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Automatic Control

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Abstract

The thesis treats model reduction for linear time-varying systems. Time-varying models appear in many fields, including power systems, chemical engineering, aeronautics, and computational science. They can also be used for approximation of time-invariant nonlinear models. Model reduction is a topic that deals with simplification of complex models. This is important since it facilitates analysis and synthesis of controllers.

The thesis consists of two parts. The first part provides an introduction to the topics of time-varying systems and model reduction. Here, notation, standard results, examples, and some results from the second part of the thesis are presented.

The second part of the thesis consists of four papers. In the first paper, we study the balanced truncation method for linear time-varying state-space models. We derive error bounds for the simplified models. These bounds are generalizations of well-known time-invariant results, derived with other methods. In the second paper, we apply balanced truncation to a high-order model of a diesel exhaust catalyst. Furthermore, we discuss practical issues of balanced truncation and approximative discretization. In the third paper, we look at frequency-domain analysis of linear time-periodic impulse-response models. By decomposing the models into Taylor and Fourier series, we can analyze convergence properties of different truncated representations. In the fourth paper, we use the frequency-domain representation developed in the third paper, the harmonic transfer function, to generalize Bode's sensitivity integral. This result quantifies limitations for feedback control of linear time-periodic systems.

Contents

Preface

The topic of linear time-varying systems is classical in the fields of engineering and mathematics. Some of the pioneers are the 19th-century scientists Floquet, Hill, and Mathieu. The motivation for their work often came from real problems, such as studies of the orbit of the moon. Many modern engineering applications also fit into this topic. Examples include power systems, chemical engineering, aeronautics, and computational science.

Model reduction is also a well-known topic, with close connections to approximation theory and system identification. Model reduction deals with reduction of model complexity. The practical relevance comes from the fact that low-complexity models are easier to analyze and to design controllers for.

In the thesis, we combine the two areas and obtain results for model reduction of linear time-varying systems.

Organization of the Thesis

Chapter 1 contains an introduction to time-varying systems and some examples. We review two model structures and define what we mean by a time-varying system. Chapter 2 contains an introduction to model reduction for time-varying systems. In particular, we define the purpose of model reduction and review some of the methods that are commonly used. Chapters 1 and 2 provide the context for the contributions, with pointers to many of the results that are derived in Papers I–IV. Chapter 3 gives some concluding remarks and suggestions for future work. Papers I–IV contain the main contributions of the thesis.

How to read the thesis. The thesis can be read in (at least) two different ways.

If the reader is not familiar with the topics, it is suggested that he or she starts with Chapters 1 and 2, in which notation, problem formulations, and some standard results are presented. It is also attempted to show how the contributions of the thesis fit into the "big picture".

If the reader is familiar with the topics, it is suggested that he or she reads Papers I–IV first. The papers are self-contained and include all technical details. After this the reader may proceed with Chapters 1 and 2 to get the author's personal view on the topic and how the results of the papers are related.

Notice that the references in the introductory chapters often refer to easily accessible proofs and results rather than papers where the results were first given.

Contributions of the Thesis

Paper I

Sandberg, H. and A. Rantzer (2004): "Balanced truncation of linear time-varying systems." *IEEE Transactions on Automatic Control*, **49:2**, pp. 217–229.

In this paper, we study balanced truncation of linear time-varying systems in discrete and continuous time. Based on relatively simple calculations with time-varying Lyapunov equations/inequalities we are able to derive both upper and lower error bounds for the truncated models. These results generalize well-known time-invariant formulas derived with other methods. The case of time-varying state dimension is considered. Input-output stability of all truncated balanced realizations is also proven. Finally, we apply the method to a high-order model.

Related publications are

Sandberg, H. (2002): "Linear time-varying systems: Modeling and reduction." Licentiate thesis ISRN LUTFD2/TFRT--3229--SE. Department of Automatic Control, Lund Institute of Technology, Sweden.

Sandberg, H. and A. Rantzer (2002): "Balanced model reduction of linear time-varying systems." In *Proceedings of the 15th IFAC World Congress*. Barcelona, Spain.

Sandberg, H. and A. Rantzer (2002): "Error bounds for balanced truncation of linear time-varying systems." In *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 2892–2897. Las Vegas, Nevada.

Paper II

Sandberg, H. (2004): "A case study in model reduction of linear time-varying systems." Submitted to *Automatica*.

In this paper, we apply the balanced truncation procedure for time-varying linear systems, both in continuous and in discrete time. The methods are applied to a linear approximation of a diesel exhaust catalyst model. The reduced-order systems are obtained by using certain projections instead of direct balancing. An approximative zero-order-hold discretization of continuous-time systems is described, and a new a priori approximation error bound for balanced truncation in the discrete-time case is obtained. The case study shows that there are several advantages to work in discrete time. It gives simpler implementation with fewer computations.

A related publication is

Sandberg, H. (2004): "A case study in model reduction of linear time-varying systems." In *Proceedings of the 2nd IFAC Workshop on Periodic Control Systems*, pp. 249–254. Yokohama, Japan.

Paper III

Sandberg, H., E. Möllerstedt, and B. Bernhardsson (2004): "Frequency-domain analysis of linear time-periodic systems." Under review for *IEEE Transactions on Automatic Control*.

In this paper, we study convergence of truncated representations of the frequency-response operator of a linear time-periodic system. The frequency-response operator is frequently called the harmonic transfer function. We introduce the concepts of input, output, and skew roll-off. These concepts are related to decay rates of elements in the harmonic transfer function. A system with high input and output roll-off may be well approximated by a low-dimensional matrix function. A system with high skew roll-off may be approximated by an operator with only few diagonals. Furthermore, the roll-off rates are shown to be determined by certain properties of Taylor and Fourier expansions of the periodic systems. Finally, we clarify the connections between the different methods for computing the harmonic transfer function that are suggested in the literature.

A related publication is

Sandberg, H., E. Möllerstedt, and B. Bernhardsson (2004): "Frequency-domain analysis of linear time-periodic systems." In *Proceedings of the American Control Conference*, pp. 3357–3362. Boston, Massachusetts.

Paper IV

Sandberg, H. and B. Bernhardsson (2004): "A Bode sensitivity integral

for linear time-periodic systems.” Submitted to *IEEE Transactions on Automatic Control.*

Bode’s sensitivity integral is a well-known formula that quantifies some of the limitations in feedback control for linear time-invariant systems. In this paper, we show that there is a similar formula for linear time-periodic systems. The harmonic transfer function is used to prove the result. To state the result we introduce the notion of roll-off 2, which means that the first time-varying Markov parameter is equal to zero. Then it follows that the harmonic transfer function is an analytic operator and a trace class operator. This is needed to prove the result.

A related publication is

Sandberg, H. and B. Bernhardsson (2004): “A Bode sensitivity integral for linear time-periodic systems.” In *Proceedings of the 43rd IEEE Conference on Decision and Control.* Paradise Island, Bahamas.

Other Publications

The following publications are also cited in the introductory chapters. They are not directly related to Papers I–IV, but treat linear time-varying systems.

Iftime, O., R. Kaashoek, H. Sandberg, and A. Sasane (2004): “A Grassmanian approach to the Hankel norm approximation problem.” In *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems.* Leuven, Belgium.

Sandberg, H. (1999): “Nonlinear modeling of locomotive propulsion system and control.” Master’s thesis ISRN LUTFD2/TFRT--5625--SE. Department of Automatic Control, Lund Institute of Technology, Sweden.

Sandberg, H. and E. Möllerstedt (2000): “Harmonic modeling of the motor side of an inverter locomotive.” In *Proceedings of the 9th IEEE Conference on Control Applications*, pp. 918–923. Anchorage, Alaska.

Sandberg, H. and E. Möllerstedt (2001): “Periodic modelling of power systems.” In *Proceedings of the 1st IFAC Workshop on Periodic Control Systems*, pp. 91–96. Cernobbio-Como, Italy.

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Many thanks to Erik Möllerstedt, who is co-author of the third paper and the work on power systems. In fact, it was Erik together with Markus Meyer (Bombardier Transportation) and Andrew Paice (ABB), who first lured me into the topic of time-varying systems. Today I am happy for that.

Many thanks to Björn Westerberg (Scania) who gave me a much desired model to test the model reduction techniques on, and to Andrey Ghulchak and Magnus Fontes for teaching me mathematics.

The five years of work leading up to the thesis have been very instructive and enriching. This is mainly because of my fellow PhD students, the professors, and the staff at the Department of Automatic Control in Lund. Thank you for creating such a stimulating working atmosphere. Especially I would like to mention a few: Johan Åkesson is not only a robust roommate, but also a good friend. Johan read the introductory chapters at an early stage, and has helped to improve them a lot. Anton Cervin has improved both my taste in music and the language of the thesis. I would also like to thank Rasmus Olsson for our (endless?) discussions on the connections between model reduction and chemometrics, and Leif Andersson for all computer support. Thank you also to Karl Johan Åström, Per Hagander, Ola Slätteke, Stefan Solyom, and Brad Schofield for comments on the introduction.

When I have been fed up with signals and systems, the lunch breaks with the Sparta-gang (Rolf, Rasmus, Magnus, and Jacob) have always been a good source for new inspiration. Our invaluable secretaries: Eva, Britt-Marie, and Agneta, have also always been very friendly and helpful.

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My final thanks go to my family and friends. Your support through the years has been a necessary condition for it all. A special thank you goes to Lena, for love and understanding.

Henrik

1

Introduction to Time-Varying Systems

In this chapter, we define what we mean by time-varying models, and look at some of their basic features. The chapter is meant to provide a presentation of the modeling framework of Papers I–IV without going into too much technical detail, and to present some examples. References are given to more complete presentations where so is suitable.

The chapter starts with a presentation of the two main model structures that we study in the thesis: state-space models and input-output models. State-space models are mainly used in Papers I and II, and input-output models are mainly used in Papers III and IV. At the end of the chapter we discuss lifting, and there are some examples to illustrate the models.

1.1 State-Space Models

A system G modeled by a state-space model is given as the differential equation

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), t), \quad x(t_0) = x_0, \\ y(t) &= g(x(t), u(t), t), \quad t \in [t_0, t_f] \subseteq \boldsymbol{R}, \end{aligned} \tag{1.1}$$

where $x(t) \in \boldsymbol{R}^n$ is the state, $u(t) \in \boldsymbol{R}^m$ is the input (or control signal), and $y(t) \in \boldsymbol{R}^p$ is the output (or measurement signal) of the system. x_0 is the initial state of the system. State-space models are the topic of many books. A standard reference is [Khalil, 2002].

By adding the assumption that f is locally Lipschitz continuous in the state x, existence and uniqueness of solutions to (1.1) can be shown, see [Khalil, 2002]. It will be assumed that there exist unique solutions to the state-space models that appear. We also use the following classification of state-space models.

DEFINITION 1.1—CLASSIFICATION OF STATE-SPACE MODELS
A state-space model in continuous time is

(a) *time invariant* if f and g do not depend on t,

(b) *time varying* if it is not time invariant,

(c) *time periodic* if there exists a $T > 0$ such that

$$\begin{aligned} f(x,u,t) &= f(x,u,t+T), \\ g(x,u,t) &= g(x,u,t+T), \end{aligned}$$

for all x, u, t.

□

State-space models for systems evolving in discrete time can be constructed in a similar way. Then we have

$$\begin{aligned} x(k+1) &= f(x(k),u(k),k), \quad x(k_0) = x_0, \\ y(k) &= g(x(k),u(k),k), \quad k \in [k_0, k_f] \subseteq \boldsymbol{Z}. \end{aligned} \tag{1.2}$$

The classification of discrete-time models is analogous to the continuous-time case.

From a strictly physical point of view, one would perhaps not expect models of natural processes to be time varying, given that the laws of physics do not change with time. However, from a local perspective, it may very well happen that a model based on first principles becomes time varying. A classical example is a model of a rocket that has decreasing mass since it burns fuel, see [Rugh, 1996]. Also, many nonlinear time-invariant models may be locally approximated by a time-varying model of a simpler (linear) structure.

It should be clear that by making a linear time-varying coordinate transformation, $x = T(t)\tilde{x}$, a time-invariant state-space model with state x can be turned into a time-varying state-space model with state $\tilde{x}$. Conversely, by extending the state space we can also make a time-varying model $\dot{x} = f(x,t)$ time invariant:

$$\begin{bmatrix} \dot{x} \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} f(x, x_t) \\ 1 \end{bmatrix} \Longrightarrow \dot{\tilde{x}} = \tilde{f}(\tilde{x}). \tag{1.3}$$

Notice that if f has some desirable property, such as linearity in the state, $\tilde{f}$ might lose this property. For this reason we will not use the state extension (1.3) in the thesis.

Hence, the classification of a model can be changed. However, the state transformations we use in Papers I and II do typically not change the classification. Lifting, which is discussed in Section 1.3, is another form of transformation that is applied to the input and output.

Linear Approximations

In the following, we will mostly restrict ourselves to linear models. This can be motivated by the fact that many nonlinear state-space models can be approximated by linear models. The formulas for linearization of a continuous-time state-space model are given next. The formulas in discrete time are completely analogous. The linearization procedure is applied to a nonlinear model in Paper II.

The first step in the linearization is to find a nominal solution $x_0(t)$, $u_0(t)$, $y_0(t)$ to (1.1). If the nominal solution is constant,

$$f(x_0, u_0, t) = 0,$$

for all t, the solution is called an *equilibrium point*. If the model is initialized at an equilibrium point and no extra input is applied, then the state will remain at x_0.

Next, new variables that represent the deviations from the nominal solution are introduced:

$$\Delta x(t) = x(t) - x_0(t), \quad \Delta u(t) = u(t) - u_0(t), \quad \Delta y(t) = y(t) - y_0(t).$$

If we assume that the mappings f and g are continuously differentiable, then we may construct a linear system of differential equations,

$$\begin{aligned} \Delta\dot{x}(t) &= A(t)\Delta x(t) + B(t)\Delta u(t), \quad \Delta x(t_0) = \Delta x_0, \\ \Delta y(t) &= C(t)\Delta x(t) + D(t)\Delta u(t), \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} A(t) &= \frac{\partial f}{\partial x}(x_0(t), u_0(t), t), \qquad & B(t) &= \frac{\partial f}{\partial u}(x_0(t), u_0(t), t), \\ C(t) &= \frac{\partial g}{\partial x}(x_0(t), u_0(t), t), \qquad & D(t) &= \frac{\partial g}{\partial u}(x_0(t), u_0(t), t). \end{aligned}$$

The linear system is described by the matrices $A(t) \in \boldsymbol{R}^{n\times n}$, $B(t) \in \boldsymbol{R}^{n\times m}$, $C(t) \in \boldsymbol{R}^{p\times n}$, and $D(t) \in \boldsymbol{R}^{p\times m}$. The linear state-space model is time-invariant if and only if all the matrices are constant.

A nonlinear time-invariant state-space model that is linearized around an equilibrium point is described by the four constant matrices: A, B, C, and D. In control-oriented literature it is often assumed that a model is given in this form. Notice, however, that even if the nonlinear state-space model is time invariant, the linear approximation is in general time varying if the nominal solution is not constant. In Papers I and II it is assumed that the models are given in this more general linear time-varying form. Some examples of linear time-varying approximations are given in Section 1.4.

If $D(t) = 0$, the solution of the system (1.4) can be written in the form

$$\Delta y(t) = C(t)\Phi_A(t,t_0)\Delta x_0 + \int_{t_0}^{t} C(t)\Phi_A(t,\tau)B(\tau)\Delta u(\tau)d\tau, \quad t \in [t_0, t_f], \tag{1.5}$$

where $\Phi_A(t,\tau)$ is the transition matrix for $\dot{x} = A(t)x$:

$$\frac{\partial}{\partial t}\Phi_A(t,\tau) = A(t)\Phi_A(t,\tau), \quad \Phi_A(\tau,\tau) = I\,.$$

For time-invariant models we have $\Phi_A(t,\tau) = e^{A(t-\tau)}$. Corresponding formulas hold in discrete time. These results are given in [Rugh, 1996].

There exist many results that guarantee that the trajectories of (1.4) are close to the trajectories of (1.1) in a neighborhood of the nominal solution. Results of a qualitative nature come from the Lyapunov stability theory, see for instance [Khalil, 2002]. There are also quantitative bounds on the error between the trajectories of the linear approximation and the nonlinear system, see for instance [Desoer and Vidyasagar, 1975].

1.2 Input-Output Models

A different modeling framework is offered by input-output models. Traditionally, state-space models come from mechanics, and the input-output models come from electrical engineering. Input-output models are treated in-depth in, for example, the books [Desoer and Vidyasagar, 1975; Zadeh and Desoer, 1979].

An input-output model of a system G is a mapping that to each input u assigns an output y,

$$G : U \to Y,$$

where

$$U \subseteq \{u : u(\cdot) : I \to \boldsymbol{R}^m\},$$
$$Y \subseteq \{y : y(\cdot) : I \to \boldsymbol{R}^p\},$$

and $I = [t_0, t_f] \subseteq \boldsymbol{R}$ or $I = [k_0, k_f] \subseteq \boldsymbol{Z}$.

State-space models (with inputs and outputs) with unique solutions can be represented as input-output models. But there are examples of input-output models that are not possible to represent as state-space models. For example, systems with time delays or partial differential equations do not fit into the state-space framework unless one uses theory for infinite-dimensional systems, see [Curtain and Zwart, 1995].

Linear impulse-response models. A linear model $G : U \to Y$ fulfills the relation

$$G(\alpha \cdot u_1 + \beta \cdot u_2) = \alpha \cdot G(u_1) + \beta \cdot G(u_2),$$

for all $\alpha, \beta \in \boldsymbol{R}$ and $u_1, u_2 \in U$. In the following, parentheses are not used for linear models, i.e., $G(u) = Gu$. Linear systems can often be represented in integral or summation forms. We call these representations impulse-response models. Impulse-response models in continuous time are in the form

$$y = Gu : \quad y(t) = \int_I g(t,\tau)u(\tau)d\tau, \quad t \in I, \tag{1.6}$$

and we can formally compute the *impulse response* g of G as

$$g(t,\tau) = \lim_{\sigma \to 0} (Gw_{\tau,\sigma})(t), \quad t, \tau \in I, \tag{1.7}$$

where $w_{\tau,\sigma}(t)$ converges to the shifted Dirac impulse function $\delta(t-\tau)$ in the sense of distributions as $\sigma \to 0$. For the integral (1.6) to be well defined for absolutely integrable inputs, the impulse response should be essentially bounded. There are several technicalities involved here, see [Sontag, 1990] for details. To justify the equality in (1.7), the order of integration and other limits has to be interchanged. Details and further assumptions on G that are needed are given in, for example, [Sandberg, 1988].

Notice that the linear state-space model (1.5) is an impulse-response model if the system is initially at rest, and

$$g(t,\tau) = \begin{cases} C(t)\Phi_A(t,\tau)B(\tau), & t \geq \tau, \\ 0, & t < \tau. \end{cases} \tag{1.8}$$

An impulse-response model that can be written in state-space form is said to have a *realization*. This is often convenient and will be further discussed in Sections 1.4 and 2.4.

In discrete time, summation replaces integration

$$y = Gu : \quad y(k) = \sum_{i \in I} g(k,i)u(i), \quad k \in I, \tag{1.9}$$

and the impulse response g of the linear system G is obtained as

$$g(k,i) = (G\delta_i)(k), \quad k,i \in I,$$

where $\delta_i(k) = \delta(k-i)$ is the shifted discrete impulse function. Also here do technicalities arise when the set I is infinite, and we need to verify that the sum converges, see [Sontag, 1990]. We often write the summation (1.9) when $I = [k_0, k_f]$ as the matrix-vector multiplication

$$y_{[k_0,k_f]} = G_{[k_0,k_f]} u_{[k_0,k_f]}, \tag{1.10}$$

or written explicitly,

$$\begin{pmatrix} y(k_0) \\ y(k_0+1) \\ \vdots \\ y(k_f) \end{pmatrix} = \begin{pmatrix} g(k_0,k_0) & g(k_0,k_0+1) & \dots & g(k_0,k_f) \\ g(k_0+1,k_0) & g(k_0+1,k_0+1) & & \\ \vdots & & \ddots & \vdots \\ g(k_f,k_0) & & \dots & g(k_f,k_f) \end{pmatrix} \begin{pmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k_f) \end{pmatrix}.$$

We now give some basic definitions for impulse-response models.

DEFINITION 1.2—CLASSIFICATION OF IMPULSE-RESPONSE MODELS
Assume that G is a linear impulse-response model in continuous time. Then G is

(a) *causal* if $g(t,\tau) = 0$, $t < \tau$,

(b) *time invariant* if $g(t,\tau) = g(t-\tau, 0) =: g(t-\tau)$,

(c) *time varying* if it is not time invariant,

(d) *time periodic* if $g(t+T, \tau+T) = g(t,\tau)$, for some $T > 0$,

for all t and τ in I. □

An analogous classification is used in discrete time.

Causality means that future inputs do not influence outputs of the past. A discrete-time causal system takes the form of a lower-triangular matrix

$$G_{[0,\infty]} = \begin{pmatrix} g(0,0) & & & 0 \\ g(1,0) & g(1,1) & & \\ g(2,0) & g(2,1) & g(2,2) & \\ \vdots & & & \ddots \end{pmatrix}. \tag{1.11}$$

The state-space models with impulse response (1.8) are always causal. Causality, as defined above, holds for most models of physical systems.

However, it is sometimes useful to consider non-causal systems, see for example Paper III.

Time invariance means that a given input signal has the same effect whenever it is applied, the output will only be a shifted version in time. In discrete time, a time-invariant system takes the form of a Toeplitz matrix

$$G_{[0,\infty]} = \begin{pmatrix} g(0) & g(-1) & g(-2) & \dots \\ g(1) & g(0) & g(-1) & \ddots \\ g(2) & g(1) & g(0) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \qquad (1.12)$$

A linear time-invariant state-space model gives a time-invariant impulse-response model (1.8).

Norms of Signals and Systems

We will work with sets U, Y that are normed linear spaces. In continuous time $I \subseteq \boldsymbol{R}$ and the spaces used are typically $L_p(I, \boldsymbol{R}^m)$, where $p \geq 1$. The signal u belongs to $L_p(I, \boldsymbol{R}^m)$ if and only if the norm

$$\|u\|_p := \|u\|_{L_p(I,R^m)} = \left(\int_I |u(t)|^p dt \right)^{1/p}$$

is finite, where $|\cdot|$ is the regular Euclidean norm on $\boldsymbol{R}^m$: $|u(t)|^2 = u^T(t)u(t)$. Often we only write L_p when m and I are known, or not essential to the discussion. Notice that p has previously also been used for the dimension of the output. Traditionally, both are denoted by p and it should be clear from the context what p means.

In discrete time we have $I \subseteq \boldsymbol{Z}$. Here the spaces used are typically $\ell_p(I, \boldsymbol{R}^m)$, where $p \geq 1$. The signal u belongs to $\ell_p(I, \boldsymbol{R}^m)$ if and only if the norm

$$\|u\|_p := \|u\|_{\ell_p(I,R^m)} = \left(\sum_{k \in I} |u(k)|^p \right)^{1/p}$$

is finite. For simplicity we often write ℓ_p instead of $\ell_p(I, \boldsymbol{R}^m)$.

It will be useful to measure the size of input-output models. This is done by introducing norms not only on signals but also on systems.

Norms of linear systems. The induced norm of the input-output model $G : U \to Y$ is defined as

$$\|G\|_{U \to Y} = \sup_{u \in U \setminus \{0\}} \frac{\|G(u)\|_Y}{\|u\|_U}. \qquad (1.13)$$

When the interval I has infinite horizon, and the norm (1.13) is finite, then G is bounded-input-bounded-output (BIBO) stable.

Next, we give a collection of methods for estimating the system norms on L_p and ℓ_p,

$$\|G\|_p := \|G\|_{L_p \to L_p}, \quad \|G\|_p := \|G\|_{\ell_p \to \ell_p}. \tag{1.14}$$

This serves as a comparison with the methods that are used in the papers. The first method applies to impulse-response models, and is based on results found in [Desoer and Vidyasagar, 1975].

THEOREM 1.1—INDUCED L_p-NORM
Assume that $G : L_p \to L_p$ has an impulse-response representation (1.6). If there are constants c_1, c_∞ such that

$$\sup_{t\in I} \int_I |g(t,\tau)| d\tau = c_\infty < \infty, \quad \sup_{\tau\in I} \int_I |g(t,\tau)| dt = c_1 < \infty,$$

then

$$\|G\|_\infty = c_\infty, \quad \|G\|_1 = c_1, \quad \|G\|_p \le c_1^{1/p} c_\infty^{1/q}, \tag{1.15}$$

where $p \in [1, \infty]$ and $1/p + 1/q = 1$. □

There is a completely analogous result in discrete time for the induced ℓ_p-norm $\|G\|_p$. We do not state it here. Both proofs are based on the Hölder's inequality, see for example [Desoer and Vidyasagar, 1975]. In the case of multi-input-multi-output systems, $|g(t,\tau)|$ should be read as the induced norm on $\boldsymbol{R}^m$.

Discrete-time systems (1.9) may be written as matrix-vector multiplication (1.10). If I is a finite set, then standard matrix norms can be used to compute $\|G\|_p$ in the cases $p = (1, 2, \infty)$. Notice that in the case $p = 2$, the induced norm corresponds to a calculation of the largest singular value of a matrix.

In Papers III and IV, there is a standing assumption that the impulse responses are causal and have uniform exponential decay. That is, there are positive constants K and κ such that

$$|g(t,\tau)| \le K \cdot e^{-\kappa(t-\tau)}, \quad t \ge \tau. \tag{1.16}$$

One reason for this assumption is that it then holds that $\|G\|_p \le K/\kappa$ for all p.

We will often work with the case $p = 2$, since L_2 and ℓ_2 are Hilbert spaces. Theorem 1.1 only gives an upper bound on $\|G\|_2$. If G is a *compact* *operator* on L_2 or ℓ_2, see for example [Young, 1988], then we can obtain the norm as

$$\|G\|_2 = \lambda_{\max}^{1/2}(G^*G), \tag{1.17}$$

where G^* is the adjoint operator and $\lambda_{\max}$ the largest eigenvalue. This corresponds to the computation of the largest singular value of G. However, models with an infinite time horizon, $I = (-\infty, \infty)$ or $I = [0, \infty)$, are typically *not* compact.

In Paper III, we calculate an *harmonic transfer function* $\hat{G}(j\omega)$ of time-periodic systems G. Here ω is the angular frequency. The harmonic transfer function allows us to compute the norm $\|G\|_2$ in a fashion similar to (1.17), even if G is defined on infinite time horizons. The harmonic transfer function is further discussed in Section 1.3. Let $\mathcal{C}_e^2$ be the set of twice differentiable and exponentially decaying impulse responses, see Definition 1 in Paper III. Then the following theorem may be derived.

THEOREM 1.2—PAPER III
Assume that $G : L_2 \to L_2$, $I = (-\infty, \infty)$, is a time-periodic causal impulse-response model (1.6) in $\mathcal{C}_e^2$, with period T. Then we can define an infinite-dimensional compact operator on ℓ_2, $\hat{G}(j\omega)$, such that

$$\|G\|_2 = \sup_{\omega \in I_0} \lambda_{\max}^{1/2}\left(\hat{G}^*(j\omega)\hat{G}(j\omega)\right),$$

where $I_0 = (-\pi/T, \pi/T]$. □

REMARK 1.1—TRANSFER FUNCTIONS FOR TIME-INVARIANT SYSTEMS
Compare the harmonic transfer function with the standard transfer function $\hat{g}(j\omega)$ for time-invariant systems. That is, $\hat{g}(j\omega) = \mathcal{F}[g](j\omega)$, the Fourier transform of the impulse response. Then,

$$\|G\|_2 = \sup_{\omega \in R} |\hat{g}(j\omega)|,$$

see, for example, [Zhou and Doyle, 1998]. □

If the linear system has a state-space realization, then we can obtain estimates of $\|G\|_2$ by means of a generalization of the famous *bounded real lemma*, see, for example [Green and Limebeer, 1995]. With this procedure, we can get an arbitrarily good estimate of the L_2-induced norm by solving differential Riccati equations or inequalities, see [Tadmor, 1990]. The following formulation is based on the results in [Lall, 1995].

THEOREM 1.3—BOUNDED REAL LEMMA
Assume that $G : L_2 \to L_2$ with $I = [0, \infty)$, and that G has a bounded linear state-space realization $(A(t), B(t), C(t), 0)$, with $x(0) = 0$. Assume furthermore that $\dot{x} = A(t)x$ is uniformly exponentially stable.

If there is a bounded (symmetric) solution $P : [0, \infty) \to \boldsymbol{R}^{n\times n}$ to

$$\dot{P} + A^T P + PA + \gamma^{-2} PBB^T P + C^T C \le 0, \tag{1.18}$$

then $\|G\|_2 \le \gamma$.

Conversely, if $\|G\|_2 < \gamma$, then there exists a bounded positive-semi-definite solution P to (1.18). □

Theorem 1.3 can be generalized to the case when $D(t) \neq 0$, see [Lall, 1995]. Theorem 1.3 is an example of that there often are explicit schemes that are relatively simple to implement numerically for systems expressed in linear state-space form. We shall see more examples of this throughout the thesis.

Even though Theorem 1.3 is simple in principle, one needs a bisection algorithm to find a good bound γ. Because each iteration involves solving a potentially large differential inequality (1.18), this might be hard to do in practice. In Papers I and II, we instead search for other simple bounds that come directly from the model reduction procedure that is studied.

1.3 Lifting

Lifting is a common operation on linear time-varying models. Essentially the idea is to find isomorphic transformations on the input and output spaces of the model, and thereby gain a model that in the new coordinates has some desirable property, such as time invariance.

Consider, for example, the causal input-output model $G : U \to Y$ with period 2,

$$\begin{pmatrix} y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ \vdots \end{pmatrix} = \begin{pmatrix} g_1(0) & & & & & 0 \\ g_1(1) & g_2(0) & & & & \\ g_1(2) & g_2(1) & g_1(0) & & & \\ g_1(3) & g_2(2) & g_1(1) & g_2(0) & & \\ g_1(4) & g_2(3) & g_1(2) & g_2(1) & g_1(0) & \\ \vdots & & & & & \ddots \end{pmatrix} \begin{pmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \\ u(5) \\ \vdots \end{pmatrix}.$$

Assume that the model is BIBO-stable and defined on scalar sequences: $U = Y = \ell_2(\boldsymbol{N}, \boldsymbol{R})$ and $g_i(k) \in \boldsymbol{R}$. Let us define the isomorphic transformation $\mathcal{T} : \ell_2(\boldsymbol{N}, \boldsymbol{R}) \to \ell_2(\boldsymbol{N}, \boldsymbol{R}^2)$,

$$\begin{pmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \\ \vdots \end{pmatrix} \xrightarrow{\mathcal{T}} \begin{pmatrix} \tilde{u}(1) \\ \tilde{u}(2) \\ \vdots \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} u(1) \\ u(2) \end{pmatrix} \\ \begin{pmatrix} u(3) \\ u(4) \end{pmatrix} \\ \vdots \end{pmatrix}.$$

We have that $\tilde{G} = \mathcal{T} G \mathcal{T}^{-1} : \tilde{U} \to \tilde{Y}$ with a matrix representation

$$\begin{pmatrix} \tilde{y}(1) \\ \tilde{y}(2) \\ \tilde{y}(3) \\ \vdots \end{pmatrix} = \begin{pmatrix} \tilde{g}(0) & & & 0 \\ \tilde{g}(1) & \tilde{g}(0) & & \\ \tilde{g}(2) & \tilde{g}(1) & \tilde{g}(0) & \\ \vdots & & & \ddots \end{pmatrix} \begin{pmatrix} \tilde{u}(1) \\ \tilde{u}(2) \\ \tilde{u}(3) \\ \vdots \end{pmatrix},$$

where $\tilde{g}(k) \in \boldsymbol{R}^{2\times 2}$. Since the matrix is on Toeplitz form, the model on the lifted signal spaces has a *time-invariant* structure, see Definition 1.2. Instead of a periodic single-input-single-output system, we have obtained a time-invariant multi-input-multi-output system. Notice that $\tilde{g}(0)$ needs to be a lower triangular 2×2 matrix for the system to be causal. Hence, for lifted systems, there is a constraint on the direct term.

Lifting is a simple algebraic manipulation, yet powerful. Surveys of different forms of lifting methods are given in [Bittanti and Colaneri, 1999; Bittanti and Colaneri, 2000]. Most controller synthesis and model reduction techniques are designed for time-invariant systems. Often these results can be applied to lifted periodic systems. See, for example, [Colaneri, 1991; Bamieh and Pearson, 1992; Voulgaris *et al.*, 1994] where H_2/H_∞-optimal controllers for periodic systems are developed using lifting techniques. Lifting can also be used on periodic systems defined in continuous time, and on systems in state-space form. A complication in continuous time is that the elements in the lifted space become infinite dimensional. In the appendix of Paper I, a lifted state-space model is used for sampling of Lyapunov differential equations.

Lifting can also be performed in the frequency domain. In fact, this is how the harmonic transfer function $\hat{G}(j\omega)$ in Theorem 1.2 is obtained. The harmonic transfer function is used extensively for T-periodic continuous-time models in Papers III and IV. By using lifting in the frequency domain, the interaction of frequencies in input and output can be easily analyzed, and the theory of analytic functions can be used. For an example of this, see the next section on sensitivity integrals.

A method closely related to lifting is a block-diagonal operator reformulation of linear time-varying systems, see [Dullerud and Lall, 1999]. This method also brings the time-varying system into a formally time-invariant structure. In this operator-theoretic framework, controller synthesis [Dullerud and Lall, 1999; Farhood and Dullerud, 2002] as well as model reduction [Lall and Beck, 2003] can be done using semidefinite programming.

Sensitivity integral. As an example on how lifting is used in the thesis, we have the following application.

The harmonic transfer function $\hat{G}(j\omega)$ is under certain assumptions an analytic operator-valued function, see Paper IV. This may be used to generalize Bode's sensitivity integral to periodic systems. Bode's sensitivity integral for time-invariant systems can be found in, for example, [Zhou and Doyle, 1998].

THEOREM 1.4—PAPER IV
Under the assumptions of Theorem 1.2, that $(I+G)^{-1}$ is a bounded operator on L_2, and that $g(t,t)=0$ for all t, we have

$$\int_0^{\pi/T} \log|\det(I+\hat{G}(j\omega))^{-1}|\,d\omega = 0.$$

□

Theorem 1.4 states that the sensitivity function $|\det(I+\hat{G}(j\omega))^{-1}|$ cannot be made arbitrarily small for all frequencies ω. If it is smaller than 1 for some frequencies, it must necessarily be larger than 1 for other frequencies. This is called the waterbed effect. Theorem 1.4 is a fundamental limitation for feedback control of periodic systems.

REMARK 1.2—ROLL-OFF 2
The condition $g(t,t)=0$ for all t, in Theorem 1.4, is in Paper III defined as a condition for *input and output roll-off* 2 of G. In Paper IV it is simply called *roll-off* 2. Large parts of Paper III are devoted to studies of different forms of roll-off for linear time-varying systems. Notice that in the traditional time-invariant Bode sensitivity integral, there is also a condition of roll-off 2 on the transfer function. □

1.4 Examples

The contributions of the Papers I, III, and IV are mainly of a theoretical nature. In this section, we give a list of more or less well-known examples of applications of time-varying modeling and control. In particular, it should be noticed that time-varying models come up in a wide variety of situations.

The last example, the example on power systems, is treated in more detail, since it is based on work co-written by the author.

Hill, Mathieu, and Floquet

Hill, Mathieu, and Floquet are the names of three scientists that are intimately related to the theory of time-periodic systems. They were all active in the 19th century.

The *Hill equation* is a linear differential equation

$$\begin{aligned} \ddot{y}(t) + p(t)y(t) &= 0, \\ p(t+T) &= p(t), \end{aligned} \tag{1.19}$$

with a periodic parameter p. The *Mathieu equation* is on the same form with a sinusoidal parameter, often parameterized as

$$p(t) = r - 2q\cos\omega_0 t, \quad \omega_0 = 2\pi/T,$$

where r and q are constants. Rewriting the equations after introducing the state $x = [y \quad \dot{y}]^T$, we see that this indeed is a time-varying linear state-space model if $q \neq 0$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \dot{x} = A(t)x, \tag{1.20}$$

without inputs and outputs.

The Hill equation was derived for modeling of the *lunar perigee*, i.e., the motion of the moon around the earth [Hill, 1886]. The Mathieu equation appears in several areas. For example, when the wave equation is solved in elliptic regions [Mathieu, 1868], but also in the analysis of a pendulum attached to an oscillating pivot point, see Figure 1.1. There are several books dedicated to this type of equation and its practical applications. There are nice surveys in [Farkas, 1994; Wereley, 1991].

It is important to understand that allowing A to be time varying in (1.20), may have dramatic consequences. Figure 1.2 shows that the behavior of the trajectories can be complex as compared to a time-invariant system of equal order. If A is constant, the stability of the system is easily determined by a computation of the eigenvalues. To check the stability of the time-periodic version, we need more advanced tools, such as the *Floquet decomposition* [Floquet, 1883]. The result by Floquet states that the transition matrix of a linear time-periodic system may decomposed into

$$\Phi_A(t,\tau) = P(t)e^{Q(t-\tau)}P^{-1}(\tau),$$

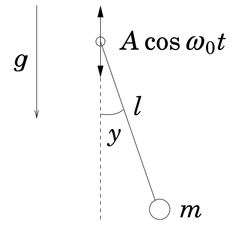

Figure 1.1 An undamped pendulum of length l and mass m, connected to a pivot point that is oscillating vertically, may be modeled by the Mathieu equation for small angles y. The approximation $\sin y \approx y$ is used. The parameters in the Mathieu equation are $r = g/l$ and $q = A\omega_0^2/2l$.

