Convergence Concepts for Adaptive Structures

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ABSTRACT

When an adaptive system is working in a time invariant environment, it is desirable that the control law converges to a limit. In this report some techniques to establish convergence with probability one are introduced and discussed. A self turning regulator and an automatic classifier are used as examples.

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1. ADAPTIVE STRUCTURES

An adaptive system is defined in [8] as follows:

"An adaptive system is provided with a means of continuously monitoring its own performance or optimum condition and a means of modifying its own parameters by closed loop action so as to approach this optimum."

Adaptive control is motivated for processes which undergo large variations in their dynamics. A common example is an aeroplane [8]. It is not possible to use the same control law for different flight conditions. An adaptive control system solves this problem by monitoring the flight situation and modifying its control law accordingly.

A typical element in an adaptive system is a real-time identifier, which continuously estimates the current values of the process parameters. In [7] several identification algorithms, capable of monitoring time-varying parameters are investigated.

The control law is then based on the estimates obtained in this way.

An interesting special case of an adaptive control system is when the process parameters are known to be constant but the values are unknown. Such situations arise in many fields of application. One solution is to first identify the process parameters and then compute and implement a suitable control law, a scheme that must be repeated each time some change in the process occurs. Another, more attractive solution is to use an adaptive control system, into which is fed the a priori information that the process parameters are constant. In [1] is discussed one such control system. This consists of a real time least squares estimator, the estimates of which are used to compute a minimum

Other examples of adaptive structures working in time invariant environment are automatic classifiers [3]. The terms self learning systems and unsupervised pattern recognition are also used in the same meaning.

An automatic classifier is a system into which is fed a sequence of objects, characterized as points in a feature space. The system then classifies the objects into a fixed number of classes.

At each time the classifier has a rule (separating surfaces in the feature space) how to classify a given object. Since no a priori knowledge of the class structures is available, the classifying rule has to be continuously updated according to the outcomes of the classification.

In [2] some specific automatic classifiers are discussed.

In these and similar cases the important question is whether the control laws (decision rules) converge to a limit. Only after this question has been answered it is relevant to investigate the properties of the limit control. However, the convergence problem is in general difficult to solve, because of the feedback structure of the adaptive system. In general a non-linear, time-varying stochastic difference (or differential) equation governs the behaviour of the system.

In this report some techniques to solve the convergence problem are introduced and discussed. In chapter 2 is made an attempt to describe the behaviour of a general adaptive system. Chapter 3 is devoted to theorems that give conditions on the adaptive systems, described as in chapter 2, that assure convergence with probability one. Some theorems that may prove useful when applying the technique are given in chapter 4. In chapter 5, finally, the examples mentioned above are investigated as regards convergence.

2. AN APPROACH TO GENERAL DESCRIPTION OF ADAPTIVE STRUCTURES

A schematic picture of an adaptive control system is given in fig. 2.1

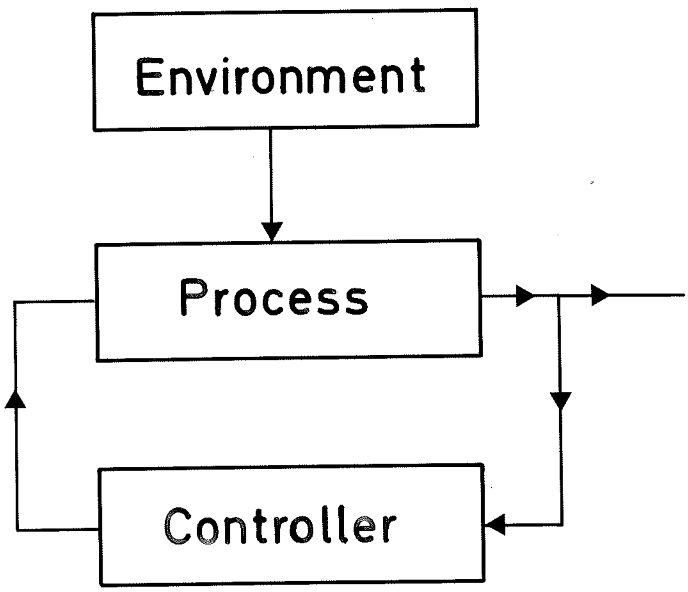

Fig. 2.1

The controller determines the input to the process from the current output and previous inputs and outputs. We may think of the controller as a rule how to form inputs to the process, where the rule itself is varying and depends on old inputs and outputs.

Heuristically, the current control rule is a reflection of the current knowledge of the system.

It is therefore feasible to separate the controller into two functions:

1) One part that condenses the information from old inputs and outputs into an estimate of the current situation of the process. This part we will call the estimator.

2) One part that uses this estimate to determine a control rule, a "control vector" which in turn is used to determine the input to the process. This part we will call the controller.

This structure is illustrated in fig. 2.2, which is taken from [8].

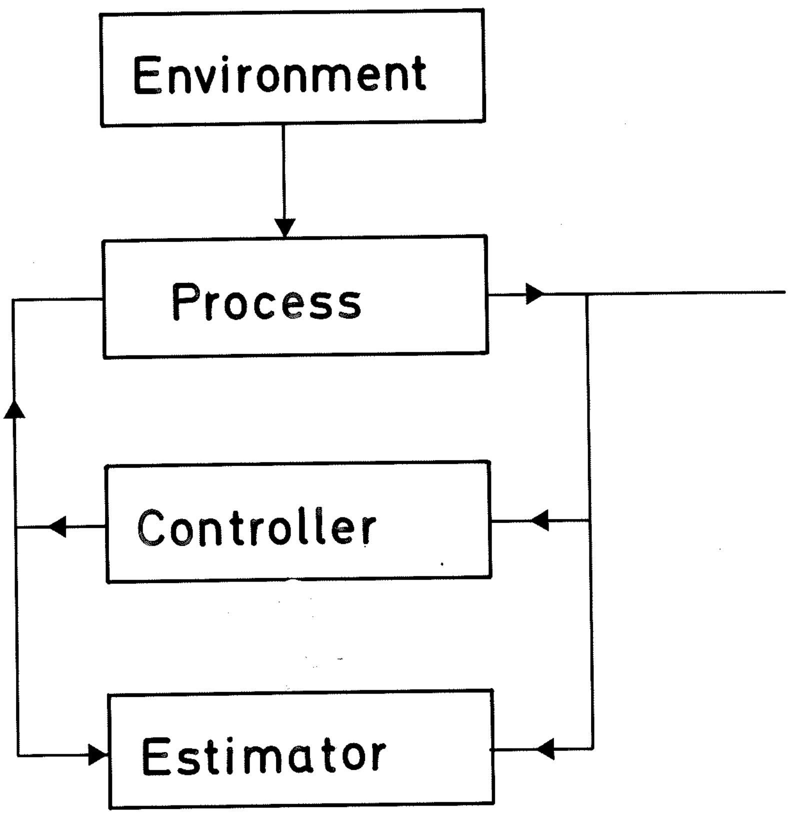

Fig. 2.2

In the examples in chapter 1 the control vector is the parameters of the minimum variance control law and a parameter vector describing the separating hypersurfaces in the feature space, respectively.

A more detailed description of how a specific adaptive system works should include rules how to form estimates from input and output and how to form inputs to the process from current and old estimates and outputs.

These rules could be defined by giving certain algorithms.

Instead, we will here make an "algorithm-free", stochastic, approach to the description.

There are several stochastic processes (all of which may be vector valued) connected with the system in fig. 2.2:

- The "environmental" or noise process
  $\{e_k, k = 0, 1\ldots\}$

- The estimate process, i.e. the sequence of estimates
  $\{x_k, k = 0, 1\ldots\}$

- The "control vector" process
  $\{c_k, k = 0,1\ldots)\}$
  c_k is meant to be the state of the controller, the rule how to form inputs from outputs, rather than the input itself.

Denote the corresponding σ-algebras by
$\mathcal{E}_k$: The σ-algebra generated by $\{e_o\ldots e_k\}$

$\mathcal{X}_k$: " $\{x_o\ldots x_k\}$

$\mathcal{C}_k$: " $\{c_o\ldots c_k\}$

All those processes are of course interrelated. In fact, all processes are generated by the noise process (and possibly the r.v. x_o) in the meaning that c_k and x_k belong to $\mathcal{E}_k$. This means that the outcomes of c_k and x_k are known as soon as the outcomes of $e_o \ldots e_k$ are knwon. Furthermore, c_k belongs to $\mathcal{X}_k$. In fact c_k can be taken to belong to the σ-algebra generated by $\{x_k\}$. A complete description of the adaptive system should thus give the rules how x_k and c_k are formed from $e_o \ldots e_k$. This would include knowledge of how the process parameters depends on $e_o \ldots e_k$.

