is then a non-decreasing, right continuous function, whose derivative F' exists almost everywhere. Furthermore, since F is non-decreasing it has at most denumerable discontinuities, which are all points where F(x) makes a jump. Conversely, any such function defines a non-negative finite function r(k) through (4). With suitable conventions as to points of discontinuity the function F is given by

$$F(x) = r(0)x + 2 \sum_{k=1}^{\infty} \frac{r(k)}{k} \sin kx$$

Proofs of these statements can be found in [4] and [7].

For the continuous case the corresponding results are known as the Bochner-Khinchine theorem.

In terms of u F can also be expressed as

$$F(x) = \lim_{N\to\infty} \int_0^x \frac{1}{N} \left| \sum_{k=0}^{N} (u(k) - \bar{u}) e^{ik\rho} \right|^2 d\rho \qquad (5)$$

The support of F is denoted by

$$\text{supp } F = \left\{ x \,|\, -\pi < x \leq \pi \quad \forall \varepsilon > 0 \quad F(x+\varepsilon) - F(x-\varepsilon) > 0 \right\}$$

The support could equivalently be defined as the smallest closed set outside which the distribution F' vanishes. Since F is an odd function, supp F will be symmetric about the origin except possibly for the point $x = \pi$.

The following formula will be much used in the sequel:

Let $a = \mathrm{col}\,(a_0, a_1, \ldots, a_{N-1})$. Then

$$a^{*}R_N a = \sum_{k,s=0}^{N-1} \bar{a}_k a_s r(k-s) = \int_{-\pi}^{\pi} \sum_{k,s}^{N-1} \bar{a}_k a_s e^{i(k-s)x} dF(x) =$$

$$= \int_{-\pi}^{\pi} \left| \sum_{0}^{N-1} a_k e^{-ikx} \right|^2 dF(x) \qquad (6)$$

<u>Remark</u>

If u is a realization of a second order, ergodic stochastic process, then (1) and (2) can be identified with the mean value and autocovariance function for the process. In this case F is the spectral distribution function for the process.

3. PERSISTENT EXCITATION OF FINITE ORDER.

Using equation (6) the following relationship between properties of R_n and $F(x)$ is obtained.

THEOREM 1

A necessary and sufficient condition for u to be persistently exciting of order n is that supp F contains at least n points.

Proof

Necessity

Since u(t) is persistently exciting of order n, R_n is positive definite. Thus

$$a^* R_n a = \int_{-\pi}^{\pi} \left| \sum_{k=0}^{n-1} a_k e^{ikx} \right|^2 dF(x) > 0 \quad \text{for all } a. \qquad (7)$$

(Since R_n is a symmetric matrix, it is immaterial whether we choose a to be real or complex.)

Assume that supp F has less than n points. By choosing the vector a the n-1 zeroes of

$$\sum_{0}^{n-1} a_k z^k$$

can be placed anywhere in the complex plane. Now choose a such that

$$\sum_{k=0}^{n-1} a_k e^{ikx}$$

is zero for $x \in \operatorname{supp} F$. The integral (7) then vanishes. Hence a contradiction and supp F has at least n points.

Sufficiency

Assume that supp F contains at least n points. Hence

$$\sum_{k=0}^{n-1} a_k e^{ikx_o} = m > 0$$

for at least one point $x_o \in \operatorname{supp} F$ for any choice of a.

If x_o is an isolated point of supp F, it is a jump point and the integral (7) consequently gets the strictly positive contribution $m \cdot \left(F(x_o+) - F(x_o-)\right)$. If x_o is not an isolated point, F is strictly increasing in a neighbourhood of x_o, and since

$$\sum_{k=0}^{n-1} a_k e^{ikx}$$

is continuous, the integrand is strictly positive in a neighbourhood of x_o. Hence (7) gets a strictly positive contribution from an interval around x_o. In any case (7) is thus non zero for all choices of a (different from the null vector) and the positive definiteness of R_n follows.

Corollary

The signal u is persistently exciting of order n but not of order n+1 if and only if supp F contains exactly n points. In this case F is a jump function with n jumps, and the spectrum F' is a sum of n delta functions.

Example 1

Let $u(t) = \sin \omega_0 t$, $t = 0, 1, 2, \ldots$ $|\omega_0| < \pi$.