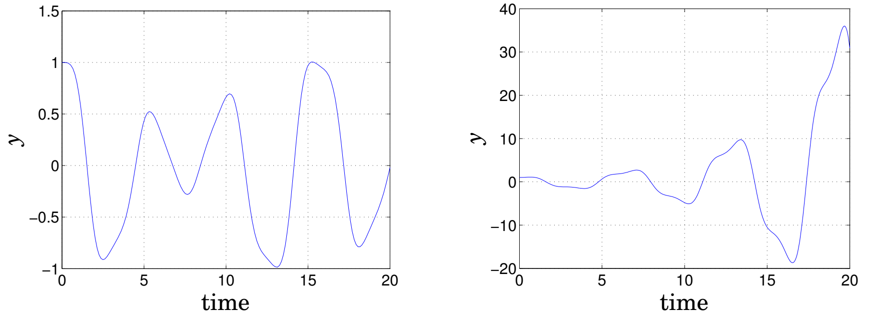

Figure 1.2 Simulation of the Mathieu equation. The parameters are chosen as $\omega_0 = 2$, $r = 2$, and $y(0) = 1$. In the left plot $q = 1.0$, yielding a stable system. In the right plot, $q = 1.4$ and the system is unstable. Both examples show that the trajectories are much more complex than for a second-order linear time-invariant system.

where $P(t)$ is a bounded T-periodic matrix and Q is a constant matrix. By a simple coordinate change, $\tilde{x} = P^{-1}(t)x$, (1.20) is transformed into

$$\dot{\tilde{x}} = Q\tilde{x}.$$

This is a time-invariant problem, and the stability is determined by the eigenvalues of Q. The problem with the Floquet decomposition is that it is a relatively complicated matter to compute $P(t)$ for general matrices $A(t)$. This requires the transition matrix Φ_A, which in most cases cannot be written on a closed form. Hence, to obtain Q and P, (1.20) generally needs to be solved over a period using numerical ODE solvers. It should be pointed out that for the Hill equation there are many analytical results available. This is due to the simple structure and dimension of $A(t)$, see

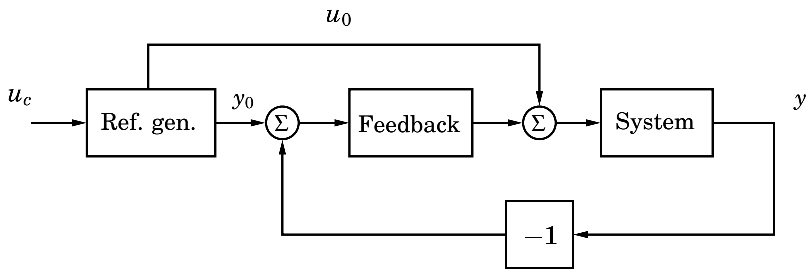

Figure 1.3 A simplified servo control problem. A reference generator takes commands u_c and generates input and output references u_0, y_0 that the system ideally should follow. When the real system deviates from the desired trajectory because of external disturbances or model errors in the reference generator, a feedback compensator generates corrections. If the feedback loop is stable, the output will remain close to the output reference.

[Farkas, 1994].

The Servo Control Problem

A linear approximation of a nonlinear model is only valid in some neighborhood of a nominal solution, as is discussed in Section 1.1. This might seem to severely restrict the use of linear approximations. However, for control purposes, linear models often provide a nice trade-off between model simplicity and accuracy. The reasons for this are:

- A controlled system typically operates close to some pre-determined trajectory.
- A well designed feedback controller makes the system robust to model errors.
- For linear models there are good controller synthesis methods.

In Figure 1.3 a typical servo control problem is illustrated. The reference generator is an open-loop controller that is sensitive to errors in the model. It may be designed by using the complete system model and optimization tools, see for example [Bryson and Ho, 1975]. For each reference u_c, we should have a nominal reference trajectory u_0 and y_0.

The feedback controller should stabilize the system and make corrections for external disturbances and model errors in the open-loop controller. If the reference trajectory is time varying, the system can be approximated by a linear time-varying model, as discussed in Section 1.1. The feedback controller for a linear time-varying model may be designed by using, for example, LQG- or H_∞-control techniques, see [Bryson and Ho, 1975; Başar and Bernhard, 1991; Lall, 1995].

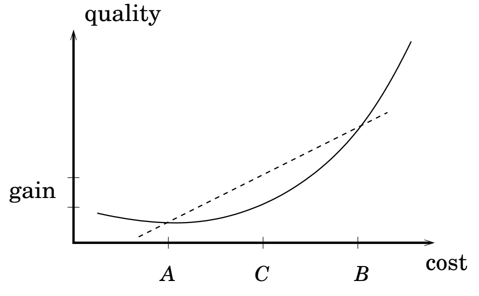

Figure 1.4 If the quality-cost diagram for operation of a simple, non-dynamic, process has the above convex appearance, then it may be favorable to use periodic control. By switching periodically between the operating points A and B, we have an average cost of C, and obtain an average increase in quality, as indicated in the figure.

Periodic Control

In the preface of [Bittanti and Colaneri, 2001], the editors write:

> *"Over the last decades, periodic control ideas and techniques have reached a notable degree of maturity. Many applications have been developed in various fields, including aerospace and power systems, robotics, telecommunication networking management, digital control devices, biological and biomedical signal and systems, earth signals and environmental data modelling, physics, etc. ..."*

Often systems under control are operated close to some constant operating point. However, there are several examples where there is a lot to gain by forcing the system into a periodic orbit, even though the system dynamics is time invariant. The survey [Bittanti and Colaneri, 1999] offers a nice exposition of several such applications, and [Bittanti and Colaneri, 2001; Katayama, 2004; Colaneri, 2004] contain several recent applications. The diagram in Figure 1.4 illustrates how periodic control can be used for a simple, non-dynamic, example.

More complex examples are airplanes that periodically change altitude to decrease the overall fuel consumption, see [Speyer, 1996]. Also, periodic operation has been observed to increase the performance of certain chemical reactors, see [Bailey, 1973]. Another example is animal and human locomotion: legs are operated in a periodic pattern to move effectively. A biped robot is described in the paper [Chevallereau *et al.*, 2003]. The reason why periodic control schemes are not implemented more often seems to be the increasing controller complexity: It is harder to track a periodic

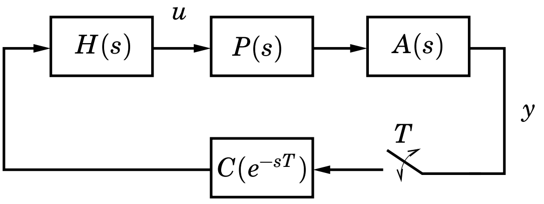

Figure 1.5 A sampled-data system with a T-periodic sampler. The process with transfer functions $P(s)$ is controlled by the discrete-time controller $C(e^{-sT})$. $A(s)$ is an anti-aliasing filter and $H(s)$ is a hold circuit.

trajectory than a steady-state operating point.

There are also some theoretically appealing properties of periodic control. Two examples are the output stabilization and the simultaneous stabilization problem of time-invariant plants. These hard problems can be solved if a periodic feedback gain is allowed. See the discussion and references in [Bittanti and Colaneri, 1999]. Another example is in [Khargonekar *et al.*, 1985], where it is shown how the gain margin can be increased with periodic control of time-invariant plants. However, a problem is that a periodic controller introduces harmonics in the loop. It is shown in [Zhang *et al.*, 1997] that, in general, better or equal H_2/H_∞-performance can be achieved with time-invariant control for time-invariant plants.

Sampled-data systems. Computer-controlled systems are common examples of periodically controlled systems. The periodicity comes from the periodic sampling of measurement signals and the periodic update of the actuation signal, see Figure 1.5. However, if we sample the system that changes periodically only once per period, and we always do it at the same relative instant in the period, then the system appears to be time invariant. This is a common approach for sampled-data systems, and methods such as the z-transform can be used, see [Åström and Wittenmark, 1997]. A problem with this method is that the signals in between the sampling instants may behave badly, and we may encounter *intersample ripple*.

Problems with intersample ripple can be avoided if the system is treated as a periodic system. A periodic model of a sampled-data system has a special structure. In particular one can often obtain closed-form solutions to controller synthesis problems. This is not the case for generic periodic systems. Sampled-data systems is a large research area; some references are [Bamieh and Pearson, 1992; Yamamoto and Khargonekar, 1996; Dullerud, 1996; Araki *et al.*, 1996; Rosenwasser and Lampe, 2000; Wittenmark *et al.*, 2002].

Computational Linear Algebra

Time-varying modeling may also be used to reduce floating-point operations. In [Dewilde and van der Veen, 1998] computational linear algebra is treated. There a triangular matrix $G_{[0,N]}$ of the type (1.11) is not considered as a dynamical system, but as any triangular matrix that we want to apply standard linear algebra operations on. For example, if we want to compute the product of the triangular matrix $G_{[0,N]}$ and a vector $u_{[0,N]}$,

$$y_{[0,N]} = G_{[0,N]}u_{[0,N]},$$

then the number of floating-point operations is $O(N^2)$ using direct calculations. The idea is to find a state-space realization of $G_{[0,N]}$

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(0) = 0, \\ y(k) &= C(k)x(k) + D(k)u(k), \end{aligned} \tag{1.21}$$

with the smallest state dimension $x(k) \in \boldsymbol{R}^n$ possible. This realization problem has close connections to the model reduction problem in Section 2.4 and Papers I and II. Using the realization (1.21) to compute $y_{[0,N]}$ the number of floating-point operations is $O(n^2N)$, which is a significant reduction if $n^2 \ll N$. Furthermore, direct storage of $G_{[0,N]}$ requires $O(N^2)$ words. To store the realization (1.21) we only need $O(n^2N)$ words.

Another example is the computation of the inverse matrix $G_{[0,N]}^{-1}$, or rather, to find the solution $u_{[0,N]}$ to $y_{[0,N]} = G_{[0,N]}u_{[0,N]}$ for a given $y_{[0,N]}$. If $G_{[0,N]}$ is a sparse triangular matrix, then the inverse matrix $G_{[0,N]}^{-1}$ is in general not sparse. That is, the inversion does not preserve the structure of the matrix. On the other hand, if a realization of $G_{[0,N]}$ is available, the solution $u_{[0,N]}$ can be computed recursively as

$$\begin{aligned} x(k+1) &= [A(k) - B(k)D^{-1}(k)C(k)]x(k) - B(k)D^{-1}(k)y(k), \quad x(0) = 0, \\ u(k) &= D^{-1}(k)C(k)x(k) + D^{-1}(k)y(k). \end{aligned}$$

Thus, the state-space realization of the inverse has the same state dimension as $G_{[0,N]}$. If the elements of $G_{[0,N]}$ are scalars, $D(k)$ are just scalars and are simple to invert.

The realization (1.21) is a recursive implementation of $G_{[0,N]}$, where the state $x(k)$ carries information into the next step. For this to be possible, the matrix must be triangular (causal or anti-causal). For non-triangular matrices the method can be applied if an initial decomposition into triangular matrices is made. A limitation of the method is that n should be small. This happens if certain generalized Hankel matrices with elements from $G_{[0,N]}$ have low rank, see [Rugh, 1996; Dewilde and van der Veen, 1998] and Section 2.4.

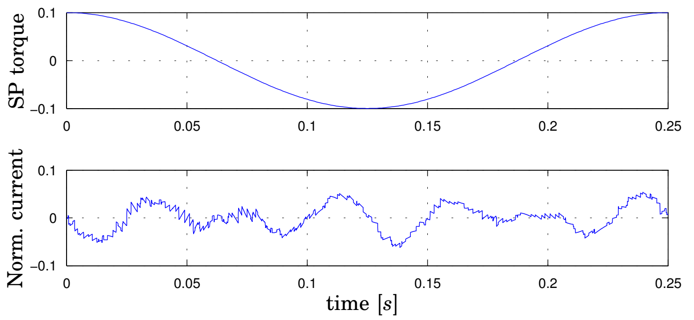

Figure 1.6 Data from a simulation of a converter-controlled asynchronous engine in an inverter locomotive, see [Sandberg, 1999]. The input is the set point (SP) to the torque controller. The output is the normalized current flowing through the engine. It is seen that the output contains multiples of the harmonic in the input. The ripple in the current comes from the converter.

Periodic Modeling of Power Systems

This example is treated in more detail than the previous examples. The reason for this is that the author has co-written some papers on the topic. The following presentation is based on the work in [Möllerstedt, 2000; Möllerstedt and Bernhardsson, 2000; Sandberg and Möllerstedt, 2001; Sandberg and Möllerstedt, 2000]. There the harmonic transfer function is also used.

The periodicity of currents and voltages makes AC power systems an ideal application for linear time-periodic system theory. These systems are driven by a voltage of well-defined frequency and amplitude. Since only relatively small deviations from this nominal voltage are allowed, the dynamics of these systems are well captured by models which are linearized around the nominal operating trajectory. This leads to linear time-periodic (LTP) models.

Actively controlled power electronic devices like power converters are powerful actuators. Power flows can be changed in a fraction of a cycle. Because of the switching dynamics, there is coupling between different frequencies. Consequently, to fully utilize the possibilities brought by the power electronics, and to avoid overly conservative solutions, harmonics and frequency coupling should be considered. For reasons of simplicity and tradition, however, linear time-invariant (LTI) models that only capture the dynamics of the fundamental frequency component are still often used for the analysis. In Figure 1.6 and Figure 1.7, two examples from [Sandberg and Möllerstedt, 2000; Sandberg, 1999] are shown where there are clear coupling between frequencies in input and output. The plot in

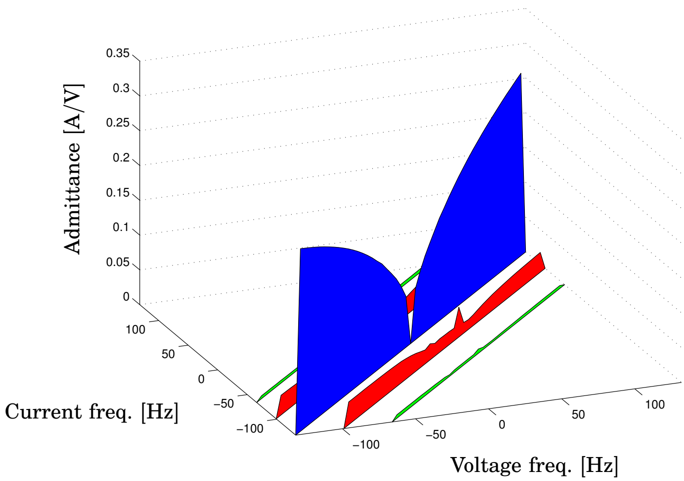

Figure 1.7 The frequency coupling between the voltage and the current flowing into the converter-controlled engine of an inverter locomotive is visualized in the plot. The figure can be interpreted as the magnitude of the harmonic transfer function that is used in Papers III and IV.

Figure 1.6 shows multiples of the frequency of the input signal in the output signal. Such behavior can be captured by LTP models. The plot in Figure 1.7 may be interpreted as the magnitude of the harmonic transfer function that is derived in Paper III and used in the Theorems 1.2 and 1.4.

Next, we show how a simple LTP model of a converter-controlled system can be derived.

Converters. A power converter is a nonlinear coupling device between two electric systems. They are often built using GTO (Gate Turn Off)-thyristors with switching frequency up to 500 Hz. IGBTs (Insulated Gate Bipolar Transistors) can also be used with switching frequency up to 10 kHz. Most common is that the converter is used to connect an AC system to a DC system. The AC side and DC side dynamics can generally be captured with linear time-invariant models that are straightforward to derive. The problem is to obtain a good description of the coupling between the two sides, one that facilitates analysis and design of the complete system.

Ideal converters. An ideal single phase converter is shown in Figure 1.8. The ideal converter has no losses and no energy storage. The

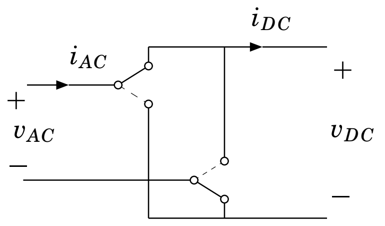

Figure 1.8 An ideal converter. The switches are used to control the power flow through the converter.

basic goal of the converter control is to shape the AC voltage so that the desired power is fed through the converter. This is done by proper switching. From Figure 1.8 it can be concluded that

$$v_{AC}(t) = s(t)v_{DC}(t), \tag{1.22}$$

where the switch function $s(t)$ can be assigned the values 1, -1, and 0, since ideal switching is assumed. The desired AC voltage is smooth (sinusoidal), and must be approximated by using pulse width modulation, for instance.

The switch function also gives a relation between AC current and DC current

$$i_{DC}(t) = s(t)i_{AC}(t). \tag{1.23}$$

Since an ideal converter has no losses and no energy storage, the instantaneous power on the DC side and the AC side must be equal, that is,

$$P_{DC}(t) = v_{DC}(t)i_{DC}(t) = v_{AC}(t)i_{AC}(t) = P_{AC}(t).$$

The current relation (1.23) can also be derived from this power balance.

Linearizing the converter. The local behavior of the converter in the neighborhood of a nominal periodic solution $\{v^0_{DC}(t), i^0_{AC}(t), s^0(t)\}$ is well described by a linear approximation of (1.22) and (1.23)

$$\begin{pmatrix} \Delta v_{AC}(t) \\ \Delta i_{DC}(t) \end{pmatrix} = \begin{pmatrix} 0 & s^0(t) & v^0_{DC}(t) \\ s^0(t) & 0 & i^0_{AC}(t) \end{pmatrix} \begin{pmatrix} \Delta i_{AC}(t) \\ \Delta v_{DC}(t) \\ \Delta s(t) \end{pmatrix}.$$

Since the nominal solution, around which the system is linearized, is periodic, the converter is represented by a periodic gain matrix. Ideally v^0_{DC} is constant and v^0_{AC} sinusoidal of the fundamental frequency. From

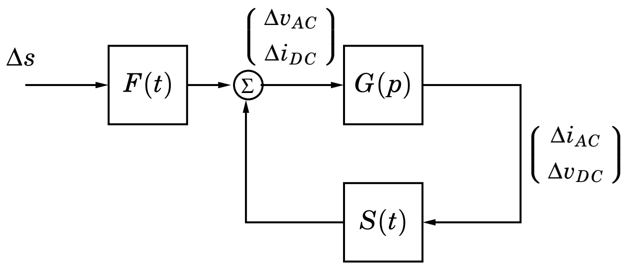

Figure 1.9 A feedback model of a power converter that connects an AC power system to a DC power system. $S(t)$ and $F(t)$ are time-periodic gain matrices. $G(p)$ is a dynamical model of the AC and DC power systems.

(1.22) it is seen that $s^0(t)$ should also be sinusoidal. However, since $s(t)$ only takes discrete values $(\pm 1, 0)$, this is only an approximation.

Often the lines of the converter are connected to systems that are well described by lumped linear time-invariant models (rational expressions in the differentiation operator p), at least close to the nominal solution. Hence, Δi_{AC} and Δv_{DC} are obtained as

$$\begin{aligned} \Delta i_{AC}(t) &= Y_{AC}(p)\Delta v_{AC}(t), \\ \Delta v_{DC}(t) &= Z_{DC}(p)\Delta i_{DC}(t), \end{aligned}$$

where $Y_{AC}(p)$ is the admittance of the AC system and $Z_{DC}(p)$ is the impedance of the DC system. The resulting closed-loop is shown in Figure 1.9 with

$$S(t) = \begin{pmatrix} 0 & s^0(t) \\ s^0(t) & 0 \end{pmatrix}, \quad F(t) = \begin{pmatrix} v^0_{DC}(t) \\ i^0_{AC}(t) \end{pmatrix}, \quad G(p) = \begin{pmatrix} Y_{AC}(p) & 0 \\ 0 & Z_{DC}(p) \end{pmatrix},$$

where $S(t)$ and $F(t)$ are T-periodic. If the models of the AC/DC systems are finite dimensional, a periodic state-space model of the system can be obtained. However, there are common examples of infinite-dimensional models, for example long transmission lines. Then an input-output model of the system may be more convenient to use.

The modeling idea presented here has been used in several papers to model and analyze inverter locomotives. See for example [Möllerstedt and Bernhardsson, 2000; Sandberg and Möllerstedt, 2000]. A modern inverter locomotive, see Figure 1.10, has two internal power converters connected via a DC-link, a so called back-to-back configuration. There is one net side converter that draws the desired amount of power from the grid into the DC-link and thereby keeps the DC-link voltage constant. The other converter is the motor side converter. It controls the torque of the

Figure 1.10 A modern inverter locomotive. (By courtesy of Bombardier Transportation.)

motor that drives the locomotive by drawing the right amount of power from the DC-link. The back-to-back configuration is a common solution in modern variable-speed drives and offers great flexibility. For example, it can operate on power grids of different base frequencies and be used to minimize the reactive power on the grid.

1.5 Summary

In this chapter, the two basic modeling frameworks used in the thesis, state-space and impulse-response models, were reviewed. Along with introduction of notation, we have also stated some theorems that are used in the papers, or are derived there. Next, we summarize how the contributions of Papers I–IV relate to this chapter.

In Paper I, the linear time-varying state-space models in Section 1.1 are used both in discrete and in continuous time. This paper treats model reduction, and this is the topic of the next chapter.

In Paper II, we apply the methods of Paper I. In particular the results are used on a nonlinear state-space model of a diesel exhaust catalyst. To apply the methods we use the linearization procedure of Section 1.1. Since we linearize the model around a time-varying trajectory, this results in a time-varying model. The linear model is treated in continuous time and in discrete time. We also develop an approximative discretization procedure that can save computations.

In Paper III, the linear impulse-response models in Section 1.2 are used. Paper III contains a detailed treatment on lifting of time-periodic models in the frequency domain. This gives the harmonic transfer function that is used in Theorem 1.2.

In Paper IV, we apply the theory of Paper III. We prove that by using the harmonic transfer function, we can generalize Bode's classical sensitivity integral, to obtain Theorem 1.4. The result is also applied to the Mathieu equation that is discussed in Section 1.4.

2

Introduction to Model Reduction

An introduction to model reduction is given in this chapter. We discuss the meaning of the term "a simple model" in the first section. In the following sections, we look at common approaches to model reduction. Balanced truncation, which is the topic of Papers I and II, is treated in more detail than the other methods. In particular, an alternative presentation is given. For frequency-domain analysis, two series expansions are developed in Papers III and IV. By truncating these series, an alternative form of model reduction is obtained. This is discussed in the last section of the chapter.

2.1 Reduction Criterion

The goal of model reduction is to derive a simple model from a complex model, while the models remain close. To do this we need to mathematically define the meaning of the words "simple" and "close".

When are two models close? Two models G and $\hat{G}$ are often considered close when some induced norm $\|G-\hat{G}\|$ is small. It is also common to introduce weights W_1, W_2, so that two models are considered close if $\|W_1(G-\hat{G})W_2\|$ is small, see [Zhou and Doyle, 1998]. This is the definition used in this thesis, and the norm is often the induced 2-norm. This is because the norm $\|\cdot\|_2$ is good for robustness studies and often can be bounded relatively easy, as we shall see.

If we use $\|G-\hat{G}\|$ as measure, then closeness is an open-loop concept. But two models can be quite different in the this sense, and still behave similarly when they are controlled by the same feedback controller. Accordingly, another measure is how close models are from a feedback perspective. For this, the gap between the graphs of the input-output operators may be used. Gap metrics for time-varying and periodic systems are studied in, for example, [Ravi *et al.*, 1992; Cantoni, 1998].

State-space models. For linear time-invariant state-space models with minimal realizations

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \quad x(k) \in \boldsymbol{R}^n, \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{2.1}$$

the standard reduction criterion is the state dimension n, called the McMillan degree of the model, or simply the order of the model. A reduced model with degree $\hat{n} < n$ is considered simpler than the original model. One justification for this is that a simpler model, in this sense, consists of fewer equations. Then the number of computations to obtain the trajectories of the model will typically decrease. This is important for on-line applications, such as controllers. A second justification is that the computational complexity of Riccati-equation-based controller-synthesis methods (H_2/H_∞, LQG) increases as $O(n^3)$. If we can reduce the number of states to half, the number of floating-point operations reduces by a factor eight. Hence, a low-order model is important also for off-line controller synthesis. Model reduction for time-invariant models (2.1) is a large topic. See, for example, the book [Obinata and Anderson, 2001] and the survey [Antoulas, 1999].

For linear time-varying state-space models

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(k) \in \boldsymbol{R}^{n(k)}, \\ y(k) &= C(k)x(k) + D(k)u(k), \end{aligned} \tag{2.2}$$

one can also define the complexity of the model as the (time-varying) state dimension $n(k)$. Indeed, the computational complexity decreases with the state dimension. However, the complexity of the time variability of the realization $(A(k), B(k), C(k), D(k))$ is then not taken into account. One may argue that a time-varying model with n_1 states, is more complex than a time-invariant model with $n_2 > n_1$ states. For example, the space to store the realization of a time-varying model increases with the time horizon (if it is not periodic). For a time-invariant model the storage space does not depend on the time horizon.

In Papers I and II, we associate complexity of the model with the number of states. One simply has to check the time variability of the low-order candidates that the balanced-truncation method generates. If the time variability is considered too complex, one can try a higher order approximation. In Papers III and IV, we use Fourier expansion methods. The complexity of the time variability can then be measured by the number of Fourier coefficients needed.

Ideal model reduction. A direct approach to the model reduction problem for a state-space model is to minimize the error $\|G - \hat{G}\|_2$, subject to an order constraint on the model $\hat{G}$. However, to the best knowledge of the author, this problem is a nonconvex optimization problem, which makes it hard to solve in practice. It is, however, possible to give necessary and sufficient conditions for the existence of an approximation of given order. The following theorem is derived in [Lall and Beck, 2003].

THEOREM 2.1—[LALL AND BECK, 2003]
Assume that G is stable and has a realization (2.2), initially at rest. Then there exists a reduced-order model $\hat{G}$ of G, with a realization of order $\hat{n}(k) \leq n(k)$ and $\|G - \hat{G}\|_2 \leq \gamma$, if and only if there exist bounded positive-definite solutions $P(k)$ and $Q(k)$ to

$$\begin{aligned} A(k)P(k)A^T(k) - P(k+1) + B(k)B^T(k) < 0, \\ A^T(k)Q(k+1)A(k) - Q(k) + C^T(k)C(k) < 0, \end{aligned} \tag{2.3}$$

such that, for all k

$$\lambda_{\min}(P(k)Q(k)) = \gamma^2, \tag{2.4}$$

with multiplicity $n(k) - \hat{n}(k)$. □

The conditions (2.3) are the reachability and observability Lyapunov inequalities, which appear frequently in Paper I. They form a convex condition and can be solved by semidefinite programming. The problem is to enforce the multiplicity condition (2.4), which is a nonconvex constraint. The theorem shows that the eigenvalues of the product $P(k)Q(k)$, the Hankel singular values, are intimately connected to the approximation problem.

Since it is hard to use Theorem 2.1 directly, various sub-optimal approaches can be taken. Some of them offer approximation guarantees, such as balanced truncation and Hankel approximation, and others such as principal orthogonal decomposition are more heuristic methods. We discuss these methods in the following sections.

Input-output models. Input-output models can often be identified with a possibly infinite matrix $G_{[k_0,k_f]}$, as discussed in the previous chapter. One way to measure the complexity of the model is to use the rank of this matrix. This is not entirely suitable since when the models are defined on infinite time horizons, the models are typically not compact operators on some Hilbert space. Hence, the models cannot be arbitrarily well approximated by finite-rank matrices. This is further discussed in Section 2.2.

A better idea than to reduce the rank of $G_{[k_0,k_f]}$ is to find a low-order realization of the input-output model, as discussed in Section 1.4. Hankel-norm approximation or balanced truncation can be used to solve this, see

Sections 2.4 and 2.5. Another idea is to rewrite the input-output model as a sum of simpler models and then truncate small terms. This idea is explored in Section 2.6.

Continuous time vs. discrete time. Most models of physical processes are formulated in continuous time. For implementation on a computer a discretization is required. One has to decide if the model reduction shall be performed before or after the discretization. For the balanced truncation procedure, it is argued in Paper II that it is better to first discretize the time-varying model, and then reduce it. The main reason for this is that continuous-time balanced truncation is computationally more expensive.

2.2 SVD and Causality Issues

Singular value decomposition (SVD) is a common method for approximation of matrices. The SVD is a standard tool in matrix analysis. See for example [Golub and Van Loan, 1996] for details on its computation and properties. Below we give some of the properties that are useful to us.

Given an $m \times n$ matrix A of rank q, there exist unitary matrices

$$\begin{aligned} U &= [\, u_1 \quad \dots \quad u_m \,] \in \boldsymbol{R}^{m\times m}, \\ V &= [\, v_1 \quad \dots \quad v_n \,] \in \boldsymbol{R}^{n\times n}, \end{aligned}$$

such that

$$A = U\Sigma V^T, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \in \boldsymbol{R}^{m\times n},$$

where

$$\begin{aligned} &\Sigma_1 = \operatorname{diag}\{\sigma_1, \sigma_2, \dots, \sigma_q\}, \\ &\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_q > 0. \end{aligned}$$

σ_i is the ith singular value of A, and u_i, v_i are the corresponding singular vectors. The SVD is often used together with the induced 2-norm and the Frobenius norm

$$\|A\|_2 = \overline{\sigma}(A) = \sigma_1, \quad \|A\|_F = \sqrt{\operatorname{trace}(A^T A)} = \sqrt{\sum_{i=1}^{q} \sigma_i^2}.$$

Low-rank approximations. A common problem in approximation theory is to find a rank $r < q$ approximation B of A. In the 2-norm we have that

$$\min_{\operatorname{rank} B \leq r} \|A - B\|_2 = \sigma_{r+1}, \tag{2.5}$$

and a *not necessarily unique* minimizer is given by a truncated dyadic expansion of A

$$B = \sum_{i=1}^{r} \sigma_i u_i v_i^T = U_r \Sigma_r V_r^T, \tag{2.6}$$

$$U_r = [\, u_1 \quad \dots \quad u_r \,], \quad \Sigma_r = \operatorname{diag}\{\sigma_1, \dots, \sigma_r\}, \quad V_r = [\, v_1 \quad \dots \quad v_r \,].$$

The same problem in the Frobenius norm gives

$$\min_{\operatorname{rank} B \leq r} \|A - B\|_F = \sqrt{\sum_{i=r+1}^{q} \sigma_i^2}, \tag{2.7}$$

and the problem has a *unique* solution (if the singular values are distinct). The solution is again given by (2.6).

Naive application of SVD to input-output models. For input-output models, the complexity of the model may be measured by the rank of the matrix $G_{[0,N]}$. It is then natural to try to reduce the rank of this matrix. The following example shows a problem with such an approach.

EXAMPLE 2.1—DIRECT SVD APPROACH

Assume that a causal discrete-time input-output model on $I = [0, 4]$ is given as

$$G_{[0,4]} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 0 & 0 \\ 0.7 & 0.7 & 1 & 0 & 0 \\ 0.8 & 0.8 & 0.9 & 1 & 0 \\ 0.6 & 0.7 & 0.8 & 0.6 & 1 \end{pmatrix}.$$

It has rank 5. Since this is an input-output model it is suitable to find an approximation that is close in an induced-norm sense. We choose the truncated dyadic expansion (2.6) since it is optimal in induced 2-norm. The singular values are

$$2.8131, \; 1.1003, \; 0.7714, \; 0.6767, \; 0.6189.$$

A rank 1 approximation with 2-norm error 1.1003 is

$$\hat{G}_{[0,4]} = \sigma_1 u_1 v_1^T = \begin{pmatrix} 0.3196 & 0.3041 & 0.2832 & 0.1826 & 0.1071 \\ 0.5597 & 0.5325 & 0.4959 & 0.3198 & 0.1876 \\ 0.7197 & 0.6847 & 0.6377 & 0.4112 & 0.2413 \\ 0.9364 & 0.8909 & 0.8296 & 0.5350 & 0.3139 \\ 0.8478 & 0.8066 & 0.7512 & 0.4844 & 0.2842 \end{pmatrix}.$$

This model is simpler in the sense that we only need to store one singular number and the corresponding singular vectors. The problem with this model is that it is *not causal*, since it is not lower triangular. □

Causality is typically lost when SVD is used directly on input-output models. Another problem is that when the time horizon of the model is infinite, then the input-output operator is generally not compact, and the singular values of the operator do not tend to zero. Hence, the model cannot be arbitrarily well approximated by finite-rank models.

In spite of these problems, SVD is a cornerstone in many model reduction techniques. We are going to briefly review some of them next. The problems above are resolved by re-arrangement of the data and/or by utilizing that there are several solutions to the minimization problem (2.5).

2.3 Proper Orthogonal Decomposition

Proper orthogonal decomposition (POD) is a method to reduce the order of state-space models. It may be applied also to nonlinear models and partial differential equations. POD is, for example, often used to reduce the order of fluid dynamics models [Holmes *et al.*, 1996]. POD is also called Karhunen-Loéve expansion and principal component analysis (PCA).

POD typically does not offer guarantees that the reduced model is close to the original model. The main idea is to use data from trajectories of the model and to find a low-dimensional subspace that captures most of the state dynamics. The data can be obtained from measurements or from simulations.

Given a state-space model

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t) \in \boldsymbol{R}^n, \tag{2.8}$$

one collects snapshots of the state over a time horizon $[t_0, t_N]$ when a typical input signal is applied to the system. These are put in a matrix X,

$$X = [\, x(t_0) \quad x(t_1) \quad \dots \quad x(t_N)\,].$$

The next step is to find a low-dimensional subspace in $\boldsymbol{R}^n$ where most of the snapshots $x(t_i)$ lie. If the approximation error is to be minimized in a least-squares sense, then the problem can be expressed using the Frobenius norm as in (2.7). After inspection of the singular values of X, we can choose a suitable subspace dimension $\hat{n}$. The approximation (2.6) is then used

$$X = U\Sigma V^T \approx U_{\hat{n}}\Sigma_{\hat{n}}V_{\hat{n}}^T.$$

If the approximation is good, we have that $x(t_i) \approx U_{\hat{n}}\hat{x}(t_i)$, for some vector $\hat{x}(t_i) \in \boldsymbol{R}^{\hat{n}}$. By simple truncation of "small states", so-called Galerkin projection, one obtains a reduced-order candidate as

$$\dot{\hat{x}} = U_{\hat{n}}^T f(U_{\hat{n}}\,\hat{x}(t), u(t), t), \quad \hat{x}(t) \in \boldsymbol{R}^{\hat{n}}. \tag{2.9}$$

Notice that the solutions $U_{\hat{n}}\hat{x}(t)$ of (2.9) are not guaranteed to be close to the solutions $x(t)$ of (2.8). In some cases one can prove that if the original model is stable around an equilibrium, then the projected model is also stable around the same equilibrium, see [Prajna, 2003].

2.4 Balanced Truncation

Balanced truncation is a popular method to reduce the order of linear time-invariant systems. The idea was introduced in [Moore, 1981]. Time-varying balanced truncation is the topic of Papers I and II. Balanced truncation can be understood in many ways. One way is to consider subspaces in the state space. This was the perspective taken in [Moore, 1981], and has close connections to POD. Building on these ideas one can generalize balanced truncation to nonlinear systems, see [Lall *et al.*, 2002; Hahn *et al.*, 2003].

In Paper I, balanced truncation is described as truncation of state-space models where "small" states are removed. This is fine, since this is how balanced truncation often is used in practice. This formulation does not, however, capture the connection between balanced truncation and other SVD methods. Next, we give an interpretation of time-varying balanced truncation based on SVD and on data given as an input-output model. This result is derived in [Shokoohi and Silverman, 1987].

Assume that the discrete-time input-output model $G_{[0,N]}$ is causal. We can then construct the generalized Hankel matrix

$$H(k) = \begin{pmatrix} g(k,k-1) & g(k,k-2) & \dots & g(k,0) \\ g(k+1,k-1) & g(k+1,k-2) & \dots & g(k+1,0) \\ \vdots & \vdots & \ddots & \vdots \\ g(N,k-1) & g(N,k-2) & \dots & g(N,0) \end{pmatrix}, \tag{2.10}$$

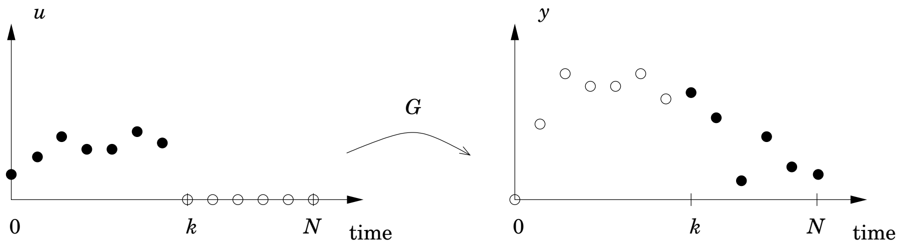

Figure 2.1 The generalized Hankel matrix $H(k)$ maps inputs $u(0),\dots,u(k-1)$ to the outputs $y(k),\dots,y(N)$ through the dynamics of the model G.

where $0 < k \le N$. If the model is time invariant, $H(k)$ will indeed be a standard Hankel matrix.

It is well known, see [Shokoohi and Silverman, 1987; Dewilde and van der Veen, 1998], that the rank of $H(k)$,

$$n(k) = \operatorname{rank} H(k), \tag{2.11}$$

determines the number of states needed at time k for a minimal realization of $G_{[0,N]}$. Using that $g(k,i) = C(k)\Phi_A(k,i+1)B(i)$, where $k > i$, we notice that the Hankel matrix can be factorized into two rank $n(k)$ matrices

$$H(k) = O(k)R(k) = \begin{bmatrix} C(k) \\ C(k+1)\Phi_A(k+1,k) \\ \vdots \\ C(N)\Phi_A(N,k) \end{bmatrix} \times \begin{bmatrix} B(k-1) & \Phi_A(k,k-1)B(k-2) & \dots & \Phi_A(k,1)B(0) \end{bmatrix}. \tag{2.12}$$

The first factor $O(k)$ is called the observability matrix, and the second factor $R(k)$ the reachability matrix. The Hankel matrix has an interpretation of a mapping of past inputs into future outputs, see Figure 2.1. We have $y_{[k,N]} = H(k)\tilde{u}_{[0,k-1]}$, using the notation in (1.10) and where $\tilde{u}_{[0,k-1]}$ is a reversely ordered $u_{[0,k-1]}$. We may write the mapping as

$$x(k) = R(k)\tilde{u}_{[0,k-1]}, \quad y_{[k,N]} = O(k)x(k),$$

where $x(k) \in \mathbf{R}^{n(k)}$ is the state of the system at time k. The state $x(k)$ is the least amount of information of the past that is needed to describe future outputs. The coordinate system of the state space is determined by the factorization of $H(k)$. For a minimal realization it holds

$$A(k) \in \boldsymbol{R}^{n(k+1)\times n(k)}, \quad B(k) \in \boldsymbol{R}^{n(k+1)\times m}, \quad C(k) \in \boldsymbol{R}^{p\times n(k)}.$$

Notice that the state dimension may be time varying. To explicitly obtain a realization, given an input-output model, we can proceed as follows.