Such a description may, however, be unnecessary complex. In most applications x_k and c_k depend on the noise history only via some old values of x and c. In terms of algorithms, this means that there exist recursive algorithms, or stochastic difference-equations for x_k and c_k. In terms of the present approach we rather consider the random variables

$$E(x_{k+1} - x_k | \mathcal{X}_k) = g_k(x_k, .., x_1)$$

As mentioned above, in many cases g_k depends only on a fixed (finite) number of variables:

$$g_k = g_k(x_k, ., x_{k-r})$$

We could call the sequence of r.v's

$$Y_k = E(x_{k+1} - x_k | \mathcal{X}_k) = g_k(x_k , ..x_1)$$

an <u>estimator-oriented description</u> of the adaptive system.

Similarily, the r.v's

$$Z_k = E(c_{k+1} - c_k | \mathcal{C}_k) = h_k(c_k, .., c_1)$$

form a <u>controller-oriented description</u>. Notice that the above descriptions do not require a complete knowledge of the process parameters.

Although the controller and the estimator are treated very similarly, they do not have symmetrical structure since c_k belongs to $\mathcal{X}_k$, but x_k does not belong to $\mathcal{C}_k$.

In case the process parameters are constant, but unknown, an alternative approach is possible.

Suppose the information-flow A in fig. 2.2 is broken so that the controller all the time is fed with a constant estimate X^o. If the process parameters are constant it is then reasonable to assume that the estimates converge to some limit $G(x^o)$, depending on x^o.

We may call the function $G(x)$ a long-time range, estimator-oriented description of the adaptive system.

Similarily, assume that the flow A is disconnected when the control is c^o and remains so until the estimator has converged to some value x^1. This limit estimate is then fed into the controller and a new control vector $H(c^o)$ results. The function $H(c)$ then forms a long-time range, controller-oriented description of the system. Although $G(x)$ and $g_k(x_k,..,x_{k-r})$ ($H(c)$ and $h_k(c_k,.,c_{k-r})$) conceptionally related, the relationship is quantitative only for a certain class of estimators (controllers):

Definition:

An estimator (controller) is said to belong to class A if

i) The estimate (control vector) at time $m+n, x_{m+n}$ (c_{m+n}) is a linear combination of the estimate (control vector) at time n, and the estimate (control vector) it would arrive at a time m+n if started up *) (if both estimator and controller are started up *)) at time n: $X_m^{(n)}$ ($C_m^{(n)}$)

$$X_{m+n} = (1 - \alpha_{m,n})X_b + \alpha_{m,n} X_m^{(n)} \qquad 0 \leq \alpha_{m,n} \leq 1$$

(C_{m+n} similarily)

ii) $\alpha_{m,n} = t$, const $\quad 0 < t < 1,\ n \to \infty \Longrightarrow \ m \to \infty$

$\alpha_{m,n} \to 1 \quad m \to \infty \quad \forall n$

$\alpha_{m,n} \to 0 \quad n \to \infty \quad \forall m$

iii) If a constant estimate $x \in X$ is fed into the controller, the estimates converge to a limit $G(x)$ with probability one

((iii) In a constant control $c \in C$ is applied, a limit estimate x is obtained, which gives the new control $H(c)$)

*) The start-up may require some initial values of $x(c)$. If so, there are to be determined in the same way as for the initial start-up.

iv) If the estimates fed into the controller (the control) vary (ies) in a sufficently small neighbourhood of x^o (c^o), then the obtained estimates (control vectors) vary in an arbitrarily small neighbourhood of $G(x^o)$ ($H(c^o)$).

Remark 1. The lack of symmetry between estimator and controller is again obvious. The definition of property A for a controller includes several assumptions about the estimator but not vice versa.

Remark 2. If $c_k = h_1(x_k)$ (a normal case) and the limit estimate $G(..)$ depends only on the constant control applied then

$G(x) = g(h_1(x))$ for some g

and

$H(C) = h_1(g(c))$

Intuitively speaking, property A implies that the expected trajectories in X- (C-) space are straight lines when a constant control is applied. In that case, loosely,

$E(x_{k+1}-x|x_1,\ldots, x_k=x^o)$ almost equals $G(x^o)$

$(E(c_{k+1} - c_k|c_1,..c_k=c^o)$ almost equals $H(c^o)$

so that the long time range approach is closely related to the expected value approach. However, in practical situations it may be easier to determine limit estimates rather than study limit properties of expected values.

It will be shown in chapter 5 that class A is not so small a class of estimators as it might seem. Examples of class A-estimators include both real-time least squares (which is equivalent to Kalman filter identification, see [7]) and stochastic approximation. These algorithms are the real-time identification algorithms discussed in [7].

3. THEOREMS ON CONVERGENCE

In this chapter sufficient conditions for convergence of adaptive structures (estimate and control vector) are given.

3.1. Heuristic approach

First, to make an intuitive approach, we consider the sequence of estimates as a sequence of points in some estimate space.

Apparently, a crucial question for the possible convergence of the sequence is: Suppose we all previous estimates, where do we expect the next to come?

The question could be answered in different ways. If the expected value of the next estimate depends only on the actual estimate and not on the previous ones, which is the case e.g. when the estimate process is a Markov process, then there is a unique direction associated with each point in the estimate space. These directions also define an ordinary differential equation.

We will in a theorem in this section show that the convergence question of the process is related to stability properties of this differential equation. However, if the expected value of the next estimate depends not only on the actual one but also on all previous ones, several directions are associated with each point in the estimate space. There are two ways to attack this problem. In all practical situations the expected value of the next estimate depends on all previous estimates only via a fixed number of variables. Hence it is possible to include these variables in the estimate vector and arrive at the situation that was described first. However, this might yield an estimate vector of an unnecessarily large dimension. A second approach is then to accept the multiple directions associated with each point in the estimate space, and investigate the propeties of these "directional cones". One theorem will be concerned with this approach.

Obviously, the relevant question is "where will the next estimate come?" rather than "where would the limit estimate come if we fixed the control law?". On the other hand, the second question often is more easily answered. For class A algorithms, the expected path in estimate space is a straight line, when the control law is held fixed. Thus the expected direction from the actual towards the next estimate coincides with the direction towards the limit estimate for fixed control law.

For class A algorithms is it hence possible to consider limit values rather than expected values, which might be a great simplification in practical cases. One theorem will be concerned with this situation.

3.2. The case when an ODE can be associated with the convergence problem

We will first consider the case when the expected direction towards the next outcome of the process depends only the last one or on a finite number of previous ones.

To this end consider a stochastic process $(x_k, k = 0, \ldots)$, which may be taken as either the estimate or the control vector process.

Theorem 3.1 gives necessary conditions on the random variables

$E(x_{k+1}-x_k | x_o, \ldots, x_k)$

to assure convergence of x_k to a limit with probability one.

THEOREM 3.1

Consider a vector valued stochastic process $(x_k, k = 0, \ldots)$. Denote the σ-algebra generated by $(x_o, \ldots x_k)$ by $\mathcal{X}_k$

Let

$$E(x_{k+1}-x_k | \mathcal{X}_k) = \alpha_k\, h(x_k)$$

where α_k is a decreasing sequence of possible numbers such that

$$\sum_1^\infty \alpha_k = \infty \quad \text{and} \quad \sum_1^\infty \alpha_k^2 < \infty$$

Assume that

$|x_{k+1}-x_k| < \alpha_k M$ almost everywhere (a.e.) for all k

Let x_k belong to a bounded region Ω_1 a.e.

Then, if the ordinary differential equation

$$\frac{dx}{dt} = h(x) \tag{3.1}$$

is asymptotically stable with stationary point x^o. and with domain of attraction (see 4) Ω_2 where $\bar{\Omega}_1 \subset \Omega_1$,
then

$x_k \rightarrow x^o$ with probability one.