The spectrum of the signal u is known to be

$$\phi(\omega) = \frac{1}{4}\left\{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right\} \qquad -\pi < \omega \leq \pi$$

Consequently this signal is persistently exciting of order 2 but of no higher order. Notice that this result holds irrespectively of ω_0, $0 < |\omega_0| < \pi$. However, by proper choice of ω_0 the sequence of numbers $u(t)$ can be periodic with any desired period greater than 2 or even non-periodic. With $\omega_0 = \pi$ the period is 2 and the signal persistently exciting of order 1, since then only the point π belongs to supp F.

Example 2

For identification purposes often certain pseudo random signals are chosen as inputs. These signals are mostly, like e.g. the PRBS signal, periodic. They will consequently have a discrete spectrum.

Fig. 1a shows one such signal and in Fig. 1b its spectrum is given. From Fig. 1b it is inferred that the signal is persistently exciting of order 6 but of no higher order.

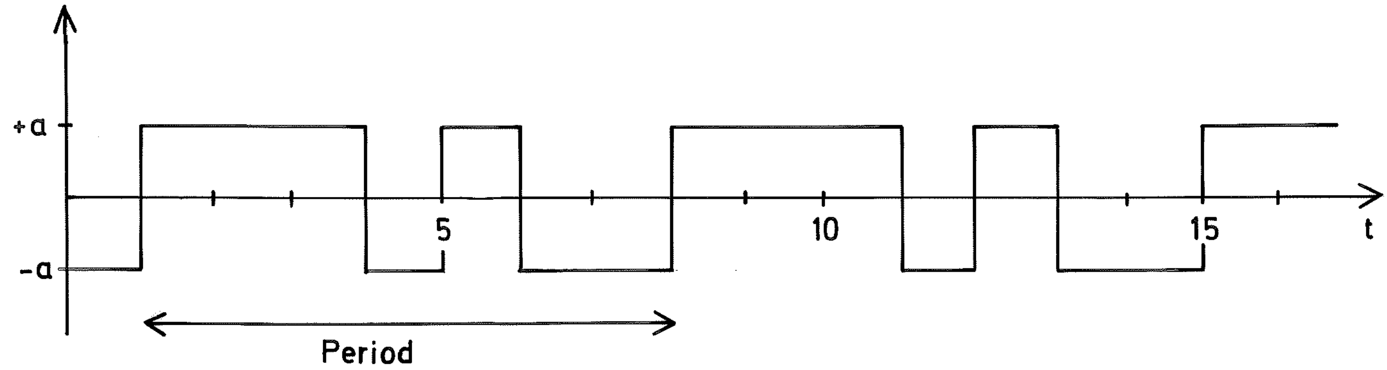

Fig. 1a.

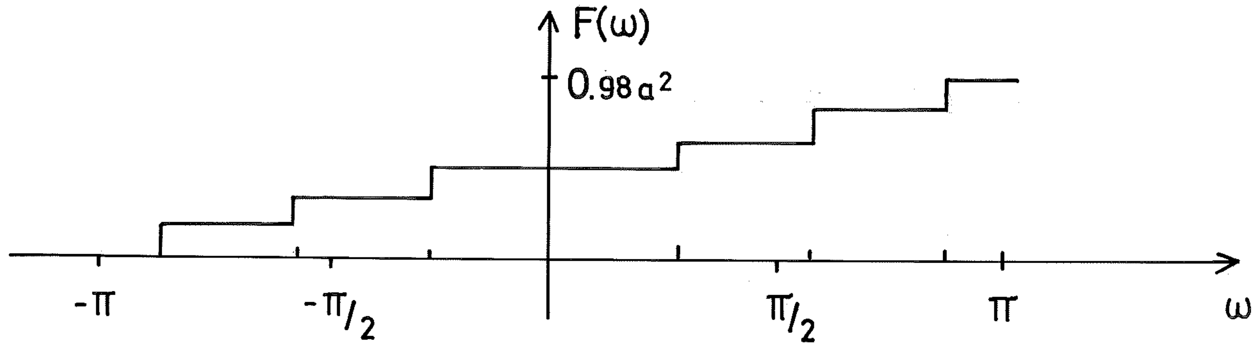

Fig. 1b.

Fig. 1 a) A PRBS-signal with period 7.

b) The spectral distribution function for this signal. The spectrum is thus a sum of 6 delta-functions. (The mean value level -1/7a has been subtracted in accordance with (2); hence there is no contribution at $\omega = 0$).