Given a Hankel matrix, each factorization (2.12) gives rise to a realization according to

$$\begin{aligned} A(k) &= O^{\dagger}(k+1)O^{\uparrow}(k) = R^{\leftarrow}(k+1)R^{\dagger}(k), \\ B(k) &= [\text{first block-columns of } R(k+1)], \\ C(k) &= [\text{first block-rows of } O(k)], \\ D(k) &= g(k,k), \end{aligned} \tag{2.13}$$

where $\dagger$ is the pseudo-inverse, $\uparrow$ is an upward block-shift of the matrix elements, and $\leftarrow$ is a block-shift to the left of the matrix elements.

The factorization can be made by using QR- or SVD-factorizations of $H(k)$. If we use SVD, also called principal component identification, as suggested in [Kung, 1978; Shokoohi and Silverman, 1987], then we obtain a *balanced realization* if we put

$$H(k) = U(k)\Sigma(k)V^T(k), \quad O(k) = U(k)\Sigma^{1/2}(k), \quad R(k) = \Sigma^{1/2}(k)V^T(k).$$

The observability and reachability Gramians that are used in Papers I and II are given by

$$\begin{aligned} Q(k) &= O^T(k)O(k) = \Sigma(k), \\ P(k) &= R(k)R^T(k) = \Sigma(k). \end{aligned} \tag{2.14}$$

Hence, the realization is equally observable and reachable. This is the reason for the term "a balanced realization". Furthermore, the nonzero elements of the diagonal Gramians $\Sigma(k)$, $\{\sigma_i(k)\}_{i=1}^{n(k)}$, are the Hankel singular values, which are identical to the singular values used in Papers I and II.

The reduction step and error bounds. Instead of first obtaining a realization as above, and then truncate it according to the methods in Paper I and II, we can directly obtain a rank $\hat{n} < n$ approximation of $H(k)$ by using a truncated SVD expansion (2.6)

$$\hat{H}(k) = U_{\hat{n}}(k)\Sigma_{\hat{n}}(k)V_{\hat{n}}^T(k), \; O_{\hat{n}}(k) = U_{\hat{n}}(k)\Sigma_{\hat{n}}^{1/2}(k), \; R_{\hat{n}}(k) = \Sigma_{\hat{n}}^{1/2}(k)V_{\hat{n}}^T(k),$$

where

$$\|H(k) - O_{\hat{n}}(k)R_{\hat{n}}(k)\|_2 = \sigma_{\hat{n}+1}(k). \tag{2.15}$$

A problem is that the low-rank factors $O_{\hat{n}}(k)$ and $R_{\hat{n}}(k)$ are not observability and reachability matrices, since there is generally not a matrix $\hat{A}(k)$ that simultaneously fulfills

$$O_{\hat{n}}(k+1)\hat{A}(k) = O_{\hat{n}}^{\uparrow}(k), \quad \hat{A}(k)R_{\hat{n}}(k) = R_{\hat{n}}^{\leftarrow}(k+1). \tag{2.16}$$

However, if $\sigma_{\hat{n}+1}(k)$ is small, then they are close to being observability and reachability matrices. By finding a least-squares solution $\hat{A}(k)$ to (2.16), we obtain a reduced-order realization from the formulas (2.13), with $O_{\hat{n}} \to O$ and $R_{\hat{n}} \to R$. It is shown in [Shokoohi and Silverman, 1987] that this procedure yields a truncated balanced realization. Hence, we can go directly from an input-output model G to a reduced model $\hat{G}$ with realization $(\hat{A}, \hat{B}, \hat{C}, D)$. The calculation of the SVD of the Hankel matrices can be implemented in a more efficient recursive fashion, see [Dewilde and van der Veen, 1998; Chahlaoui and Van Dooren, 2003].

Above we have approximated the Hankel matrix $H(k)$ with $\hat{H}(k)$. It may be surprising that it can be proven that the corresponding models G and $\hat{G}$, in general, also are close. That is, we can find functions C_1 and C_2 such that

$$C_1(\hat{n}) \leq \|G - \hat{G}\|_2 \leq C_2(\hat{n}). \tag{2.17}$$

These functions are useful when we select the approximation order $\hat{n}$. This is because it may be computationally expensive to compute $\|G - \hat{G}\|_2$ for every possible choice of $\hat{n}$. See the discussion after Theorem 1.3. The next theorem summarizes the error bounds obtained in Papers I and II. The max-min ratio $S_{[0,N]}(\sigma)$ is a product of the ratios of all local minima and maxima of $\sigma(k)$ over the interval $[0, N]$. See Definition 1 in Paper I.

THEOREM 2.2—PAPERS I AND II

Assume that G has a realization (2.2), initially at rest, on $I = [0, N]$. Then there is a lower error bound that limits what can be achieved with an approximation $\hat{G}$ with $\hat{n}$ states

$$C_1(\hat{n}) = \max_{k \in [1,N]} \sigma_{\hat{n}+1}(k),$$

where $\sigma_i(k)$ are the singular values of $H(k)$ in (2.10).

If the balanced truncation procedure is used to construct $\hat{G}$, then there are two upper error bounds.

(i)

$$C_2(\hat{n}) = 2 \sum_{i=\hat{n}+1}^{n} S_{[0,N]}(\sigma_i),$$

where $S_{[0,N]}(\sigma_i)$ is the max-min ratio of $\sigma_i(k)$ over $[0, N]$;

(ii)

$$C_2(\hat{n}) = K \cdot \max_{k\in[1,N]} \sigma_{\hat{n}+1}(k),$$

where $K = C/(1-\lambda) > 1$ and $\|\Phi_A(k,i)\|_2 \leq C \cdot \lambda^{k-i}$, $\lambda < 1$, for an input-normalized realization of G. □

The lower bound and the upper bound (i) are generalizations of well-known time-invariant results, see [Glover, 1984; Enns, 1984]. These are derived in Paper I. The upper bound (ii) is derived in Paper II. The bound (ii) is interesting since it is not a sum over all truncated singular values. A problem with the bound is that K may be large.

The balanced truncation procedure also works for continuous-time models, and details can be found in the Papers I and II.

REMARK 2.1—SUBOPTIMALITY OF BALANCED TRUNCATION
In the this description of balanced truncation, the Gramians $P(k)$ and $Q(k)$ in (2.14) fulfill

$$\begin{aligned} A(k)P(k)A^T(k) - P(k+1) + B(k)B^T(k) &= 0, \quad P(0) = 0, \\ A^T(k)Q(k+1)A(k) - Q(k) + C^T(k)C(k) &= 0, \quad Q(N+1) = 0. \end{aligned}$$

In [Lall and Beck, 2003] and in Paper I, we only require that $P(k)$ and $Q(k)$ fulfill linear matrix inequalities (LMIs) of the type (2.3). This adds more degrees of freedom to the problem. The LMIs can be solved using standard semidefinite programming.

It is pointed out in [Lall and Beck, 2003] that if there is an approximation $\|G - \hat{G}\|_2 < \gamma$, then there are Gramians that fulfill the conditions (2.3)–(2.4) of Theorem 2.1. Using these Gramians (that are hard to find in practice) in the balanced truncation procedure, yields a model with $\|G - \hat{G}\|_2 \leq 2\gamma$. Hence, in theory, by using balanced truncation with inequalities, we can always come a factor two away from an optimal approximation. □

REMARK 2.2—BALANCED TRUNCATION FOR STATE-SPACE MODELS
In the above presentation, balanced truncation is based on SVD and the initial model is an impulse-response model. It is more common in model reduction that the initial model already is in state-space form. This is the case treated in Papers I and II. In fact, it is then simpler to treat models in continuous time, infinite time horizons, and stability issues. All of these issues are treated in Paper I, and the error bound in Theorem 2.2 (i) is proven using state-space techniques.

To prove the bound (i), we need Lemmas 1–4 in Paper I. To prove the lemmas, the following Schur lemma is useful. If A_{11} is nonsingular, then

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix},$$

where $\Delta = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

In Paper II, it is discussed how one may obtain the reduced-order model in a numerically sound way. Projections $S_L^T(k)$ and $S_R(k)$ are used, such that the reduced-order model $\hat{G}$ can be obtained as

$$\begin{aligned} \hat{A}(k) &= S_L^T(k+1)A(k)S_R(k), \quad \hat{B}(k) = S_L^T(k+1)B(k), \\ \hat{C}(k) &= C(k)S_R(k). \end{aligned} \tag{2.18}$$

□

2.5 Hankel-Norm Approximation

The balanced truncation method may seem rather heuristic since the approximation $\hat{H}(k)$ of $H(k)$ in (2.15) is not of generalized Hankel structure. It is possible to develop a theory where all the approximations really are of generalized Hankel structure. We give a brief overview of this theory next, but we leave most details out. The book [Dewilde and van der Veen, 1998] contains all the details. Much of the following hinges on generalizations of the classical AAK-lemma [Adamjan *et al.*, 1971]. The AAK-lemma was used to solve the Hankel-norm approximation problem in the time-invariant case. See, for example, [Glover, 1984].

The Hankel norm of a system G is defined by

$$\|G\|_H = \max_{k \in [1,N]} \|H(k)\|_2,$$

where $H(k)$ is the generalized Hankel matrix (2.10). We have $\|G\|_H \leq \|G\|_2$ and $\|G + F\|_H = \|G\|_H$ if F is strictly anti-causal (strictly upper triangular). The main result we state in this section is derived in [Dewilde and van der Veen, 1998].

THEOREM 2.3—[DEWILDE AND VAN DER VEEN, 1998]
Let G be a causal input-output model in discrete time, and assume that none of its Hankel singular values $\sigma_i(k)$ are equal to γ. Then there exists a causal model $\hat{G}$ with a realization of order $\hat{n}(k) = \#\{i : \sigma_i(k) > \gamma\}$, and

$$\|G - \hat{G}\|_H \leq \gamma. \tag{2.19}$$

□

In [Dewilde and van der Veen, 1998], an algorithm is given for the computation of the realization of $\hat{G}$. Furthermore, all the solutions are parameterized, and the case when the time horizon N is infinite is treated.

Notice that the bound (2.19) is in the Hankel norm and not in the regular 2-norm that was used in Theorem 2.1 and (2.17). Hence, the ideal model reduction problem stated in Section 2.1 has a simple solution if the norm is changed. In a sense, the above method works similar to the SVD method described in Example 2.1, with certain important differences. As discussed in Section 2.2 the minimization problem (2.5) does in general not have an unique solution. This is utilized in the Hankel method by enforcing a condition that the causal part (the lower triangular part) of the approximation should have a realization of certain order.

For a given $\gamma > 0$, the method obtains all, not necessarily causal, approximations $\tilde{G}$ such that

$$\|G - \tilde{G}\|_2 \leq \gamma,$$

where the causal part of $\tilde{G}$ has a realization of given order. This problem has a solution if γ and the order constraints are chosen as in Theorem 2.3. After a causal/anti-causal decomposition of $\tilde{G}$, $\tilde{G} = \hat{G} + F$, we obtain the bound (2.19),

$$\|G - \hat{G}\|_H = \|G - \hat{G} - F\|_H \leq \|G - \hat{G} - F\|_2 \leq \gamma.$$

Hence, just as in Example 2.1 we obtain a non-causal approximation in the 2-norm. The difference is that the causal part of the approximation always has a low-order realization.

Even if it is common that a good approximation in the Hankel norm also is a good approximation in the induced 2-norm, it is desirable to have bounds of the type (2.17). The reason is that the induced 2-norm is more suitable for robustness studies of feedback systems. For optimal *time-invariant* Hankel-norm approximation one can derive such a bound,

$$C_2(\hat{n}) = \sum_{i=\hat{n}+1}^{n} \sigma_i, \tag{2.20}$$

see [Glover, 1984]. To the author's best knowledge, a similar bound has not been derived for time-varying Hankel-norm approximation. This is an interesting problem for future research.

Generalized Hankel-norm approximation. Hankel-norm approximation can also be performed for continuous-time models, see [Kaashoek and Kos, 1994]. In fact, the Hankel-norm approximation problem can be formulated and solved in quite abstract settings. In the paper [Iftime *et al.*, 2004] (co-written by the author), a framework that applies to both continuous-time and discrete-time models, as well as to infinite-dimensional models, is developed. To apply the theory, the objects G that shall be simplified must belong to a so-called C^*-algebra $\mathcal{R}$ with a norm $\|\cdot\|_{\mathcal{R}}$, see for example [Böttcher and Silbermann, 1990]. Furthermore, there should be a suitable index function $\nu(\cdot) : \mathcal{R} \to Z_+$. The index function is a measure of complexity. The (sub-optimal) Hankel-norm approximation problem can then be formulated as follows:

For a given $\gamma > 0$, find all $\tilde{G} \in \mathcal{R}$ such that

$$\|G - \tilde{G}\|_{\mathcal{R}} < \gamma, \quad \text{and} \quad \nu(\tilde{G}) = \hat{n}.$$

Under extra technical assumptions, this problem is solved in [Iftime *et al.*, 2004]. In the case discussed in Theorem 2.3, we can take $\mathcal{R}$ as the set of bounded (not necessarily causal) linear operators on ℓ_2, and $\nu(\tilde{G})$ as the realization order of the *causal* part of $\tilde{G}$.

2.6 Truncated Fourier and Taylor Series

Next, two other types of model reduction are studied. These methods apply to input-output models and are treated in detail in Paper III. The main idea is to write the input-output model as a series

$$G = G_1 + G_2 + G_3 + \dots,$$

where the terms G_k are simple and tend to zero quickly in some sense. We will investigate two such expansions, Taylor and Fourier expansions.

Taylor expansions. The first expansion is applied to causal impulse-response models G in continuous time

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau. \tag{2.21}$$

The smooth input belongs to the set of Schwartz functions $\mathcal{S}$, which are dense in L_p, for $1 \le p < \infty$. We define the *input Markov parameters* $a_k(t)$, and the *output Markov parameters* $b_k(t)$, where

$$a_k(t) = (-1)^{k-1} \left. \frac{\partial^{k-1} g(t,s)}{\partial s^{k-1}} \right|_{s=t}, \quad \text{and} \quad b_k(t) = \left. \frac{\partial^{k-1} g(s,t)}{\partial s^{k-1}} \right|_{s=t}.$$

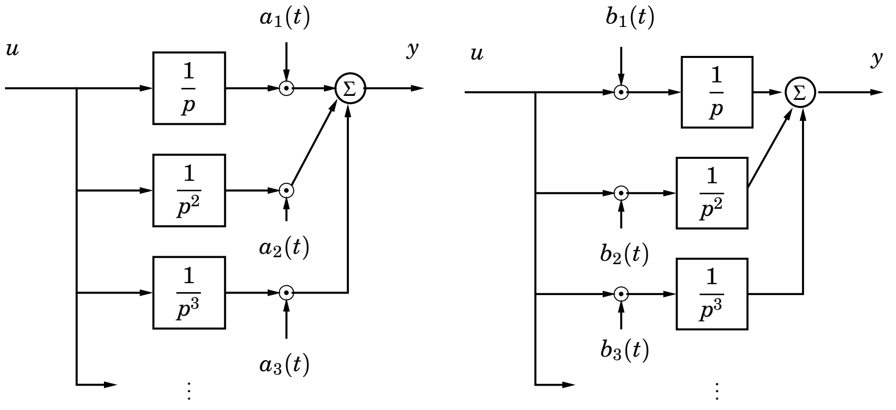

Figure 2.2 The input and output Markov parameter expansions of a time-varying system. The expansions are dual to each other, and each term is a modulated multiple integrator.

$\mathcal{C}_e^L$ is the set of L times differentiable and exponentially decaying impulse responses, see Definition 1 in Paper III, and $\frac{1}{p}$ is the integration operator. Now, the following theorem may be derived.

THEOREM 2.4—PAPER III
Assume that the causal time-varying impulse response $g(t,\tau)$ belongs to the set $\mathcal{C}_e^L$. Then for every input $u \in \mathcal{S}$, $y(t)$ in (2.21) can be expressed in either of the following two ways.
— Input Markov parameter expansion

$$y(t) = a_1(t)\frac{u(t)}{p} + a_2(t)\frac{u(t)}{p^2} + \ldots + a_L(t)\frac{u(t)}{p^L} + \left(G^{Ri}\frac{u}{p^L}\right)(t),$$

where $a_1(t),\ldots,a_L(t)$ are the input Markov parameters, and G^{Ri} is a bounded linear operator on L_2.
— Output Markov parameter expansion

$$y(t) = \frac{1}{p}b_1(t)u(t) + \frac{1}{p^2}b_2(t)u(t) + \ldots + \frac{1}{p^L}b_L(t)u(t) + \left(\frac{1}{p^L}G^{Ro}u\right)(t),$$

where $b_1(t),\ldots,b_L(t)$ are the output Markov parameters, and G^{Ro} is a bounded linear operator on L_2. □

The expansions can be visualized as in Figure 2.2. The terms in the expansions are simple, since they are modulated multiple integrators of the type

$$G_k = a_k(t)\frac{1}{p^k}, \quad \text{or} \quad G_k = \frac{1}{p^k}b_k(t),$$

for $1 \leq k \leq L$. The expansions are dual to each other.

The reason for calling them Taylor expansions is that for time-invariant systems, they both correspond to a Taylor expansion of the transfer function $G(s)$ around $s = \infty$. In the time-invariant case, $a_k(t)$ and $b_k(t)$ are constant and equal to the standard Markov parameters of the system.

The expansions in Theorem 2.4 are suitable for studies of the high-frequency behavior of the system. For time-varying systems it is well known that there is coupling between frequencies in input and output, see Section 1.4. We introduce an ideal high-pass filter Q_Ω, with frequency characteristics

$$\hat{Q}_\Omega(j\omega) = \begin{cases} 1, & |\omega| > \Omega, \\ 0, & |\omega| \leq \Omega. \end{cases}$$

We can now prove that there is a constant C_1 for the input Markov parameter expansion, such that

$$\|(G - G_1 - G_2 - \ldots - G_N)Q_\Omega\|_2 \leq \frac{C_1}{\Omega^{N+1}}, \quad N < L.$$

For the output Markov parameter expansion there is a constant C_2, such that

$$\|Q_\Omega(G - G_1 - G_2 - \ldots - G_N)\|_2 \leq \frac{C_2}{\Omega^{N+1}}, \quad N < L.$$

The output expansion is suitable if we want to study the high-frequency content of the output for any input. The input expansion is suitable if we want to study the response of the system to high-frequency inputs. This may be seen as model reduction with the weight Q_Ω, see Section 2.1.

REMARK 2.3—INPUT ROLL-OFF AND OUTPUT ROLL-OFF
The Taylor expansions are used in Paper III to study the convergence rate of truncated harmonic transfer functions $\hat{G}(j\omega)$. It is shown that the decay rates of elements in the harmonic transfer function in the up-down and left-right directions are determined by how many of the input and output Markov parameters that are zero. These decay rates are labeled input and output roll-off. □

Fourier expansions. The following expansions are applied to causal impulse-response models (2.21), with the additional assumption that they should be *periodic*, see Definition 1.2. Then there is a period $T > 0$, such that

$$g(t, \tau) = g(t + T, \tau + T),$$

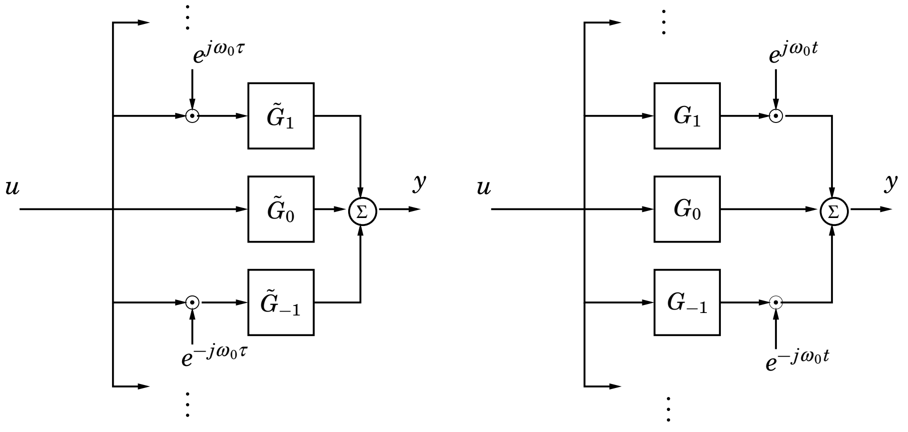

Figure 2.3 The Fourier expansions. The coefficients G_k and $\tilde{G}_k$ are time-invariant impulse-response models.

for all t, τ.

The periodic systems that we consider belong to a Hilbert space H_2, with the scalar product

$$\begin{aligned}\langle G, H\rangle_{H_2} &= \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty}\overline{g(r+\tau,\tau)}h(r+\tau,\tau)drd\tau \\ &= \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty}\overline{g(t,t-r)}h(t,t-r)drdt.\end{aligned}$$

H_2 is combination of the Hilbert spaces $L_2[0,T]$ and $L_2[0,\infty)$. A periodic system belongs to H_2 if and only if the norm

$$\|G\|_{H_2}^2 = \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty}|g(r+\tau,\tau)|^2 drd\tau = \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty}|g(t,t-r)|^2 drdt$$

is finite. The idea is now to apply generalized Fourier expansion on H_2. It is shown in Paper III that this leads to the following theorem.

THEOREM 2.5—PAPER III
Assume that G belongs to H_2. Then the causal impulse response of G can be expressed as

$$\begin{aligned} g(t,\tau) &= \sum_{k=-\infty}^{\infty}\tilde{g}_k(t-\tau)e^{jk\omega_0\tau}, \\ g(t,\tau) &= \sum_{k=-\infty}^{\infty}g_k(t-\tau)e^{jk\omega_0 t}, \end{aligned} \tag{2.22}$$

with convergence in H_2, and where $\omega_0 = 2\pi/T$. The Fourier coefficients are given by

$$\tilde{g}_k(r) = \frac{1}{T}\int_0^T e^{-jk\omega_0\tau} g(\tau + r, \tau)d\tau, \quad g_k(r) = \frac{1}{T}\int_0^T e^{-jk\omega_0 t} g(t, t-r)dt,$$

where $r \geq 0$. □

Since the Fourier coefficients in (2.22) only depend on the difference $t-\tau$, they can be interpreted as time-invariant impulse-responses of models $\tilde{G}_k$ and G_k. Then G can be expressed as a sum, as suggested in Figure 2.3. The Fourier expansions may be truncated,

$$G_{[N]} = \sum_{|k|\leq N} e^{jk\omega_0 t} G_k = \sum_{|k|\leq N} \tilde{G}_k e^{jk\omega_0 \tau}.$$

It is discussed in Paper III how quickly the Fourier expansion converges in the induced L_p-norm, for $1 \leq p \leq \infty$. In particular, conditions are given for when these expansions converge at a rate k,

$$\|G - G_{[N]}\|_p \leq C \cdot N^{-k},$$

for some constant C.

REMARK 2.4—THE HARMONIC TRANSFER FUNCTION
The Fourier expansions are used in Paper III to define the harmonic transfer function $\hat{G}(j\omega)$. After the Fourier expansions, frequency-domain analysis of a time-periodic system essentially reduces to frequency-domain analysis of its time-invariant components. □

2.7 Summary

In this chapter, we have formulated the model reduction problem. We have also described several different methods to solve the problem. Next, we summarize how the contributions of Papers I–IV relate to this chapter.

In Paper I, the balanced truncation method is treated. The main contribution of this paper is the derivation of the lower bound and the first upper bound in Theorem 2.2. This is done in a framework that works in both continuous and in discrete time. These bounds are direct generalizations of standard time-invariant results. Furthermore, the case of time-varying state-space dimension is treated, and it is shown that BIBO stability is preserved under truncation.

In Paper II, the methods in Paper I are applied to a linear 24th-order state-space model. Using the projection technique mentioned in Remark 2.2, it is shown that a first-order model is enough to approximate the full model. The advantages and disadvantages with continuous-time and discrete-time models are also discussed. An approximative discretization procedure is introduced as a step in the model reduction. Finally, the bound in Theorem 2.2 (ii) is proven. This bound is interesting since it is of a different structure than the twice-the-sum-of-the-tail formula. A problem with the bound is the constant K, which may be large and hard to estimate.

In Paper III, the Taylor and Fourier expansions in Section 2.6 are derived in detail. The convergence rates of the truncated series are investigated. For this, the concepts of input, output and skew roll-off are introduced. Furthermore, relations between these concepts and the time-varying Markov parameters are derived. These studies are valuable for studies of truncated harmonic transfer functions.

Paper IV is not so much concerned with model reduction. See the summary of Chapter 1. But Paper IV relies on the Taylor and Fourier expansions developed in Paper III for the proof of the analyticity property of the harmonic transfer function.

3

Concluding Remarks

In the thesis, the model reduction problem for linear time-varying systems is addressed. It is argued that linear time-varying models are useful in many areas of engineering. Some examples are presented in the introduction. In the introduction, we also review state-space models, impulse-response models, and common model reduction techniques. Furthermore, some of the results in Papers I–IV are highlighted and compared to other available results.

Results

The results in Papers I–IV can be summarized as follows:

- Error bounds for balanced truncation of linear time-varying systems are derived. The results generalize well-known time-invariant results, derived with other methods. An upper error bound that is not of the traditional "twice-the-sum-of-the-tail" type is also obtained.
- Practical issues for balanced truncation of linear time-varying systems are discussed. Approximative discretization and projections are used.
- Taylor and Fourier expansions of linear time-varying impulse-response models are derived.
- Convergence criteria for the harmonic transfer function of a linear time-periodic system are derived. The concepts of input, output, and skew roll-off are introduced.
- A Bode sensitivity integral for linear time-periodic systems is derived by using the harmonic transfer function.

Suggestions for Future Work

Most of the results that have made balanced truncation a popular method for time-invariant models, now have their counterparts in the time-varying setting. See, for example, [Shokoohi and Silverman, 1987; Lall and

Beck, 2003] and Paper I. But still interesting problems remain. One problem is analysis of the tightness of available error bounds. Theorem 2.2 (ii) indicates that there are error bounds that only depend on the maximum truncated singular value. It is interesting to see if this bound can be improved.

Another interesting problem is to study how the methods can be applied to high-order models. Balanced truncation is a method that is relatively computationally expensive. When the model has thousands of states, balanced truncation is often not possible to use. It is interesting to investigate if the method can be modified to make it feasible in these cases, and still guarantee that good approximations are obtained.

For the harmonic transfer function there are also several interesting problems remaining. For example, is it possible generalize more time-invariant transfer function relations?

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Paper I

Balanced Truncation of Linear Time-Varying Systems

Henrik Sandberg and Anders Rantzer

Abstract

In this paper, balanced truncation of linear time-varying systems is studied in discrete and continuous time. Based on relatively basic calculations with time-varying Lyapunov equations/inequalities we are able to derive both upper and lower error bounds for the truncated models. These results generalize well-known time-invariant formulas. The case of time-varying state dimension is considered. Input-output stability of all truncated balanced realizations is also proven. The method is finally successfully applied to a high-order model.

1. Introduction

This paper treats model reduction of time-varying linear systems. Time-varying linear systems are of interest not only for modeling of time-varying physical processes, but also because of the fact that *time-invariant* nonlinear systems can be well approximated by *time-varying* linear systems around nominal trajectories. Linear time-varying systems have attained much attention lately, see for example the survey over periodic systems in [Bittanti and Colaneri, 1999] and references therein.

1.1 Problem Statement

We will assume that a linear system G is given, either in continuous or discrete time. The system should have a finite-dimensional realization with n states. The objective is to find a system $\hat{G}$ with $\hat{n}$ states that approximates G well, where $\hat{n}$ ideally should be much smaller than n. One objective is to find simple candidates $\hat{G}$ for given G and $\hat{n}$. Another objective is to find simple functions $C_1(\cdot)$ and $C_2(\cdot)$, error bounds, such that

$$C_1(\hat{n}) \leq \|G - \hat{G}\| \leq C_2(\hat{n}), \tag{1}$$

as this simplifies the selection of $\hat{G}$. The operator norm will be the induced 2-norm. Notice that we can always compute $\|G-\hat{G}\|$ to any wanted degree of accuracy once $\hat{G}$ is chosen. However, this is computationally expensive and involves bisection algorithms and solving time-varying Riccati-equations, see for instance [Tadmor, 1990], which is hardly something we would like to do for each candidate $\hat{G}$. So bounds of the type (1) are helpful. Moreover, we would like essential properties of the original system G, such as stability, to be preserved for each candidate $\hat{G}$.

1.2 Previous Work

To reduce the order of linear time-invariant systems, balanced truncation is often used. Balanced realizations were introduced in [Mullis and Roberts, 1976], but were first used for the purpose of model reduction in [Moore, 1981]. A sufficient condition for asymptotic stability of truncated models was later given in [Pernebo and Silverman, 1982]. Since then an error bound has been proven, [Enns, 1984; Glover, 1984], which gives a simple bound on the worst case error between the original and truncated model and justifies the approximation. The bound was first derived for continuous-time systems, but it also holds for discrete-time systems as proven in [Al-Saggaf and Franklin, 1987]. The bound is a sum of truncated Hankel singular values and the result is now considered to be standard and is included in most courses on robust control and identification.

Balanced realizations for time-varying linear systems have also received attention, see for example [Shokoohi *et al.*, 1983; Verriest and Kailath, 1983], for some early references. For the related class of linear parameter-varying (LPV) systems, balanced truncation has been studied in for example [Wood, 1996]. However, until recently no error bound has been given for the time-variable case. To obtain bounds, methods for uncertain systems could be utilized, see for example [Beck *et al.*, 1996]. However, these bounds would be conservative as the known time-variance is encapsulated in an uncertainty ball.

The first explicit error bound for balanced discrete time-varying models, to the authors' best knowledge, was given in [Lall *et al.*, 1998] and later refined in [Lall and Beck, 2003]. There, an operator-theoretic framework was used to give bounds similar to those that apply to time-invariant models. For discrete time-periodic linear systems bounds have been proven in [Longhi and Orlando, 1999; Varga, 2000]. There, a special form of lifting isomorphism was used.

1.3 Contributions of This Paper

In this paper we will work directly with the time-varying observability and reachability Lyapunov inequalities [linear matrix inequalities (LMIs)] in both continuous and discrete time. It will be seen that it is natural to allow the state-space dimension to vary in size over time. In fact, a time-varying state dimension may be required for a minimal realization as is shown and used in for example [Gohberg *et al.*, 1992] and [Varga, 2000]. The approach will give fairly simple calculations and more general error bounds (1) than in the previously mentioned references. In particular we will allow for time-varying Gramians, which is not treated in [Lall *et al.*, 1998; Lall and Beck, 2003]. As special cases we will recover the known bounds for both time-invariant and time-varying systems. Furthermore, the method will give new results on input-output stability of the reduced models.

The ability to vary the state-space dimension over time is not only of interest for technical reasons. In for example stiff problems, such as chemical reactions, it is frequent that in the initial phase, many complex reactions take place and that the dynamics then slows down. It is then reasonable to have a model with many states in the initial phase and then switch to a low-order model after some time. The analysis presented will help to decide when to switch the number of states and also how much loss in accuracy a certain choice might give.

1.4 Organization

The organization of the paper is as follows. In Sections 2 and 3, notation for discrete and continuous-time systems will be introduced, along with two lemmas on observability and reachability. The lemmas will form the basis of the following analysis. In Section 4, we will define what a balanced system is and how we, with the help of the lemmas, can attain simple upper error bounds. In Section 5, input-output stability of all truncated models is proved. In Section 6, a lower error bound for truncated models is given. In Section 7, an example of how balanced model truncation works in practice is given. In Appendix B it is shown how sampling of a continuous-time system can be combined with model truncation.

2. Discrete-Time Systems

As some aspects of the calculations are simpler for discrete-time systems, we will start at that end. It should, however, be pointed out that everything presented here will later also be done for continuous-time systems.

2.1 Preliminaries and Notation

The linear systems G that we consider are assumed to have a finite-dimensional state-space realization

$$G : \begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) & x(0) = 0 \\ \quad y(k) = C(k)x(k) + D(k)u(k) \end{cases} \tag{2}$$

with m inputs and p outputs. It will be useful to utilize time-varying state-space dimension as commented in the introduction. It is known that minimal realizations of linear systems in general have this property; see [Gohberg *et al.*, 1992]. However, it will also be a useful technical tool for reducing the order of systems where the state-space dimension originally is constant over time. Let the state-space dimension at time k be $n(k)$. The signals and matrices then have the dimensions

$$\begin{array}{lll} A(k) \in \boldsymbol{R}^{n(k+1)\times n(k)} & B(k) \in \boldsymbol{R}^{n(k+1)\times m} & x(k) \in \boldsymbol{R}^{n(k)} \\ C(k) \in \boldsymbol{R}^{p\times n(k)} & D(k) \in \boldsymbol{R}^{p\times m} & y(k) \in \boldsymbol{R}^{p} \\ & & u(k) \in \boldsymbol{R}^{m}. \end{array}$$

We will assume that all the matrices are real, bounded, and defined for $k \in [0, T]$. Sometimes we will have $T = +\infty$, and then the system is assumed to be stable. Notice that as the model order may vary with k, $A(k)$ is not necessarily a square matrix but rather rectangular. We could also let the number of inputs and outputs vary over time, but we avoid this for the sake of notational simplicity.

The signals will belong to the Hilbert space $\ell_2[0,T]$. We will utilize the weighted Euclidean norm as defined by

$$|x(k)|_P^2 = x^T(k)P(k)x(k)$$

with a positive-definite matrix $P(k) \in \boldsymbol{R}^{n(k)\times n(k)}$, and also the weighted ℓ_2-norm

$$\|x\|_P^2 = \sum_{k=0}^{T} |x(k)|_P^2. \tag{3}$$

Discrete-time signals x over a time interval $[0,\infty)$ belong to $\ell_2^n[0,\infty)$ iff the norm (3) is finite for $P(k) = I$ with $T = +\infty$. If we want to emphasize that the norm is taken over the interval $[0,T]$, we will write $\|x\|_{P,[0,T]}$, but the interval will normally be clear from the context. Linear systems as defined in (2) can be identified with a linear operator $G : \ell_2^m[0,T] \to \ell_2^p[0,T]$. The operator is *bounded* iff the induced norm

$$\|G\| = \sup_{\|u\|\le 1} \|Gu\|$$

is bounded. Often we will make an upper estimate of $\|G\|$ by finding a constant $C(G) > 0$ such that

$$\|y\| \le C(G) \cdot \|u\|$$

for all admissible u.

The system we would like to obtain, $\hat{G}$, will be called a reduced-order system. It will have the state-space dimension $\hat{n}(k)$ where $\hat{n}(k) \le n(k)$ for all k. We will construct $\hat{G}$ from a truncation of the realization of G. The following partitions will be used:

$$\begin{aligned}
A(k) &= \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix} & A_{11}(k) &\in \boldsymbol{R}^{\hat{n}(k+1)\times \hat{n}(k)} \\
B(k) &= \begin{bmatrix} B_1(k) \\ B_2(k) \end{bmatrix} & B_1(k) &\in \boldsymbol{R}^{\hat{n}(k+1)\times m} \\
C(k) &= [\, C_1(k) \quad C_2(k) \,] & C_1(k) &\in \boldsymbol{R}^{p\times \hat{n}(k)} \\
x^T(k) &= [\, x_1^T(k) \quad x_2^T(k) \,] & x_1(k) &\in \boldsymbol{R}^{\hat{n}(k)}.
\end{aligned}$$

If the realization (2) is chosen such that the states $x_2(k)$ are "small" in some sense, a reasonable reduced-order candidate is obtained by truncating the corresponding states

$$\hat{G} : \begin{cases} \hat{x}(k+1) = A_{11}(k)\hat{x}(k) + B_1(k)u(k) & \hat{x}(0) = 0 \\ \hat{y}(k) = C_1(k)\hat{x}(k) + D(k)u(k) & \hat{x}(k) \in \boldsymbol{R}^{\hat{n}(k)}. \end{cases} \tag{4}$$

The auxiliary signal

$$\hat{z}(k+1) = A_{21}(k)\hat{x}(k) + B_2(k)u(k) \tag{5}$$

will naturally show up later. It is not needed to evaluate the map $\hat{G}$. $\hat{z}(k)$ has dimension $\boldsymbol{R}^{n(k)-\hat{n}(k)}$ and is defined when truncation has occurred, i.e., $\hat{n}(k) < n(k)$. As $\hat{z}$ is not necessarily defined for all k, it will be useful to collect the time points where it does exist in a set $\mathcal{T}$

$$\mathcal{T} = \{k : \hat{z}(k) \text{ exists}\}. \tag{6}$$

Furthermore, let us define $\hat{z}(0) = 0$ if $\hat{n}(0) < n(0)$.

If the systems G and $\hat{G}$ are supposed to have a similar input-output behavior when the above truncation scheme is used, it is important that the coordinate system in the realization of G is well chosen. As we will see, such coordinate systems exist in many cases. A change in coordinate system, $x(k) = T(k)\tilde{x}(k)$, for invertible $T(k)$, will transform the realization according to

$$\begin{aligned}&\{A(k),\, B(k),\, C(k),\, D(k)\} \xrightarrow{T(k)} \{\tilde{A}(k),\, \tilde{B}(k),\, \tilde{C}(k),\, \tilde{D}(k)\}\\ &= \{T^{-1}(k+1)A(k)T(k),\, T^{-1}(k+1)B(k),\, C(k)T(k),\, D(k)\}.\end{aligned} \tag{7}$$

2.2 The Observability Lyapunov Inequality

Consider the *Lyapunov observability inequality*

$$A^T(k)Q(k+1)A(k) + C^T(k)C(k) \le Q(k), \quad k \in [0, T]. \tag{8}$$

$Q(k)$ is often called an *observability Gramian*. The positive-semidefinite solutions $Q(k)$, $k = 0 \dots T{+}1$, bound the amount of energy there will be in the output for a given initial state $x(0)$ of the system G with zero input

$$|x(T+1)|_Q^2 + \|y\|_{[0,T]}^2 \le |x(0)|_Q^2.$$

The inequality can, however, also be used to calculate the ℓ_2-norm of the difference in the outputs from G and $\hat{G}$ when both systems are driven by the same input signal. To see this, assume there is a positive-semidefinite solution $Q(k)$ to (8) with the block-diagonal structure

$$Q(k) = \begin{bmatrix} Q_1(k) & 0 \\ 0 & q(k)\cdot I_{n(k)-\hat{n}(k)} \end{bmatrix} \in \boldsymbol{R}^{n(k)\times n(k)} \tag{9}$$

for $k = 0 \dots T+1$ and $q(k)$ scalar. Then rewrite (8) for each k in the following way:

$$\begin{bmatrix} A(k) \\ I \end{bmatrix}^T \begin{bmatrix} Q(k+1) & 0 \\ 0 & -Q(k) \end{bmatrix} \begin{bmatrix} A(k) \\ I \end{bmatrix} + C^T(k)C(k) \le 0. \tag{10}$$

If we apply the same input signal u to (2) and (4) we obtain the trajectories x and $\hat{x}$. Use the trajectories to calculate the difference

$$\begin{bmatrix} x_1(k) - \hat{x}(k) \\ x_2(k) \end{bmatrix} \in \boldsymbol{R}^{n(k)}.$$

Multiply (10) for each k from the right with the difference and from the left with its transpose. We then obtain

$$\begin{bmatrix} x_1(k+1) - \hat{x}(k+1) \\ x_2(k+1) - \hat{z}(k+1) \\ x_1(k) - \hat{x}(k) \\ x_2(k) \end{bmatrix}^T \begin{bmatrix} \boldsymbol{Q}(k+1) & 0 \\ 0 & -\boldsymbol{Q}(k) \end{bmatrix} \times \begin{bmatrix} x_1(k+1) - \hat{x}(k+1) \\ x_2(k+1) - \hat{z}(k+1) \\ x_1(k) - \hat{x}(k) \\ x_2(k) \end{bmatrix} + |y(k) - \hat{y}(k)|^2 \leq 0$$

which is the same as

$$\Delta \left| \begin{bmatrix} x_1(k) - \hat{x}(k) \\ x_2(k) \end{bmatrix} \right|_Q^2 - 2q(k+1)\hat{z}^T(k+1)x_2(k+1) + |\hat{z}(k+1)|_q^2 + |y(k) - \hat{y}(k)|^2 \leq 0 \quad (11)$$

using the structure (9) of $\boldsymbol{Q}(k)$. The forward difference operator Δ is defined as

$$\Delta r(k) = r(k+1) - r(k)$$

on a scalar sequence $\{r(k)\}$. Now we can state the following lemma:

LEMMA 1—OBSERVABILITY
If there is a solution $\boldsymbol{Q}(k)$ with the structure (9) to the Lyapunov inequality (8) on the interval $[0, T+1]$, then the solutions of (2) and (4) satisfy the following.