Proof

According to the converse Lyapunov theorem on global stability (see e.g. Krasovskij 5) there exists a positive definite function $v(x)$ in Ω_2, with continuous partial derivatives of any order, such that

$\frac{d}{dt}\, v(x)$ is negative definite in Ω_1 (the time derivative is to be taken along trajectories of (3.1)). Furthermore $0 \leq v(x) \leq 1$.
Consider $v_k = v(x_k)$!

The idea of the proof is that v_k is "almost" a supermartingale. If v_k were supermartingale it would converge to a limit a.e., and it would not be too difficult to show that such a limit necessarily must equal zero with probability one. Now, v_k is a supermartingale only ourside a decreasing region around $v = 0$, so that it seems reasonable to expect $\underline{\lim}_{k\to 0}\, v_k = 0$.

It then remains to show that $\overline{\lim}_{k\to 0}\, v_k$ cannot be strictly positive. Intuitively, this is clear, since then v_k would have to "strive against the stream" an increasing number of steps, infinitely many times.

More formally, consider

$$E(v_{k+1}-v_k|\mathcal{X}_k) = E(v_x^T(x_k)(x_{k+1}-x_k)|\mathcal{X}_k)+ E(x_{k+1}-x_k)^T v_{xx}(\xi)$$

$$(x_{k+1}-x_k)|\mathcal{X}_k) = v_x^T(x_k)E(x_{k+1}-x_k\ |\mathcal{X}_k)+ \alpha_k\ K(x_k,x_{k+1}) =$$

$$= \alpha_k\{v_x^T(x_k)h(x_k)+K(x_k,x_{k+1})\} = \alpha_k\{\dot{v}(x_k)+K(x_k,x_{k+1})\} \qquad (3.2)$$

$$\leq \alpha_k^2\cdot C$$

since $|K(x_k,x_{k+1})| < C\cdot\alpha_k$ and $\dot{v}(x_k) \leq 0$.

Furthermore, as long as $\dot{v}(x_k) \leq -2C\alpha_k$ $\{v_k,\mathcal{X}_k\}$ is a supermartinagale i.e. the left member of (3.2) is non-positive.

Define

$$A_k = \{x|\dot{v}(x) \geq -2C\alpha_k\}$$

and

$$\beta_k = \sup_{x \in A_k} v(x)$$

Thus $v(x_k) > \beta_k \Rightarrow \dot{v}(x_k) \leq -2C\alpha_k$ (3.3)

Now, $\alpha_k \searrow 0$ implies $A_k \searrow \{x^o\}$, for otherwise we could find a sequence $x_k; x \in A_k$ with $x_k \to x^1 \neq x^o$ and $\dot{v}(x_k) \to 0$ as $k \to \infty$.

Consequently $\beta_k \searrow 0$

Form from v_k the new sequence of r.v's $w_k^{(n)}$:

$$w_k^{(n)} = \begin{cases} 1 & \text{if } k \leq n \\ v_k & \text{if } k > n \end{cases}$$

Introduce the optimal r.v. γ_n which takes the value k if $w_k^{(n)} < \beta_k$ and the optimal variable $\gamma_n^k = \min(\gamma_n, k)$. Obviously γ_n^k is optimal relative to $\mathcal{X}_k$. Consider the stopped process

$$z_k^{(n)} = w_{\gamma_n^k}^{(n)}$$

Fig. 3.1 shows how this is formed from v_k:

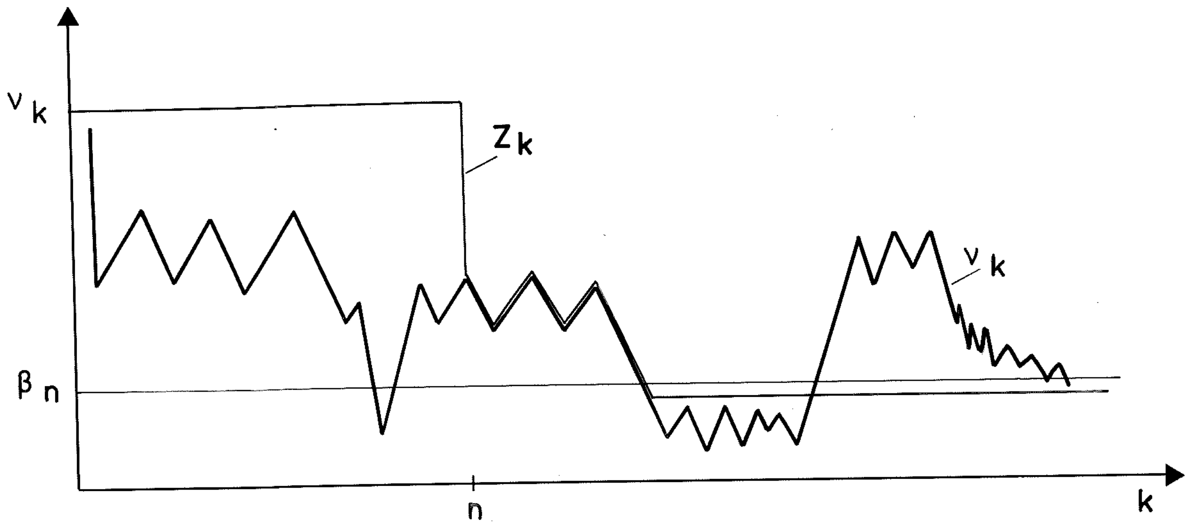

Fig 3.1

$Z_k^{(n)}$ is easily verified to be a supermartingale. It consequently a.e. tends to a limit $Z^{(n)}$, which is a r.v..

Define the set

$$B_n = \{\omega | Z^{(n)}(\omega) > \beta_n\} = \{\omega | \gamma_n = \infty\}$$

Now,

$$E(Z_k^{(n)} - Z_o^{(n)}) = E(\sum_{j=0}^{k-1} (Z_{j+1}^{(n)} - Z_j^{(n)}))=$$

$$= E(\sum_{j=0}^{k-1} E(Z_{j+1}^{(n)} - Z_j^{(n)} | \mathcal{X}_j)) \qquad (3.4)$$

Denote

$$E(Z_{j+1}^{(n)} - Z_j^{(n)} | \mathcal{X}_j) = d_j$$

$$d_j = \begin{cases} 0 \text{ if } j \leq n \\ \leq \dot{V}(x_j)\cdot\alpha_j/2 & \text{if } n < j < \gamma_n \\ 0 \text{ if } j \geq \gamma_n \end{cases}$$

Since $0 \leq Z_k^{(n)} \leq 1$ (3.4) is thus a uniformly limited sum of non-positive numbers.

This implies that $\left|\sum_{j=0}^{\infty} E(d_j)\right| < \infty$

On B_n d_j is less equal $-2C\alpha_n \alpha_j$ for all j.

If $P(B_n) > 0$ $\overline{\lim_{j\to\infty}} \frac{1}{\alpha_j} E(d_j) = \delta < 0$ and

$$-\sum_{j=N}^{\infty} E(d_j) \geq -\sum_{N}^{\infty} \alpha_j \delta/2 = -\delta/2 \sum_{N}^{\infty} \alpha_j = \infty$$

since $\Sigma\alpha_j$ diverges, which is a contradiction. Consequently $P(B_n) = 0$. That is $Z^{(n)} \leq \beta_n$ a.e. and consequently $\forall n$ there exists with probability 1 an $N, \infty > N > n$ such that $v_N < \beta_N$

$\beta_n \searrow 0$ then implies $\underline{\lim}_{k\to\infty} v_k = 0$ a.e.

Furthermore γ_n is a.e. finite.

Now consider the process

$$w_k = v_k + \sum_{j=k}^{\infty} \alpha_j^2 C$$

$$E(w_{k+1} - w_k | \mathcal{F}_k) = \alpha_k\{(\dot{v}(x_k) + K(x_k, k_{k+1})\} - \alpha_k^2 C \leq 0.$$

Hence w_k is a positive supermatingale and $\lim_{k\to\infty} w_k = w$ exists with probability one. But $\underline{\lim}_{k\to\infty} w_k = \underline{\lim}_{k\to\infty} v_k = 0$ and

Hence $v_k \to 0$ with probability 1.

But $v_k = v(x_k)$ and v is positive definite. Hence $x_k \to x^o$ a.e.

Q.E.D.