4. SIGNALS, THAT ARE PERSISTENTLY EXCITING OF ANY FINITE ORDER.

For identification purposes signals that are persistently exciting of any finite order are of special interst, since they allow for models of any order. According to the previous section such signals are characterized in the frequency domain as signals for which supp F contains an infinite number of points. The behavious of R_n as n tends to infinity, however, requires further analysis.

For example, the least squares estimate uses a matrix inverse. For noisy data this inverse exists if and only if the input is persistently exciting of the model order. Now, even if all R_n are positive definite the smallest eigenvalue may tend to zero as $n \to \infty$, so the matrix may be impossible to invert numerically for large n.

We will therefore distinguish between the following three classes for a signal u, that is persistently exciting of any finite order.

Let

$$a = \text{col}\ (1, a_1, a_2, \ldots, a_{n-1})$$

Class A: $\inf \frac{a^T R_n a}{a^T a} = m > 0$

Class B: $\inf \frac{a^T R_n a}{a^T a} = 0$ and $\inf a^T R_n a > 0$

Class C: $\inf a^T R_n a = 0$ and R_n positive definite for all n

where the infimum is to be taken over a and n.

Clearly, class A precisely corresponds to the case when $R_n - mI$ is non-negative definite for all n (m independent of n).

There is a close relationship between the number of m for class A and the spectrum of the signal.

THEOREM 2

u belongs to class A, with greatest lower bound m, if and only if $F'(x) \geq m$ almost everywhere.

Proof

Define a function $r_1(k) = r(k) - m\delta_{0,k}$. Clearly,

$$r_1(k) = \int_{-\pi}^{\pi} e^{ikx} dF(x) - m \int_{-\pi}^{\pi} e^{ikx} dx = \int_{-\pi}^{\pi} e^{ikx} dF_1(x)$$

where $F_1(x) = F(x) - mx$ is a right continuous function.

Now, if $F'(x) \geq m$, then F_1 is non-decreasing and r_1 is non-negative definite according to Section II. Consequently, R_∞ is of class A.

Conversely, if $r_1(k)$ is non-negative definite, F_1 is the corresponding uniquely determined right continuous function and consequently it is non-decreasing. Hence $F'(x) \geq m$ almost everywhere (i.e. where the derivative exists).

COROLLARY

Suppose that N observations of the signal $\{u(t), t = 0, 1, \ldots, N-1\}$ are available.

Let $\hat{r}(k)$ be an estimate of the autocovariance function $r(k)$:

$$\hat{r}(k) = \begin{cases} \frac{1}{N} \sum_{s=0}^{N-k-1} u(s) \quad u(s+k) & N > k > 0 \\ 0 & k \geq N \end{cases}$$

and let $\hat{R}_m$ denote the corresponding covariance matrix. (The mean value is without loss of generality set to zero.)

Form the periodogram estimate of the spectrum:

$$\hat{f}(x) = \frac{1}{N} \left| \sum_{k=0}^{N-1} u(k)e^{-ikx} \right|^2$$

Then $\hat{R}_m \geq \delta I$ (meaning that $\hat{R}_m - \delta I$ is non-negative definite) for all m if and only if $\hat{f}(x) \geq \delta$ for all x.

Proof

Straightforward calculation yields

$$\hat{r}(k) = \int_{-\pi}^{\pi} \hat{f}(x)e^{ikx}dx$$

and the corollary follows from the theorem.

For the distinction between classes B and C we use a result to be found in text-books on analytic functions, see e.g. [5]. As discussed in the next section the result is well-known in prediction theory [3].

THEOREM 3

u is of class B if and only if

$$\int_{-\pi}^{\pi} \log F'(x)dx \text{ exists } (> -\infty)$$

and $\inf F'(x) = 0$.

5. SUMMARY AND CONNECTION WITH PREDICTION THEORY.

Together with the classes A, B and C defined in the previous section we introduce the class D:n for signals that are persistently exciting of order n, but not of order n+1.

Then Theorems 1, 2 and 3 make it possible to associate every signal u with properties (1) and (2) with one of these classes.

The classes can be characterized in the time domain as well as in the frequency domain. The time domain characterization is more suitable for parametric identification purposes, since the covariance matrices then arise naturally. On the other hand frequency domain characterization, i.e. properties of the spectrum of the signal, is easier to understand intuitively. It is also more suitable for considerations on certain transformations of the signal.

Classes A, B and C correspond to signals that are persistently exciting of any finite order. However, only class A gives uniform lower bounds on the eigenvalues of R_n. The classes A and B have, as shown in the next section, strong invariance properties with respect to linear filtering, and if seems reasonable to call signals belonging to these two classes persistently exciting of infinite order.