(i)

$$\left| \begin{bmatrix} x_1(T+1) - \hat{x}(T+1) \\ x_2(T+1) \end{bmatrix} \right|_Q^2 + \|y - \hat{y}\|_{[0,T]}^2 + \sum_{k \in \mathcal{T}} \left(|\hat{z}(k)|_q^2 - 2q(k)\hat{z}^T(k)x_2(k) \right) \leq 0 \quad (12)$$

where equality holds if (8) was solved with equality.

(ii) For every nonincreasing positive scalar sequence $\{a(k)\}_{k=0}^{T}$ we have

$$\|y-\hat{y}\|^2_{a,[0,T]} - \sum_{k\in\mathcal{T}} a(k-1)2q(k)\hat{z}^T(k)x_2(k) \le 0. \qquad (13)$$

□

Proof. (i) Sum the inequalities (11) over the interval $k = 0\dots T$ and notice the cancelling terms.

(ii) Multiply (11) with $a(k)$ for each k, and sum over $k = 0\dots T$. For nonincreasing $a(k)$ the partially cancelling terms become nonnegative numbers. The sum over $\mathcal{T}$ is the only sign-indefinite term, which leads to the inequality (13). ■

As seen if $\mathcal{T} = \emptyset$ the difference in output is zero, as $G = \hat{G}$. All terms in (12) are necessarily nonnegative except the terms $\hat{z}^T(k)x_2(k)$. These terms are the price we pay for truncating states. One might think that if the numbers $q(k)$ are small for $k \in \mathcal{T}$, then $\|y-\hat{y}\|$ will be small. Indeed, if the states $x_2(k)$ are unobservable there is a solution $Q(k)$ such that $q(k) = 0$ and $\|y-\hat{y}\| = 0$. Thus a small $q(k)$ could indicate that k should be included in the set $\mathcal{T}$ and that the corresponding states $x_2(k)$ should be truncated. However, we should remember that $q(k)$ is only a weight. A sufficient condition for a small $\|y-\hat{y}\|^2$ is that $|x_2(k)|_q^2$ is small for all k in $\mathcal{T}$. This can be seen by completing the squares in the sum (12). Then we see that $\|y-\hat{y}\|^2$ is bounded by $\sum_{k\in\mathcal{T}} |x_2(k)|_q^2$. However, this is not a bound of the type (1). To obtain such a bound we will make a dual analysis, which is the topic of Section 2.3.

2.3 The Reachability Lyapunov Inequality

Here, it will be seen how far away the states in G and $\hat{G}$ can be forced with the input signal u. The following inequality will be called the *Lyapunov reachability inequality*

$$A(k)P(k)A^T(k) + B(k)B^T(k) \le P(k+1), \quad k \in [0,T]. \qquad (14)$$

$P(k)$ is often called a *reachability Gramian*. Assume there is a positive-definite block-diagonal solution to (14)

$$P(k) = \begin{bmatrix} P_1(k) & 0 \\ 0 & p(k)\cdot I_{n(k)-\hat{n}(k)} \end{bmatrix} \in \boldsymbol{R}^{n(k)\times n(k)} \qquad (15)$$

with $k = 0 \ldots T{+}1$ and $p(k)$ scalar. Notice that (14) is equivalent to

$$\begin{bmatrix} A(k) & B(k) \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P^{-1}(k{+}1) & 0 \\ 0 & -P^{-1}(k) \end{bmatrix} \times \begin{bmatrix} A(k) & B(k) \\ I & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \quad (16)$$

Now, assume we apply the same input signal u to G and $\hat{G}$. We then obtain the system trajectories x and $\hat{x}$. Multiply (16) for each k with

$$\begin{bmatrix} x_1(k) + \hat{x}(k) \\ x_2(k) \\ 2u(k) \end{bmatrix} \in \boldsymbol{R}^{n(k)+m}$$

from the right and with its transpose from the left. This gives

$$\Delta \left\| \begin{bmatrix} x_1(k) + \hat{x}(k) \\ x_2(k) \end{bmatrix} \right\|_{P^{-1}}^2 + 2p^{-1}(k+1)\hat{z}^T(k+1)x_2(k+1) + |\hat{z}(k+1)|_{p^{-1}}^2 \le 4|u(k)|^2 \quad (17)$$

if the structure of $P(k)$ is used. Now, the following lemma can be stated.

LEMMA 2—REACHABILITY
If there is a solution $P(k)$ to the inequality (14) with the structure (15) on the interval $[0, T{+}1]$ then the solutions to (2) and (4) satisfy the following.
(i)

$$\left\| \begin{bmatrix} x_1(T{+}1) + \hat{x}(T{+}1) \\ x_2(T{+}1) \end{bmatrix} \right\|_{P^{-1}}^2 + \sum_{k \in \mathcal{T}} \left(|\hat{z}(k)|_{p^{-1}}^2 + 2p^{-1}(k)\hat{z}^T(k)x_2(k) \right) \le 4\|u\|_{[0,T]}^2. \quad (18)$$

(ii) For every positive nonincreasing scalar sequence $\{b(k)\}_{k=0}^T$

$$\sum_{k \in \mathcal{T}} b(k{-}1) 2p^{-1}(k)\hat{z}^T(k)x_2(k) \le 4\|u\|_{b,[0,T]}^2. \quad (19)$$

□

Proof. As in Lemma 1. Use (17) instead of (11). ■

The lemma gives boundaries on the reachable set in the state-space for fixed amounts of input energy. Notice that when $\mathcal{T} = \emptyset$ (18) reduces to the well-known result

$$|x(T+1)|^2_{P^{-1}} \leq \|u\|^2_{[0,T]} + |x(0)|^2_{P^{-1}}$$

as $x(k) = \hat{x}(k)$ for all k. Also notice that the sum in (19) potentially can cancel the sum in (13), namely if

$$a(k-1)q(k) = b(k-1)p^{-1}(k) \tag{20}$$

for all $k \in \mathcal{T}$. We have obtained a bound on the terms $\hat{z}^T(k)x_2(k)$ and this will be utilized Section 4.

As we will utilize the truncation recursively in the following it is convenient that the realization of $\hat{G}$, $\{A_{11}(k), B_1(k), C_1(k), D(k)\}$, fulfills the Lyapunov inequalities (8) and (14), with $Q_1(k)$ and $P_1(k)$ respectively. This can be seen from straightforward calculations.

3. Continuous-Time Systems

The previous ideas in discrete time goes through in continuous time without much alternation. However, we have to be somewhat careful when the number of states changes over time.

3.1 Preliminaries and Notation

The linear operator G will now operate on the Hilbert space $L_2[0,T]$, that is $G : L_2^m[0,T] \to L_2^p[0,T]$. A measurable signal x belongs to $L_2^n[0,T]$ iff the norm

$$\|x\|^2_P = \int_0^T |x(t)|^2_P \, dt$$

is finite for $P(t) = I$. The norm $\|G\|$ is the standard induced norm. We assume there is a finite-dimensional realization of G

$$G : \begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad x(0) = 0 \\ y(t) = C(t)x(t) + D(t)u(t). \end{cases} \tag{21}$$

The matrices and signals have the same dimensions as in discrete time, we will for now assume that the state dimension is $n(t) = n$ and is constant over time. We will assume that the matrices are continuous and bounded over time in all their entries. With these conditions existence and uniqueness of solutions to (21) is guaranteed, see for example [Rugh, 1996]. When the infinite time-horizon case is studied the system is assumed to be stable.

If we use the same matrix partitions as before we can define the $\hat{n}$th-order reduced-order system $\hat{G}$

$$\hat{G}: \begin{cases} \dot{\hat{x}}(t) = A_{11}(t)\hat{x}(t) + B_1(t)u(t) & \hat{x}(0) = 0 \\ \hat{y}(t) = C_1(t)\hat{x}(t) + D(t)u(t). \end{cases} \tag{22}$$

The auxiliary error signal $\hat{z} \in \boldsymbol{R}^{n-\hat{n}}$ becomes

$$\hat{z}(t) = A_{21}(t)\hat{x}(t) + B_2(t)u(t). \tag{23}$$

As we assume constant state dimension for now, the set $\mathcal{T}$ is the interval $[0, T]$.

Coordinate transformations $x(t) = T(t)\tilde{x}(t)$ with a continuously differentiable $T(t)$, non-singular for all t, give

$$\begin{aligned} &\{A(t), B(t), C(t), D(t)\} \xrightarrow{T(t)} \{\tilde{A}(t), \tilde{B}(t), \tilde{C}(t), \tilde{D}(t)\} \\ &= \{T^{-1}(t)[A(t)T(t) - \dot{T}(t)], T^{-1}(t)B(t), C(t)T(t), D(t)\}. \end{aligned} \tag{24}$$

so that the input-output map is invariant.

3.2 The Observability Lyapunov Inequality

The *observability Lyapunov inequality* takes the form

$$Q(t)A(t) + A^T(t)Q(t) + \dot{Q}(t) + C^T(t)C(t) \leq 0 \tag{25}$$

in continuous time. We can perform the same analysis as in Section 2.2 by noting that (25) can be written as

$$\begin{bmatrix} A(t) \\ I \end{bmatrix}^T \begin{bmatrix} 0 & Q(t) \\ Q(t) & \dot{Q}(t) \end{bmatrix} \begin{bmatrix} A(t) \\ I \end{bmatrix} + C^T(t)C(t) \leq 0. \tag{26}$$

As in Section 2.2 we get:

LEMMA 3—OBSERVABILITY
If there is a solution $Q(t)$ with the structure (9) to the Lyapunov inequality (25) on the interval $[0, T]$, then the solutions of (21) and (22) satisfy the following.

(i)

$$\left\| \begin{bmatrix} x_1(T) - \hat{x}(T) \\ x_2(T) \end{bmatrix} \right\|_Q^2 + \|y - \hat{y}\|^2 - \int_0^T 2q(t)\hat{z}^T(t)x_2(t)dt \leq 0 \tag{27}$$

where equality holds if (25) was solved with equality.

(ii) For every nonincreasing positive continuous scalar $a(t)$ we have

$$\|y - \hat{y}\|_a^2 - \int_0^T a(t)2q(t)\hat{z}^T(t)x_2(t)dt \le 0. \tag{28}$$

□

Proof. As Lemma 1 but use (26) instead of (10). Replace summation with integration. ■

3.3 The Reachability Lyapunov Inequality

The *reachability Lyaponov inequality* takes the form

$$A(t)P(t) + P(t)A^T(t) - \dot{P}(t) + B(t)B^T(t) \le 0 \tag{29}$$

in continuous time. If there is a positive-definite solution $P(t)$, (29) is equivalent to

$$\begin{bmatrix} A(t) & B(t) \\ I & 0 \end{bmatrix}^T \begin{bmatrix} 0 & P^{-1}(t) \\ P^{-1}(t) & \frac{d}{dt}P^{-1}(t) \end{bmatrix} \begin{bmatrix} A(t) & B(t) \\ I & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{30}$$

The analog to Lemma 2 becomes the following.

LEMMA 4—REACHABILITY

If there is a solution $P(t)$ to the inequality (29) with the structure (15) on the interval $[0, T]$ then the solutions to (21) and (22) satisfy the following.

(i)

$$\left\| \begin{bmatrix} x_1(T) + \hat{x}(T) \\ x_2(T) \end{bmatrix} \right\|_{P^{-1}}^2 + \int_0^T 2p^{-1}(t)\hat{z}^T(t)x_2(t)dt \le 4\|u\|^2. \tag{31}$$

(ii) For every positive nonincreasing continuous scalar $b(t)$

$$\int_0^T b(t)2p^{-1}(t)\hat{z}^T(t)x_2(t)dt \le 4\|u\|_b^2. \tag{32}$$

□

Proof. As Lemma 2. Use (30) instead of (16). Replace summation with integration. ■

3.4 Continuous-Time Systems with Time-Varying State Dimension

It is possible to analyze systems where the state dimension varies over time, i.e., $n(t)$ takes integer values but changes with time. This will be useful in Section 4 as we then do not need to distinguish between discrete- and continuous-time systems.

Assume that G has n states and that $\hat{G}$ has $\hat{n}_1$ states until time t^-, and then switches to $\hat{n}_2$ states at t^+, i.e., an instant switch. The question is what to do with new states and also with the ones that disappear. Furthermore, are Lemmas 3 and 4 still valid?

From t^- to t^+ the control signal u will not have time to influence the states as the input energy becomes zero on this interval of zero measure. The dynamics of the original system G becomes

$$x(t^+) = A^J x(t^-) \quad A^J = I_n$$

i.e., nothing happens with the states. The truncated realizations $A_{11}^J \in \boldsymbol{R}^{\hat{n}_2 \times \hat{n}_1}$ become

$$\begin{aligned} A_{11}^J &= \begin{bmatrix} I_{\hat{n}_1} \\ 0 \end{bmatrix} \qquad \hat{n}_2 > \hat{n}_1 \quad \text{or} \\ A_{11}^J &= [\, I_{\hat{n}_1} \quad 0 \,] \quad \hat{n}_2 < \hat{n}_1. \end{aligned} \tag{33}$$

So new states should just be initialized to zero. If there are continuous solutions $Q(t)$ and $P(t)$ to the inequalities (25) and (29) we can readily use them as solutions to the discrete-time Lyapunov equations for the jump

$$\begin{aligned} (A^J)^T Q(t^+) A^J &= Q(t^-) \quad \text{and} \\ A^J P(t^-)(A^J)^T &= P(t^+) \end{aligned} \tag{34}$$

which are fulfilled with $Q(t^-) = Q(t^+) = Q(t)$ and $P(t^-) = P(t^+) = P(t)$. For each jump, we therefore get the following addition to Lemma 3:

$$\left|\left| \begin{bmatrix} x_1(t^+) - \hat{x}(t^+) \\ x_2(t^+) \end{bmatrix} \right|\right|_Q^2 - \left|\left| \begin{bmatrix} x_1(t^-) - \hat{x}(t^-) \\ x_2(t^-) \end{bmatrix} \right|\right|_Q^2 + |\hat{z}(t^+)|_q^2 - 2q(t^+)\hat{z}^T(t^+)x_2(t^+) = 0,$$

with $\hat{x}(t^-) \in \boldsymbol{R}^{\hat{n}_1}$ and $\hat{x}(t^+) \in \boldsymbol{R}^{\hat{n}_2}$. The two first terms get canceled by the boundary terms of the integrals from the constant state modes before and after the switch in the lemma. So, the only real contribution is the two last terms. For Lemma 4, the additions become

$$\left|\left| \begin{bmatrix} x_1(t^+) + \hat{x}(t^+) \\ x_2(t^+) \end{bmatrix} \right|\right|_{P^{-1}}^2 - \left|\left| \begin{bmatrix} x_1(t^-) + \hat{x}(t^-) \\ x_2(t^-) \end{bmatrix} \right|\right|_{P^{-1}}^2 + |\hat{z}(t^+)|_{p^{-1}}^2 + 2p^{-1}(t^+)\hat{z}^T(t^+)x_2(t^+) \le 0.$$

Again the only real contribution is the two last terms. The remaining sign-indefinite terms

$$q(t^+)\hat{z}^T(t^+)x_2(t^+) \quad \text{and} \quad p^{-1}(t^+)\hat{z}^T(t^+)x_2(t^+)$$

can be canceled by proper choice of $a(t)$ and $b(t)$ as will be discussed in the Section 4.

The conclusion is that if the jump transitions (33) are used there is no real change to the results in Lemmas 3 and 4 and the set $\mathcal{T}$ can be defined exactly as in the discrete-time case, (6), and we may replace the integrals $\int_0^T$ in Lemmas 3 and 4 by $\int_{\mathcal{T}}$.

REMARK 1—DISCONTINUITIES IN $\hat{y}$
With the proposed scheme we see that when new states are added, i.e., $\hat{n}_2 > \hat{n}_1$, $\hat{y}$ will be continuous at the switching instant as the new states are initialized to zero. Moreover $\hat{z}(t^+)$ is zero. In the other case when $\hat{n}_2 < \hat{n}_1$, $\hat{y}$ can be discontinuous at the switching instant as states are thrown away, and then $|\hat{z}(t^+)|^2 > 0$. □

REMARK 2—DISCONTINUOUS STATE TRANSFORMATIONS
The technique here can also be used when one, at some time instant, would like to make an instantaneous state transformation, i.e., $T(t)$ is discontinuous. Then the jump transition matrix A^J becomes the solution to

$$T(t^+)A^J = T(t^-),$$

and all the calculations in this section can be redone with this jump matrix A^J. The corresponding Lyapunov equations to (34) become

$$\begin{aligned}(A^J)^T\tilde{Q}^+(t^+)A^J &= \tilde{Q}^-(t^-) \quad \text{and}\\ A^J\tilde{P}^-(t^-)(A^J)^T &= \tilde{P}^+(t^+).\end{aligned}$$

$\tilde{\cdot}^-$ and $\tilde{\cdot}^+$ denote matrices given in the coordinate systems $T(t^-)$ and $T(t^+)$, respectively. How the solutions $P(t)$ and $Q(t)$ are transformed is discussed in Section 4, (37). □

4. Balanced Realizations and Error Bounds

Sections 2 and 3 rely heavily on the ability to obtain block-diagonal solutions to the inequalities (8), (14), (25), and (29), respectively. Often this is possible to obtain. In particular, if there are any solutions $P(\cdot) > 0$ and

$Q(\cdot) > 0$ for all time instants in some realization of G, then there exists a *balanced realization* of G where the Lyapunov inequalities take the form

$$\begin{aligned} \tilde{A}^T(k)\Sigma(k+1)\tilde{A}(k) - \Sigma(k) + \tilde{C}^T(k)\tilde{C}(k) \leq 0 \\ \tilde{A}(k)\Sigma(k)\tilde{A}^T(k) - \Sigma(k+1) + \tilde{B}(k)\tilde{B}^T(k) \leq 0 \end{aligned} \tag{35}$$

in discrete time, and in continuous time with some extra regularity conditions

$$\begin{aligned} \Sigma(t)\tilde{A}(t) + \tilde{A}^T(t)\Sigma(t) + \dot{\Sigma}(t) + \tilde{C}^T(t)\tilde{C}(t) \leq 0 \\ \tilde{A}(t)\Sigma(t) + \Sigma(t)\tilde{A}^T(t) - \dot{\Sigma}(t) + \tilde{B}(t)\tilde{B}^T(t) \leq 0. \end{aligned} \tag{36}$$

with the diagonal solution (balanced Gramians)

$$\Sigma(\cdot) = \tilde{P}(\cdot) = \tilde{Q}(\cdot) = \text{diag}\{\sigma_1(\cdot), \sigma_2(\cdot), \ldots, \sigma_n(\cdot)\} > 0.$$

A linear system G with a realization fulfilling (35) or (36) with a Gramian Σ is called a *balanced system*. σ_i will be denoted as the singular value corresponding to the state x_i in a particular balanced system. Notice that it is always possible to permute the singular values in Σ. Normally one chooses to put the elements in descending order so that

$$\sigma_1(\cdot) \geq \sigma_2(\cdot) \geq \ldots \geq \sigma_n(\cdot) > 0.$$

As the singular values change in size over time it may be that the ordering must be changed at some time instants to maintain the aforementioned order. This can be done with an instantaneous coordinate transformation (permutation), see Remark 2 in Section 3.4. However, as we will see, the ordering is not critical to our discussion. But in general it makes good sense to put small singular values last in the Σ-matrix. By defining a balanced realization with inequalities instead of equalities it becomes non-unique, and the singular values are non-unique. This was introduced in [Hinrichsen and Pritchard, 1990; Beck *et al.*, 1996], and has several appealing properties including the possibility of tighter error bounds and that every truncated realization remains balanced.

If we have solutions $Q(\cdot)$ and $P(\cdot)$ in a given coordinate system we can obtain the needed coordinate transformation $T(\cdot)$ to obtain a balanced realization. This is the topic of many papers in discrete time; see, for example, [Shokoohi and Silverman, 1987; Varga, 2000], and the references therein. In continuous time, we need regularity conditions on the realization to guarantee the existence of a well-behaved balancing transformation. In [Verriest and Kailath, 1983], for instance, analyticity of the realization is assumed. In [Shokoohi *et al.*, 1983; Shokoohi *et al.*, 1984], uniform observability and controllability is assumed. How in practice to obtain $T(t)$ in continuous time is not obvious, as we need $T(t)$ and also $\dot{T}(t)$ on an interval. Pointwise we can always obtain a $T(\cdot)$ as we will see. An approximate approach to obtain $T(t)$ over an interval is presented in [Imae *et al.*, 1992].

We will not go into much detail here, as this is done in the references previously mentioned, let us just notice that under the coordinate transformation (7) and (24) the solutions to the Lyapunov inequalities transform as

$$\begin{aligned} \tilde{Q}(\cdot) &= T^T(\cdot)Q(\cdot)T(\cdot) \\ \tilde{P}(\cdot) &= T^{-1}(\cdot)P(\cdot)T^{-T}(\cdot) \end{aligned} \tag{37}$$

so that the eigenvalues of their product is invariant. Therefore we can calculate the singular values for a realization with Gramians P and Q as

$$\sigma_i^2(\cdot) = \lambda_i(P(\cdot)Q(\cdot)) = \lambda_i(\tilde{P}(\cdot)\tilde{Q}(\cdot))$$

at each time-instant and also obtain a balancing coordinate system $T(\cdot)$.

As a first step toward error-bounds for truncated balanced realizations let us note that from Lemmas 1 and 2 and Lemmas 3 and 4 we get the following bound:

PROPOSITION 1—CANCELLING CONDITION
If the nonincreasing weights $a(\cdot)$ and $b(\cdot)$ are chosen so that for all time-instants k or t in $\mathcal{T}$

$$\begin{aligned} a(k-1)q(k) &= b(k-1)p^{-1}(k) \quad &\text{(Discrete time)} \\ a(t)q(t) &= b(t)p^{-1}(t) \quad &\text{(Continuous time)} \end{aligned} \tag{38}$$

then

$$\|y - \hat{y}\|_a \le 2\|u\|_b. \tag{39}$$

□

Proof. Add Lemma 1 (ii) with Lemma 2 (ii) and notice that the sign-indefinite terms are canceled if $a(k)$ and $b(k)$ fulfill the previous condition. Analogous in continuous time. ■

4.1 Monotonically Balanced Systems

We will proceed by formulating an error bound for truncated balanced realizations which looks familiar to the well-known time-invariant result in [Enns, 1984; Glover, 1984]. We will first look at balanced systems where the singular values are monotonic in time, as this is the simplest nontime-invariant case. It is useful to group equal singular values together as this makes the error bound sharper. If there are $N(\cdot)$ unique singular values use the notation

$$\Sigma(\cdot) = \text{diag}\{\sigma_1(\cdot)I_{s_1}, \ldots, \sigma_N(\cdot)I_{s_N}\}$$

where $s_1(\cdot)+\ldots+s_N(\cdot) = n(\cdot)$. Now, the following result is easily obtained.

THEOREM 1—MONOTONICALLY BALANCED SYSTEMS
Suppose the system G has a balanced realization on the interval $[0, T]$ with $\Sigma(\cdot) = \text{diag}\{\Sigma_1(\cdot), \Sigma_2(\cdot)\}$

$$\begin{aligned}\Sigma_1(\cdot) &= \text{diag}\{\sigma_1(\cdot)I_{s_1}, \ldots, \sigma_r(\cdot)I_{s_r}\}\\ \Sigma_2(\cdot) &= \text{diag}\{\sigma_{r+1}(\cdot)I_{s_{r+1}}, \ldots, \sigma_N(\cdot)I_{s_N}\}\end{aligned}$$

where each singular value $\sigma_i(\cdot), \quad i = r+1\ldots N$ is either nonincreasing or nondecreasing over time.

The truncated $(s_1 + \cdots + s_r)$-order system $\hat{G}$ is then balanced by $\Sigma_1(\cdot)$ and

$$\|G - \hat{G}\| \leq 2 \sum_{i=r+1}^{N} \sup_{t\in[0,T]} \sigma_i(t). \tag{40}$$

□

Proof. Start by removing the states with the singular value σ_N, and call this truncated system $\hat{G}_1$. Thus put $p = q = \sigma_N$. By assumption there are two possibilities: σ_N is nonincreasing or nondecreasing. First, consider the nonincreasing case. Then, choose $b(t) = \sigma_N^2(t)$ and $a(t) = 1$ in Proposition 1 ($\mathcal{T} = [0, T]$, use $a(k-1)$ and $b(k-1)$ in discrete time) and notice that the cancelling condition is fulfilled. In the nondecreasing case, choose $a = \sigma_N^{-2}$ and $b = 1$. It follows that

$$\begin{aligned}\|y - \hat{y}_1\| &\leq 2\|u\|_{\sigma_N^2}, \quad \text{or}\\ \|y - \hat{y}_1\|_{\sigma_N^{-2}} &\leq 2\|u\|\end{aligned}$$

which leads to $\|G - \hat{G}_1\| \leq 2\sup_t \sigma_N(t)$. Next notice that $\hat{G}_1$ is still balanced with the rest of Σ, Σ_{N-1}. We proceed iteratively and remove σ_{N-1} from $\hat{G}_1$, and repeat the scheme until the system $\hat{G} = \hat{G}_{N-r}$ is reached. Finally use the triangular inequality:

$$\|G - \hat{G}\| = \|G - \hat{G}_1 + \hat{G}_1 + \ldots + \hat{G}_{N-r-1} - \hat{G}\| \leq 2 \sum_{i=r+1}^{N} \sup_{t\in[0,T]} \sigma_i(t).$$

■

REMARK 3—TIME-INVARIANT $\sigma(\cdot)$
For time-invariant asymptotically stable systems we can find time-invariant solutions $\Sigma(\cdot) = \Sigma$ to (35) and (36), which become algebraic Lyapunov inequalities. We then recover the well-known error bound for time-invariant systems. Also for time-varying systems we may find time-invariant solutions. If we look for solutions to the LMIs (35) and (36) with the constraint $\Sigma_2(\cdot) = \Sigma_2$ using standard semidefinite programming techniques, we obtain the error bound first shown in [Lall *et al.*, 1998; Lall and Beck, 2003]. In [Lall and Beck, 2003] it was shown that there always exists a solution $\Sigma(\cdot) = \Sigma$ to (35), so that the problem is always feasible. However, if the time horizon $[0, T]$ of the problem is large, the LMIs are of high dimension and become computationally expensive to solve. □

4.2 Nonmonotonically Balanced Systems

For many systems we expect the balanced Gramians $\Sigma(\cdot)$ to be nonmonotonic in time. We might try to resolve this by changing the boundary conditions to the Lyapunov equations until a monotonic solution is found, and then use Theorem 1. Alternatively, we may search for time-invariant solutions, as commented in Remark 3. In any case, we would still like to have a bound for nonmonotonic solutions, and this will be derived in this section. The following definition will be useful:

DEFINITION 1—THE MAX-MIN RATIO OF σ
Let the singular value $\sigma(\cdot)$ be defined on the interval $\mathcal{T} = [t_0, t_f]$, and let it have M local *maximums* for $t > t_0$, located at $t_0 < t_1^{\max} < \ldots < t_M^{\max} \le t_f$. Then there will be M local *minimums* so that

$$t_0 \le t_1^{\min} < t_1^{\max} < \ldots < t_M^{\min} < t_M^{\max} \le t_f$$

where $\sigma(t_i^{\min})$ is the local minimum immediately before $\sigma(t_i^{\max})$ for $i = 1 \ldots M$. The max-min ratio of σ is defined as

$$\begin{aligned} S_{\mathcal{T}}(\sigma) &= \sigma(t_0) \prod_{i=1}^{M} \frac{\sigma(t_i^{\max})}{\sigma(t_i^{\min})}, \qquad && M > 0 \\ S_{\mathcal{T}}(\sigma) &= \sigma(t_0), && M = 0. \end{aligned}$$

□

Now, we can formulate a general error bound that applies both to monotonically and nonmonotonically balanced systems.

THEOREM 2—GENERAL ERROR BOUND
Let $l(i)$ be any function that is defined for $i = 1 \dots L$ and takes integer values in the range $1 \dots n$, where n is the number of states in G.

The error between the balanced system G and its truncated balanced realization $\hat{G}$, where the states $x_{l(i)}$ have been truncated on the time intervals $\mathcal{T}_i$, $i = 1 \dots L$, is bounded by

$$\|G - \hat{G}\| \leq 2 \sum_{i=1}^{L} S_{\mathcal{T}_i}(\sigma_{l(i)}) \tag{41}$$

and $\hat{G}$ is balanced.

If the singular value for some other state x_k, $k \neq l(i)$, coincides with one in the sum (41), then x_k can be truncated over the same interval without inducing extra error. □

Proof. Start to truncate all states with the singular value $\sigma_{l(1)}$ over $\mathcal{T}_1$. Permute the states so that we can use Proposition 1. Then $p(\cdot) = q(\cdot) = \sigma_{l(1)}(\cdot)$. We need to find nonincreasing a and b such that

$$a(\cdot)\sigma^2_{l(1)}(\cdot) = b(\cdot).$$

If $\sigma_{l(1)}$ is initially nonincreasing put $b(t) = \sigma^2_{l(1)}(t)$ and $a(t) = 1$ (use $b(k-1)$ and $a(k-1)$ in discrete time). If $\sigma_{l(1)}$ reaches a local minimum at $t_1^{\min} < t_f$ define $b(t) = \sigma^2_{l(1)}(t_1^{\min})$ and $a(t) = \sigma^2_{l(1)}(t_1^{\min})/\sigma^2_{l(1)}(t)$ for $t > t_1^{\min}$. A local maximum will be reached, either at the end of the interval or before, so $t_1^{\max}$ exists. We can continue to define $a(t)$ and $b(t)$ as before, i.e., one is always constant and the other decreasing. When the whole interval $\mathcal{T}_1$ is covered we have from Proposition 1

$$a(t_f)\|y - \hat{y}_1\|^2 = \inf_{t \in \mathcal{T}_1} a(t)\|y - \hat{y}_1\|^2 \leq 4 \sup_{t \in \mathcal{T}_1} b(t)\|u\|^2 = 4b(t_0)\|u\|^2$$

and, therefore

$$\|G - \hat{G}_1\| \leq 2\sqrt{\frac{b(t_0)}{a(t_f)}} = 2S_{\mathcal{T}_1}(\sigma_{l(1)}).$$

If $\sigma_{l(1)}$ is initially nondecreasing an analogous treatment is applicable.

Finally, we can continue recursively with $i = 2 \dots L$ and use the triangular inequality to obtain the final result, just as in Theorem 1. ■

REMARK 4—LARGE MAX-MIN RATIOS
The max-min ratio may in some cases be an unnecessarily conservative bound. This is the case when $\sigma(t_i^{\max})/\sigma(t_i^{\min})$ is a large number. Then it is advisable to split the interval $\mathcal{T}$ into two intervals: $\mathcal{T}_1 = [t_0, t_i^{\min}]$ and $\mathcal{T}_2 = [t_i^{\min}, t_f]$, and truncate the state in two steps. We can always divide every time interval $\mathcal{T}$ into smaller ones so that the singular value is monotonic in each subinterval, and remove them recursively. □

EXAMPLE 1—THE MONOTONIC CASE
Theorem 1 follows from Theorem 2. Notice that for monotonic singular values $\sigma(\cdot)$, $S_{\mathcal{T}}(\sigma) = \sup_{\mathcal{T}} \sigma(\cdot)$. So, we have

$$
\begin{array}{lll}
l(1) = r+1 & \mathcal{T}_1 = [0, T] & S_{\mathcal{T}_1}(\sigma_{r+1}) = \sup_{t \in \mathcal{T}_1} \sigma_{r+1}(t) \\
\vdots & \vdots & \vdots \\
l(L) = N & \mathcal{T}_L = [0, T] & S_{\mathcal{T}_L}(\sigma_N) = \sup_{t \in \mathcal{T}_L} \sigma_N(t).
\end{array}
$$

□

EXAMPLE 2—CONTINUOUS-TIME SYSTEM
Assume we have a third-order balanced continuous-time system G over the time interval $[0, 1]$. The realization has the dimensions

$$A(t) \in \boldsymbol{R}^{3\times 3} \quad B(t) \in \boldsymbol{R}^{3\times 1} \quad C(t) \in \boldsymbol{R}^{1\times 3}$$

and the balanced Gramian is $\Sigma(t) = \operatorname{diag}\{\sigma_1(t), \sigma_2(t), \sigma_3(t)\}$, so that σ_i is the singular value of state x_i. The singular values are plotted in Figure 1. If we truncate the state x_3 over $[0, 1]$ we obtain the system $\hat{G}_1$. As σ_3 is monotonic, we can use Theorem 1

$$\|G - \hat{G}_1\| \le 2 \sup_t \sigma_3(t) = 0.8.$$

Alternatively, we use Theorem 2 and get the same value

$$\|G - \hat{G}_1\| \le 2S_{[0,1]}(\sigma_3) = 2\sigma_3(0)\frac{\sigma_3(1)}{\sigma_3(0)} = 0.8.$$

If we then want to truncate x_2 over $[0, 1]$ from $\hat{G}_1$, to get $\hat{G}_2$, we have the bound

$$\|\hat{G}_1 - \hat{G}_2\| \le 2S_{[0,1]}(\sigma_2) = 2\sigma_2(0)\frac{\sigma_2(0.2)}{\sigma_2(0)} = 0.7$$

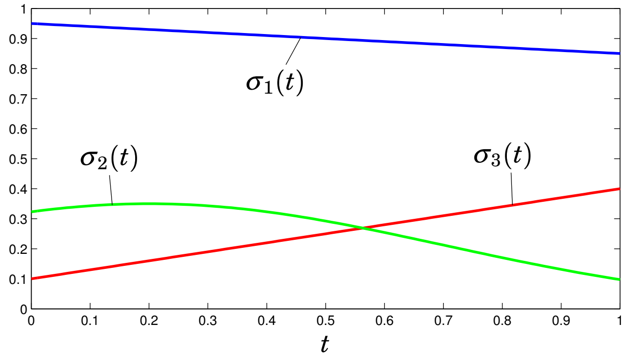

Figure 1. The singular values for Example 2. σ_2 and σ_3 are truncated in the example.

as the only maximum is $\sigma_2(0.2)$, and the minimum immediately before is $\sigma_2(0)$. Therefore, the error between the first-order system $\hat{G}_2$ and G is bounded by

$$\|G - \hat{G}_2\| \leq 0.8 + 0.7 = 1.5.$$

□

As noted in Remark 4 it is important how the intervals $\mathcal{T}_i$ are chosen and how much the singular values vary over that interval. It may very well be that we need to let the state dimension vary in order for the error to be smaller than some chosen threshold. Finally, notice that the max-min ratio is just a bound resulting from a particular choice of $a(\cdot)$ and $b(\cdot)$. There are other choices and bounds; see [Sandberg and Rantzer, 2002] for an entirely different but more complex choice.

4.3 Periodic Systems

Periodic systems are very important special cases of time-varying systems. For instance, we obtain such a system when a nonlinear system is linearized around a limit cycle. Periodic systems have realizations where

$$A(\cdot) = A(\cdot + T) \qquad B(\cdot) = B(\cdot + T)$$
$$C(\cdot) = C(\cdot + T) \qquad D(\cdot) = D(\cdot + T)$$

for some time-period T. These systems have received much attention in the literature, see, for instance, [Bittanti and Colaneri, 1999; Möllerstedt, 2000], and the references therein. For stable balanced periodic systems we can find periodic Gramians

$$\Sigma(\cdot) = \Sigma(\cdot + T)$$

which solve (35) and (36) with equality, see [Kano and Nishimura, 1996; Varga, 2000]. These solutions are clearly not monotonic. A problem with applying Theorem 2 directly to these solutions is that for each new period included in $\mathcal{T}_i$, the bound grows. Still we would like to let $t_f \to \infty$ for many periodic systems. In [Lall *et al.*, 1998; Longhi and Orlando, 1999; Varga, 2000] a bound for balanced discrete time-periodic systems is presented. We can also derive this bound:

COROLLARY 1—BALANCED DISCRETE TIME-PERIODIC SYSTEMS
If the balanced system G has a Gramian $\Sigma(k+T) = \Sigma(k)$ for all k and some T, and $\Sigma(k)$ is partitioned as in Theorem 1, then its truncation $\hat{G}$ is balanced with $\Sigma_1(k)$ and

$$\|G - \hat{G}\| \leq 2 \sum_{k=1}^{T} \sum_{i=r(k)+1}^{N(k)} \sigma_i(k) \tag{42}$$

over the infinite horizon $[0, \infty)$. □

Proof. Use Proposition 1. Remove first the states with the singular value $\sigma_N(1)$. As the system is periodic we can simultaneously remove these states at $1, 1+T, 1+2T, \ldots$. So, $\mathcal{T} = \{1, 1+T, 1+2T, \ldots\}$. The constant values $a(k) = 1$ and $b(k) = \sigma_N^2(1)$ for all k fulfill the cancelling condition. Continue then recursively over the whole period and use then the triangular inequality. ■

This might seem to be a satisfactory error bound. However, if the period is long (T large) this bound gets large very quickly if states are removed over the whole period. In particular, if we sample a continuous-time periodic system then the bound gets less useful the faster we have sampled the system. In the limit case, when we use the result directly on a continuous-time system, the bound is always infinity. More on sampling is given in Appendix B. A better technique to obtain a bound in this case may be to utilize the inequalities in (35) and (36) and to look for *time-invariant* diagonal solutions Σ; see Remark 3.