Coroallary

It follows from the proof that even without the condition $\sum_{k=}^{\infty} \alpha_k^2 < \infty$ there a.e. exists a subsequence

$$x_{n_k}(\omega) \to x^o$$

Remark 1

Let x_k be generated by a simple stochastic approximation algorithm:

$$x_{k+1} = x_k + \alpha_k(Z_k - x_k)$$

where $E(Z_k) = \Theta$ and Z_k independent of $x_o, x_1, \ldots x_k$.

We get

$$E(x_{k+1} - x_k | \mathcal{X}_k) = \alpha_k(\Theta - x_k)$$

The corresponding ordinary differential equation is

$$\dot{x} = -x + \Theta$$

which surely is globally asymptotically stable with stationary point .
Hence the convergence of x_k to Θ with probability 1 is obtained from theorem 3.1.

Remark 2

The function $v(x_k)$ resembles the concept of stochastic Lyapunov function introduced by Kushner [6] . However, as shown in the proof, $v(x_k)$ is not really a supermartinagale and thus no stochastic Lyapunov function. Neither need x_k be a Markov process.

The assumption that the expected difference

$$E(x_{k+1} - x_k | \mathcal{X}_k)$$

shall depend only on x_k is unnecessarily limiting. Since α_k decreases towards zero it is seen that any fixed number of consequtive outcomes will come arbitrarily close to each other.

Hence, we have the following theorem:

THEOREM 3.2

Suppose the assumptions of theorem 3.1 hold, with instead

$$E(x_{k+1} - x_k | \mathcal{X}_k) = \alpha_k H(x_k, x_{k-1}, \dots, x_{k-r})$$

Assume that, with $h(x) = H(x, x, \dots x)$

$$|H(x_1, x_2, \dots, x_{r+1}) - h(x)| \leq L \cdot \sup_{i=1,\dots,r+1} |x_i - x| \quad , x \in \Omega_1$$

Then the conclusion of theorem 3.1 holds.

PROOF

We have instead of (3.2)

$$E(v_{k+1} - v_k | \mathcal{X}_k) = \alpha_k \{\dot{v}(x_k) + v_x^T(x_k) f(x_n, x_{n-1}, \dots, x_{n-r})$$

$$+ K(x_n, x_{n+1})\}$$

where

$$|f(x_n, x_{n-1}, \ldots, x_{n-r})| = |H(x_n, x_{n-1}, \ldots, x_{n-r}) - h(x_n)| \leq$$

$$\leq L \cdot \sup_j |x_j - x_n| \leq L \cdot M \cdot r \cdot \alpha_{n-r}$$

$v_x(x_n)$ is uniformly bounded, say

$$|v_x(x_n)| \leq P \text{ for } x_n \in \Omega_1$$

Then the right member of (3.2) is

$$\alpha_k \cdot \alpha_{k-r}(C + P \cdot L \cdot r \cdot M)$$

rather than

$$\alpha_k^2 \cdot C$$

but if we take

$$C' = C + P \cdot L \cdot r \cdot M$$

we can continue as in the proof of theorem 3.1.
and

$$\sum_j \alpha_j \alpha_{j-r} \leq \sum_j \alpha_{j-r}^2 < \infty$$

Q.E.D.

3.3. The case when multiple directions are associated with each point

If the expected difference

$$E(x_{k+1} - x_k | \mathcal{X}_k)$$

depends on all previous x_j , there is no differential equation (3.1) associated with the convergence problem . However, sufficient conditions for convergence can be established by assuming the existence of a "Lyapunov function" for the "directional cones".

THEOREM 3.3

Consider a vector valued stochastic process (x_k, k=0,1,...). Denote the σ-algebra generated by $(x_o,\ldots,x_k)$ by $\mathcal{X}_k$.
Let

$$E(x_{k+1} - x_k | \mathcal{X}_k) = \alpha_k \, h_k(x_k, x_{k-1},\ldots,x_o)$$

where α_k is a decreasing sequence of positive numbers such that

$$\sum_1^\infty \alpha_k = \infty \qquad \text{and} \qquad \sum_1^\infty \alpha_k^2 < \infty$$

Assume that

$$|x_{k+1} - x_k| < \alpha_k \cdot M \text{ a.e. for all k.}$$

Let x_k belong to a bounded region Ω_1 a.e.

Assume that there exists a twice continuously differentiable positive definite function v(x) in an open region Ω_2, $\overline{\Omega}_1 \subset \Omega_2$ i.e.

$$\begin{cases} v(x^o) = 0 \\ v(x) > 0 \qquad x \in \Omega_2 \quad x \neq x^o \end{cases}$$

such that

$v_x^T(x)h_k(x,z_1,z_2,\dots,z_k)$ is

negative definite in x, independently of z_i, i.e.

$$v_x^T(x)h_k(x_1,z_1,z\ ,\dots,z_k) = \begin{cases} = 0 & x = x^o \\ < \delta(x) < 0 & \text{for } x \in \Omega_2 \text{ and } x \neq x^o \text{ for all } z \end{cases} \qquad \forall z$$

Then $x_k \to x^o$ with probability one.

PROOF

If we denote $v_x^T(x)h_k(x,z_1,z_2,\dots z_k) = \dot{v}(x)$ the proof of theorem 3.1 can be applied with no changes.

A disadvantage of this condition is that it is not necessary one. This is probably true for the criterion in theorem 3.1 and 3.2. If (3.1)in some significant way fails to be globally asymptotically stable then x_k does not converge to a constant limit with probability one.

We have so far only considered conditioneing with respect to σ -algebras generated by the process itself. In practical cases it might be easier to condition with respect to larger σ-algebras.

Theorem 3.4 solves this problem in the spirit of theorem 3.3.

THEOREM 3.4

Consider a vector valued stochastic process (x_k, k=0,1,...). Let $\mathcal{G}_k$ be a sequence of σ-algebras such that

$\mathcal{G}_{k+1} \supset \mathcal{G}_k$ and $x_k \in \mathcal{G}_k$

Assume that

$$E(x_{k+1} - x_k | \mathcal{G}_k) = \alpha_k G_k(z_k, z_{k-1}, \ldots, z_o) \cdot h(x_k)$$

where α_k is a sequence of positive numbers such that

$$\sum_1^\infty \alpha_k = \infty \quad \text{and} \quad \sum_1^\infty \alpha_k^2 < \infty$$

Assume that

$$|x_{k+1} - x_k| < \alpha_k M \text{ a.e. for all } k$$

Let x_k belong to a bounded region Ω_1 a.e.

Assume that there exists a twice continuously differentiable, positive definite function $v(x)$ in an open region Ω_2, $\bar{\Omega}_1$ Ω_2 i.e.

$$\begin{cases} v(x^o) = 0 \\ v(x) > 0 \quad \text{for } x \quad \Omega_2 \text{ and } x \neq x^o \end{cases}$$

such that

$$v_x^T(x)\, G_k(z_k, z_{k-1}, \ldots, z_o) \cdot h(x)$$

is negative definite in x i.e.

$$v_x^T(x) G_k(z_k, \ldots, z_o) h(x) \begin{cases} = 0 \text{ for } x = x^o \text{ or } z \in \Omega_k^o \\ < 0 \text{ for } x \in \Omega_2 \text{ and } x \neq x_o \text{ and } z \notin \Omega_k^o \end{cases}$$

where $P(\Omega_k^o | \mathcal{X}_k) = 0$ a.e. where $\mathcal{X}_k$ is the σ-algebra generated by $(x_o, \ldots, x_k)$

Then

$x_k \to x^o$ with probability one.

PROOF

Obviously

$$E(v_x^T(x_k)G_k(z_k, z_{k-1}, \dots, z_o)h(x_k) | \ _k) \leq w(x_k) < 0$$

for $x \in \Omega_2$, $x \neq x^o$

Since $\mathcal{X}_k \subset \mathcal{G}_k$ we get

$$E(v(x_{k+1})-v(x_k) | \mathcal{X}_k) = E\{E(v(x_{k+1})-v(x_k) | \mathcal{G}_k) | \mathcal{X}_k\} =$$

$$= E\{E(v_x^T(x_k)(x_{k+1} - x_k) | \mathcal{G}_k) + E(x_{k+1}-x_k)^T v_{xx}(\xi)(x_{k+1}-x_k) | \mathcal{G}_k) | \mathcal{X}_k\}$$

$$= E\{\alpha_k \ v_x^T(x_k)G_k(z_k, z_{k-1}, \dots, z_o)h(x_k) + \alpha_k K(x_k, x_{k+1}) | \mathcal{X}_k\}$$

$$\leq \alpha_k \{w(x_k) + K(x_k, x_{k+1})\}$$

as in (3.2)

We may now proceed as in the proof of theorem 3.1 with

$w(x_k)$ instead of $\dot{v}(x_k)$

In spite of the rather special structure of this theorem it turns out to be useful in practical situations, as shown in chapter 5.