Similar distinctions apply in prediction theory. It was shown by Wold [7] that classes A and B, characterized by

$$\lim_{n\to\infty} \det R_{n+1}/\det R_n > 0$$

correspond to stochastic processes that cannot be predicted with any desired degree of accuracy. The criterion

$$\int_{-\pi}^{\pi} \log F'(x)dx > -\infty$$

for these classes is due to Kolmogorov [6].

Class D:n consists of signals that can be predicted exactly if n former values are known. However, as indicated in example 1 there is no direct implication between periodicity and persistent excitation other than that a periodic signal with period n is not persistently exciting of order n. Class C is kind of a limiting case of D:n as n tends to infinity. These signals can also be predicted exactly, but it requires knowledge of infinitely may previous values.

Classes A and B consist of signals that, apart from a possible component from class C or class D:n, can be considered as filtered white noise. This is the Wold decomposition theorem [7]. The signals belong to A if and only if the filter in question has no zeroes on the unit circle.

In Table 1 the properties of the classes are summarized.

Table 1.

Class	Characterization in the frequency domain	Characterization in the time domain	Persistent excitation	Predictability
A	$F'(x) \geq m$ almost everywhere	$R_n - mI$ non negative definite for all n. ($\inf \frac{a^T R_n a}{a^T a} = m$)	Persistenly exciting of infinite order	Cannot be predicted with arbirtary accuracy
B	$\int_{-\pi}^{\pi} \log F'(x)dx > -\infty$ $\inf F'(x) = 0$	$\inf \frac{a^T R_n a}{a^T a} = 0$ $\inf a^T R_n a > 0$	Persistently exciting of infinite order	Cannot be predicted with arbitrary accuracy
C	Supp F contains infinitely many points $\int_{-\pi}^{\pi} \log F'(x)dx = -\infty$	$\inf a^T R_n a = 0$ R_n positive definite for all n	Persistently exciting of any finite order	Can be predicted with any desired accuracy if infinitely many former values are known
D:n	F(x) a jump function with n jumps	R_{n+1} singular R_n positive definite	Persistently exciting of order n but not of order n+1	Can be predicted exactly if n former values are known

6. PERSISTENT EXCITATION OF FILTERED SIGNALS.

As an example of the application of the results obtained in the previous sections we will here consider what happens when the signal u is digitally filtered through exponentially stable filters:

$$y(r) = \sum_{k=0}^{\infty} h(k)u(r-k) \tag{8}$$

$$|h(k)| \leq \alpha^k \qquad \alpha < 1$$

The function

$$H(z) = \sum_{k=0}^{\infty} h(k)z^k$$

will thus be analytic on the unit circle. In particular the set of zeroes of H(z) on the unit circle has no cluster point. Consequently $H(e^{ix})$ has only a finite number of zeroes in the interval $-\pi < x \leq \pi$. Furthermore,

$$\int_{-\pi}^{\pi} \log|H(e^{ix})|^2 dx$$

is integrable. (The latter property is true also for weaker conditions on h(k), e.g. $\Sigma\, h(k)^2 < \infty$, see [5].) Let $F_y(x)$ and $F_u(x)$ be the spectral distribution functions for y and u respectively. A straightforward calculation using (2) yields

$$F_y(x) = \int_{-\pi}^{x} |H(e^{i\xi})|^2 dF_u(\xi) \tag{9}$$

Furthermore

$$F'_y(x) = |H(e^{ix})|^2 F'_u(x) \quad \text{a.e.} \tag{10}$$

From (9) we infer that x belongs to supp $F_y(x)$ if and only if either x is an inner point of supp $F_u(x)$ or x belongs to supp $F_u(x)$ and $|H(e^{ix})| > 0$.

From these observations we conclude that:

THEOREM 4

Let u and y be related through (8). If H(z) has no zeroes on the unit circle then y belongs to the same class A, B, C or D:n as u. In particular: if u is persistently exciting of order n then so is y.

If no restriction on the zeroes of H(z) is made, then the transitions $A \to B$ and $D:n \to D:k$ $(k \leq n)$ and only these are possible.

<u>Remark</u>

The invariance of class C is a result of our restrictions on h(k). If we require only $\Sigma\, h(k)^2 < \infty$ then also the transition $C \to D:n$ is possible. However, we may not, even with such a filter, cross the border line $B \to C$. This is natural in light of the prediction interpretation of Section V.

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