5. Input-Output Stability of Truncated Systems

One of the advantages of the analysis so far is that it has not been necessary to worry about stability. The only thing we need is a diagonal solution $\Sigma(\cdot)$ over some interval $[0, T]$. We could for instance reduce an unstable plant over a finite interval and still get error bounds. Many balanced truncation schemes in the literature require asymptotic stability of the plants.

Still, in order for our methodology to be good, a truncated realization of a stable system G should also be stable in some sense. That this is indeed the case will be shown here in the continuous-time case. The discrete-time case is analogous. Assume from now on that there exist constants

$$0 < \underline{\sigma} I \leq \Sigma(t) \leq \overline{\sigma} I < \infty \tag{43}$$

for $0 \leq t < \infty$. From the reachability Lyapunov inequality, we have

$$\begin{aligned} x^T(t)\Sigma^{-1}(t)x(t) &\leq \|u\|^2_{[0,t]} \\ \hat{x}^T(t)\Sigma_1^{-1}(t)\hat{x}(t) &\leq \|u\|^2_{[0,t]}. \end{aligned} \tag{44}$$

So for all $u \in L_2^m[0,\infty)$ both x and $\hat{x}$ will be bounded. Even if the states in both G and $\hat{G}$ are bounded, it is not clear that if G is input-output stable (finite L_2-gain), that $\hat{G}$ will be input-output stable. But with the results from Sections 2–4 we have the following theorem.

THEOREM 3—INPUT-OUTPUT STABILITY
If the balanced system G is input-output stable and there are constants $\overline{\sigma}$ and $\underline{\sigma}$ satisfying (43), then all states x are bounded for all u in $L_2^m[0,\infty)$, and every truncated system $\hat{G}$ is also input-output stable and the states $\hat{x}$ are bounded. □

Proof. See Appendix A. ■

This result might seem contradictory to the result in [Pernebo and Silverman, 1982], which says that we get guaranteed asymptotic stability on $\hat{G}$ if Σ_1 and Σ_2 have no entries in common. But in the theorem above we concentrate on input-output stability. To see the effects consider, the following example from [Zhou and Doyle, 1998].

EXAMPLE 3—[ZHOU AND DOYLE, 1998]
The continuous-time system with the transfer function

$$\frac{s^2 - s + 2}{s^2 + s + 2} \quad \text{and realization}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & -1 & \sqrt{2} \\ \hline 0 & \sqrt{2} & 1 \end{array}\right]$$

is balanced with $\Sigma = I$. The $\{\sigma_2\}$-truncated system

$$\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right]$$

is clearly not asymptotically stable but the pole is neither observable nor controllable, and the system is *input-output stable* and $\hat{x}_1$ will be bounded, more precisely 0, for all u. Theorem 3 says that this will always happen when truncating a balanced system. (Obviously, in this case, a better approximation is just to keep $D = 1$, as we then get a zero-order model and the same error bound.) □

The result may seem unnecessary as we can truncate states that have equal singular values without extra cost. But the result shows that we do not need to worry about singular values that are equal for some time-instants, we will not lose input-output stability. The example also shows that a truncated system may have a non-minimal realization. The theorem, however, guarantees it is well behaved.

6. Lower Bound on the Approximation Error

When doing optimal Hankel-norm approximation of time-invariant systems a lower bound on the Hankel-norm for approximations of different system order (McMillan degree) is obtained, see [Glover, 1984; Green and Limebeer, 1995]. As the Hankel-norm is always smaller than or equal to the induced L_2-norm we also get a bound on the best possible approximation in this norm. We will see that a similar analysis is possible for linear time-varying systems. Let us consider finite-horizon linear systems G in continuous time and the following Lyapunov equations:

$$A^T(t)Q(t) + Q(t)A(t) - \dot{Q}(t) + C^T(t)C(t) = 0 \tag{45}$$

$$A(t)P(t) + P(t)A^T(t) + \dot{P}(t) + B(t)B^T(t) = 0 \tag{46}$$

$$Q(t_f) = 0 \quad P(t_0) = 0 \tag{47}$$

$$t \in (t_0, t_f) : Q(t) > 0 \quad P(t) > 0. \tag{48}$$

Inequalities (48) mean that the realization of G is completely reachable and observable. Notice that we here have dropped the Lyapunov inequalities for equalities. This is not a severe restriction. In practice one often solves the equalities as a first step anyhow, as it is less computationally expensive than solving the strict inequalities with semidefinite programming, and because it often gives good enough upper error bounds.

If we can balance the equations (45)–(48), the balanced Gramian will have the interesting property $\Sigma(t_0) = \Sigma(t_f) = 0$. Balanced finite-horizon systems of this sort were throughly studied in [Verriest and Kailath, 1983]. Among other things it was shown that if $\{A(t), B(t), C(t)\}$ are analytic functions in t, then the coordinate transformation $T(t)$ needed to obtain a balanced realization $\{\tilde{A}(t), \tilde{B}(t), \tilde{C}(t)\}$ exists, and is a Lyapunov transformation in every compact subset of (t_0, t_f). The entries of the balanced realization will tend to infinity at the boundaries t_0 and t_f. For practical computations it seems to be reasonable to embed the interval of interest, $[0, T]$, in a sufficiently large interval $[t_0, t_f]$.

Let us look at the linear system G on the time interval $[t_0, t_f]$, and divide the interval into two parts: $[t_0, \tau]$ and $[\tau, t_f]$. If we have a solution $Q(t)$ to the observability Lyapunov equation (45) we can compute the norm $\|y\|_{[\tau,t_f]}$ simply if $u(t) = 0$ for $t > \tau$ and $x(\tau)$ is known. Then

$$x^T(\tau)Q(\tau)x(\tau) = \int_{\tau}^{t_f} |y(t)|^2 dt = \|y\|^2_{[\tau,t_f]}.$$

Analogously we have results for the reachability equation (46) and from linear optimal control theory. There is a minimum control signal $u^*(t)$ (in L_2-sense) that takes the state from $x(t_0) = 0$ to any $x(\tau)$ that fulfills

$$x^T(\tau)P^{-1}(\tau)x(\tau) = \int_{t_0}^{\tau} |u^*(t)|^2 dt = \|u^*\|^2_{[t_0,\tau]}$$

see for example [Green and Limebeer, 1995]. Now, define the Hankel-norm $\|G\|_{H,\tau}$ at time τ and calculate it as

$$\begin{aligned}\|G\|^2_{H,\tau} &= \sup_{u\neq 0} \frac{\|y\|^2_{[\tau,t_f]}}{\|u\|^2_{[t_0,\tau]}} \\ &= \sup_{x(\tau)} \frac{x^T(\tau)Q(\tau)x(\tau)}{x^T(\tau)P^{-1}(\tau)x(\tau)} = \overline{\sigma}(P(\tau)Q(\tau))\end{aligned}$$

where $u(t) = 0$ for $t > \tau$. As $P(\tau) > 0$ and $Q(\tau) > 0$ we can find a balancing coordinate transformation at time τ from (37), so we have $\overline{\sigma}(P(\tau)Q(\tau)) = \overline{\sigma}(\Sigma^2(\tau)) = \sigma_1^2(\tau)$, because the Hankel-norm is invariant under coordinate transformations. Also, notice that $\|G\|_{H,t_0} = \|G\|_{H,t_f} = 0$ and that

$$\|G\|_{H,\tau} \leq \|G\| = \sup_{u\neq 0} \frac{\|y\|_{[t_0,t_f]}}{\|u\|_{[t_0,t_f]}} \tag{49}$$

for all τ in $[t_0, t_f]$. Next, define the Hankel-operator of G at time τ, $\Gamma_{G,\tau}$, as the past to future restriction of G

$$\Gamma_{G,\tau} : L_2^m[t_0, \tau] \xrightarrow{G} L_2^p[\tau, t_f].$$

We see that $\|\Gamma_{G,\tau}\| = \|G\|_{H,\tau}$. The operator $\Gamma_{G,\tau}$ has finite rank, namely n. We here only consider constant-state-dimensional systems as full-order systems G often come in this form. For each choice of τ there is a singular-value decomposition (Schmidt decomposition; see [Young, 1988; Green and Limebeer, 1995]) of $\Gamma_{G,\tau}$

$$\Gamma_{G,\tau} u = \sum_{i=1}^{n} \sigma_i(\tau) \langle u, v_i \rangle w_i$$

where $\{v_i\}_1^n$ is a set of orthonormal functions in $L_2^m[t_0, \tau]$ and $\{w_i\}_1^n$ is a set of orthonormal functions in $L_2^p[\tau, t_f]$. $\sigma_i(\tau) = \lambda_i^{1/2}(P(\tau)Q(\tau))$ are the singular values. $\langle \cdot, \cdot \rangle$ is the standard scalar product on $L_2^m[t_0, \tau]$

$$\langle u, v \rangle = \int_{t_0}^{\tau} u^T(s) v(s) ds.$$

We can now state the following theorem:

THEOREM 4—LOWER ERROR-BOUND
Suppose G is a linear system with a finite-horizon nth-order realization with Gramians that fulfill (45)–(48). Let the singular values be ordered so that $\sigma_1(t) \geq \ldots \geq \sigma_n(t) > 0$ for each t. Then for any linear system $\hat{G}$ of order $r < n$ it holds that

$$\|G - \hat{G}\|_{H,\tau} \geq \sigma_{r+1}(\tau) \tag{50}$$

for all $\tau \in [t_0, t_f]$. Furthermore

$$\|G - \hat{G}\| \geq \max_t \sigma_{r+1}(t). \tag{51}$$

□

Proof. The operator $\Gamma_{\hat{G},\tau}$ has rank r. If we use the Schmidt vectors v_i from $\Gamma_{G,\tau}$ as basis there exist numbers $\alpha_i \neq 0$ such that the signal $v = \sum_{i=1}^{r+1} \alpha_i v_i$ gives $\Gamma_{\hat{G},\tau} v = 0$. Now

$$\|(\Gamma_{G,\tau} - \Gamma_{\hat{G},\tau})v\|^2 = \|\Gamma_{G,\tau} v\|^2 = \left\| \sum_{i=1}^{r+1} \alpha_i \sigma_i(\tau) w_i \right\|^2$$
$$= \sum_{i=1}^{r+1} \alpha_i^2 \sigma_i^2(\tau) \geq \sigma_{r+1}^2(\tau) \sum_{i=1}^{r+1} \alpha_i^2 = \sigma_{r+1}^2(\tau) \|v\|^2.$$

This gives (50), and (49), then gives (51). ■

If we make a one-step truncation ($L = 1$) of a finite-horizon balanced system G and truncate the states with the singular value $\sigma_N(t)$, we get

$$\max_t \sigma_N(t) \leq \|G - \hat{G}\| \leq C \cdot \max_t \sigma_N(t)$$

where $C \geq 2$ depends upon the monotonicity conditions as discussed in Theorem 2. Therefore we can often expect a very good approximation in this type of one-step reductions. For multi-step reductions ($L > 1$), the approximation may be much less close to an optimal approximation, just as for standard balanced truncation for time-invariant systems. However, notice that we have not proven that there exists an approximation that really obtains the lower bound, so we do not know exactly how far away the optimum is. In [Lall and Beck, 2003] a sufficient and necessary condition for the existence of a system $\hat{G}$ of order r for a given approximation error γ is given. The condition is however nonconvex and hard to check. The discussion here only justifies the balanced truncation procedure when the lower and upper bounds are close to each other.

7. Example: Reduction of Diesel Exhaust Catalyst Model

Until now there has been no computations that show that the suggested methods really give rise to good low-order approximations in practice. In fact, there has been a fair amount of theoretical work done in the literature on time-varying balancing, but the authors have not found many real examples. Here we will give a brief overview of the results for an example, just to show that the computations are feasible.

We will look at a model taken from [Westerberg *et al.*, 2003]. This is a model of a diesel exhaust catalyst. In one end of the catalyst the exhausts from the diesel engine comes in. The exhausts are blended with some extra diesel fuel (HC). The amount of added diesel fuel is the control input in this example. In the catalyst the exhausts and the diesel react and at the other end the concentration of NO_x will have decayed.

The given model consists of 28 nonlinear stiff differential equations which describe concentrations and temperatures throughout the catalyst. To get a single-input-single-output system we choose the added amount of HC at the inlet as input, and the concentration of NO_2 at the outlet as output. If we are only interested in these aspects of the system, then we can directly drop four of the states in the nonlinear model. To apply the methods of this paper we need to linearize the system. In order to get a time-varying system we linearize the system around a pulsating input signal (three pulses) over a finite horizon, so that the system does not

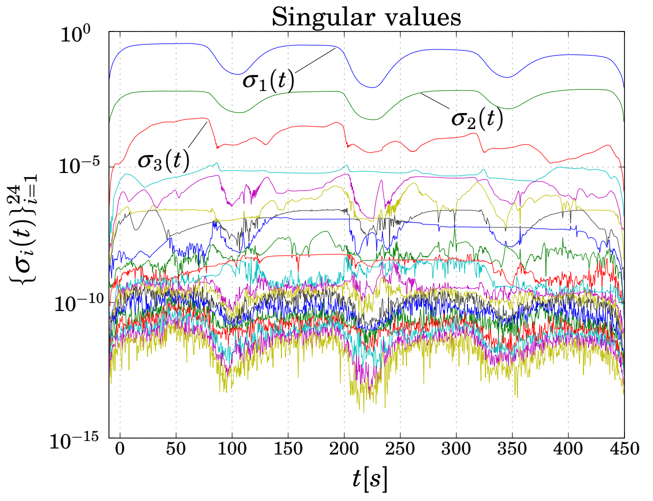

Figure 2. The singular values for the linear time-varying system G, which approximates the diesel exhaust catalyst over the time interval -10–450 s. One singular value is dominating, which predicts that one state is needed to make the approximation.

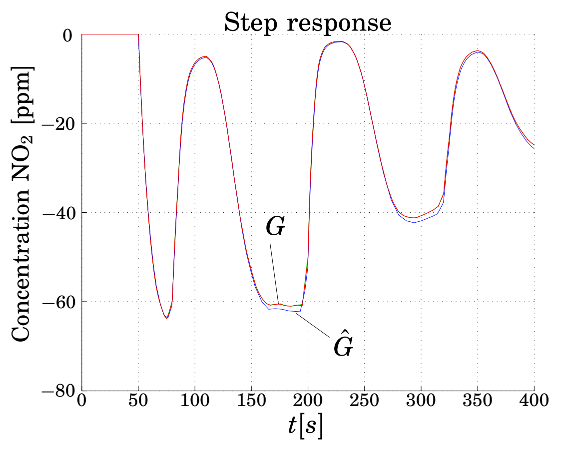

Figure 3. The step responses for the 24th-order linear time-varying system G and its first-order approximation $\hat{G}$.

reach steady-state. We then get a time-varying linear system G with 24 states around a nominal trajectory.

To find a balanced realization and the singular values we need to solve two time-varying Lyapunov inequalities. As $n = 24$ this involves rather heavy computations. We choose to first find solutions to the system (45)–(47), with $t_0 = -10$ s and $t_f = 450$ s. The singular values, $\sigma_i(t) = \lambda_i^{1/2}(P(t)Q(t))$, are plotted in Figure 2. The plot is in logarithmic scale and we notice that one singular value ($\sigma_1(t)$) is dominating. The three pulses in the nominal solution can be seen as three drops in the singular values. To reduce the computation time we have chosen the ODE-solver tolerance (for `ode15s` in MATLAB) so that only the two largest singular values have good accuracy. To find a balanced coordinate system we use the relation (37) on a time grid $\{t_k\}$. As the eigenvectors of $P(t_k)Q(t_k)$ typically give a badly conditioned coordinate transformation, we have chosen to use the numerically sound Schur method developed for time-invariant systems in [Safonov and Chiang, 1989] at each grid point to obtain a well-behaved coordinate transformation $T(t_k)$. Linear interpolation is used between the grid points.

We have shown in this paper that we can truncate states that have a small singular value without inducing large errors. For a first-order approximation ($\hat{n} = 1$) the upper error bound is essentially $2 \cdot S_{[-10,450]}(\sigma_2)$, if we assume that the other much smaller singular values also really only have four maximums. Now, as $\sigma_{2,\max} \approx 6 \cdot 10^{-3}$ and $\sigma_{2,\min} \approx 1 \cdot 10^{-3}$ we get that $S_{[-10,450]}(\sigma_2) \approx (6 \cdot 10^{-3})^4/(1 \cdot 10^{-3})^3 = 1296 \cdot 10^{-3}$. This is an overly conservative bound. Instead one should divide into time intervals as suggested in Remark 4. So another, and better, bound is given by $2 \cdot (S_{[-10,110]} + S_{[110,225]} + S_{[225,340]} + S_{[340,450]}) \approx 2 \cdot 4 \cdot \sigma_{2,\max} = 48 \cdot 10^{-3}$. As we derived a lower bound in Section 6 we can say

$$6 \cdot 10^{-3} \leq \|G - \hat{G}\| \leq 48 \cdot 10^{-3}, \tag{52}$$

for the first-order approximation $\hat{G}$. In Figure 3 we see a step response test for G and $\hat{G}$. The error in this particular case is $7.2 \cdot 10^{-3}$, which shows that a typical error is in the same order of magnitude as the worst-case bounds in (52). Notice that the step responses here are very different from what is obtained from *time-invariant* linear systems. If we instead use a second-order approximation, $\hat{n} = 2$, there is no visible error in the step response test.

We have succeeded in finding a low-order approximation for a nontrivial high-order linear time-varying system. The drawback is that solving for $P(t)$ and $Q(t)$ is computationally heavy, although it is feasible for n of this order of magnitude.

8. Conclusion

In this paper, we have from basic analysis of the reachability and observability Lyapunov inequalities analyzed the effects of truncation of states for linear systems, in both continuous and discrete time. The analysis also covers the case when the state dimension varies over time. This is valuable as systems may need a different amount of states for different time intervals to be well approximated.

In particular, we have studied balancing of time-varying systems. From the solutions to the two Lyapunov inequalities, the Gramians, we obtain a balanced coordinate system, often well suited for truncation, and singular values. The singular values give an upper bound on the L_2-induced error for truncated models. Furthermore, we obtain a lower error bound also expressed in the singular values. Both bounds are generalizations of well-known results for time-invariant systems.

Stability was not a main issue in the paper, as we can make approximations over a finite time horizon. Nevertheless, we proved that if a full-order system is input-output stable, then every truncated balanced realization of it will also be input-output stable.

Finally, a brief example showed that the methods are possible to use in practice. A 24th-order linear time-varying approximation of a diesel exhaust catalyst was truncated to a first-order system with almost no error.

Future work should include finding sharper error bounds. Especially in the infinite time-horizon case with nonmonotonic singular values. Furthermore, numerical issues should be considered. The method requires knowledge of the Gramians of the system, which restricts the use of the method. The Gramians may be too computationally expensive to obtain for high-order systems.

Appendix A: Proof of Theorem 3

We will prove that input-output stability is maintained every time Proposition 1 is used to truncate a system. Under the given assumptions there are constants so that

$$
\begin{aligned}
&0 < \delta_a \le a(t) \le \varepsilon_a < \infty \qquad && 0 < \delta_P \le P(t) \le \varepsilon_P < \infty \\
&0 < \delta_b \le b(t) \le \varepsilon_b < \infty \qquad && 0 < \delta_Q \le Q(t) \le \varepsilon_Q < \infty
\end{aligned}
$$

for all t. The calculations will be made in continuous time, but they are very similar in discrete time. Upon adding Lemma 3 (ii) and Lemma 4 (ii) we obtain

$$
\left\| \begin{bmatrix} x_1(T) + \hat{x}(T) \\ x_2(T) \end{bmatrix} \right\|^2_{bP^{-1}} + \left\| \begin{bmatrix} x_1(T) - \hat{x}(T) \\ x_2(T) \end{bmatrix} \right\|^2_{aQ} + \|y - \hat{y}\|^2_a \le 4\|u\|^2_b.
$$

Using the inequality $\|x+y\|_a^2 \geq \frac{1}{2}\|x\|_a^2 - \|y\|_a^2$, we get

$$\frac{1}{2}|\hat{x}(T)|^2_{aq+bp^{-1}} + \frac{1}{2}\|\hat{y}\|_a^2 \leq 4\|u\|_b^2 + \|y\|_a^2 + |x_1(T)|^2_{aQ_1+bP_1^{-1}} - |x_2(T)|^2_{aq+bp^{-1}}.$$

If $u \in L_2[0,\infty)$ and G is input-output stable we know that the terms $\|u\|_b^2$ and $\|y\|_a^2$ are bounded. Because of the relations (44) and $0 < aq + bp^{-1} \leq \varepsilon_a\varepsilon_Q + \varepsilon_b\delta_P^{-1} < \infty$ for all t we see that the terms involving $\hat{x}(T)$, $x_1(T)$ and $x_2(T)$ are bounded for all T. Therefore we conclude that $\hat{y} \in L_2[0,\infty)$.

Appendix B: Sampled Lyapunov Equations

In this paper, we have treated systems in both discrete and continuous time. Models from physics and engineering often come in the form of differential equations. For control purposes, however, systems have to at some point be transformed into discrete time if implementation on computers is intended. We will see that this discretization can be done at the same time as the model reduction is performed.

The first step towards discretization in time is to find a different system representation. We will use so-called *lifting*, see for instance [Bamieh and Pearson, 1992]. This transformation is an isomorphic isometry, i.e., the transformation preserves the system structure and norm. We will call the discretization time points $\{t(k)\}$. The inputs and the outputs of the lifted system belong to the signal spaces

$$\bar{u}(k) \in L_2^m[t(k), t(k{+}1)] \quad \bar{y}(k) \in L_2^p[t(k), t(k{+}1)].$$

The lifted n-state continuous-time system G is given by

$$\begin{aligned} \bar{x}(k+1) &= \bar{A}(k)\bar{x}(k) + \bar{B}(k)\bar{u}(k) \\ \bar{y}(k) &= \bar{C}(k)\bar{x}(k) + \bar{D}(k)\bar{u}(k) \end{aligned}$$

where

$$\bar{A}(k) = \Phi(t(k{+}1), t(k)) \tag{53}$$

$$\bar{B}(k)\bar{u}(k) = \int_{t(k)}^{t(k+1)} \Phi(t(k{+}1), s)B(s)\bar{u}(k;s)ds \tag{54}$$

$$\bar{C}(k;t) = C(t)\Phi(t, t(k)) \tag{55}$$

$$\begin{aligned} \bar{D}(k)\bar{u}(k)(t) = &\int_{t(k)}^{t} C(t)\Phi(t,s)B(s)\bar{u}(k;s)ds \\ &+ D(t)\bar{u}(k;t), \end{aligned} \tag{56}$$

and $\Phi(t,\tau)$ is the fundamental solution of $\dot{x} = A(t)x$. The operators act on the following spaces

$$\begin{aligned}
\bar{A}(k) &: \boldsymbol{R}^n \to \boldsymbol{R}^n \\
\bar{B}(k) &: L_2^m[t(k), t(k+1)] \to \boldsymbol{R}^n \\
\bar{C}(k) &: \boldsymbol{R}^n \to L_2^p[t(k), t(k+1)] \\
\bar{D}(k) &: L_2^m[t(k), t(k+1)] \to L_2^p[t(k), t(k+1)].
\end{aligned}$$

We will need the adjoint operators. These act on the dual spaces. As all involved spaces are Hilbert spaces we can represent all elements in the dual space with elements in the primal space. The adjoints we need are given by

$$\bar{A}^*(k) = \Phi^T(t(k+1), t(k)) \tag{57}$$

$$\bar{B}^*(k;t) = B^T(t)\Phi^T(t(k+1), t) \tag{58}$$

$$\bar{C}^*(k)\bar{y}(k) = \int_{t(k)}^{t(k+1)} \Phi^T(s, t(k))C^T(s)\bar{y}(k;s)ds. \tag{59}$$

We will now see that if we have solutions to the continuous-time Lyapunov equations, $Q(t)$ and $P(t)$, we can use them at the sampling instants for the lifted system. Consider the observability Lyapunov equation in continuous time for $t \in [t(k), t(k+1)]$

$$Q(t)A(t) + A^T(t)Q(t) + \dot{Q}(t) + C^T(t)C(t) = 0$$

and its solution

$$\Phi^T(t(k+1), t)Q(t(k+1))\Phi(t(k+1), t) + \int_t^{t(k+1)} \Phi^T(s,t)C^T(s)C(s)\Phi(s,t)ds = Q(t). \tag{60}$$

Using the lifting operators putting $t = t(k)$, the solution (60) can be written as a discrete Lyapunov equation with the solution $Q(t(k))$

$$\bar{A}^*(k)Q(t(k+1))\bar{A}(k) + \bar{C}^*(k)\bar{C}(k) = Q(t(k)). \tag{61}$$

We get analogous results for the reachability Lyapunov equation

$$A(t)P(t) + P(t)A^T(t) - \dot{P}(t) + B(t)B^T(t) = 0$$

with the solution

$$\Phi(t,t(k))P(t(k))\Phi^T(t,t(k)) \\ + \int_{t(k)}^{t} \Phi(t,s)B(s)B^T(s)\Phi(t,s)ds = P(t) \quad (62)$$

which with the lifting operators becomes

$$\bar{A}(k)P(t(k))\bar{A}^*(k) + \bar{B}(k)\bar{B}^*(k) = P(t(k+1)). \quad (63)$$

So, we can first compute continuous-time solutions $Q(t)$ and $P(t)$, and then choose suitable sampling instants $\{t(k)\}$ and compute the pointwise balancing transformation $T(t(k))$ to balance (61) and (63). Then we can truncate the lifted system and obtain error bounds as we have done before with discrete Lyapunov equations. Finally, finite-dimensional bases should be chosen to approximate the infinite-dimensional signal spaces. For instance, zero-order hold could be used for the signals $\bar{u}(k)$.

As an alternative, we could for instance first do zero-order hold sampling of the continuous-time system G and then balance the resulting discrete-time system. We would then not obtain the same approximation as above and the error bound will be in induced ℓ_2-sense, not in induced L_2-sense as in the lifting approach.

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Paper II

A Case Study in Model Reduction of Linear Time-Varying Systems

Henrik Sandberg

Abstract

In this paper, we apply the balanced truncation procedure for time-varying linear systems, both in continuous and in discrete time. The methods are applied to a linear approximation of a diesel exhaust catalyst model. The reduced-order systems are obtained by using certain projections instead of direct balancing. An approximative zero-order-hold discretization of continuous-time systems is described, and a new a priori approximation error bound for balanced truncation in the discrete-time case is obtained. The case study shows that there are several advantages to work in discrete time. It gives simpler implementation with fewer computations.

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1. Introduction

Balanced truncation is a method that is often used for order reduction of linear time-invariant systems. The method was introduced for this purpose in [Moore, 1981]. The simplicity of the method, the stability guarantees [Pernebo and Silverman, 1982], and the simple error bound on truncated models [Glover, 1984; Enns, 1984], have contributed to make the method popular.

Balanced truncation for time-varying linear systems have also obtained attention, see for example [Shokoohi *et al.*, 1983; Verriest and Kailath, 1983] for some early references. Recently there has been an increased interest in balanced truncation of time-varying systems. In [Longhi and Orlando, 1999; Lall and Beck, 2003; Sandberg and Rantzer, 2004] different error bounds were obtained, and in [Varga, 2000; Chahlaoui and Van Dooren, 2002] numerical issues were discussed. Many of the nice properties of balanced truncation for time-invariant systems now have their counterparts in the time-varying setting. However, there has not been many published tests of the method in the time-varying case. In this paper an attempt to apply balanced truncation on a diesel exhaust catalyst model is discussed in some detail. The model is described in [Westerberg *et al.*, 2003]. The same model was also used for model reduction in [Sandberg and Rantzer, 2004].

The contribution of this paper is mainly the comparison between balanced truncation in discrete time and continuous time. Also coordinate projections that simplify the model reduction are presented. These projections come from a generalization of the results in [Safonov and Chiang, 1989]. The projections have been used for time-varying discrete-time systems, see for example [Varga, 2000; Chahlaoui and Van Dooren, 2002], but seems not to have been stated in continuous time before. Furthermore, a simple a priori error bound on the approximation error of balanced truncation is obtained. The error bound has an appealing structure and seems to hold only in discrete time. It does not depend on the time-variability of the singular values, which the bound in [Sandberg and Rantzer, 2004] in general does.

In the following, $|x|_P^2 = x^T P x$ for vectors x and positive-definite matrices P, $|A| = \overline{\sigma}(A)$ on matrices A, and $\|u\|$ denotes the standard norm on signals u in L_2 or ℓ_2. $\|G\|$ denotes the induced L_2- or ℓ_2-norm of operators G. Two operators (or systems) F and G are input-output equivalent if $\|F - G\| \equiv 0$.

2. Balanced Truncation in Continuous Time

It is assumed that the given linear time-varying system G has a finite-dimensional realization (A, B, C, D). Hence, G can be represented as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(0) = 0, \\ y(t) &= C(t)x(t) + D(t)u(t), \quad t \in [0, T], \end{aligned} \tag{1}$$

with states $x(t) \in \boldsymbol{R}^n$, inputs $u(t) \in \boldsymbol{R}^m$, and outputs $y(t) \in \boldsymbol{R}^p$. Alternatively the notation $G := \mathcal{S}(A, B, C, D)$ is used. The realization (A, B, C, D) is assumed to be bounded and continuous. The resulting linear system defines a bounded linear operator $y = Gu$ on $L_2[0, T]$. The objective of the model reduction is to find a system $\hat{G}$ with state dimension $\hat{n} < n$ and a small approximation error $\|G - \hat{G}\|$.

The first step towards $\hat{G}$, using the balanced truncation method, is to solve the observability and reachability differential Lyapunov equations

$$\begin{aligned} Q(t)A(t) + A^T(t)Q(t) + \dot{Q}(t) + C^T(t)C(t) = 0, \\ A(t)P(t) + P(t)A^T(t) - \dot{P}(t) + B(t)B^T(t) = 0, \end{aligned} \tag{2}$$

with boundary conditions $Q(T) = 0$ and $P(0) = 0$. $P(t) \in \boldsymbol{R}^{n \times n}$ and $Q(t) \in \boldsymbol{R}^{n \times n}$ are often called the reachability and observability Gramians, respectively. From the Gramians numbers corresponding to the familiar Hankel singular values that are used for time-invariant systems can be defined as

$$\sigma_i^2(t) = \lambda_i(P(t)Q(t)) \geq 0, \quad i = 1 \dots n, \tag{3}$$

and they are ordered such that $\max_t \sigma_1(t) \geq \ldots \geq \max_t \sigma_n(t) \geq 0$. The singular values are invariant under (Lyapunov) coordinate transformations, and hence are the same for all realizations of G.

The next step is to find a *balanced realization*. A balanced realization fulfills (2) with $P(t) = Q(t) = \Sigma(t) = \text{diag}\{\sigma_1(t), \ldots, \sigma_n(t)\}$. Under certain regularity conditions on the given realization (A, B, C, D), see [Shokoohi *et al.*, 1983; Verriest and Kailath, 1983], a time-varying coordinate transformation $x = T_B(t)x_B$ can be found, such that the transformed realization (A_B, B_B, C_B, D) is balanced. Then introduce the partitions

$$\begin{aligned} A_B &= \begin{bmatrix} A_{B11} & A_{B12} \\ A_{B21} & A_{B22} \end{bmatrix}, \quad B_B = \begin{bmatrix} B_{B1} \\ B_{B2} \end{bmatrix}, \\ C_B &= [\, C_{B1} \quad C_{B2} \,]. \end{aligned}$$

A truncated balanced realization is denoted by $\hat{G}_B$ and is given by

$$\begin{aligned} \dot{\hat{x}}(t) &= A_{B11}(t)\hat{x}(t) + B_{B1}(t)u(t), \quad \hat{x}(0) = 0, \\ \hat{y}(t) &= C_{B1}(t)\hat{x}(t) + D(t)u(t), \qquad t \in [0, T], \end{aligned} \tag{4}$$

with $\hat{x}(t) \in \boldsymbol{R}^{\hat{n}}$. This realization keeps the most dominant states from an input-output point-of-view, see [Shokoohi *et al.*, 1983]. Although this approximation method is somewhat heuristic, there do exist approximation error bounds expressed in the truncated $\sigma_i(t)$, see for example [Sandberg and Rantzer, 2004].

The balancing coordinate transformation T_B often has bad numerical properties already in the time-invariant case. It was suggested in [Safonov and Chiang, 1989] that one should use non-balancing projections instead. The procedure presented there can be generalized to the time-varying case. If one, based on the sizes of the singular values (3), chooses to keep $\hat{n}$ states, then two matrices

$$S_L(t) \in \boldsymbol{R}^{n\times\hat{n}}, \quad S_R(t) \in \boldsymbol{R}^{n\times\hat{n}}, \tag{5}$$

should be found, that for all t in $[0, T]$ fulfill

(i) $S_L^T(t)S_R(t) = I_{\hat{n}}$,

(ii) the columns of $S_R(t)$ span the right eigenspace of $P(t)Q(t)$ corresponding to $\sigma_1(t)\ldots\sigma_{\hat{n}}(t)$,

(iii) the rows of $S_L^T(t)$ span the left eigenspace of $P(t)Q(t)$ corresponding to $\sigma_1(t)\ldots\sigma_{\hat{n}}(t)$.

Then the following proposition can be derived.

PROPOSITION 1—CONTINUOUS TIME
Assume that there exists a balanced approximation $\hat{G}_B$ of G. Assume also that there is a pair of projections, $S_L(t), S_R(t)$, that are continuously differentiable and fulfill (i)–(iii). Then the approximation $\hat{G}$ given by

$$\hat{G} := \mathcal{S}\,(S_L^T[AS_R - \dot{S}_R], S_L^T B, CS_R, D) \tag{6}$$

is input-output equivalent to $\hat{G}_B$. □

By using the proposition one does not need to construct a balanced realization of G. The projections S_L and S_R may be obtained on a time grid $\{t_k\}$ from the Gramians $P(t_k)$ and $Q(t_k)$ using the Schur-projection method developed for time-invariant systems in [Safonov and Chiang, 1989]. In between the grid points, $S_L(t)$ and $S_R(t)$ may be smoothly interpolated. One can allow for isolated jump discontinuities as will be discussed next.

2.1 Time-Varying State Dimension

It is possible to have a time-varying state dimension in the approximation $\hat{G}$. This may be of interest, for example, when a singular value is only

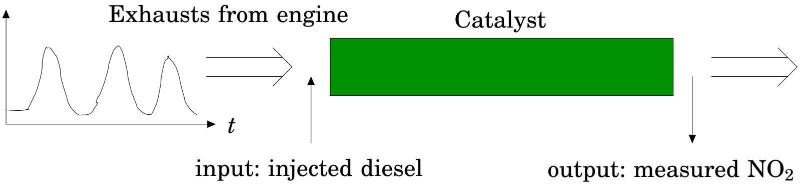

Figure 1. A schematic figure of the diesel exhaust catalyst. By injecting a small amount of extra diesel into the exhausts before it enters the catalyst, the amount of NO_x can be decreased.

small on parts of the interval $[0, T]$. If at time $t < t_i$ the system $\hat{G}$ should have $\hat{n}_1$ states, and at time $t > t_i$ it should have $\hat{n}_2$ states, there is a jump transition at the switching instant t_i

$$\boldsymbol{R}^{\hat{n}_2} \ni \hat{x}(t_i+) = S_L^T(t_i+)S_R(t_i-)\hat{x}(t_i-), \tag{7}$$

where $\hat{x}(t_i-) \in \boldsymbol{R}^{\hat{n}_1}$, $S_R(t_i-) \in \boldsymbol{R}^{n\times\hat{n}_1}$, and $S_L^T(t_i+) \in \boldsymbol{R}^{\hat{n}_2\times n}$. Equation (7) is also useful when the projections (5) have discontinuities at certain time instants t_i. For discrete-time systems a change of state-space dimension is implemented by non-quadratic transition matrices, see for example [Varga, 2000] and Section 5. Equation (7) is the corresponding non-quadratic jump transition matrix in continuous time.

3. Case Study: Continuous Time

Here a diesel exhaust catalyst model taken from [Westerberg *et al.*, 2003] is studied, see Figure 1. This example was also briefly studied in [Sandberg and Rantzer, 2004]. The input is the amount of injected diesel, and the output is the amount of NO_2 that is emitted into atmosphere. The system is linearized around a pulsating trajectory (three pulses) over a finite horizon of 460 seconds. This is a test cycle that is used to study the emission of the engine under different working conditions.

To find the singular values (3) the two time-varying Lyapunov equations (2) need to be solved. Since $n = 24$ this involves rather heavy computations. The three largest singular values are plotted in Figure 2. The singular values σ_4–σ_{24} are all smaller than 10^{-5}. The ODE-solver tolerance has deliberately been turned down to save computation time. Since $\hat{n}$ will be chosen as $\hat{n} = 1$ or $\hat{n} = 2$ it is enough that the two largest singular values are smooth. If the singular values that are kept are not smooth, one cannot expect that the corresponding projections S_L and S_R are smooth. The projections $S_L(t)$ and $S_R(t)$ are computed as described in

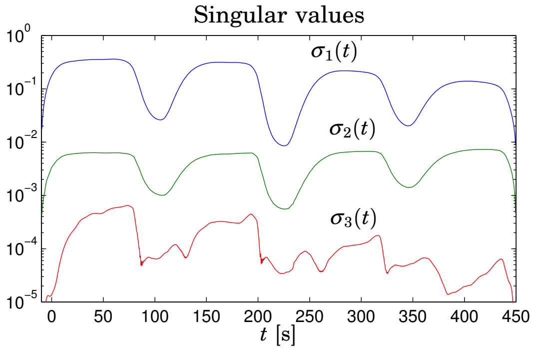

Figure 2. The three largest singular values for the linear time-varying system G, which approximates the diesel exhaust catalyst. (Reproduced with permission from [Sandberg and Rantzer, 2004]. ©2004 IEEE)

Section 2. The time grid $\{t_k\}$, is chosen to be an equidistant 801-point grid on the interval $[0, 400]$ and the Schur-method is used. The smoothness of the interpolated projections $S_L(t)$ and $S_R(t)$ is then verified.