3.4. Extensions of theorem 3.1 - 3.4

In theorems 3.1, 3.2 and 3.4 we have assumed that the function

$$E(x_{k+1} - x_k | \mathcal{X}_k)$$

does not depend on k. This restriction can be removed as follows:

EXTENSION 1 TO THEOREM 3.1 - 3.4

Assume that the right members of the conditional random variables of theorems 3.1 - 3.4 have an additional term $\beta_k \tilde{g}_k(x_k,\ldots,x_o)$

Assume that

$$|\tilde{g}_k(x_k,\ldots,x_o)| < L \quad \text{for } x_k,\ldots,x_o \in \Omega_1$$

and that

$$\sum_{k=1}^{\infty} \beta_k < \infty$$

Then all the theorems still hold

PROOF

The additional term causes the right member in (3.2) to be

$$C\alpha_k^2 + L\beta_k$$

Still

$$\sum_1^{\infty} (C\alpha_k^2 + L\beta_k) < \infty$$

Furthermore we have

$$E(v_{k+1}-v_k|\mathcal{X}_k) = \alpha_k\{\dot{v}(x_k) + K(x_k,x_{k+1}) + \frac{\beta_k}{\alpha_k}\tilde{g}(x_k,\ldots,x_o)\}$$

But $K(x_k,x_{k+1}) + \frac{\beta_k}{\alpha_k}\tilde{g}(x_k,\ldots,x_o)$
still tends to zero as $k\to\infty$ so the proofs are not affected.

The theorems so far have been concerned with the question of convergence with probability one. If the domain of attraction Ω_2 does not include $\bar{\Omega}_1$, we may conclude convergence with probability less than one.

EXTENSION 2 TO THEOREM 3.1 - 3.4

Assume that the domain of attraction Ω_2 in theorem 3.1 - 3.4 does not necessarily include $\bar{\Omega}_1$. Then the conclusion is that

$$x_k \to x^o$$

with at least the same probability as x_k remains in any closed subset $\tilde{\Omega}$ of Ω_2.

PROOF

Consider the same stochastic process, v_k as before and option it with respect to the event that x_k leaves $\tilde{\Omega}$ $(v_k > 1-\delta)$ to obtain the stopped process $\tilde{v}_k$. Then apply the proofs to this process bearing in mind that the probability that it actually is not stopped is equal to the probability that x_k does not leave $\tilde{\Omega}$.

3.5. A theorem for class A-algorithms

We now turn to the long-time range approach. If the limit values for a process x_k for a fixed control law exist $(=G(x))$ as assumed for class A algorithms, then it is very reasonable to expect that

$$E(x_{k+1} - x_k|\mathcal{X}_k)$$

either equals or tends to

$$-x + G(x)$$

as $k \to \infty$.

Hence theorem 3.1 or extension 1 of it is applicable. However, it may practically be easier to establish the limit values without reference to and examination of the conditional variables.

Therefore we give a separate theorem, with an independent proof for the long-time range formulation.

THEOREM 3.5

Consider a class A-algorithm, updating the vector x_k(which may be either the estimate or the control vector).

Let the corresponding limit function be F(x). Assume that **x** belongs to a bounded region Ω_1 a.e.

If the ordinary differential equation

$$\frac{dx}{dt} = -x + F(x) \qquad (3.6)$$

is asymptotically stable with stationary point x and domain of attraction Ω_2, $\bar{\Omega}_1$ Ω_2, then $x_k \to x$ with probability one.

PROOF

The proof very much resembles the proof of theorem 3.1. As there, we infer the existence of a Lyapuniv function v(x) and study the $v_k = v(x_k)$ for realization of x_k. First we show that

$$\underline{\lim}_{n \to \infty} v_n = 0$$

Assume that $\underline{\lim}_{n \to \infty} v_n = \delta > 0$

Take a subsequence v_{n_k} such that $v_{n_k} \to \delta$ as $k \to \infty$. Then extract another subsequence n_k from this, such that $x_{n'_k} \to x^o$, which is possible due to the Bolzano-Weirstrass theorem.

The point x^o has the properties

$$v(x^o) = \delta$$

$$\dot{v}(x^o) = -\delta^o < 0$$

Let $\tilde{x}(t)$ be the solution of (3.6) with

$$\tilde{x}(0) = x^o$$

$$\tilde{x}(t) = x^o + t(F(x^o) - x^o) + o(t) \qquad t \to 0$$

Denote

$$x^o + t(F(x^o) - x^o) = \hat{x}(t)$$

Then

$$v(\hat{x}(t)) = v(x^o) - \delta^o t + o(t) \qquad t \to 0 \qquad (3.7)$$

For sufficiently small t, $0 < t < t^o$, the ordo term is less than $\frac{1}{2}\delta^o t$
Now since $\underline{\lim}_{k\to\infty} v_k = \delta = v(x^o)$
we have for $n'_k > N_o(t)$

$$v_{n'_k+m} > v(x^o) - \frac{1}{4}\delta^o t \qquad \forall_m,\ n'_k > N_o(t)$$

This, together with (3.7) gives

$$|v(\hat{x}(t)) - v(x_{n'_k+m}) > \frac{\delta^o t}{4} \quad \forall_m,\ n'_k > N_o(t),\ t < t^o \qquad (3.8)$$

The function v is Lipschitz continuous in Ω_1. Let the Lipschitz constant be K_L

Then from (3.8)

$$|\hat{x}(t) - x_{n'_k+m}| > \frac{\delta^o t}{4K_L} \qquad \forall_m,\ n'_k > N_o(t), \quad t < t^o$$

or, using the definition of $\hat{x}(t)$

$$\frac{\delta^o t}{4K_L} < |x^o + t(F(x^o) - x^o) - x_{n'_k+m}| \tag{3.9}$$

Since x_k is obtained from a class A algorithm:

$$x_{n'_k+m} = (1 - \alpha_{m,n'_k})x_{n'_k} + \alpha_{m,n'_k} x_m^{(n'_k)}$$

With this inserted in (3.9)

$$\frac{\delta^o t}{4K_L} < |x^o - x_{n'_k}| + |tx^o - \alpha_{m,n'_k} x_{n'_k}| + $$
$$+ |\alpha_{m,n'_k} x_m^{(n'_k)} - tF(x^o)|$$

Now, for a given t, we can choose

$$n'_k > N_1(t) \geq N_o(t)$$

such that x^o is arbitrarily close to $x_{n'_k}$ and such that $m = m(n'_k, t)$ can be chosen so that α_{m,n'_k} is arbitrarily close to t.

Hence

$$|x_m^{(n'_k)} - F(x^o)| > \frac{\delta^o}{8K_L} \quad \text{for } n'_k > N_1(t), \quad m = m(n'_k, t) \text{ and}$$
$$t \leq t_o \tag{3.10}$$

Now for a fixed control law, corresponding to $x = x^o$, $x_m^{(n)} \to F(x^o)$ a.e. as $n \to \infty$ and $m = m(n,t)$, t fixed.

Furthermore according to property (iv) of a class A algorithm, if the control law varies in a sufficiently small region, $x_m^{(n)}$ will lie arbitrarily close to $F(x^o)$ for large n. But this is in contradiction to (3.10) since a sufficiently small t guarantees that x_{m+n_k} is sufficiently close to x^o and thus that

$x_m^{(n')}$ is arbitrarily close to $F(x^o)$

for large n'_k.

Hence $\underline{\lim}_{k\to\infty} v_k = 0$

Suppose now, that

$\overline{\lim}_{k} v_k = \delta > 0$

We will lead this into contradiction in the same way as above.

Since $\underline{\lim}_{k\to\infty} v_k = 0$

the strip $[\frac{\delta}{2}, \frac{\delta}{2} + \delta^o(t)]$, $\delta^o(t) < \delta/2$
will be crossed in each direction infinitely many times. Let v_{n_k} be a sequence of upcrossings, i.e. v_{n_k} enters the strip at $\delta/2$ and leaves it for $n=n_k+m(n_k)$ at $\frac{\delta}{2} + \delta^o(t)$.

Choose a convergent subsequence x_{n_k} of x_{n_k}. Let $x_{n'_k} \to x^o$ and proceed as above.

Thus we arrive at the conclusion that

$\overline{\lim}_{k\to\infty} v_k = 0$ and hence $\lim_{k\to\infty} v_k = 0$.

This is true for all realizations that fulfill the class A properties. But since these are valid with probability one also

$\lim_{k\to\infty} v_k = 0$ and hence $x_k \to x^*$ as $k \to\infty$

with probability one.
Q.E.D.

4. CONNECTION BETWEEN CONTROLLER-ORIENTED AND ESTIMATOR-ORIENTED DESCRIPTIONS

In terms of the long time range description of chapter 2, an intuitive approach to the convergence problem could be described as follows. Suppose that the control law c_k is held fixed for a long time. When the estimates have converged a new control law is determined from the limit estimates: this gives $c_{k+1} = H(c_k)$.

Similarily for the estimates $x_{k+1} = G(x_k)$.

If now the "long time range algorithm" $c_{k+1} = H(c_k)$ converges to a limit it seems reasonable to assume that also the actual algorithms converge. Such arguments are given in [1] and [2].

In fact, theorem 3.5 shows that for certain algorithms the assertion is correct. The criterion could also be replaced by the weatrer one, to consider stability properties of the ODE

$$\dot{c} = - c + H(c)$$

There is in general no free choice which describtion to choose when applying theorem 3.5. Usually it is the estimator which most easily can be written as a class A-algorithm. On the other hand the control vector often has lower dimension than the estimate. It would thus be preferable to work with the controller rather than with the estimator.

In this chapter we will discuss whether and when the two ordinary differential equations

$$\dot{x} = - x + G(x) \tag{4.1}$$
$$\dot{c} = - c + H(c) \tag{4.2}$$

are equivalent with respect to stability properties.

To do so we introduce some more structure in the problem. In chapter 2 it was remarked that normally the control vector is determined from the actual estimate, which has higher dimension:

$c_k = f(x_k) \qquad f : R^n \to R^m \qquad n \geq m$

Furthermore the limit estimate depends only on the control chosen so that

$G(x) = g(f(x)) \qquad g : R^m \to R^n$

Hence also

$H(c) = f(g(c))$.

We are thus left with the problem to discuss the connection between stability properties of

$$\dot{x} = -x + g(f(x)) \tag{4.3}$$

and

$$\dot{c} = - c + f(g(c)) \tag{4.4}$$

Obviously, if x is a solution of (4.3) f(x) is not a solution of (4.4). So (4.4) does not describe how the control law is expected to develop, even if (4.3) yields the expected sequence of estimates.

Nevertheless, if x is a solution of (4.3) f(x) is related to solution of (4.4) and conversely.

The scheme of this chapter will be as follows. First possible convergence points of the algorithms for estimate and control vector are interrelated using the equation (4.3) and (4.4).

Then local stability properties of the ODE's are discussed and finally two theorems on global stability properties are given

THEOREM 4.1

Let the estimator be of class A.
Assume that the control vector $c_k = \tilde{f}(x_k, \ldots x_{k-r})$, where f is continuous.
Let $f(x) = \tilde{f}(x, x, \ldots x)$. Then
a) If $c_n \to c^*$ as $n \to \infty$

then i) $x_n \to g(c^*)$ as $n \to \infty$

ii) c^* is a stationary point of (4.4)

iii) $g(c^*)$ is a stationary point of (4.3)

b) Conversely, if $x_n \quad x^*$ as $n \to \infty$

then i) $c_n \to f(x^*)$ as $n \to \infty$

ii) x^* is a stationary point of (4.3)

iii) $f(x^*)$ is a stationary point of (4.4)

PROOF

a) i) Take $\varepsilon > 0$, since g is continuous $\exists\, \delta$ such that $|c_n - c^*| < \delta \Rightarrow |g(c_n) - g(c^*)| < \varepsilon/2$

Since $c_n \to c^*$ $\exists\, N_1$ such that $n \geq N_1 \Rightarrow |c_n - c^*| < \delta$

Now for class A-estimators,

$$x_{N+m} = (1 - \alpha_{N,m})x_N + \alpha_{Nm}\, x_m^{(N)}$$

$\lim\limits_{m\to\infty} x_{N+m} = \lim\limits_{m\to\infty} x_m^{(N)}$ if existing and (propertiy iv)

$x_m^{(N)} = \Sigma\, \alpha_i x_m^{(N)}(c_i)$ $\quad |c_i - c^*| < \delta \quad \Sigma\alpha_i = 1$

$|x_m^{(N)}(c_i) - g(c_i)| < \varepsilon/2$ for $m > N_2$ and $|c_i - c^*| < \delta$

Hence

$$|x_m^{(N)} - g(c^*)| < \Sigma\, \alpha_i |x_m^{(N)}(c_i) - g(c^*)| \leq$$

$$\leq \Sigma\, \alpha_i \{|x_m^{(N)}(c_i) - g(c_i)| + |g(c_i) - g(c^*)|\} \leq \varepsilon$$

(N_1 depends only on ε; N_2 depends only on ε)

Let $m\to\infty$, then

$$\overline{\lim_{m\to\infty}} |x_m^{(N)} - g(c^*)| \leq \varepsilon \to \overline{\lim_{\to\infty}} |x_k - g(c^*)| \leq \varepsilon \Rightarrow \lim_{k\to\infty} x_k = g(c^*)$$

a)ii) $c_n = \tilde{f}(x_n, \ldots x_{n-r})$

The left member tends to c^*. The right member tends to $\tilde{f}(g(c^*), g(c^*) \ldots) =$
$= f(g(c\ ))$ (f continuous)

Hence $c^* = f(g(c^*))$ and c^* is a s.p. of (4.4)

a)iii) $c^* = f(g(c^*)$ $g(c^*) = g(f(g(c^*)))$
$g(c\ )$ is a s.p. of (4.3)

b-assertions follow from the part just proved.
Q.E.D.

Theorem 4.1 relates possible convergence points of c_k to possible convergence points of x_k.

There is also a relationsship between all stationary points of (4.3) and (4.4). To establish that an elementary property of linear mappings is required.

LEMMA 1

Let A be a n x k matrix and B a k x n matrix. Suppose $n \geq k$. Then AB has the same eigenvalues as BA and besides n-k eigenvalues = 0.

PROOF

Let λ be a non zero eigenvalue of BA with eigenvector e, i.e.

$BAe = \lambda e$, obviously $Ae \neq 0$

Then

$$AB(Ae) = A(BAe) = A(e) = Ae$$

i.e. Ae is an eigenvector with eigenvalue of AB. Similarily any non-zero eigenvalue of AB is also an eigenvalue of BA, and the lemma follows Q.E.D.

THEOREM 4.2

Let $f: R^n \to R^k$ and $g: R^k \to R^n$ be continuously differentable

Consider the ODE's

$$\dot{x} = -x + g(f(x)) \tag{4.3}$$

$$\dot{c} = -c + f(g(c)) \tag{4.4}$$

There is then a 1-1 correspondence between stationary points of (4.3) and (4.4).

Furthermore, the linearized ODE's around any two corresponding stationary points have identical characters.

PROOF

Let x^o be a stationary point (s.p.) of (4.3) i.e. $x^o = g(f(x^o))$. This implies $f(x^o) = f(g(f(x^o)))$ and $c^o = f(x^o)$ is a s.p. of (4.4). Similarily if c^1 is a s.p. of (4.4) $x^1 = g(c^1)$ is a s.p. of (4.3).

Furthermore two different s.p. of (4.3) x^o, x^1, correspond to different s.p. of (4.4). If not $c^o = f(x^o) = c^1 = f(x^1)$. This implies that $g(c^o) = g(f(x^o)) = x^o = g(c^1) = g(f(x^1)) = x^1$ which is a contradiction.

Now linearize around a s.p. x^o and $c^o = f(x^o)$ respectively.

$$\Delta\dot{x} = (g'|_{(x^o)} \circ f|'_{x^o} - I)\,\Delta x$$

$$\Delta\dot{c} = (f'|_{g(c^o)} \circ g'|_{c^o} - I)\,\Delta c$$

But $g(c^o) = x^o$ and $f(x^o) = c^o$. Hence, according to the lemma the linearized ODE's have the same eigenvalues (the Δx-eq. may have sone in -1 in addition) and the theorem follows.
Q.E.D.