To assess the quality of the approximations, some simulations of the models are made. In Figure 3 a step response test for G is shown, and two approximations $\hat{G}_1$ and $\hat{G}_{1,2}$. $\hat{G}_1$ is a first-order approximation, and $\hat{G}_{1,2}$ is a first-order approximation up to $t = 225$, then the state dimension is increased to two, with a transition (7). One reason for increasing the state dimension may be a desire for increased accuracy at the end of the simulation. To switch at this time instant is suitable, since $\sigma_2(t)$ starts to increase then, as seen in Figure 2.

Already $\hat{G}_1$ must be considered to be a good approximation of G, even though it has only one state. $S_L(t)$ and $S_R(t)$ show that the single state $\hat{x}$ in $\hat{G}_1$ is a time-varying weighted average of the concentrations of NO in four locations in the catalyst. A person with insight in the dynamics may realize that these are the relevant states in G. However, the procedure automatically yields a suitable linear combination of these states that can be used as single state $\hat{x}$.

The approximation error plot in Figure 3 shows that there are errors of magnitude 2 ppm. The maximum error occurs when $\sigma_2(t)$ is large. By switching to a second-order model at $t = 225$ and to keep $\sigma_2(t)$ over a maximum, as is done in $\hat{G}_{1,2}$, one can effectively push down the error. Notice that there is a small transient in $\hat{y}_{1,2}$ when the state dimension increases. The error bounds derived in [Sandberg and Rantzer, 2004] guarantee that this transient cannot be too large. After the transient has died out, the second-order model performs much better.

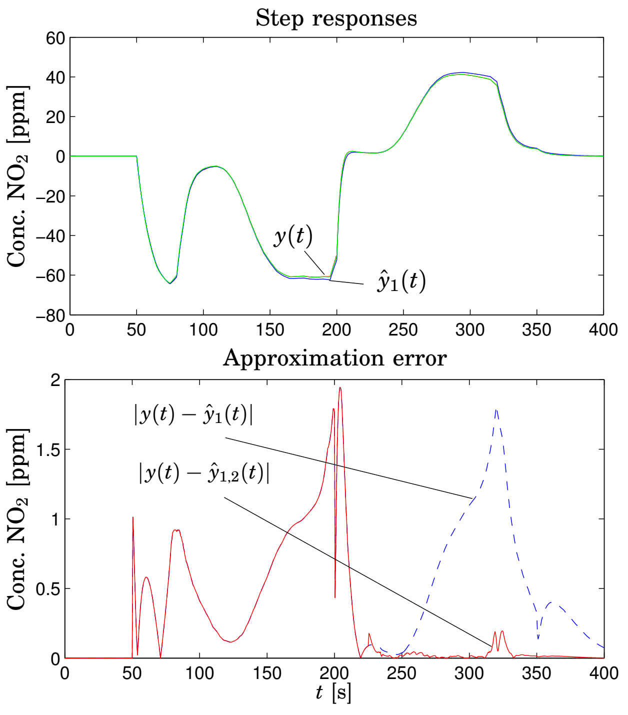

Figure 3. (Upper plot) The step responses for the 24th-order linear time-varying system G and a first-order approximation $\hat{G}_1$. There is a step up at $t = 50$ and an equal step down at $t = 200$. At $t = 350$ the input becomes zero. (Lower plot) The absolute error between the approximations and the full-order model. At $t = 225$ the state dimension is increased from one to two for $\hat{G}_{1,2}$.

There are a number of problems with the described method. For example, it is computationally expensive to solve the Lyapunov differential equations (2). In this case it takes hours to solve the equations on a Pentium 4. It is also cumbersome to interpolate the projections smoothly. For these reasons, the same example is tested in discrete time next.

4. Discretization and Error Bounds

Here it will be seen that it is often numerically more convenient to discretize the model (1) before the truncation is made. Apart from the practical issues, there are also stronger stability results for balanced truncation in discrete time, see [Shokoohi and Silverman, 1987]. It will also be seen that a simple bound on $\|G - \hat{G}\|$ can be derived. The bound does not seem to have a counterpart in continuous time.

In this paper, an approximative zero-order-hold (ZOH) sampling of (1) is made. The idea is to choose the sampling instants $\{t_k\}$ such that in each interval $[t_k, t_{k+1}]$, the realization does not vary a lot. The following approximation is used:

$$\begin{aligned} A(k) &:= \Phi_{A^c}(t_{k+1}, t_k) \approx \exp(A^c(t_k) \cdot (t_{k+1} - t_k)), \\ B(k) &:= \int_{t_k}^{t_{k+1}} \Phi_{A^c}(t_{k+1}, \tau) \cdot B^c(\tau) d\tau \\ &\approx \int_{t_k}^{t_{k+1}} \exp(A^c(t_k) \cdot (t_{k+1} - \tau)) d\tau \cdot B^c(t_k), \\ C(k) &:= C^c(t_k), \quad D(k) := D^c(t_k), \end{aligned}$$

where superscript c denotes the continuous-time realization and Φ_{A^c} is the continuous-time transition matrix. If the realization is time invariant over each sampling interval, the approximation is exact, see [Rugh, 1996].

Next, assume that $A^c(t)$ is not time invariant but continuously time varying. Use the decomposition $A^c(t) = A^c(t_k) + \Delta A^c(t)$, where $|\Delta A^c(t)| \leq M_{t_k}(h_k)$ for all t in the interval $[t_k, t_{k+1}]$. Here $h_k = t_{k+1} - t_k$ and M_{t_k} is a bound with the property $M_{t_k}(h_k) \searrow 0$ as $h_k \to 0$. Then it holds that

$$|A(k) - \exp(A^c(t_k) \cdot h_k)| = O(M_{t_k}(h_k) \cdot h_k), \quad h_k \to 0.$$

The convergence follows from a Peano-Baker series expansion of the transition matrix Φ_{A^c}. A similar result holds for $B(k)$.

The approximation method above may be crude. For the method to work well it is essential that the sampling instants are chosen properly (sufficiently dense). However, the error from the approximative discretization may be acceptable since more errors will in any case be introduced during the balanced truncation. The idea of model reduction is to derive a simple model with a small approximation error. But one should not forget that it should also be relatively simple to obtain the reduced model. To compute the exact ZOH-sampled model is computationally expensive, since $\Phi_{A^c}(t, \tau)$ is needed, and the extra accuracy gained by the exact formulas may be lost when balanced truncation is applied afterwards.

In the following, any bounded time-varying realization (A, B, C, D) is considered, not only the ones obtained from the approximate discretization above. The discrete-time realization of G is given by

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k), \quad x(0) = 0, \\ y(k) &= C(k)x(k) + D(k)u(k), \quad k \in [0, T], \end{aligned} \tag{8}$$

and the corresponding Lyapunov equations are

$$\begin{aligned} A^T(k)Q(k+1)A(k) + C^T(k)C(k) &= Q(k), \\ A(k)P(k)A^T(k) + B(k)B^T(k) &= P(k+1), \end{aligned} \tag{9}$$

with $Q(T+1) = 0$ and $P(0) = 0$. According to [Shokoohi and Silverman, 1987], a balanced realization can be found in discrete time if the realization is uniformly controllable and observable. Such a powerful existence condition for a balanced realization does not exist in continuous time.

4.1 An Error Bound

A simple diagonal state rescaling of a balanced realization leads to an *input normalized* realization G_N with states x_N and Gramians $Q(k) = \Sigma^2(k)$ and $P(k) = I$, $k \in [1, T]$. Let us call the truncated normalized realization $\hat{G}_N$. $\hat{G}_N$ is input-output equivalent to a truncated balanced realization $\hat{G}_B$. Let us partition the state-space,

$$x_N^T(k) = [\, x_{N1}^T(k) \quad x_{N2}^T(k) \,],$$

and conformally $\Sigma(k) = \text{diag}\{\Sigma_1(k), \Sigma_2(k)\}$ with

$$\begin{aligned} \Sigma_1(k) &= \text{diag}\{\sigma_1(k), \ldots, \sigma_{\hat{n}}(k)\}, \\ \Sigma_2(k) &= \text{diag}\{\sigma_{\hat{n}+1}(k), \ldots, \sigma_n(k)\}, \end{aligned}$$

and then truncate the states $x_{N2}(k)$ for $k \in \mathcal{T} \subseteq [1, T]$. The state of $\hat{G}_N$ is called $\hat{x}$. According to Lemma 1 in [Sandberg and Rantzer, 2004], the input-normalized realization fulfills

$$\left\| \begin{bmatrix} x_{N1}(T+1) - \hat{x}(T+1) \\ x_{N2}(T+1) \end{bmatrix} \right\|_{\Sigma^2}^2 + \|y - \hat{y}\|_{[0,T]}^2 \le \sum_{k \in \mathcal{T}} |x_{N2}(k)|_{\Sigma_2^2}^2, \tag{10}$$

after a completion of squares. Here $y = G_N u = Gu$ and $\hat{y} = \hat{G}_N u = \hat{G}_B u$. Actually the first term on the left hand side is zero since $\Sigma(T + 1) = 0$. If the sum on the right hand side of (10) can be bounded with a constant times $\|u\|$, then one has an upper bound on $\|G - \hat{G}_B\|$. Under the assumption that the singular values are ordered in decreasing order for all k, the following upper estimate holds

$$\sum_{k \in \mathcal{T}} |x_{N2}(k)|_{\Sigma_2^2}^2 \le \left(\sup_{k \in \mathcal{T}} \sigma_{\hat{n}+1}^2(k) \right) \cdot \|x_N\|_{[0,T]}^2. \tag{11}$$

The mapping from u to x_N is given by

$$x_N(k) = \sum_{i=0}^{k-1} \Phi_{A_N}(k, i+1) B_N(i) u(i).$$

Let us assume that (8) is exponentially stable. Because balancing is a topologically equivalent coordinate transformation, the input normalized realization will be exponentially stable if and only if the original realization is exponentially stable. Then there are constants $C_N > 0$, $0 < \lambda_N < 1$, such that $|x_N(k)| \leq C_N \cdot \lambda_N^{k-i} \cdot |x_N(i)|$, for $u = 0$.

From the reachability Lyapunov equation it follows that in general the state is bounded by $|x(i)|^2_{P^{-1}} \leq \|u\|^2_{[0,i-1]}$ for all u. If an impulse input $u(k) = \delta(k-i+1)$ and an input-normalized system is used this reduces to $|x_N(i)|^2 \leq 1$. From this initial position the state is bounded exponentially and the states can be summed over the horizon

$$\sum_{k=i}^{T} |x_N(k)| \leq C_N \cdot \sum_{k=i}^{\infty} \lambda_N^{k-i} = \frac{C_N}{1-\lambda_N}. \tag{12}$$

This is a bound that holds for all i and as $T \to \infty$. It is also a bound on the induced ℓ_2-norm on the map $u \mapsto x_N$, because the ℓ_1-norm of a uniform bound of an impulse response gives an upper bound on induced ℓ_2-norm. Together with (10) and (11), (12) gives us the bound (14) in Proposition 2.

A discrete-time counterpart of Proposition 1 can be stated. The notation $(S_L^T)^+(k) = S_L^T(k+1)$ is used.

PROPOSITION 2—DISCRETE-TIME
Assume that G has a uniformly controllable and observable realization, and that there is a pair of projections $S_L(k)$ and $S_R(k)$ that fulfill (i)–(iii). Then $\hat{G}$ given by

$$\hat{G} := \mathcal{S} \left\{ (S_L^T)^+ A S_R, (S_L^T)^+ B, C S_R, D \right\} \tag{13}$$

is input-output equivalent to $\hat{G}_B$.

Furthermore, when $T = \infty$ and G is exponentially stable, there is a constant $K = C_N/(1-\lambda_N)$, independent of $\hat{n}$, such that

$$\|G - \hat{G}_B\| = \|G - \hat{G}\| \leq K \cdot \sup_{k \in \mathcal{T}} \sigma_{\hat{n}+1}(k). \tag{14}$$

□

The discrete-time projection (13) has been suggested before. See for example [Varga, 2000; Chahlaoui and Van Dooren, 2002].

There are two nice features with the bound (14): It is not a sum over all truncated singular values as in the regular error bound for balanced truncation, and it only contains the maximum truncated singular value. Previous bounds for time-varying singular values include monotonicity conditions, see [Lall and Beck, 2003; Sandberg and Rantzer, 2004]. Since the bound only contains the first of the truncated singular value, the result may be useful for the approximate balancing methods that use low-rank approximations of the Gramians, see [Chahlaoui and Van Dooren, 2002].

The problem with the bound is that the constant K may be hard to estimate. Furthermore, λ_N may be close to one. This makes K large and potentially the bound very conservative. The continuous-time counterpart of (14) is not as nice, since the corresponding inequality (10) is different, see [Sandberg and Rantzer, 2004].

REMARK 1

If the calculations leading up to (14) are repeated using a balanced and not a normalized realization, then a bound

$$\|G - \hat{G}\| \leq \frac{C_B}{1-\lambda_B}\sqrt{\sup_k \sigma_1(k)}\sqrt{\sup_{k\in\mathcal{T}} \sigma_{\hat{n}+1}(k)}$$

is obtained, where C_B and λ_B depend on the balanced system matrix $A_B(k)$. This bound may be smaller than (14), but has a more complex structure. Which bound that is better to use in general should be further studied. □

5. Case Study: Discrete Time

The approximative discretization described in Section 4 is applied to the continuous-time model in Section 3 using a uniform sampling period of $h_k = 5$ s. The discretized model is evaluated by performing the same step responses as in Figure 3. The error between the continuous-time model and the discretized model is seen in Figure 4. The discretization error is of the same magnitude as the approximation error in continuous time. If this is not considered to be acceptable, the sampling period should be made shorter.

The singular values of the discretized model are qualitatively very similar to the ones in Figure 2. It should be pointed out, however, that they are not just a sampled version. Also, it takes only seconds to solve the Lyapunov equations (9) with $n = 24$, whereas it takes hours to solve (2) with reasonable accuracy.

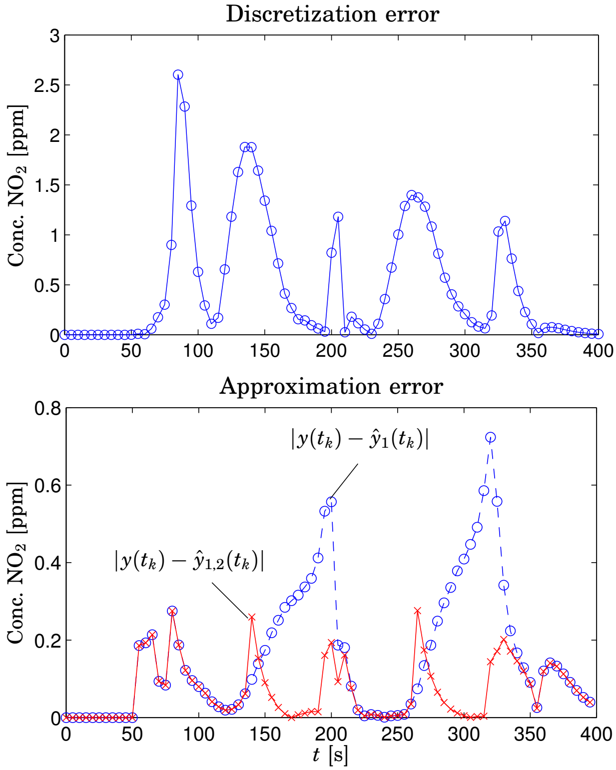

Figure 4. (Upper plot) The discretization error is the absolute difference between the step response in Figure 2 and the same step response with the discretized model. (Lower plot) The approximation error between the discrete-time system and its truncated balanced realizations. There are state-dimension switches at $t = 140$, 195, 265, 320. The total error between the reduced discrete-time model and the full continuous-time model is obtained by adding the error plots at each instant.

It is simple to implement a state-dimension change in discrete time. At each sample k, one chooses how many states that should be included in the approximation. If a particular singular value is smaller than a previously deleted one, one can truncate the corresponding state without increasing the error bound (14). If the projections $S_L(k)$ and $S_R(k)$ fulfill (i)–(iii), then the realization will automatically take care of the state-dimension switches, since the state matrices $S_L^T(k+1)A(k)S_R(k)$ will be non-square.

In Figure 4, the different truncated models are compared to the full model. In the variable-order model a second state is added during the two last maxima of $\sigma_2(k)$. Just as in the continuous-time case it is noticed that a state-dimension switch gives a transient in the output signal. Also it is seen that if one adds the discretization and the approximation error, one gets about the same total error as the continuous-time approximation. However, the discrete-time model is simpler to obtain.

The worst-case error bound in Remark 1 is smaller than (14) for this model. But still it is a factor 100 greater than the observed approximation error in the step response test in Figure 4. This shows that the error for a particular input signal may be much smaller than the bounds.

6. Conclusion

The balanced truncation method has been applied on a time-varying linear model using a projection method. A continuous-time and a discrete-time approach have been compared. Both approaches gave approximately the same total approximation error, but the discrete-time calculations are computationally cheaper and simpler. Also a new a priori error bound for balanced truncation in discrete time was derived.

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Paper III

Frequency-Domain Analysis of Linear Time-Periodic Systems

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Abstract

In this paper, we study convergence of truncated representations of the frequency-response operator of a linear time-periodic system. The frequency-response operator is frequently called the harmonic transfer function. We introduce the concepts of input, output, and skew roll-off. These concepts are related to the decay rates of elements in the harmonic transfer function. A system with high input and output roll-off may be well approximated by a low-dimensional matrix function. A system with high skew roll-off may be represented by an operator with only few diagonals. Furthermore, the roll-off rates are shown to be determined by certain properties of Taylor and Fourier expansions of the periodic systems. Finally, we clarify the connections between the different methods for computing the harmonic transfer function that are suggested in the literature.

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Notation. Signals defined in continuous time on an interval I belong to the Lebesgue spaces $L_r(I)$, where $r \geq 1$, and the standard norms on these spaces will be denoted by $\|\cdot\|_{L_r(I)}$. When the interval I is clear from the context, it will be left out in the notation. We denote square-summable sequences by ℓ_2, and the norm by $\|\cdot\|_{\ell_2}$. The set of L times continuously differentiable real functions in some open set Ω is denoted by $\mathcal{C}^L(\Omega)$. p denotes the differentiation operator, and $1/p$ integration. $\boldsymbol{R}$ denotes the real axis, $\boldsymbol{R}_+$ the non-negative real axis, and $\boldsymbol{Z}$ the set of all integers. j is the imaginary unit, and $j\boldsymbol{R}$ is the imaginary axis.

1. Introduction

In this paper, we study linear operators G defined on signals u in L_r, where $r \geq 1$,

$$y = Gu.$$

We restrict ourselves to the set of bounded operators G, i.e., operators with finite induced norm

$$\|G\|_{L_r \to L_r} = \sup_{\|u\|_{L_r} \leq 1} \|Gu\|_{L_r}. \tag{1}$$

We assume in the following that the given operator G is bounded and has a time-domain representation with a *causal* impulse response

$$g(t,\tau) = 0, \quad \text{for all} \quad t < \tau,$$

and

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)\,d\tau, \tag{2}$$

where $u(t)$ and $y(t)$ belong to $L_r(-\infty,\infty) = L_r(\boldsymbol{R})$. Conditions for representability of an operator as an integral equation (2) are given in, for instance, [Sandberg, 1988]. It is well known that systems with finite-dimensional state-space realizations as well as infinite-dimensional models such as time-delay systems may be written on the form (2). We will often make the assumption that the impulse response has *uniform exponential decay*. This means that there are positive constants κ_1 and κ_2 such that

$$|g(t,\tau)| \leq \kappa_1 \cdot e^{-\kappa_2(t-\tau)}, \quad \text{for all} \quad t \geq \tau.$$

In particular this assumption implies boundedness of G, since $\|G\|_{L_r \to L_r} \leq \kappa_1/\kappa_2$, for all $r \geq 1$.

If there is a real positive number T such that

$$g(t+T, \tau+T) = g(t, \tau), \quad \text{for all} \quad t \geq \tau, \tag{3}$$

then the operator (or the system it represents) is said to be *periodic* with period T. The impulse response of a *time-invariant* system satisfies

$$g(t, \tau) = g(t-\tau, 0), \quad \text{for all} \quad t \geq \tau.$$

A system with a finite-dimensional state-space realization can be written as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t). \end{aligned} \tag{4}$$

The impulse response of the system is given by

$$g(t, \tau) = C(t)\Phi_A(t, \tau)B(\tau) + D(t)\delta(t-\tau),$$

where $\Phi_A(t, \tau)$ is the transition matrix for $\dot{x} = A(t)x$, see [Rugh, 1996]. If the system is periodic, then all the matrices in the realization (4) are T-periodic.

In the following, $u(t)$ and $y(t)$ will be scalar signals. That is, we treat single-input-single-output systems. However, this is just for notational convenience. Everything can be done for multi-input-multi-output systems with only minor modifications.

It is well known, see, for example, [Wereley, 1991; Zhou and Hagiwara, 2002a], that frequency-domain representations of periodic systems are infinite-dimensional operators. The main goal of this paper is to study the convergence properties of different truncations of this operator. In particular, we obtain the relations between the frequency-domain operator and Taylor and Fourier expansions of the impulse response. Furthermore, we clarify the connections to state-space approaches, so-called harmonic balance methods.

1.1 Previous Work

The study of periodic systems has a long history in applied mathematics and control. One reason for the interest in periodic systems is that natural and man-made systems often have the periodicity property (3). Some examples are oscillators used in communication systems, planets and satellites in orbit, rotors of wind mills and helicopters, sampled-data systems, and AC power systems. There is an excellent survey of periodic systems and control in [Bittanti and Colaneri, 1999].

Frequency-domain analysis of linear time-periodic systems in continuous time has been studied by several authors in the past. A classical reference is the work by Zadeh, see [Zadeh, 1950], where the steady-state response of a time-varying system to harmonics is used to define a time-varying transfer function, the *parametric transfer function* (PTF). The PTF has been further developed and used in [Rosenwasser and Lampe, 2000; Lampe and Rosenwasser, 2003]. The PTF is a scalar function that depends on two variables: time and frequency.

A second frequency-domain representation can be obtained by using time-domain lifting on the time-periodic system, and then applying the z-transform, see [Wereley, 1991; Colaneri, 2000]. Colaneri called this function the *transfer function operator* (TFO). This is an integral operator with a kernel depending on time and frequency. An alternative derivation of the TFO comes from studies of the steady-state response of the system to an *exponentially modulated periodic* (EMP) signal. The EMP signals serve as good test functions, since EMP signals are mapped to EMP signals by periodic systems.

A third approach was taken by Wereley and Hall in [Wereley and Hall, 1990; Wereley, 1991]. They applied an harmonic balance method to state-space systems with EMP inputs. That is, periodic matrices and signals are expanded into Fourier series and harmonics are equated. This method yields a transfer function that depends only on frequency. The function was called the *harmonic transfer function* (HTF). The HTF is an infinite-dimensional operator. The infinite dimensionality can be seen as the price that is paid for the removal of the time dependence in the transfer function. For example, it was shown in [Lampe and Rosenwasser, 2001] that a Fourier expansion of the PTF in the time direction yields the elements of the HTF.

All of the above transfer functions can be used for studies of periodic systems, for example, to compute norms. It is important to understand that all of these transfer functions are equivalent. The PTF and the TFO are time dependent. If this time dependence is expressed in an harmonic basis of the type $\{e^{jk2\pi t/T}\}_{k\in Z}$, we essentially obtain the HTF. Relations of this sort are treated in [Yamamoto and Araki, 1994; Yamamoto and Khargonekar, 1996]. In this paper, we investigate what happens when higher harmonics in this basis are truncated. An alternative approximation method is, for example, to use fast sampling in time. Such approximations are discussed in [Yamamoto *et al.*, 1997]. It should also mentioned be in this context that the frequency-domain operator of a discrete-time periodic system becomes finite-dimensional, see [Goodwin and Feuer, 1992; Hwang, 1997].

The above listing of frequency-domain methods is not complete. There are more representations, see, for example, [Ball *et al.*, 1995; Cantoni, 1998].

In the area of sampled-data systems a lot of related work has been done. The PTF has been applied to sampled-data systems in [Rosenwasser and Lampe, 2000]. A method similar to the TFO, that is, a lifting and z-transform approach, has been used in, for example, [Bamieh and Pearson, 1992; Yamamoto and Khargonekar, 1996]. An approach similar to the HTF has been used in [Araki *et al.*, 1996; Dullerud, 1996]. A nice property of sampled-data systems is that often closed-form solutions are obtained. This is not the case for generic periodic systems. The literature on sampled-data systems is vast and many more references can be found in the above work.

We mainly work with the HTF in this paper. From the above discussion, it follows that this is not a severe restriction. The HTF has successfully been used by several authors for different applications in the past. For example, for identification of helicopter dynamics, see [Hwang, 1997], for vibration damping in helicopters, see [Bittanti and Cuzzola, 2002], and for stability and robustness analysis in switched power systems, see [Möllerstedt and Bernhardsson, 2000a; Möllerstedt and Bernhardsson, 2000b]. A nice feature with the HTF is that we can directly extract Bode-type diagrams, from the diagonals of the HTF, that describe the cross-coupling of frequencies. This can be used to detect resonances that involve several frequencies, see [Möllerstedt and Bernhardsson, 2000b].

Another reason for studies of the HTF is that it has recently obtained a lot of theoretical attention. Formally, we can work with the HTF just as with a standard transfer function. Hence, formulas for H_2/H_∞-norms are completely analogous to the time-invariant formulas, see [Wereley, 1991; Zhou and Hagiwara, 2002a; Zhou and Hagiwara, 2002b]. But the HTF is also useful for studies of attainable H_2-performance, see [Zhang and Zhang, 1997], generalization of the Nyquist criterion [Hall and Wereley, 1990], and for generalization of Bode's sensitivity integral [Sandberg and Bernhardsson, 2004b; Colaneri, 2004].

1.2 Computation of the HTF

Despite all of this work, there are still open issues about the HTF. In particular, how the HTF should be computed. Three approaches have been taken, to the authors' knowledge.

In the first approach, see [Wereley, 1991; Zhou and Hagiwara, 2002a], it is assumed that G has a state-space realization (4), and that a Floquet transformation has been performed. Then the matrix $A(t)$ is time invariant, and explicit formulas for the elements in the HTF can be given as a series of the Fourier coefficients of $B(t), C(t)$, and $D(t)$.

The second approach is also a state-space approach, see [Zhang and Zhang, 1997]. The elements of the HTF are given implicitly via an inversion of an unbounded quasi-Toeplitz operator, with the Fourier coefficients $\{A_k\}_{k\in Z}$ of the state matrix $A(t)$ on the diagonals. It has been claimed that this yields the HTF when the dimension of finite-dimensional truncations of the operator grows towards infinity. However, to the best knowledge of the authors of this paper, how and when this convergence works has not been properly explained. This second approach is interesting since it does not require a Floquet transformation on the state-space model, and allows us to work directly with the Fourier coefficients of the state-space realization. We call this method the *truncated harmonic balance method*.

A third approach was used in [Möllerstedt and Bernhardsson, 2000b]. This approach was based on an impulse-response model (2) of the periodic system. In particular, it was shown how the elements of the HTF can be computed via a Fourier expansion of the impulse response. This approach is interesting since it only uses input-output data of the model. However, the calculations in [Möllerstedt and Bernhardsson, 2000b] were formal and many details and possibilities were not treated.

In this paper, we clarify the connections between the above approaches. We show that we, at least in the limit, obtain the same operator no matter what approach that is used. Hence, it is justified to use the term HTF in all of the above cases.

Furthermore, in the above references, with the exception of the work in [Araki *et al.*, 1996; Zhou and Hagiwara, 2002b], the convergence of truncated HTFs has not been a major issue. In this paper, on the other hand, convergence issues are in focus, and we will improve the convergence rate bounds. Convergence analysis is essential since all implementations use some sort of truncation.

1.3 Organization and Contributions

In Section 2, we derive Taylor expansions of time-varying systems. The expansions are around "infinite frequency", and the coefficients become time-varying Markov parameters. Two different expansions are studied. We introduce the concepts of input and output roll-off of a time-varying system, and state an equivalence between the Markov parameters and the roll-off concepts. This is a generalization of standard results for time-invariant systems.

In Section 3, we derive Fourier expansions of time-periodic systems. The generalized Fourier coefficients become time-invariant systems. Similar ideas were suggested in, for example, [Zadeh, 1950; Möllerstedt and Bernhardsson, 2000b]. Here we identify a Hilbert space, H_2, where the Fourier series converge. Furthermore, we derive conditions under which truncated Fourier series converge in induced L_r-norm, and introduce the concept of skew roll-off.

In Section 4, we define the HTF based on the impulse response, as was done in [Möllerstedt and Bernhardsson, 2000b]. Its definition is straightforward after a Fourier expansion. We show that if the periodic system has high input and output roll-off, then it can be well approximated by a finite low-dimensional matrix function. If the system has high skew roll-off, then it can be approximated by an operator with only few diagonals.

In Section 5, we develop error bounds for the closed-loop operator $(I + G)^{-1}$, when it is computed from truncated HTFs. We relate the error to the roll-off concepts of G.

In Section 6, we show that the HTF defined in Section 4 is identical to the HTF defined in [Wereley, 1991; Zhou and Hagiwara, 2002a]. We also study the truncated harmonic balance method. It is seen that by a minor modification of the method, we can show that it converges to the desired operator. The convergence may, however, be quite slow.

An early version of this paper, that contains some of the results is [Sandberg *et al.*, 2004].

2. Taylor Expansions of Time-Varying Systems

We will obtain a frequency-domain description of G. Many times the input signal $u(t)$ and output signal $y(t)$ in $L_2(\boldsymbol{R})$ are represented by their Fourier transforms $\hat{u}(j\omega)$ and $\hat{y}(j\omega)$, where ω is the angular frequency. This presents no problems since $L_2(\boldsymbol{R})$ is isomorphic with $L_2(j\boldsymbol{R})$ under the Fourier transform, see for example [Dym and McKean, 1972]. We use the standard norms as follows:

$$\|u\|_{L_2} = \|u(\cdot)\|_{L_2(R)} = \left(\int_{-\infty}^{\infty} |u(t)|^2 dt\right)^{1/2} \tag{5}$$

$$= \|\hat{u}(\cdot)\|_{L_2(jR)} = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega\right)^{1/2}, \tag{6}$$

and the equality of (5) and (6) follows from Plancherel's theorem.

2.1 Markov Parameters for Time-Varying Systems

As a first step in the analysis, we make an expansion of the convolution integral (2) which resembles a Taylor expansion. This is motivated by the Markov parameters for time-invariant systems. That is, a transfer function $\hat{g}(j\omega)$ of a time-invariant system, where $g(t)$ is the impulse response, can under certain regularity conditions be Taylor expanded as

$$\hat{g}(j\omega) = \frac{g(0)}{j\omega} + \frac{g'(0)}{(j\omega)^2} + \frac{g''(0)}{(j\omega)^3} + \ldots, \quad \text{as} \quad |\omega| \to \infty. \tag{7}$$

The Markov parameters are $\{g(0), g'(0), g''(0), \ldots\}$ and they determine the response to high-frequency signals. If the first Markov parameters are zero, then high-frequency signals are attenuated quickly. The Markov parameters also play an essential role in the realization theory of linear systems, see for example [Rugh, 1996].

To make a similar expansion of time-varying impulse responses, we start by putting restrictions on the impulse response, in order for the computations to be justified. The set of continuously differentiable and exponentially bounded impulse responses will appear frequently throughout this article.

DEFINITION 1—THE SET $\mathcal{C}_e^L$

A causal time-varying (not necessarily periodic) real impulse response $g(t,\tau)$ belongs to the set $\mathcal{C}_e^L$ if

(E1) $g(t,\tau)$ belongs to $\mathcal{C}^L(\Omega)$, where $\Omega = \{(t,\tau) : t > \tau\}$;

(E2) all the partial derivatives of $g(t,\tau)$ up to order L can be continuously extended to the boundary $t = \tau$;

(E3) all the partial derivatives of $g(t,\tau)$ up to order L has uniform exponential decay for $t \geq \tau$.

□

EXAMPLE 1—STATE-SPACE MODELS

Assume that $D(t) = 0$ for the state-space model (4). Then we can check that state-space models belong to $\mathcal{C}_e^L$ if the model is exponentially stable, $B(\cdot)$ and $C(\cdot)$ belong to $\mathcal{C}^L(\boldsymbol{R})$, and $A(\cdot)$ belongs to $\mathcal{C}^{L-1}(\boldsymbol{R})$. Furthermore, all the matrices should be bounded over $\boldsymbol{R}$. □

It will be useful to consider signals in the space of Schwartz functions $\mathcal{S}$,

$$\mathcal{S} = \{u(t) : u(\cdot) \in \mathcal{C}^\infty(\boldsymbol{R}) \text{ and } t^\alpha p^\beta u(t) \text{ is bounded for all } \alpha, \beta \geq 0\},$$

where p is the differentiation operator. The set $\mathcal{S}$ is dense in $L_r(\boldsymbol{R})$, for $1 \leq r < \infty$, and the Fourier transform of an element in $\mathcal{S}$ is again in $\mathcal{S}$, see [Hörmander, 1990].

To obtain expansions of the type (7) we proceed by using integration by parts. Choose an input signal $u \in \mathcal{S}$, and notice that a differentiation of the product $g(t,\tau)\int_{-\infty}^{\tau} u(s)ds$ gives

$$\frac{\partial}{\partial\tau}\left(g(t,\tau)\int_{-\infty}^{\tau} u(s)ds\right) = \frac{\partial g}{\partial\tau}(t,\tau)\int_{-\infty}^{\tau} u(s)ds + g(t,\tau)u(\tau). \tag{8}$$

If we integrate this equality in the τ-direction over the interval $(-\infty, t]$ we obtain $y(t)$, having (2) in mind, as

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau = g(t,t)\frac{u(t)}{p} - \int_{-\infty}^{t} \frac{\partial g}{\partial\tau}(t,\tau)\frac{u(\tau)}{p}d\tau. \tag{9}$$

$1/p$ is the integration operator: $u(t)/p = \int_{-\infty}^{t} u(s)ds$. The above computations are allowed if $g \in \mathcal{C}_e^1$.

One should notice that (9) is an expansion of the time-varying system (2) into a sum of one time-varying direct term, $g(t,t)$, and a new stable time-varying system with $\frac{\partial g}{\partial \tau}(t,\tau)$ as impulse response. The new input is the integrated former input: u/p. The procedure can be repeated both in the τ- and in the t-direction. After the following definition we obtain the general expansion formulas.

DEFINITION 2—INPUT AND OUTPUT MARKOV PARAMETERS
For a system with impulse response $g(t,\tau)$ in $\mathcal{C}_e^L$, the *input Markov parameters* are defined as

$$\left\{ g(t,t),\ -\frac{\partial g}{\partial \tau}(t,t),\ \frac{\partial^2 g}{\partial \tau^2}(t,t),\ldots,(-1)^{L-1}\frac{\partial^{L-1} g}{\partial \tau^{L-1}}(t,t) \right\}, \tag{10}$$

and, the *output Markov parameters* are defined as

$$\left\{ g(t,t),\ \frac{\partial g}{\partial t}(t,t),\ \frac{\partial^2 g}{\partial t^2}(t,t),\ldots,\frac{\partial^{L-1} g}{\partial t^{L-1}}(t,t) \right\}, \tag{11}$$

for $t \in \boldsymbol{R}$. □

REMARK 1—INTERPRETATION OF DERIVATIVES
The derivatives in the output Markov parameters should be interpreted as follows:

$$\frac{\partial^k g}{\partial t^k}(t,t) = \lim_{\varepsilon \to 0+} \left(\left. \frac{\partial^k}{\partial s^k} g(s,t) \right|_{s=t+\varepsilon} \right),$$

for $k \in [0, L-1]$. The limits exist and are continuous in t by assumption (E2). A similar definition holds for the input Markov parameters. □

For time-varying systems in $\mathcal{C}_e^L$, the Markov parameters are bounded continuously time-varying functions.

REMARK 2—MARKOV PARAMETERS FOR TIME-INVARIANT SYSTEMS
For time-invariant systems, with impulse response $g(t,\tau) = g(t-\tau,0)$, the input and output Markov parameters coincide with the traditional Markov parameters, and are constant and equal. □

THEOREM 1—TAYLOR EXPANSIONS OF TIME-VARYING SYSTEMS
Assume that the causal time-varying impulse response $g(t,\tau)$ belongs to the set $\mathcal{C}_e^L$. Then for every input $u \in \mathcal{S}$, the output $y = Gu$ given by (2) can be expressed in either of the following two ways.
— Input Markov parameter expansion

$$y(t) = a_1(t)\frac{u(t)}{p} + a_2(t)\frac{u(t)}{p^2} + \ldots + a_L(t)\frac{u(t)}{p^L} + (-1)^L \int_{-\infty}^{t} \frac{\partial^L g}{\partial \tau^L}(t,\tau)\frac{u(\tau)}{p^L}d\tau, \quad (12)$$

where $a_1(t), \ldots, a_L(t)$ are the input Markov parameters (10).
— Output Markov parameter expansion

$$y(t) = \frac{1}{p}b_1(t)u(t) + \frac{1}{p^2}b_2(t)u(t) + \ldots + \frac{1}{p^L}b_L(t)u(t) + \frac{1}{p^L}\int_{-\infty}^{t} \frac{\partial^L g}{\partial t^L}(t,\tau)u(\tau)d\tau, \quad (13)$$

where $b_1(t), \ldots, b_L(t)$ are the output Markov parameters (11). □

Proof. To prove (12) repeat the integration by parts procedure from (8)–(9) on $\frac{\partial g}{\partial \tau}u/p^2$, which gives

$$\int_{-\infty}^{t} \frac{\partial g}{\partial \tau}(t,\tau)\frac{u(\tau)}{p}d\tau = \frac{\partial g}{\partial \tau}(t,t)\frac{u(t)}{p^2} - \int_{-\infty}^{t} \frac{\partial^2 g}{\partial \tau^2}(t,\tau)\frac{u(\tau)}{p^2}d\tau.$$

Substitute this into (9). This may be repeated on $\frac{\partial^m g}{\partial \tau^m}u/p^{m+1}$ for $m = 2 \ldots L-1$ and we obtain (12).