Although local properties thus are identical, we must consider only special cases to establish results on global properties of (4.3) and (4.4).

LEMMA 2

Let $\dot{x} = f(x)$, where $f(x)$ is continuously differentiable be asymptotically stable with domain of attraction Ω_1. Assume that x is known to belong to some bounded region Ω_2, $\bar{\Omega}_2 \subset \Omega_1$. Then $\dot{x} = f(x+y(t))$ where $y(t) \searrow 0$ $t\to\infty$ is asymptotically stable with a domain of attraction that includes Ω_1.

PROOF

Let $V(x)$ be a Lyapunov function to $\dot{x} = f(x)$. Then $V(x)$ is p.d. in Ω_1 and $\dot{V}(x)$ n.d. in Ω_2. (The existence of V is acertained by Zubov's theorem, see eg [4]). Let $\tilde{x}$ be a solution of $\dot{x} = f(x+y(t))$

Then

$$\dot{V}(\tilde{x}) = V_{\tilde{x}}^T\dot{\tilde{x}} = V_{\tilde{x}}^T f(\tilde{x}+y(t)) = V_{\tilde{x}}^T f(\tilde{x}) + V_{\tilde{x}} f_x y(t) + o(y(t)) \quad y(t) \to 0$$

The first term is n.d. in Ω_2. The second term tends to zero as $t\to\infty$. Denote the region in which $\dot{V}(x)$ is strictly negative with Ω_t

Obviously $\Omega_t \rightarrow \Omega_2/\{x^o\}$ as $t \rightarrow \infty$ (V_x and f_x are uniformly bounded).

This acertains that $V(\tilde{x}) \rightarrow 0$ and $\tilde{x} \rightarrow x^o$.

Q.E.D.

THEOREM 4.3

Let $f: R^n \rightarrow R^m$

and

$g: R^m \rightarrow R^n$

Suppose g is linear and has full rank and that x, and y belong to bounded sets.

Then

$\dot{x} = -x + g(f(x))$ is glob. as stable if and only if

$\dot{y} = -y + f(g(y))$ is glob. as. stable.

PROOF

Let $g(y) = Ay$ where A is a linear operator

The proofs are slightly diferent for the cases

$n \geq m$ and $m \trianglerighteq n$.

i) $n \trianglerighteq m$

Decompose $R^n = V \oplus V^\perp$, V = range (A)

Decompose x accordingly

$x = Az + y \qquad y \in V^\perp \qquad z \in R^m$

We get

$\dot{x} = A\dot{z} + \dot{y} = - Az - y + Af(Az + y)$

$A(\dot{z} + z - f(Az+y)) + \dot{y} + y = 0$

But $\dot{y} \in V^\perp$ since $y(t) \in V^\perp$ $\forall t$ and $V^\perp$ is a linear subspace

Hence

$\dot{y} + y = 0$
$A(\dot{z}+z-f(Az+y)) = 0$

Since A has full rank we get

$$\begin{cases} \dot{y} = -y \\ \dot{z} = -z+f(g(z) + y) \end{cases}$$

It remains to prove that

$\dot{z} = -z + f(g(z) + y_o e^{-t})$ gl. as. stable $\forall y_o$
is equivalent to
$\dot{z} = -z + f(g(z))$ is gl. as. stable

But one way is trivial and the other follows from Lemma 2.

ii) $m \geq n$

Decompose $R^n = V \oplus V^1$; V = Nullspace (A) and continue as above.

THEOREM 4.4

Let $f: R^n \to R$ have the property that $f(x_o + t(x_1 - x_o))$ belongs to the interval

$[f(x_o), f(x_1)]$ for all t, $0 \leq t \leq 1$ x_o and x_1. Then if the one dimensional ordinary differential equation

$$\dot{c} = -c + f(g(c)) \tag{4.5}$$

is globally asymptotically stable then so is the n-dimensional ODE

$$\dot{x} = -x + g(f(x)) \tag{4.6}$$

PROOF

Let the stationary point of (4.5) be $c = c^o$. Then $V_1(c) = (c-c^o)^2$ is a Lyapunov function for (4.5), i.e.

$(c-c^o)(f(g(c)) - c) < 0$ for $c \neq c^o$

We will show that

$$V_2(x) = (f(x) - c^o)^2$$

is a Lyapunov function for (4.6)

Consider

$$f(x^1 + t(-x^1 + g(f(x^1)))$$

Since this value lies between $f(x^1)$ and

$f(g(f(x^1)))$ for all t $0 < t < 1$

$$\text{sign}\left[\frac{d}{dt}\left\{f\ x^1 + t(-x^1 + g(f(x^1))\right\}_{t=0}\right] = \text{sign}[(f(g(x^1))) - f(x^1)]$$

But

$$\frac{d}{dt} f\left\{x^1 + t(-x^1 + g(f(x^1))\right\}_{t=0} = (\nabla f(x^1) \,|\, [-x^1 + g(f(x^1))])$$

and since

$$\frac{d}{dt} V_2(x) = 2(f(x) - c^o)(\nabla f(x^1) \,|\, [-x^1 + g(f(x^1))])$$

we get

$$\frac{d}{dt} V_2(x) < 0 \text{ for } f(x) \neq c^o$$

From theorem 4.2 (4.6) has a uniquely determined stationary point x^o.

The only case when a solution $x(t)$ of (4.6) other than x^o may fulfill

$$f(x(t)) \equiv c^o$$

is when $\dot{x} = -x + g(f(x(t))) = -x + x^o$. But then $x(t)$ obviously tends to $x^o = g(c^o)$. Hence the as mptotic convergence of solutions of (4.6) towards x^o follows.

5. APPLICATIONS

In this chapter we will apply the convergence theorems of chapter 3 to investigate the convergence properties of some adaptive systems.

When faced with a specific adaptive system several ways to prove convergence are possible. If the control vector algorithm is such that the expected value of the next outcome depends on all previous ones, then it is natural to first try to apply theorem 3.3 or 3.4. In case it is difficult to find a suitable Lyapunov function, it is reasonable to consider the estimate process and, possibly by extending the dimension of the estimate, make theorem 3.1 or 3.2 applicable. Another possibility, which is closely related, is to rewrite the estimate as a class A algorithm and apply theorem 3.5. In these cases we are left with an ordinary differential equation, the stability properties of which yield all information needed to establish convergence or non-convergence (cf. remark in section 3.3) of the algorithm . Theorem 4.1 to 4.4 may be useful for the investigation of stability properties.

We will now apply this technique to investigate convergence properties of two specific adaptive systems.

Example 1. Self tuning regulator

In [1] is discussed a regulator for a system with unknown but constant parameters.

The model of the system is

$$A(q)y(t) = B(q)u(t-k)+\lambda C(q)e(t)$$

where $A(q) = q^n + a_1 q^{n-1} + \dots + a_n$

$B(q) = b_1 q^{n-1} + \dots + b_n$

$C(q) = q^n + \dots + c_n$

and where q is the forward shift operator. Parameters $a_1, \dots a_n$, $b_1 \dots b_n$ are estimated recursively with a real-time least squares algorithm and then the control law

$$u(t) = \frac{1}{\hat{b}_1}[\hat{a}_1 y(t) + \dots \hat{a}_n y(t-n+1) - \hat{b}_2 u(t-1) - \dots - \hat{b}_n u(t-n+1)]$$

is applied. (The estimates of a_i and b_i are denoted $\hat{a}_i$ and $\hat{b}_i$).

Consequently we may take $\Theta = [\hat{a}_1 \dots \hat{a}_n \hat{b}_1 \dots \hat{b}_n]$ as the control vector.

In [7] it is shown that the estimates Θ are given recursively by

$$\Theta(t+1) = \Theta(t) + K(t)[y(t) - \phi(t-1)\Theta(t)] \qquad (5.1)$$

$$K(t) = P(t)\phi(t-1)^T\{R_2 + \phi(t-1)P(t)\phi^T(t-1)\}^{-1}$$

$$P(t+1) = P(t) - K(t)\{R_2 + \phi(t-1)P(t)\phi^T(t-1)\}K(t)^T$$

$$\phi(t) = [-y(t) \quad u(t) - y(t-1), \dots -y(t-n+1)]$$

An identical algorithm results from the Kalman filter approach.

To apply theorems 3.1 to 3.4 for the control vector process we consider

$$\Theta(t+1) - \Theta(t) = K(t)\{y(t) - \phi(t-1)\Theta(t)\}$$

The expected value of this quantity given all previous values of Θ, actually depends on all these, since $K(t)$ does. Hence only theorems 3.3 and 3.4 are applicable.

However, by a proper choice of estimates also theorems 3.1 and 3.2 can be applied:

Choose as estimate vector

$$x(t) = [\hat{r}_y^t(0), \hat{r}_y^t(n), \hat{r}_{uy}^t(-n), \dots \hat{r}_{uy}^t(n-1), \hat{r}_u^t(0), \dots \hat{r}_u^t(n-1)]^T$$

where $\hat{r}$ are the covariance estimates at time t

$$\hat{r}_{uy}^t(k) = \frac{1}{t}\sum_{r=1}^{t} u(r+k)y(r)$$

Then

$$x(t+1)-x(t) = \begin{bmatrix} \frac{1}{t+1}(y^2(t+1) - r_y^{(t)}(0)) \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{t+1}(u(t+1)u(t-n) - r_u^t(n-1)) \end{bmatrix}$$

and

$$E(x(t+1) - x(t) \mid x(t),\ldots x(0)) = f(x(t))$$

so that theorem 3.1 is applicable for the estimate process.

In fact the estimate process is a class A-algorithm:
Property i) holds trivially with $\alpha_{m,n} = \frac{m}{n+m}$ from which ii) follows.

A constant estimate x gives a constant control law

$$\Theta = \begin{bmatrix} R_y & R_{uy} \\ R_{yu} & R_u \end{bmatrix}^{-1} \begin{bmatrix} \hat{r}_y(1) \\ \vdots \\ \hat{r}_{yu}(N) \end{bmatrix}$$

(R_y, R_{uy} and R_u are covariance matrices, se [7].

With this control applied x converges with probability 1 to a continuous function G(x). Property iv) follows from the continuity of x in Θ and of G(x) in x.

We will now consider convergenc of the algorithm in the special case

$$y(t) + ay(t-1) = b\,u(t-1) + e(t)$$

Since this structure is not identifiable from closed loop data, b is taken as an arbitrary constant, say 1.

Assume that $|y(t)| < M$ a.e.

i) Using theorem 3.4

In this case we obtain from eq (5.1)

$$\hat{a}(t+1) - \hat{a}(t) = \frac{y(t)y(t-1)P(t)}{R_2+y(t-1)^2P(t)}$$

where

$$\frac{1}{p(t)} = \sum_{s=0}^{t} y^2(s)$$

Take as the sequence of σ -algebras

$$\mathcal{F}_t = \{y(0) \ldots, y(t-1)\}$$

Then $P(t) \in \mathcal{F}_t$ and

$$E(y(t)|\mathcal{F}_t| = (a-b\hat{a})y(t-1)$$

Hence

$$E(\hat{a}(t+1) - \hat{a}(t)|\mathcal{F}_t) = \frac{y^2(t-1)P(t)}{R_2+y^2(t-1)P(t)} (a - \hat{a}b)$$

In terms of theorem 3.4 we can take

α_k as $1/k$

Take as Lyapunov function

$$V(x) = (x - a/b)^2$$

Then

$$V'(x)G_t(y_{t-1},\ldots,y_o)h(x) = -\frac{y^2(t-1)P(t)}{R_2+y^2(t-1)P(t)} (a-xb)^2 \frac{1}{b}$$

which is < 0 unless

$x = a/b$

or

$y^2(t-1)P(t) = 0$

But, obviously

$$P(\{y^2(t-1)P(t) = 0\} | \mathcal{F}_t) = 0 \quad \text{a.e.}$$

and hence

$x(=a)$ a/b with probability one.

<u>Remark.</u>

To be true, $P(t)$ depends also on $y(t)$, so that taking expected values, given $y(k)$ only up to $y(t-1)$, the result is not exactly $P(t)$. It is seen that this does not affect the arguments.

Formally:

$$P(t) = P(t-1) - \frac{y(t-1)^2}{(\sum_{k=0}^{t} y^2(k))(\sum_{k=0}^{t-1} y^2(k))}$$

so that the error is $0(1/t^2)$.

Since $\sum_1^\infty 1/t^2 < \infty$, extension 1 of theorem 3.4 is applicable.

ii) Using theorem 3.5

According to what was said above we obtain a class A algorithm if we take as estimate vector

$$x = \begin{bmatrix} \hat{r}_u(0) \\ \hat{r}_{uy}(0) \\ \hat{r}_{uy}(1) \end{bmatrix}$$

The control law $\hat{a}$ is formed as

$$\hat{a} = \frac{\hat{r}_u(0) - \hat{r}_{uy}(1)}{\hat{r}_{uy}(0)}$$

Suppose the control law is fixed a_o; $u(t) = a_0 y(t)$

Then

$$\hat{r}_{uy}(0) \to r_u(0)/a_0$$

$$\hat{r}_u(0) \to a_0^2 r_y(0) = a_o^2/(1-(a-ba_o)^2)$$

$$\hat{r}_{uy}(1) \to a_o r_y(1) = a_o(ba_o-a)/(1-(a-ba_o)^2)$$

and the new control is

$$a_1 = \frac{a_o^2 - a_o(ba_o-a)}{a_o} = a_o - (ba_o-a)$$

Applying theorem 3.5, we obtain that the convergence of the algorithm depends on the stability of the corresponding ODE:

$$\dot{x}_1 = -x_1 + \frac{a_o^2}{1-(a-ba_o)^2}$$

$$\dot{x}_2 = -x_2 + \frac{a_o}{1-(a-ba_o)^2} \qquad (5.2)$$

$$\dot{x}_3 = -x_3 + \frac{a_o(ba_o-a)}{1-(a-ba_o)^2}$$

where $a_o = \dfrac{x_1-x_3}{x_2}$

This is the estimator oriented ODE. The corresponding controller oriented differential equation is

$$\dot{y} = a-by \qquad (5.3)$$

The functions g and f introduced in chapter 4 are in this case

$$f(x_1, x_2, x_3) = \frac{x_1-x_3}{x_2}$$

$$g_1(g) = \frac{y^2}{1-(a-by)^2}$$

$$g_2(y) = \frac{y}{1-(a-by)^2}$$

$$g_3(y) = \frac{y(by-a)}{1-(a-by)^2}$$

From theorem 4.2 and the simple properties of (5.3) we have that the only singular point of (5.2) is

$$x_o = ((\frac{a}{b})^2, \frac{a}{b}, 0)$$

and this is an asymptotically stable singular point if b 0.

Furthermore, in this case we can also obtain results on global stability:

Since $f(x_1,x_2,x_3) = \dfrac{x_1-x_3}{x_2}$

$f(x_o+t(x_1-x_o)) = \dfrac{At+B}{Ct+D}$ with suitable A,B,C,D.

This function has no extrema for $0 < t < 1$. Therefore global asymptotic stability for (5.2) follows from the same property of (5.3) and theorem 4.4.

Hence, the convergence of x towards x_o and a towards a/b with probability one is again acertained, this time from theorem 3.5.

<u>Example 2. Automatic classifiers</u>

In [2] is discussed some automatic classifiers for samples that belong to one of two random variables.

In the one dimensional case they work as follows.

The sample x is compared with a decision value c, and is classified as class A if greater and class B if smaller than c. The value c is determined as

$$c = \frac{\hat{m}_A + \hat{m}_B}{2} + \alpha \left(\sqrt{\hat{v}_A} - \sqrt{\hat{v}_B} \right)$$

where m_A is the estimated mean value of those samples classified as A v_A is correspondingly the estimated variance. m_A, etc are updated with a simple stochastic approximation algorithm. To apply theorem 3.1 introduce the estimate vector

$$x = \begin{bmatrix} m_A \\ m_B \\ v_A \\ v_B \end{bmatrix}$$

$$E(x_{k+1} - x_k \mid x_o \ldots x_k) = \frac{1}{k+1} f(x_k, c_k)$$

Since c_k is a function of x_k theorem 3.1 is applicable and the corresponding ODE determined. It depends on the actual distribution of the random variables. In [2] the corresponding controller-oriented ODE is investigated for a number of distribution functions.

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