The first step in proving (13) is to differentiate (2),

$$y'(t) = g(t,t)u(t) + \int_{-\infty}^{t} \frac{\partial g}{\partial t}(t,\tau)u(\tau)d\tau,$$

and then to integrate over $(-\infty, t]$ in the t-direction

$$y(t) = \frac{1}{p}g(t,t)u(t) + \frac{1}{p}\int_{-\infty}^{t} \frac{\partial g}{\partial t}(t,\tau)u(\tau)d\tau. \quad (14)$$

Repeat this procedure on the virtual output $y_m = \int \frac{\partial^m g}{\partial t^m}u d\tau$ for $m = 1 \ldots L-1$ and substitute into (14). The above computations are allowed under the given assumptions. Equation (13) may also be proven from (12) by a duality argument, see Remark 3. ■

REMARK 3—DUALITY OF INPUT AND OUTPUT MARKOV PARAMETERS
The adjoint G^* of $G : L_2 \to L_2$, where $g(t,\tau) \in \mathcal{C}_e^L$, is given by

$$z(\tau) = \int_\tau^\infty g(t,\tau)w(t)dt,$$

and one can make Taylor expansions of this relation as well

$$z(\tau) = g(\tau,\tau)\frac{w(\tau)}{p^*} + \frac{\partial g}{\partial t}(\tau,\tau)\frac{w(\tau)}{(p^*)^2} + \dots,$$
$$z(\tau) = \frac{1}{p^*}g(\tau,\tau)w(\tau) - \frac{1}{(p^*)^2}\frac{\partial g}{\partial \tau}(\tau,\tau)w(\tau) + \dots,$$

where $w(\tau)/p^* = \int_\tau^\infty w(s)ds$. So with the interchange $t \leftrightarrow \tau$ the input Markov parameters of G are the output Markov parameters of G^*, and vice versa. □

REMARK 4—MIXED MARKOV PARAMETERS
It is possible to mix the input and output expansions so that Markov parameters of the type

$$\frac{\partial^{m+n} g}{\partial \tau^m \partial t^n}(t,t),$$

appear in the expansions. The input, output, and mixed Markov parameters are all entries in the Hankel matrices used in the realization theory of linear time-varying systems, see [Silverman, 1971; Rugh, 1996]. □

2.2 Input Roll-Off and Output Roll-Off

We will truncate the representations of signals and systems, and therefore it is interesting to study how the systems treat high-frequency signals. Equation (7) shows that there is a relation between the Markov parameters and high-frequency behavior for time-invariant systems. This relation will be further explored for time-varying systems.

A projection operator on L_2 which we call P_Ω is now defined. Its representation in the frequency domain is given by

$$\widehat{(P_\Omega y)}(j\omega) = \begin{cases} \hat{y}(j\omega), & |\omega| \le \Omega, \\ 0, & |\omega| > \Omega. \end{cases}$$

Notice that P_Ω is not causal in the time domain, and $\|P_\Omega\|_{L_2 \to L_2} = 1$. It is also convenient to define $Q_\Omega = I - P_\Omega$. For a truncated system to be a good approximation, we need some sort of roll-off, corresponding to strict properness for linear time-invariant systems, see [Zhou and Doyle, 1998]. For this we define a square truncation next. This term is motivated in Section 4.1.

DEFINITION 3—SQUARE TRUNCATION
$P_{\Omega_1} G P_{\Omega_2}$ is called a square truncation of G. □

We call a system strictly proper if the square truncated system converges to G,

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_2 \to L_2} \to 0 \quad \text{as} \quad \Omega_1, \Omega_2 \to \infty.$$

To give sufficient conditions for properness, we notice that we can make the bound

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_2 \to L_2} \leq \|(I - P_{\Omega_1})G\|_{L_2 \to L_2} + \|G(I - P_{\Omega_2})\|_{L_2 \to L_2}.$$

DEFINITION 4—INPUT AND OUTPUT ROLL-OFF
If for a bounded operator G there are positive constants C_1 and k_1 such that

$$\|(I - P_\Omega)G\|_{L_2 \to L_2} \leq C_1 \cdot \Omega^{-k_1},$$

then G is said to have *output roll-off* k_1, and if there are positive constants C_2 and k_2 such that

$$\|G(I - P_\Omega)\|_{L_2 \to L_2} \leq C_2 \cdot \Omega^{-k_2},$$

then G is said to have *input roll-off* k_2. □

For systems with output roll-off k_1 and input roll-off k_2 we have strict properness and the following rate of convergence for square truncated operators $P_{\Omega_1} G P_{\Omega_2}$,

$$\|G - P_{\Omega_1} G P_{\Omega_2}\|_{L_2 \to L_2} \leq C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}. \tag{15}$$

Some simple properties for calculations with systems with input/output roll-off are stated in the following proposition.

PROPOSITION 1—INPUT AND OUTPUT ROLL-OFF
The following rules apply to systems with roll-off.

(i) If H has output roll-off k_1 and G is bounded, then HG has output roll-off of k_1. If G has input roll-off k_2 and H is bounded, then HG has input roll-off of k_2.

(ii) If G has a time-invariant impulse response, that is $g(t, \tau) = g(t-\tau, 0)$ for all $t \geq \tau$, then if G has output roll-off k_1, it also has input-roll off k_1, and vice versa. Moreover $|\hat{g}(j\omega)| \leq C \cdot |\omega|^{-k}$ where $k = k_1 = k_2$ and $C = C_1 = C_2$.

(iii) If H is a time-invariant system with output roll-off k_1 and G has output roll-off k_2, then HG has output roll-off $k_1 + k_2$. If H is a time-invariant system with input roll-off k_1 and G has input roll-off k_2, then GH has input roll-off $k_1 + k_2$.

□

Proof. Follows from Definition 4, the induced norm property

$$\|GH\|_{L_2 \to L_2} \le \|G\|_{L_2 \to L_2} \|H\|_{L_2 \to L_2},$$

and $Q_\Omega H = Q_\Omega H Q_\Omega$ for time-invariant H. ■

Since Definition 4 may be hard to check for a given operator G, it simplifies if we decompose the system into terms that are easier to analyze. The Taylor expansions are such decompositions. This can be understood by studying (7). If a transfer function $\hat{g}(j\omega)$ has roll-off k, i.e., there are constants such that $|\hat{g}(j\omega)| \le C \cdot |\omega|^{-k}$, then the first $k-1$ Markov parameters are zero. The corresponding result for time-varying systems is stated next.

THEOREM 2—MARKOV PARAMETERS AND ROLL-OFF
Assume that $g(t,\tau)$ belongs to $\mathcal{C}_e^L$ and that $L > \max\{k_1, k_2\}$. Then G has exactly

(i) output roll-off k_1, if and only if the $(k_1 - 1)$ first output Markov parameters (11) are zero for all t;

(ii) input roll-off k_2, if and only if the $(k_2 - 1)$ first input Markov parameters (10) are zero for all t.

□

Proof. We need a bound on $\|GQ_\Omega\|_{L_2 \to L_2}$ expressed in Ω to prove (ii). By definition we have

$$\|GQ_\Omega\|_{L_2 \to L_2} = \sup_{\substack{\|u\|_{L_2} \le 1 \\ P_\Omega u = 0}} \|Gu\|_{L_2}. \tag{16}$$

We start to prove the 'if'-part of (ii). The problem is approached by making an input Markov parameter expansion of $y = Gu_\Omega$, where $u_\Omega \in \mathcal{S}$ and $P_\Omega u_\Omega = 0$. Since $\mathcal{S}$ is dense in L_2, $\sup_{\|u_\Omega\|_{L_2} \le 1} \|Gu_\Omega\|_{L_2}$ is equal to (16).

Using the input expansion in Theorem 1, and the assumption that the first Markov parameters are zero, we have

$$y(t) = (-1)^{k_2-1} \frac{\partial^{k_2-1} g}{\partial \tau^{k_2-1}}(t,t) \frac{u_\Omega(t)}{p^{k_2}} + (-1)^{k_2} \int_{-\infty}^{t} \frac{\partial^{k_2} g}{\partial \tau^{k_2}}(t,\tau) \frac{u_\Omega(\tau)}{p^{k_2}} d\tau. \tag{17}$$

From (17) it follows that there is a constant C_2 such that

$$\|y\|_{L_2} \leq C_2 \left\| \frac{u_\Omega}{p^{k_2}} \right\|_{L_2} \leq C_2 \frac{\|u_\Omega\|_{L_2}}{\Omega^{k_2}}.$$

The input roll-off constant C_2 is easily obtained since the impulse response is in $\mathcal{C}_e^L$. Put $K_1 = \sup_t \left| \frac{\partial^{k_2-1} g}{\partial \tau^{k_2-1}}(t,t) \right|$, $K_2 = \kappa_1/\kappa_2$ where $|\frac{\partial^{k_2} g}{\partial \tau^{k_2}}(t,\tau)| \leq \kappa_1 e^{-\kappa_2(t-\tau)}$, and $C_2 = K_1 + K_2$. This finishes the 'if'-part of the proof.

The 'only if'-part of (ii) follows by applying a special input signal to GQ_Ω with a limited width in both the time and the frequency domain. This gives a lower bound on $\|GQ_\Omega\|_{L_2 \to L_2}$. We will for simplicity show the result for $k_2 = 1$. Similar calculations hold for $k_2 > 1$. We sketch the proof next.

Use a Gaussian wave package $u_{k,\Omega}(t)$ (belongs to $\mathcal{S}$) of frequency Ω as input signal,

$$u_{k,\Omega}(t) = \frac{1}{\sqrt{2\pi\sigma}} e^{j\Omega t} e^{-(t-m_k)^2/2\sigma_k^2},$$
$$\hat{u}_{k,\Omega}(j\omega) = e^{-(\omega-\Omega)^2\sigma_k^2/2 - jm_k(\omega-\Omega)}.$$

and fix sequences of m_k and σ_k (the location and width of the package in time) such that

$$\int_{-\infty}^{\infty} |g(t,t)u_{k,\Omega}(t)|^2 dt \to \sup_t |g(t,t)|^2 \int_{-\infty}^{\infty} |u_{k,\Omega}(t)|^2 dt, \quad k \to \infty. \tag{18}$$

Next we use that $u_{k,\Omega}$ has its energy concentrated around the frequency Ω. For fixed k, we have that

$$\left\| \frac{u_{k,\Omega}}{p^l} \right\|_{L_2} \to \frac{\|u_{k,\Omega}\|_{L_2}}{\Omega^l}, \quad \Omega \to \infty. \tag{19}$$

A second-order Taylor expansion of G exists by the assumption $L > k_2$,

$$y(t) = \underbrace{g(t,t)\frac{u_{k,\Omega}(t)}{p}}_{y_1(t)} + \underbrace{\frac{\partial g}{\partial \tau}(t,t)\frac{u_{k,\Omega}(t)}{p^2} + \int_{-\infty}^{t} \frac{\partial^2 g}{\partial \tau^2}(t,\tau)\frac{u_{k,\Omega}(\tau)}{p^2} d\tau}_{y_2(t)}.$$

From the proof of the 'if'-part, it is seen that $\|y - y_1\|_{L_2} \leq C/\Omega^2$ for some positive constant C. From (18) and (19) there is an arbitrarily small ε_k such that

$$\|y\|_{L_2} = \frac{\sup_t |g(t,t)| + \varepsilon_k}{\Omega} \|u_{k,\Omega}\|_{L_2} + O(\Omega^{-2}), \quad \Omega \to \infty. \tag{20}$$

Table 1. The first input and output Markov parameters of a time-varying system on state-space form (4).

#	Input Markov parameter	Output Markov parameter
0.	$D(t)$	$D(t)$
1.	$C(t)B(t)$	$C(t)B(t)$
2.	$C(t)[A(t)B(t) - B'(t)]$	$[C'(t) + C(t)A(t)]B(t)$
3.	$C(t)[B''(t) - 2A(t)B'(t)$	$[C''(t) + 2C'(t)A(t)$
	$-A'(t)B(t) + A^2(t)B(t)]$	$+C(t)A'(t) + C(t)A^2(t)]B(t)$

Furthermore, if we fix an $a > 1$, we have that

$$GQ_\Omega u_{k,a\Omega} \to Gu_{k,a\Omega}, \quad \Omega \to \infty.$$

This together with (20) gives a lower bound on $\|GQ_\Omega\|_{L_2 \to L_2}$ which decays as $\sup_t |g(t,t)|/a\Omega$. The 'only if'-part of case (ii) follows.

To prove case (i) of the theorem a similar chain of arguments can be used. ■

EXAMPLE 2—NON-INTEGER-VALUED ROLL-OFF

Under the assumptions of Theorem 2, input and output roll-off always become integer values. This is not necessary for all types of periodic systems. Consider, for example, sampled-data systems. From the analysis in [Araki *et al.*, 1996], it follows that the sampled-data systems considered there have at least input and output roll-off 1/2. In fact, we conjecture that a sampled-data system G_2SG_1, where S is a periodic sampler and G_1, G_2 are linear time-invariant filters, has exactly input roll-off $(k_1 - 1/2)$ and output roll-off $(k_2 - 1/2)$ where k_1 is the roll-off of G_1 and k_2 the roll-off of G_2. This is in agreement with [Araki *et al.*, 1996], since roll-off 1 is assumed for the filters there. □

EXAMPLE 3—FINITE-DIMENSIONAL STATE-SPACE MODELS

Let us assume that the system has a state-space realization (4) and that the impulse response belongs to $\mathcal{C}_e^L$, see Example 1. According to Theorem 2 the roll-off of the state-space system can be determined by checking which of the Markov parameters that are zero. The first few Markov parameters of (4) are given in Table 1. In particular they coincide with the normal Markov parameters for time-invariant systems: D, CB, CAB, $CA^2B, \ldots$

Notice that the Markov parameter conditions for output roll-off k_1 correspond to the conditions for relative degree k_1 of a periodic system, defined in [De Nicolao *et al.*, 1998]. Also notice that the kth Makov parameters may be written as $C[(L_A^*)^{k-1}B]$ and $[L_A^{k-1}C]B$, where $L_A^*B = AB - B'$ and $L_AC = CA + C'$, using notation from [De Nicolao *et al.*, 1998]. $\square$

3. Fourier Expansions of Time-Periodic Systems

Until now we have not used the periodicity condition

$$g(t,\tau) = g(t+T, \tau+T).$$

The periodicity condition can be used for Fourier expansions. The possibility of Fourier expansions of periodic systems was discussed already in [Zadeh, 1950]. Here, we will formulate the problem in a Hilbert space and discuss the convergence issues.

For this analysis we define the space H_2 for periodic systems

$$H_2 = \{G : \|G\|_{H_2} < \infty \ , \ g(t,\tau) \text{ is } T\text{-periodic and causal}\}$$

where

$$\|G\|_{H_2}^2 = \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty}|g(\tau+r,\tau)|^2 dr d\tau = \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty}|g(t,t-r)|^2 dr dt.$$

The above equality follows from the periodicity condition (3). The H_2-norm is also used in, for example, [Zhang and Zhang, 1997; Colaneri, 2000; Zhou and Hagiwara, 2002a]. H_2 is a Hilbert space with the scalar product

$$\begin{aligned}\langle G, H\rangle_{H_2} &= \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty}\overline{g(\tau+r,\tau)}h(\tau+r,\tau) dr d\tau \\ &= \frac{1}{T}\int_{t=0}^{T}\int_{r=0}^{\infty}\overline{g(t,t-r)}h(t,t-r) dr dt.\end{aligned} \tag{21}$$

H_2 is composed of the well-known separable Hilbert spaces $L_2[0,T]$ and $L_2[0,\infty)$. Orthonormal basis functions in H_2 are, for example, $\tilde{E}_{k,n}$ or $E_{k,n}$ with impulse responses

$$\tilde{e}_{k,n}(t,\tau) = e^{jk\omega_0\tau}e^{-(t-\tau)/2}L_n(t-\tau) \quad \text{or} \quad e_{k,n}(t,\tau) = e^{jk\omega_0 t}e^{-(t-\tau)/2}L_n(t-\tau),$$

where $L_n(\cdot)$ are Laguerre polynomials and $\omega_0 = 2\pi/T$. Which basis to choose depends on if the first or second form of the scalar product (21) is used. By standard results from functional analysis and on Hilbert spaces, see, for example [Kreyszig, 1978], all system in H_2 can be represented by a generalized Fourier series

$$G = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \langle \tilde{E}_{k,n}, G \rangle_{H_2} \tilde{E}_{k,n} = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} \langle E_{k,n}, G \rangle_{H_2} E_{k,n}, \tag{22}$$

where the series converge in H_2-norm.

Instead of using the expansion (22), it will be more useful for us to only use the expansion in the k-direction. This is expressed in the following theorem.

THEOREM 3—FOURIER EXPANSION OF PERIODIC IMPULSE RESPONSES IN H_2
Assume that G belongs to H_2. The impulse response of G can then be expressed as

$$g(t,\tau) = \sum_{k=-\infty}^{\infty} \tilde{g}_k(t-\tau) e^{jk\omega_0 \tau}, \tag{23}$$

$$g(t,\tau) = \sum_{k=-\infty}^{\infty} g_k(t-\tau) e^{jk\omega_0 t}, \tag{24}$$

with convergence in H_2 and where $\omega_0 = 2\pi/T$. The Fourier coefficients are given by

$$\tilde{g}_k(r) = \frac{1}{T} \int_0^T e^{-jk\omega_0 \tau} g(\tau + r, \tau) d\tau, \tag{25}$$

$$g_k(r) = \frac{1}{T} \int_0^T e^{-jk\omega_0 t} g(t, t-r) dt, \tag{26}$$

where $r \geq 0$. □

Proof. We show (23) and (25). A similar calculation gives (24) and (26). The sum in the n-direction in (22) can be written explicitly as, using that $r = t - \tau$,

$$\frac{1}{T} \int_{\eta=0}^{T} e^{-jk\omega_0 \eta} \left\{ \sum_{n=0}^{\infty} \left(\int_{\zeta=0}^{\infty} e^{-\zeta/2} L_n(\zeta) g(\eta+\zeta, \eta) d\zeta \right) e^{-r/2} L_n(r) \right\} d\eta \, e^{jk\omega_0 \tau}. \tag{27}$$

The factor in brackets {} is a Fourier expansion of $g(\cdot + \eta, \eta)$. Since G belongs to H_2, $g(\cdot + \eta, \eta) \in L_2[0, \infty)$ for almost all η. Hence, the series converges in L_2-sense to $g(\eta + r, \eta)$ for almost all η. The result follows since (27) has the structure of the terms in (23). ■

Some immediate properties of the Fourier coefficients in (23)–(24) are stated in the following corollary.

COROLLARY 1—PROPERTIES OF FOURIER COEFFICIENTS
Assume that G and H belong to H_2. Then it holds that

(i) the Fourier coefficients $\tilde{g}_k$ and g_k satisfy $\tilde{g}_k(r) = e^{jk\omega_0 r} g_k(r)$, $r \geq 0$;

(ii) if $g(t,\tau)$ is real, then $g_{-k}(r) = \overline{g_k(r)}$ for all k, $r \geq 0$;

(iii) the Fourier coefficients $\tilde{g}_k(\cdot)$ and $g_k(\cdot)$ belong to $L_2[0,\infty)$ for all k;

(iv) the scalar product (21) can be expressed as

$$\langle G, H\rangle_{H_2} = \sum_{k=-\infty}^{\infty} \langle \tilde{g}_k, \tilde{h}_k\rangle_{L_2} = \sum_{k=-\infty}^{\infty} \langle g_k, h_k\rangle_{L_2}.$$

□

By Corollary 1 it follows that there is no major difference between the expansions (23) and (24). They are essentially the same. In the following sections we will mostly work with the expansions in g_k. This choice is arbitrary.

It is also useful to introduce an orthogonal projection $P_{[N]} : H_2 \to H_2$,

$$G_{[N]} = P_{[N]} G : \quad g_{[N]}(t,\tau) = \sum_{k=-N}^{N} \tilde{g}_k(t-\tau) e^{jk\omega_0 \tau} = \sum_{k=-N}^{N} g_k(t-\tau) e^{jk\omega_0 t}. \tag{28}$$

We make the following definition.

DEFINITION 5—SKEW TRUNCATION
$G_{[N]}$, given by (28), is called an Nth-order *skew truncation* of G □

The reason for the term "skew truncation" will be more clear once we have constructed the harmonic transfer function, see Section 4.1. In particular it will be seen that this corresponds to the skew truncations in [Zhou and Hagiwara, 2002b]. A simple application of Corollary 1 (iv) gives that the skew truncations converge in H_2,

$$\|G - G_{[N]}\|_{H_2} \to 0, \text{ as } N \to \infty.$$

REMARK 5—OPTIMAL APPROXIMATIONS IN H_2
By Corollary 1 (iv), systems that do not contain the same Fourier coefficients are orthogonal in H_2, and

$$G_{[N]} \perp (G - G_{[N]}).$$

Hence, by standard results from approximation in Hilbert spaces, $G_{[N]}$ are optimal approximations. In particular, $G_{[0]}$ is the optimal approximation of a *time-invariant structure* in H_2.

Similar results on optimal approximations in H_2 follow by the analysis of achievable H_2 performance of periodic control systems in [Zhang and Zhang, 1997]. □

Next, we will study some input-output properties of the skew truncations.

3.1 Input-Output Properties

The interpretation of the convergence in the above Fourier expansions is that for impulse inputs, the outputs converge in "mean energy sense". This is a quite weak form of convergence. By strengthening the assumptions on $g(t, \tau)$, we can show stronger forms of convergence, in induced-norm sense. This is the topic of the rest of this section.

The input-output map of $y_N = G_{[N]}u$ is given by

$$\begin{aligned} y_N(t) &= \sum_{k=-N}^{N} \int_{-\infty}^{t} \tilde{g}_k(t-\tau) e^{jk\omega_0 \tau} u(\tau) d\tau \\ &= \sum_{k=-N}^{N} \int_{-\infty}^{t} g_k(t-\tau) e^{jk\omega_0 t} u(\tau) d\tau, \end{aligned} \tag{29}$$

where we have interchanged the order of integration and summation. The output $y_N(t)$ is given by a parallel connection of input or output modulated time-invariant systems.

We associate with the kth Fourier coefficients of G, causal and time-invariant system $\tilde{G}_k$ and G_k

$$\begin{aligned} \tilde{G}_k : &\quad \tilde{g}_k(t, \tau) = \tilde{g}_k(t-\tau), \quad t \geq \tau, \\ G_k : &\quad g_k(t, \tau) = g_k(t-\tau), \quad t \geq \tau. \end{aligned}$$

Hence, we can represent $G_{[N]}$ by the formal Fourier series

$$G_{[N]} = \sum_{k=-N}^{N} \tilde{G}_k e^{jk\omega_0 \tau} = \sum_{k=-N}^{N} e^{jk\omega_0 t} G_k, \tag{30}$$

where the Fourier coefficients are *time-invariant* systems. We will show that these series converge in induced norms.

Let us again use the class of continuously differentiable and exponentially bounded impulse responses, $\mathcal{C}_e^L$, defined in Section 2.

LEMMA 1—BOUNDED FOURIER COEFFICIENTS
Assume that a periodic system G has an impulse response $g(t, \tau)$ that belongs to $\mathcal{C}_e^L$. Then,

(i) G belongs to H_2;

(ii) there are positive constants α and C such that the Fourier coefficients are bounded by

$$|\tilde{g}_k(r)| = |g_k(r)| \leq \frac{C \cdot e^{-\alpha r}}{|k\omega_0|^L}, \quad |k| > 0, r \geq 0.$$

□

Proof. (i) We shall prove that G belongs to H_2. Since G belongs to at least $\mathcal{C}_e^0$, by (E3) we have $|g(t,\tau)| \leq \kappa_1 e^{-\kappa_2(t-\tau)}$. Hence, we can bound the H_2-norm

$$\|G\|_{H_2}^2 \leq \frac{1}{T}\int_{\tau=0}^{T}\int_{r=0}^{\infty} \kappa_1^2 e^{-2\kappa_2 r} dr d\tau = \frac{\kappa_1^2}{2\kappa_2}.$$

(ii) By assumption (E1) we have that

$$g(\cdot + r, \cdot) \in \mathcal{C}^L(\boldsymbol{R}), \quad \text{for all} \quad r \geq 0.$$

Make a Fourier expansion in the τ-direction of $\frac{d^L}{d\tau^L} g(\tau + r, \tau)$, and notice that

$$(jk\omega_0)^L \tilde{g}_k(r) = \frac{1}{T}\int_{\tau=0}^{T}\left[\frac{d^L}{d\tau^L} g(\tau + r, \tau)\right] e^{-jk\omega_0 \tau} d\tau.$$

Hence, we have the bound

$$|k\omega_0|^L |\tilde{g}_k(r)| \leq \max_{\tau}\left|\frac{d^L}{d\tau^L} g(\tau + r, \tau)\right| \leq C \cdot e^{-\alpha r}, \tag{31}$$

for some positive constants C and α. Such constants exist by assumption (E3). The result follows. ■

Remark 6—The constants C and α in Lemma 1

By the proof of Lemma 1, it follows that C and α should fulfill (31). For time-invariant systems we have $g(\tau+r, \tau) = g(r, 0)$. Hence, all derivatives are zero in the τ-direction and we can choose $C = 0$. This should be no surprise since $g_0(r)$ and $\tilde{g}_0(r)$ are the only non-zero Fourier coefficients for time-invariant systems. □

Using Lemma 1 we can show that following theorem.

THEOREM 4—CONVERGENCE OF SKEW TRUNCATIONS IN INDUCED L_∞-NORM
Assume that a periodic system G has an impulse response $g(t,\tau)$ that belongs to $\mathcal{C}_e^L$, where $L > 1$. Then for all inputs $u(\cdot) \in L_\infty(\boldsymbol{R})$, $y_N(t)$ in (29) converges, uniformly in t, to $y(t)$ in (2), as $N \to \infty$.

Furthermore, the rate of convergence of the Fourier series in (30) is at least $(L-1)$,

$$\|G - G_{[N]}\|_{L_\infty \to L_\infty} \leq K \cdot N^{-(L-1)}, \quad N > 0,$$

for some positive constant K. □

Proof. Using Lemma 1 we have that

$$\begin{aligned}
|y(t) - y_N(t)| &\leq 2 \left| \int_{-\infty}^{t} \sum_{k=N+1}^{\infty} g_k(t-\tau) e^{jk\omega_0 t} u(\tau) d\tau \right| \\
&\leq 2 \int_{-\infty}^{t} \sum_{k=N+1}^{\infty} \frac{C \cdot e^{-\alpha(t-\tau)}}{(k\omega_0)^L} |u(\tau)| d\tau \\
&\leq 2 \frac{C\gamma}{\alpha \omega_0^L N^{L-1}} \|u(\cdot)\|_{L_\infty(R)},
\end{aligned} \tag{32}$$

where γ is a constant such that

$$\sum_{k=N+1}^{\infty} \frac{1}{k^L} \leq \frac{\gamma}{N^{L-1}}, \quad L > 1. \tag{33}$$

Since (32) is independent of t and tends to zero as $N \to \infty$, the convergence is uniform. Furthermore, we can choose $K = 2C\gamma/\alpha\omega_0^L$. ■

We can apply Theorem 4 to input signals that are harmonics and thereby see the connection between the above Fourier expansions and the classical analysis by Zadeh.

EXAMPLE 4—HARMONIC RESPONSE AND THE PTF
The response of a periodic system to an harmonic $u(t) = e^{j\omega t}$ is now easy to obtain. Under the assumptions of Theorem 4 and by the definition of the Fourier transform we obtain

$$\begin{aligned}
y(t) = G e^{j\omega t} &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{t} g_k(t-\tau) e^{jk\omega_0 t} e^{j\omega\tau} d\tau \\
&= \left(\sum_{k=-\infty}^{\infty} \hat{g}_k(j\omega) e^{jk\omega_0 t} \right) e^{j\omega t},
\end{aligned} \tag{34}$$

where $\hat{g}_k$ is the Fourier transform of g_k. Since $g_k(\cdot) \in L_1[0,\infty) \cap L_2[0,\infty)$, we have that $\hat{g}_k(\cdot)$ is uniformly continuous and belongs to $L_2(j\boldsymbol{R})$, see [Dym and McKean, 1972]. Equation (34) shows that the response includes a countable number of frequencies $[\ldots, \omega - \omega_0,\ \omega,\ \omega + \omega_0, \ldots]$, separated by multiples of ω_0.

The *parametric transfer function* (PTF) $\hat{G}(j\omega, t)$ that is used in [Zadeh, 1950; Lampe and Rosenwasser, 2003] may be defined as the steady-state response of the periodic system G to an harmonic,

$$\hat{G}(j\omega, t) := \frac{Ge^{j\omega t}}{e^{j\omega t}}.$$

By the analysis above we realize that

$$\hat{G}(j\omega, t) = \sum_{k=-\infty}^{\infty} \hat{g}_k(j\omega) e^{jk\omega_0 t}.$$

The PTF can also be compute directly from the impulse response $g(t,\tau)$, see [Zadeh, 1950; Lampe and Rosenwasser, 2003].

The HTF that we define in Section 4, is also a frequency-domain representation of G. The difference is that the PTF is scalar but depends on time and frequency, whereas the HTF only depends on frequency. The price is that the HTF becomes infinite dimensional. The relation between the PTF and the HTF has also been discussed in [Lampe and Rosenwasser, 2001]. □

3.2 Skew Roll-Off and Convergence in Induced L_r-norm

In Section 2 we studied the convergence of square truncated systems in induced L_2-norm. Since we will work in the frequency domain in Section 4, where L_2 is important, we also study convergence of skew truncations in induced L_2-norm here.

In analogy with input roll-off and output roll-off we define skew roll-off.

DEFINITION 6—SKEW ROLL-OFF
If there are positive constants C and k such that

$$\|G - G_{[N]}\|_{L_2 \to L_2} \leq C \cdot N^{-k}, \quad N > 0,$$

then G has *skew roll-off k*. □

REMARK 7—SKEW ROLL-OFF OF TIME-INVARIANT SYSTEMS
For a time-invariant system G, it holds that $G_k = 0$ for all $k \neq 0$ and $G = G_0$. Hence, a time-invariant system has *infinite skew roll-off*. □

The following sufficient condition helps to determine skew roll-off.

THEOREM 5—CONVERGENCE OF SKEW TRUNCATIONS IN INDUCED L_r-NORM
Assume that the impulse response of a periodic system G belongs to $\mathcal{C}_e^L$, and $L > 1$. Then, there is a positive constant K such that for $1 \le r < \infty$,

$$\|G - G_{[N]}\|_{L_r \to L_r} \le K \cdot N^{-(L-1)}, \quad N > 0,$$

and G has at least skew roll-off $L - 1$. □

Proof. If the domain of G is the Schwartz functions $\mathcal{S}$, we can by Theorem 4 interchange the order of integration and summation and represent G by the Fourier series

$$G = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t} G_k,$$

where G_k are time-invariant systems. Furthermore, we have the bound

$$\|G_k\|_{L_r \to L_r} \le \|g_k(\cdot)\|_{L_1(R_+)} \le \frac{C}{\alpha |k\omega_0|^L},$$

by Lemma 1. Now, since $\mathcal{S}$ is dense in L_r, $1 \le r < \infty$, we can conclude that

$$\|G - G_{[N]}\|_{L_r \to L_r} \le 2 \sum_{k=N+1}^{\infty} \|G_k\|_{L_r \to L_r} \le 2 \frac{C\gamma}{\alpha \omega_0^L N^{L-1}}, \tag{35}$$

where γ is a given by (33). ■

4. The Harmonic Transfer Function

By including a sufficient amount of frequencies in the Fourier expansion of G, we can come arbitrarily close to G itself, as discussed in the previous section. Since we have decomposed the periodic system into time-invariant terms, the frequency-domain analysis is now straightforward.

Assume in the following that the assumptions of Theorem 5 hold. Notice that $y_N(t)$ from (29) may be written as

$$y_N(t) = \sum_{k=-N}^{N} g_k(t) e^{jk\omega_0 t} * u(t) e^{jk\omega_0 t} \tag{36}$$

where $*$ is the standard convolution product. Now pick an input u in L_2. We can apply the Fourier transform on both sides of (36), and get

$$\hat{y}_N(j\omega) = \sum_{k=-N}^{N} \hat{g}_k(j\omega - jk\omega_0) \hat{u}(j\omega - jk\omega_0). \tag{37}$$

All the Fourier transforms are well defined, since by the assumptions $g_k(\cdot) \in L_1[0,\infty) \cap L_2[0,\infty)$ and $u(\cdot) \in L_2(\boldsymbol{R})$. By Theorem 5, y_N converges to y in L_2. Furthermore, $L_2(\boldsymbol{R})$ and $L_2(j\boldsymbol{R})$ are isomorphic under the Fourier transform. Hence, for all inputs $u \in L_2$, $\hat{y}_N(j\omega)$ converges to $\hat{y}(j\omega)$ in $L_2(j\boldsymbol{R})$ as $N \to \infty$. Therefore we can put $N = \infty$ in (37) if we mean convergence in L_2-sense, and not point-wise convergence.

Next we rewrite the summation (37) by using a form of lifting on $L_2(j\boldsymbol{R})$. In [Araki *et al.*, 1996] this lifting was called the Sample-Data Fourier transform (SD-transform). The SD-transform is an isometric isomorphism between $L_2(j\boldsymbol{R})$ and a Hilbert space we denote by $L_2^Z(jI_0)$. It maps the Fourier transform into an infinite-dimensional column-vector-valued function. The SD-transform of $\hat{u}(j\omega)$ is denoted by $\hat{U}(j\omega)$ and is defined as

$$\hat{U}(j\omega) = [\dots \ \hat{u}(j\omega + j\omega_0) \quad \hat{u}(j\omega) \quad \hat{u}(j\omega - j\omega_0) \ \dots]^T .$$

Since the vector contains repeated versions of $\hat{u}(j\omega)$, it is enough to define $\hat{U}(j\omega)$ for $\omega \in I_0 = (-\omega_0/2, \omega_0/2]$ to be able to take the inverse SD-transform. We define the norm in $L_2^Z(jI_0)$ as

$$\begin{aligned}\|\hat{U}(\cdot)\|_{L_2^Z(jI_0)} &= \frac{1}{\sqrt{2\pi}}\left(\int_{I_0} \|\hat{U}(j\omega)\|_{\ell_2}^2 d\omega\right)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}}\left(\int_{I_0} \sum_{k=-\infty}^{\infty} |\hat{u}(j\omega + jk\omega_0)|^2 d\omega\right)^{1/2}.\end{aligned}$$

For signals $u \in L_2$, there are now three representations: $u(t)$, $\hat{u}(j\omega)$, and $\hat{U}(j\omega)$, and the following extended Plancherel's theorem holds,

$$\|u\|_{L_2} = \|u(\cdot)\|_{L_2(R)} = \|\hat{u}(\cdot)\|_{L_2(jR)} = \|\hat{U}(\cdot)\|_{L_2^Z(jI_0)}. \tag{38}$$

If u has finite L_2-norm, then $\hat{U}(j\omega)$ is in ℓ_2 (its elements are square summable) for almost all $\omega \in I_0$, that is $\|\hat{U}(j\omega)\|_{\ell_2} < \infty$ a.e.

If we write (37) when $N = \infty$ in matrix-vector form

$$\begin{bmatrix} \vdots \\ \hat{y}(j\omega + j\omega_0) \\ \hat{y}(j\omega) \\ \hat{y}(j\omega - j\omega_0) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(j\omega + j\omega_0) & \hat{g}_1(j\omega) & \hat{g}_2(j\omega - j\omega_0) & \\ \ddots & \hat{g}_{-1}(j\omega + j\omega_0) & \hat{g}_0(j\omega) & \hat{g}_1(j\omega - j\omega_0) & \ddots \\ & \hat{g}_{-2}(j\omega + j\omega_0) & \hat{g}_{-1}(j\omega) & \hat{g}_0(j\omega - j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}(j\omega + j\omega_0) \\ \hat{u}(j\omega) \\ \hat{u}(j\omega - j\omega_0) \\ \vdots \end{bmatrix}$$

it can be written in a compact form using the SD-transform

$$\hat{Y}(j\omega) = \hat{G}(j\omega)\hat{U}(j\omega), \quad \omega \in I_0 = (-\omega_0/2, \omega_0/2]. \tag{39}$$

We call $\hat{G}(j\omega)$ the *harmonic transfer function* (HTF) of G. This was the term used by Wereley in [Wereley, 1991]. A similar object was called the FR operator in [Araki *et al.*, 1996] in the case of sampled-data systems. The difference between these efforts is the way the elements of $\hat{G}(j\omega)$ are computed. In the sampled-data case explicit formulas are given in [Araki *et al.*, 1996]. In the time-periodic state-space case formulas are given in [Wereley, 1991; Zhou and Hagiwara, 2002a], and in the impulse response case formulas are given here and in [Möllerstedt and Bernhardsson, 2000b]. The relation between the impulse-response and state-space approaches is further discussed in Section 6.

It was assumed in the above discussion that the impulse response belongs to $\mathcal{C}_e^L$. This was done to motivate the construction of the HTF from an input-output view. However, the HTF $\hat{G}(j\omega)$ is a meaningful construction as soon as its elements, the Fourier transforms of the Fourier coefficients of G, are well defined. This is the case when G is in H_2.

The following proposition may be derived by using techniques similar to those in [Wereley, 1991; Araki *et al.*, 1996; Zhou and Hagiwara, 2002a].

PROPOSITION 2—NORM FORMULAS
Assume that the periodic system G belongs to H_2. Then the HTF $\hat{G}(j\omega)$, defined as in (39), is a Hilbert-Schmidt operator for almost all ω, and

$$\|G\|_{H_2}^2 = \frac{1}{2\pi} \int_{I_0} \operatorname{trace}[\hat{G}^*(j\omega)\hat{G}(j\omega)]d\omega. \tag{40}$$

If, furthermore, the impulse response $g(t,\tau)$ belongs to C_e^L and $L > 1$, then $\hat{G}(j\omega)$ is a bounded operator on ℓ_2 for all ω, and

$$\|G\|_{L_2 \to L_2} = \sup_{\omega \in I_0} \|\hat{G}(j\omega)\|_{\ell_2 \to \ell_2}. \tag{41}$$

□

REMARK 8—HILBERT-SCHMIDT OPERATORS
The HTF evaluated at the frequency ω, $\hat{G}(j\omega)$, is a Hilbert-Schmidt operator if

$$\operatorname{trace}[\hat{G}^*(j\omega)\hat{G}(j\omega)] < \infty.$$

In particular, the Hilbert-Schmidt operators are compact operators, see, for example, [Böttcher and Silbermann, 1990]. □

Hence, just as for the standard transfer function, the induced L_2-norm of the system is given by the supremum of the transfer function.

4.1 Square and Skew Truncations Revisited

By looking at the structure of the HTFs of square and skew truncated systems we make important connections to the work in [Zhou and Hagiwara, 2002b]. The reasons for the terms "square" and "skew" will also be obvious.

To compute the induced L_2-norm (1) of a system G with input and output roll-off, the following observation, which follows directly from (15), is useful

$$0 \leq \|G\|_{L_2\to L_2} - \|P_{\Omega_1} G P_{\Omega_2}\|_{L_2\to L_2} \leq C_1 \cdot \Omega_1^{-k_1} + C_2 \cdot \Omega_2^{-k_2}. \qquad (42)$$

Proposition 2 gives us a way to compute the induced L_2-norm, given a HTF $\hat{G}(j\omega)$. It is not essential that $\hat{G}(j\omega)$ corresponds to a causal operator for (41) to hold, it is true for every frequency domain relation (39). Hence we can apply it to the approximation $P_{\Omega_1} G P_{\Omega_2}$. The HTF of $P_{\Omega_1} G P_{\Omega_2}$ becomes a finite-dimensional matrix.

Proposition 3—Square truncated HTFs
Assume that G has an HTF $\hat{G}(j\omega)$. If we choose $\Omega_1 = (N_1 + 1/2)\omega_0$ and $\Omega_2 = (N_2 + 1/2)\omega_0$ the HTF of $P_{\Omega_1} G P_{\Omega_2}$ is given by the $(2N_1+1)\times(2N_2+1)$-matrix

$$\hat{G}_{(N_1,N_2)}(j\omega) = \begin{bmatrix} \hat{g}_{N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_1+N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{N_2}(j\omega - jN_2\omega_0) \\ \vdots & & \vdots \\ \hat{g}_{-N_1-N_2}(j\omega + jN_2\omega_0) & \dots & \hat{g}_{-N_1+N_2}(j\omega - jN_2\omega_0) \end{bmatrix},$$

and

$$\hat{P}_{\Omega_1}(j\omega)\hat{G}(j\omega)\hat{P}_{\Omega_2}(j\omega) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{G}_{(N_1,N_2)}(j\omega) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The truncated HTF converges to $\hat{G}(j\omega)$ in the induced L_2-norm according to (15) and (41). □

Hence we can represent a linear periodic system in $\mathcal{C}_e^L$ arbitrarily well with finite-dimensional matrices and compute its norm as

$$\|P_{\Omega_1} G P_{\Omega_2}\|_{L_2\to L_2} = \max_{\omega\in I_0} \overline{\sigma}\left(\hat{G}_{(N_1,N_2)}(j\omega)\right). \qquad (43)$$

By griding the frequency interval I_0 and by computing the maximum singular value we get an estimate of $\|G\|_{L_2 \to L_2}$, and the rate of convergence depends upon the roll-off of G according to (42).

If the system has a high output roll-off, then $\hat{G}(j\omega)$ decays quickly in the up-down direction, and if it has high input roll-off it decays quickly in the left-right direction. In Theorem 2 we have given the necessary and sufficient conditions for the decay rates: the more input and output Markov parameters that are zero, the quicker does the HTF decay.

To use the square truncation (43) to estimate the norm of a periodic system has been suggested by many authors. In [Araki *et al.*, 1996] it is shown that the square truncation converges at least at a rate $K \cdot N^{-1/2}$, for $(2N+1) \times (2N+1)$ truncations and some constant K. In [Araki *et al.*, 1996] sampled-data systems are studied, but similar techniques are used in [Zhou and Hagiwara, 2002a; Zhou and Hagiwara, 2002b]. As seen above in (42), the bound on convergence rate may be improved by checking the Markov parameters. In [Zhou and Hagiwara, 2002b] a bisection algorithm and a Hamiltonian matrix is used for the computation of (43).

Let us now look at the skew truncated HTFs.

PROPOSITION 4—SKEW TRUNCATED HTFS
The HTF of the skew truncation $G_{[N]}$, $\hat{G}_{[N]}(j\omega)$, consists of the $2N+1$ main diagonals of $\hat{G}(j\omega)$. □

Hence, as N increases, more diagonals are added to the skew truncated HTF. Furthermore, we now know that each diagonal represents a Fourier coefficient of G. From the Theorems 4 and 5 we know how quickly this HTF converges in induced norms. That is, we can quantify how much accuracy there is, at least, to gain by including an extra diagonal in the HTF. From Lemma 1 it is seen that the smoother the impulse response $g(\tau + r, \tau)$ is in τ, for each fixed r, the quicker does the norm bound for each added diagonal decay.

Skew truncations are used to compute the H_2-norm of periodic systems in [Zhou and Hagiwara, 2002b]. However, it is not the HTF of $G_{[N]}$ that is used there. Instead the state-space matrices $B(t)$ and $C(t)$ are skew truncated. That means that the H_2-norm is computed for the system $\bar{G}_{[N]}$, with impulse response

$$
\begin{aligned}
\bar{g}_{[N]}(t,\tau) &= C_{[N]}(t) e^{Q(t-\tau)} B_{[N]}(\tau), \\
B_{[N]}(t) &= \sum_{k=-N}^{N} B_k e^{jk\omega_0 t}, \quad C_{[N]}(t) = \sum_{k=-N}^{N} C_k e^{jk\omega_0 t},
\end{aligned}
$$

where Q is the (Floquet-transformed) state matrix $A(t)$, see the discussion in Section 1.2. The HTF of $\bar{G}_{[N]}$ has $4N+1$ diagonals, but notice that in general

$$\bar{G}_{[N]} \neq G_{[2N]}.$$

Hence, $\bar{G}_{[N]}$ are in general not H_2-optimal approximations of G, see Remark 5.

There are some other simple relations between input, output, and skew roll-off that are now easy to obtain. The following corollary states two results.

COROLLARY 2—RELATIONS BETWEEN DIFFERENT ROLL-OFF CONCEPTS
Assume that G has output roll-off k_1 and input roll-off k_2. Then

(i) the transfer functions $\hat{g}_k(j\omega)$, have at least roll-off $\max\{k_1, k_2\}$ for all k (in the classical sense);

(ii) G has at least skew roll-off $\min\{k_1, k_2\} - 1$.

□

Proof. (i). Follows by a contradiction proof. Assume that there exists a k so that $g_k(j\omega)$ has roll-off smaller than $L = \max\{k_1, k_2\}$. Then by using the definition of input/output roll-off and the induced L_2-norm formula in Proposition 2 we can derive a contradiction.

(ii). Study the HTF of $G - P_\Omega G P_\Omega$ with $\Omega = (N + 1/2)\omega_0$. Then use the norm formula in Proposition 2, and we see from the convergence of square-truncated HTFs that

$$\|G_{2N+1}\|_{L_2 \to L_2} = \sup_{\omega} |\hat{g}_{2N+1}(j\omega)| \leq C_1 \cdot \Omega^{-k_1} + C_2 \cdot \Omega^{-k_2}.$$

Hence, there is a constant K such that $\|G_k\|_{L_2 \to L_2} \leq K \cdot |k|^{-\min\{k_1, k_2\}}$. After using the triangular inequality as in (35) the result follows. ■

Notice that the roll-off estimates given in Corollary 2 may be very conservative. Consider, for example, time-invariant systems with input and output roll-off k. Corollary 2 (ii) then says that the skew roll-off is at least $k - 1$, when it in fact is infinite. In general Lemma 1 and Theorem 5 are better to use, but Corollary 2 shows that there are connections between the different roll-off concepts.

5. Closed-Loop Systems and Inverses

In this section, we study closed-loop systems with time-periodic G, see Figure 1. The closed-loop operator is $(I + G)^{-1}$. For the closed-loop system

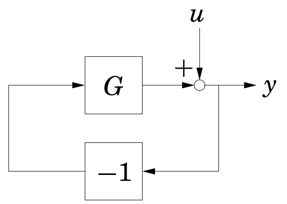

Figure 1. A closed-loop system. The relation between input and output is $y = (I+G)^{-1}u$.

to be stable in the normal sense, we need to show that $\|(I+G)^{-1}\|_{L_2\to L_2}$ is bounded and that $(I+G)^{-1}$ is causal.

To approximate G we have used the square truncation $P_{\Omega_1}GP_{\Omega_2}$ and the skew truncation $G_{[N]}$. We now investigate the properties of $(I + P_{\Omega_1}GP_{\Omega_2})^{-1}$ and $(I+G_{[N]})^{-1}$.

We should first remember that even if G is causal, the approximation $P_{\Omega_1}GP_{\Omega_2}$ is non-causal, even if it gets "less non-causal" as it converges to G. For this reason it is difficult to prove causality of $(I+G)^{-1}$ by studying $(I+P_{\Omega_1}GP_{\Omega_2})^{-1}$. However, to compute $\|(I+G)^{-1}\|_{L_2\to L_2}$, we have the following result:

Theorem 6—Approximation of $(I+G)^{-1}$ using $P_{\Omega_1}GP_{\Omega_2}$
Assume that $(I+G)^{-1}$ and $(I+P_{\Omega_1}GP_{\Omega_2})^{-1}$ are bounded operators on L_2, and that G has output roll-off k_1 and input roll-off k_2. Then the relative L_2-induced norm error is bounded

$$\frac{\|(I+G)^{-1}-(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2}}{\|(I+G)^{-1}\|_{L_2\to L_2}} \le \|(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2}(C_1\Omega_1^{-k_1}+C_2\Omega_2^{-k_2}).$$

□

Proof. First we make an orthogonal decomposition of the Hilbert space L_2, so that $L_2 = P_\Omega L_2 \oplus Q_\Omega L_2$. In this basis G takes the operator-matrix form

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where $G_{11} : P_{\Omega_2}L_2 \to P_{\Omega_1}L_2$, $G_{12} : Q_{\Omega_2}L_2 \to P_{\Omega_1}L_2$, $G_{21} : P_{\Omega_2}L_2 \to Q_{\Omega_1}L_2$, and $G_{22} : Q_{\Omega_2}L_2 \to Q_{\Omega_1}L_2$. Next we notice that

$$\|(I+G)^{-1}-(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2} \le \|(I+G)^{-1}\|_{L_2\to L_2} \times \|I-(I+G)(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2},$$

and that second factor on the right hand side in matrix form becomes

$$I - \begin{bmatrix} I+G_{11} & G_{12} \\ G_{21} & I+G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix} = -\begin{bmatrix} 0 & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} (I+G_{11})^{-1} & 0 \\ 0 & I \end{bmatrix}.$$

The norm of the first factor is bounded by $C_1\Omega_1^{-k_1}+C_2\Omega_2^{-k_2}$, and the second factor is bounded by $\|(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2}$, and the result follows. ■

The approximation $G_{[N]}$ is a finite sum of modulated time-invariant causal systems, and hence $G_{[N]}$ is a causal approximation of G. If we decompose G as $G = G_{[N]} + \Delta$, we may study the causality and boundedness of $(I+G)^{-1}$ with the help of the small-gain theorem. First notice that

$$(I+G)^{-1} = (I+G_{[N]}+\Delta)^{-1} = (I+(I+G_{[N]})^{-1}\Delta)^{-1}(I+G_{[N]})^{-1}. \quad (44)$$

Now if N is large enough and $(I+G_{[N]})^{-1}$ is stable, we have

$$\|(I+G_{[N]})^{-1}\Delta\|_{L_r\to L_r} < 1, \quad 1 \le r \le \infty, \quad (45)$$

and we can make the following Neumann series expansion with absolute convergence in induced L_r-norm

$$(I+(I+G_{[N]})^{-1}\Delta)^{-1} = \sum_{k=0}^{\infty}(I+G_{[N]})^{-k}(-\Delta)^k. \quad (46)$$

Notice that under the assumption (45), and that $(I+G_{[N]})^{-1}$ is causal, (46) is a causal operator. This follows since the terms in the series are causal and tend geometrically to zero. Bounds on $\|\Delta\|_{L_r\to L_r}$ are given in Theorems 4 and 5.

If the approximation error Δ is small, and if $(I+G_{[N]})^{-1}$ is stable, then stability of $(I+G)^{-1}$ follows. We formalize this in the following theorem.

THEOREM 7—APPROXIMATION OF $(I+G)^{-1}$ USING $G_{[N]}$
Assume that $(I+G_{[N]})^{-1}$ is a bounded and causal operator on L_r, where $1 \le r \le \infty$, and that

$$\kappa = \|(I+G_{[N]})^{-1}\|_{L_r\to L_r}\|\Delta\|_{L_r\to L_r} < 1,$$

where $\Delta = G - G_{[N]}$. Then

$$(I+G)^{-1} = \left(\sum_{k=0}^{\infty}(I+G_{[N]})^{-k}(-\Delta)^k\right)(I+G_{[N]})^{-1} \quad (47)$$

is a bounded and causal operator on L_r. Furthermore, the error is bounded

$$\frac{\|(I+G_{[N]})^{-1}-(I+G)^{-1}\|_{L_r\to L_r}}{\|(I+G_{[N]})^{-1}\|_{L_r\to L_r}} \leq \frac{\kappa}{1-\kappa}. \tag{48}$$

□

Proof. (47) follows from (44) and (46). (48) follows from a simple bound on the geometric series. ■

Theorems 6 and 7 show that conclusions about the closed-loop system can be made from studies of the truncated representations of G. Theorem 6 may be used to estimate the induced L_2-norm of the closed-loop system. If the square truncated HTFs are used, the necessary computation becomes

$$\|(I+P_{\Omega_1}GP_{\Omega_2})^{-1}\|_{L_2\to L_2} = 1/\min_{\omega\in I_0} \underline{\sigma}\left((I+\hat{G}_{(N_1,N_2)}(j\omega))\right).$$

To use Theorem 7, one needs $(I+G_{[N]})^{-1}$, which may be difficult to obtain, even though $G_{[N]}$ has a simple structure.

6. Application to State-Space Models

Now we return to the state-space systems described in (4), and show how the results in the previous sections may be applied.

6.1 Floquet-Transformed State-Space Models

Assume that a Floquet transformation, see, for example [Rugh, 1996], has been performed on the state-space realization (4) of G, and that the Fourier series

$$B(t)=\sum_{l=-\infty}^{\infty} B_l e^{jl\omega_0 t}, \quad C(t)=\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t},$$

are absolutely convergent. The state matrix is constant, $A(t)=Q$. Then the Fourier series of the impulse response $g(t,t-r)$ is given by

$$g(t,t-r)=C(t)e^{Qr}B(t-r)=\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} C_{k-l}e^{Qr}B_l e^{-jl\omega_0 r}\right)e^{jk\omega_0 t},$$

after interchange of summation order. Hence, the Fourier coefficients of G, see Section 3, are given by

$$g_k(r) = \sum_{l=-\infty}^{\infty} C_{k-l} e^{Qr} B_l e^{-jl\omega_0 r}.$$

Using the definition (39) of $\hat{G}(j\omega)$, we see that $\hat{G}(j\omega)$ is identical to the HTF defined in [Wereley, 1991; Zhou and Hagiwara, 2002a]. By the analysis of square and skew truncations of the HTF in this paper, we now also better know its convergence properties.

6.2 Convergence of the Truncated Harmonic Balance Method

As discussed in Section 1.2, it is of interest to compute the HTF of a state-space model without first applying the Floquet transform. Truncated harmonic balance was suggested as a method for this in, for example, [Zhang and Zhang, 1997]. However, to the authors' knowledge, no analysis of how and when this method converges has been presented. We will do an attempt to analyze the method here. To guarantee convergence we must change the method slightly.

Define the multiplication operator A as follows:

$$y = Ax : \quad y(t) = A(t)x(t),$$

and B and C similarly. The input-output relation of the state-space model (4) is then given by

$$y = Gu = C(pI - A)^{-1}Bu = C\left(I - \frac{1}{p+\varepsilon}(A+\varepsilon)\right)^{-1} \frac{1}{p+\varepsilon}Bu, \quad (49)$$

for all $\varepsilon > 0$, where p is the differentiation operator. The reason for introducing ε is to make all operators bounded. Equation (49) is decomposed of three simple operators

$$(\mathcal{C}, \mathcal{A}, \mathcal{B}) = \left(C, \frac{1}{p+\varepsilon}(A+\varepsilon), \frac{1}{p+\varepsilon}B\right). \quad (50)$$

The operators $\mathcal{A}$ and $\mathcal{B}$ have impulse responses of the type

$$g_{\mathcal{A}}(t,\tau) = e^{-\varepsilon(t-\tau)}(A(\tau)+\varepsilon),$$

and the HTF $\hat{\mathcal{A}}(j\omega)$ becomes

$$\begin{bmatrix} \ddots & & & & 0 \\ & \frac{1}{\varepsilon+j(\omega+\omega_0)} & & & \\ & & \frac{1}{\varepsilon+j\omega} & & \\ & & & \frac{1}{\varepsilon+j(\omega-\omega_0)} & \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & \ddots & \ddots & & \ddots \\ \ddots & A_0+\varepsilon & A_1 & A_2 & \\ \ddots & A_{-1} & A_0+\varepsilon & A_1 & \ddots \\ & A_{-2} & A_{-1} & A_0+\varepsilon & \ddots \\ \ddots & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where the second factor is a Toeplitz operator and $\{A_k\}_{k\in Z}$ are the Fourier coefficients of $A(t)$. It is now straightforward to check the input and output Markov parameters. The first Markov parameters of $\mathcal{A}$ and $\mathcal{B}$ are $A(t)+\varepsilon$ and $B(t)$, and hence, both $\mathcal{A}$ and $\mathcal{B}$ have input and output roll-off 1, by Theorem 2. If both $A(\cdot)$ and $B(\cdot)$ belong to $\mathcal{C}^L(\boldsymbol{R})$, then $\mathcal{A}$ and $\mathcal{B}$ have skew roll-off $L-1$, by Theorem 5.

Let us focus on the square truncations, since these are suggested in [Zhang and Zhang, 1997]. Under the assumptions of Theorem 6 and (15), and that the operators are bounded for all Ω, we have bounds

$$\begin{aligned} \|(I-\mathcal{A})^{-1}-(I-P_\Omega\mathcal{A}P_\Omega)^{-1}\|_{L_2\to L_2} &\le K_{\mathcal{A}}(N)=O(N^{-1}), \\ \|\mathcal{B}-P_\Omega\mathcal{B}P_\Omega\|_{L_2\to L_2} &\le K_{\mathcal{B}}(N)=O(N^{-1}), \end{aligned}$$

as $N\to\infty$, and $\Omega=(N+1/2)\omega_0$.

To approximate $\mathcal{C}$, we use skew truncations $\mathcal{C}_{[N]}$, since $\mathcal{C}$ is a multiplication operator and has no input and output roll-off. If $C(\cdot)$ is periodic and in $\mathcal{C}^L(\boldsymbol{R})$, we have that an Nth-order skew truncation converges as

$$\|\mathcal{C}-\mathcal{C}_{[N]}\|_{L_2\to L_2}\le K\cdot N^{-(L-1)}=K_{\mathcal{C}}(N),$$

for some constant K. The HTF of $\mathcal{C}_{[N]}$ is a Toeplitz operator with the Fourier coefficients $C_{-N}\dots C_N$ on the diagonals.

Hence, by square truncations of each of the operators in (50), we can approximate $P_\Omega G P_\Omega$ with

$$\widehat{P_\Omega G P_\Omega} := P_\Omega \mathcal{C}_{[N]}(I-P_\Omega\mathcal{A}P_\Omega)^{-1}[P_\Omega\mathcal{B}P_\Omega]$$

and

$$\begin{aligned} &\|P_\Omega G P_\Omega-\widehat{P_\Omega G P_\Omega}\|_{L_2\to L_2} \\ &\qquad\le c_1K_{\mathcal{A}}(N)+c_2K_{\mathcal{B}}(N)+c_3K_{\mathcal{C}}(N)=O(N^{-1}), \quad (51) \end{aligned}$$

as $N \to \infty$, for some constants c_1, c_2, c_3. In (51) it is assumed that all involved operators have bounded norm as $N \to \infty$. Notice that (51) is a worst-case bound. If the system is time invariant, all the operators are diagonal and $\widehat{P_\Omega G P_\Omega} = P_\Omega G P_\Omega$.

If we approximate G, then we obtain

$$\|G - \widehat{P_\Omega G P_\Omega}\|_{L_2 \to L_2} \leq C_1 \cdot \Omega^{-k_1} + C_2 \cdot \Omega^{-k_2} + c_1 K_{\mathcal{A}}(N) + c_2 K_{\mathcal{B}}(N) + c_3 K_{\mathcal{C}}(N).$$

The first part of the error bound depends on the input and output roll-off of G, which are determined by the Markov parameters in Table 1. The second part depends on the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, as discussed above.

To summarize, by pulling out $p + \varepsilon$ we have been able to show convergence of the truncated harmonic balance method suggested in [Zhang and Zhang, 1997]. We have also seen that the worst-case convergence rate is slow, only $O(N^{-1})$ for matrix dimensions $(2N+1) \times (2N+1)$. The advantage with the method is that we only work with the Fourier coefficients of the realization and simple linear algebra. No knowledge of the transition matrix $\Phi_A(t, \tau)$ is needed.

If the transition matrix is known, we can compute the elements in $P_\Omega G P_\Omega$ exactly by our results in Section 4. Then the convergence of square truncations may be much faster, and depends only on the input and output roll-off.

Since the convergence rate of the square-truncation method may be slow, it is an interesting problem for future research to study how the method may be improved. We can redo the above computations using skew truncations. Then, however, the matrices will not be of finite dimension.

7. Conclusion

We have studied linear time-periodic systems from a frequency-domain point of view in this paper. We started to study Taylor expansions of time-varying systems and defined input and output Markov parameters. We also introduced the concepts of input and output roll-off. These roll-off rates are determined by the Markov parameters. Next we studied Fourier expansions of periodic systems in H_2. We also gave sufficient conditions for convergence rates of truncated Fourier expansions in induced L_r-norm, and introduced the concept of skew roll-off.

After the Fourier expansion, it was straightforward to define the frequency-response operator that is called the HTF. The roll-off concepts were shown to determine the decay rates of elements in different directions of the HTF, and we were able to strengthen available convergence bounds. After studies of the closed-loop system, we applied the results to systems given on state-space form. This allowed us to give conditions under which the truncated harmonic balance method converges. This method is interesting since only the Fourier coefficients of the realization are needed. However, the convergence rate can be quite slow.

This paper has provided a systematic convergence analysis of the HTF. This is important since in all applications listed in Section 1.1, some sort of truncation is used. We have analyzed the most common approaches of truncation here. However, it is still unclear how the HTF is best truncated. Furthermore, the truncated harmonic balance method is an interesting topic for future research. The reason for this is that it does not require knowledge of the transition matrix for $A(t)$. Most other methods that apply to periodic systems require such knowledge.

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Paper IV

A Bode Sensitivity Integral for Linear Time-Periodic Systems

Henrik Sandberg and Bo Bernhardsson

Abstract

Bode's sensitivity integral is a well-known formula that quantifies some of the limitations in feedback control for linear time-invariant systems. In this paper, we show that there is a similar formula for linear time-periodic systems. The harmonic transfer function is used to prove the result. To state the result we introduce the notion of roll-off 2, which means that the first time-varying Markov parameter is equal to zero. Then it follows that the harmonic transfer function is an analytic operator and a trace class operator. This is needed to prove the result.

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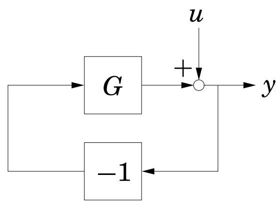

Figure 1. The sensitivity operator S is defined as $y = Su = (I + G)^{-1}u$.

1. Introduction

In recent years there has been an increased interest for the fundamental limitations in feedback control. One reason for this is that in many control design tools these limitations are not clearly visible, and an inexperienced designer can easily specify performance criteria that are not possible to attain. The articles [Stein, 2003] and [Åström, 2000] contain examples of this. There are many of these limitations in control. The connection between amplitude and phase of transfer functions and Bode's sensitivity integral formula are two examples. The limitations come from the fact that the transfer functions are *analytic functions*, and this has strong implications.

In this paper we focus on Bode's sensitivity integral. This is a standard result in control, see for example [Glad and Ljung, 2000]. The sensitivity function $S = (I + G)^{-1}$ is defined as in Figure 1. The result says that the sensitivity function cannot be small for all frequencies. If the transfer function $\hat{G}(s)$ of the open-loop system G has roll-off 2 and is stable, then we have in the multi-input-multi-output (MIMO) case that

$$\int_0^\infty \log|\det(I + \hat{G}(j\omega))^{-1}|\, d\omega = 0. \tag{1}$$

This is also called the waterbed effect. In particular, the modulus of the sensitivity, $|\det \hat{S}(j\omega)|$, cannot be less than 1 for all frequencies ω.

This trade-off holds for time-invariant linear systems. It is known that there are limitations also for linear time-varying and nonlinear systems, see for example [Zang, 2004]. However, frequency-domain methods are then often not applicable. In the paper [Iglesias, 2002], an analogue to (1) is developed for continuous-time time-varying linear systems. The sensitivity integral is interpreted as an entropy integral in the time domain, i.e., no frequency-domain representation is used. For discrete-time time-varying systems similar time-domain results are given in [Iglesias, 2001].

For time-periodic linear systems there do exist frequency-domain representations. Sampled-data systems are a special type of time-periodic systems. Fundamental limitations for sampled-data systems are studied in [Freudenberg *et al.*, 1995; Ortiz *et al.*, 2000] using transfer function techniques. We study general time-periodic systems in this paper and we use the *harmonic transfer function* (HTF), see [Wereley, 1991; Möllerstedt and Bernhardsson, 2000; Zhou and Hagiwara, 2002], which formally is a MIMO transfer function $\hat{G}(s)$ with an infinite amount of inputs and outputs. Using the convergence and existence results for the harmonic transfer function that are developed in [Sandberg *et al.*, 2004] we will be able to write (1) with $\hat{G}(j\omega)$ being the HTF. To do this we need to answer the following questions:

1. What does roll-off 2 mean for a time-periodic system?

2. In what sense is the HTF $\hat{G}(s)$ analytic?

3. What does the determinant mean for the HTF?

We do not consider open-loop unstable systems in this paper. This case is considered in [Iglesias, 2002] using exponential dichotomies. In the time-invariant case when the open-loop system is unstable, the right hand side of (1) is equal to $\pi \sum_i \operatorname{Re} p_i$, where p_i are the unstable open-loop poles, see [Freudenberg and Looze, 1985]. The authors do believe that it will be possible to generalize the method of this paper to cover also the unstable case.

During the completion of this article, the authors became aware of the independent work in [Colaneri, 2004]. The sensitivity integral derived there is similar to the one in this paper, but it applies to open-loop unstable periodic systems. However, the result is derived using other techniques, from [Iglesias, 2002], and is restricted to state-space models.

The paper is organized as follows: In Section 2, we give some of the basic results for the harmonic transfer function. The section ends with Proposition 1 which tells what roll-off 2 means. In Section 3, we review what an analytic operator is. In Proposition 2, we show that with roll-off 2 the HTF is in fact an analytic operator. In Section 4, we review the definition of the trace class operators and the operator determinant. In Proposition 3, we see that the HTF indeed is a trace class operator and that the determinant is well defined. By using the propositions of the previous sections, we can in Section 5 state the main result, which is a direct analogue of (1) for periodic systems. In Section 6, we give an example of the result.

This article is based on the conference papers [Sandberg and Bernhardsson, 2004a; Sandberg and Bernhardsson, 2004b].

2. The Harmonic Transfer Function and Roll-Off

It is shown in [Sandberg *et al.*, 2004] how the harmonic transfer function of a time-periodic system G given on impulse-response form

$$y(t) = \int_{-\infty}^{t} g(t,\tau)u(\tau)d\tau \tag{2}$$

can be computed. We repeat some results briefly here. For a *periodic system* there is a period $T \neq 0$ such that

$$g(t,\tau) = g(t+T,\tau+T). \tag{3}$$

We assume that $g(t,\tau)$ is real and has uniform exponential decay

$$|g(t,\tau)| \leq K \cdot e^{-\alpha(t-\tau)}, \quad t \geq \tau,$$

for some positive constants K and α. The operator G is then bounded on L_2.

To define the HTF of a linear periodic system G we need the following steps: First we expand the periodic impulse response (3) in a Fourier series

$$\begin{aligned} g(t,\tau) &= \sum_{l=-\infty}^{\infty} g_l(t-\tau)e^{jl\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}, \\ g_l(t-\tau) &= \frac{1}{T}\int_0^T g(r,r-t+\tau)e^{-jl\omega_0 r}dr, \end{aligned} \tag{4}$$

with convergence in L_2, see [Sandberg *et al.*, 2004]. Hence, we expand the periodic system into a sum of modulated time-invariant impulse responses $g_l(t)$. For exponentially stable systems we can apply the Laplace transform on each time-invariant impulse response $g_l(t)$,

$$\hat{g}_l(s) = \int_0^{\infty} g_l(t)e^{-st}dt, \quad \operatorname{Re} s > -\alpha. \tag{5}$$

Furthermore, we have that $\hat{g}_l(s)$ is analytic in $\operatorname{Re} s > -\alpha$ and $\hat{g}_l \in H_2 \cap H_\infty$. Now the HTF $\hat{G}(s)$ is defined as the infinite-dimensional matrix

$$\begin{bmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \hat{g}_0(s+j\omega_0) & \hat{g}_1(s) & \hat{g}_2(s-j\omega_0) & \\ \ddots & \hat{g}_{-1}(s+j\omega_0) & \hat{g}_0(s) & \hat{g}_1(s-j\omega_0) & \ddots \\ & \hat{g}_{-2}(s+j\omega_0) & \hat{g}_{-1}(s) & \hat{g}_0(s-j\omega_0) & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{6}$$

for complex numbers s in a region J_0,

$$J_0 = \{s : \operatorname{Re} s \geq 0,\ \operatorname{Im} s \in I_0\}$$
$$I_0 = (-\omega_0/2, \omega_0/2], \quad \omega_0 = 2\pi/T.$$

Notice that we only need to define the HTF for frequencies $\omega \in I_0$. The HTF $\hat{G}(s)$ is a linear infinite-dimensional operator, which is a bounded operator on the space of square-summable sequences ℓ_2 (at least for almost all $s \in J_0$).

In [Wereley, 1991; Zhou and Hagiwara, 2002; Sandberg *et al.*, 2004] it is shown that for stable systems G we can compute the induced L_2-norm as

$$\|G\|_{L_2 \to L_2} = \sup_{\|u\|_{L_2} \leq 1} \|Gu\|_{L_2} = \operatorname{ess\,sup}_{\omega \in I_0} \|\hat{G}(j\omega)\|_\infty, \tag{7}$$

where $\|\cdot\|_\infty$ is the induced ℓ_2-norm.

2.1 Roll-Off of Periodic Systems

For all numbers q we can rewrite (2) as

$$y(t)e^{-qt} = \int_{-\infty}^{t} [g(t,\tau)e^{-q(t-\tau)}]u(\tau)e^{-q\tau}d\tau. \tag{8}$$

We use the notation

$$y_q = G_q u_q,$$

where the operator G_q has impulse response $g(t,\tau)e^{-q(t-\tau)}$ and maps input signals of the type $u_q(t) = u(t)e^{-qt}$ into signals $y_q(t) = y(t)e^{-qt}$. For every fixed $q \geq 0$ we may apply the theory developed in [Sandberg *et al.*, 2004]. In particular we may apply the time-varying Markov parameter expansions.

In the following proposition we use the notation $g_x^{(a)} = \partial^a g/\partial x^a$, and p is the differential operator $pu(t) = du(t)/dt$. Furthermore, the set $\mathcal{S}$ is the set of Schwartz functions, i.e., the set of infinitely differentiable functions $u(t)$ with $t^a p^b u(t)$ bounded for $t \in \boldsymbol{R}$ and all non-negative a and b. The set $\mathcal{S}$ is dense in L_2.

PROPOSITION 1

Assume that $g(t,t) = 0$ for all t, that $g(t,\tau)$ is twice continuously differentiable in the region $t \geq \tau$, and that all the derivatives have uniform exponential decay. Then G is said to have *roll-off* 2, and for all $q \geq 0$ we may expand (8) in either of the following ways,

$$y_q(t) = -g'_\tau(t,t)\frac{1}{(p+q)^2}u_q(t) + \int_{-\infty}^{t} [g''_\tau(t,\tau)e^{-q(t-\tau)}]\frac{1}{(p+q)^2}u_q(\tau)d\tau, \quad (9)$$

$$y_q(t) = \frac{1}{(p+q)^2}g'_t(t,t)u_q(t) + \frac{1}{(p+q)^2}\int_{-\infty}^{t} [g''_t(t,\tau)e^{-q(t-\tau)}]u_q(\tau)d\tau, \quad (10)$$

when $u_q \in \mathcal{S}$. □

Proof. We prove (10). (9) may be proven similarly. By the assumptions on $g(t,\tau)$ and since $u_q \in \mathcal{S}$, y_q is absolutely (and hence uniformly) continuous and belongs to L_1. By Barbalat's lemma we conclude that $y_q(t) \to 0$ as $|t| \to \infty$. If we differentiate (8) with respect to t we obtain

$$\frac{d}{dt}y_q(t) = g(t,t)u_q(t) - qy_q(t) + \int_{-\infty}^{t} [g'_t(t,\tau)e^{-q(t-\tau)}]u_q(\tau)d\tau. \quad (11)$$

If we integrate (11) over $(-\infty, t]$ and solve for $y_q(t)$ we obtain

$$y_q(t) = \frac{1}{p+q}g(t,t)u_q(t) + \frac{1}{p+q}\int_{-\infty}^{t} [g'_t(t,\tau)e^{-q(t-\tau)}]u_q(\tau)d\tau.$$

By assumption, $g(t,t) = 0$ and the first term disappears. If we repeat the above procedure on the second term we obtain (10). ■

We say that the systems in Proposition 1 have roll-off 2, see also [Sandberg *et al.*, 2004]. This can be motivated as follows. We introduce P_Ω as an ideal (non-causal) low-pass filter with the frequency characteristic

$$\hat{P}_\Omega(j\omega) = \begin{cases} 1, & |\omega| \le \Omega \\ 0, & |\omega| > \Omega \end{cases}.$$

Proposition 1 together with the facts that $\mathcal{S}$ is dense in L_2 and that the Fourier transform of a function in $\mathcal{S}$ is again in $\mathcal{S}$, implies that if we filter the input or the output of systems G_q there are positive constants C_1, C_2, δ such that

$$\|G_q(I - P_\Omega)\|_{L_2\to L_2} \le \frac{C_1}{|\delta + q + j\Omega|^2}, \quad (12)$$

$$\|(I - P_\Omega)G_q\|_{L_2\to L_2} \le \frac{C_2}{|\delta + q + j\Omega|^2}. \quad (13)$$

To show (12) one uses (9), and to show (13) one uses (10), see [Sandberg *et al.*, 2004]. In particular, we have that $\|G_q\|_{L_2\to L_2} = O(q^{-2})$ as $q \to \infty$

and $\|G_q(I - P_\Omega)\|_{L_2\to L_2} = O(\Omega^{-2})$ and $\|(I - P_\Omega)G_q\|_{L_2\to L_2} = O(\Omega^{-2})$ for each fixed q as $\Omega \to \infty$.

The relation between the HTF of G and G_q is simple,

$$\hat{G}(q + j\omega) = \hat{G}_q(j\omega), \quad q + j\omega \in J_0,$$

so it is enough to speak of $\hat{G}(s)$. The high-pass filtering of G_q with $(I - P_\Omega)$ means that rows or columns are truncated (replaced by zeros) in $\hat{G}(s)$. If we choose $\Omega = (N + 1/2)\omega_0$ for some non-negative integer N, then $G_q(I - P_\Omega)$ has an HTF where the $2N + 1$ middle columns of $\hat{G}(s)$ are replaced by zeros. $(I - P_\Omega)G_q$ has an HTF where the $2N + 1$ middle rows of $\hat{G}(s)$ are replaced by zeros, see [Sandberg *et al.*, 2004] for details. This has consequences for the roll-off of the individual transfer functions $\hat{g}_l(s)$ as is shown in the next section.

REMARK 1
For a stable time-invariant system with smooth impulse response $g(t, \tau) = g(t - \tau)$, $t \geq \tau$, the Markov parameters are equal to $\{g(0), g'(0), g''(0), \ldots\}$. If $g(t, t) = g(0) = 0$ then we have that

$$|\hat{g}(s)| = O(|s|^{-2}),$$

as $|s| \to \infty$ and $\operatorname{Re} s \geq 0$. This is called roll-off 2 for a time-invariant system. □

3. Analytic Operators

To prove Bode's integral theorem for time-invariant systems, one uses that the transfer function is analytic and Cauchy's integral theorem. We will do something similar. The HTF is an infinite-dimensional operator and therefore we will need some of the theory for analytic operators.

There are several equivalent definitions of an analytic operator, see for example [Kato, 1976]. We say that a bounded linear operator $\hat{G}(s)$ is analytic in an open set $\Omega \subseteq \boldsymbol{C}$ if it can be expanded in a power series around each $s_0 \in \Omega$,

$$\hat{G}(s) = \sum_{k=0}^{\infty} (s - s_0)^k \hat{G}_k, \quad s \in \Omega(s_0) \subseteq \Omega,$$

with uniform convergence in the open disc $\Omega(s_0)$ in the induced ℓ_2-norm, $\|\cdot\|_\infty$. The constant operators $\hat{G}_k$ are linear bounded operators on ℓ_2. To prove that the HTF $\hat{G}(s)$ is an analytic operator we can check the following sufficient conditions [Kato, 1976]:

K1 All the elements of $\hat{G}(s)$ are analytic functions in Ω.

K2 There is a positive constant K such that $\|\hat{G}(s)\|_\infty \leq K$ for all $s \in \Omega$.

The property K1 follows by (5) and (6). The property K2 needs some extra attention. We will use the Hilbert-Schmidt norm to prove it. It is well known that the Hilbert-Schmidt norm $\|\cdot\|_2$ gives an upper bound to the induced ℓ_2-norm, i.e., $\|\hat{G}(s)\|_\infty \leq \|\hat{G}(s)\|_2$. Now, by definition

$$\|\hat{G}(s)\|_2^2 = \sum_{k,l=-\infty}^{\infty} |\hat{g}_l(s + jk\omega_0)|^2. \tag{14}$$

We will show that we can bound the sum (14) for all $s \in J_0$.

By using the roll-off formulas and the discussion about the truncation of rows and columns in Section 2.1, we can conclude that for all non-negative integers N, and $\operatorname{Re} s \geq 0$,

$$|\hat{g}_l(s)| \leq \frac{C_1 + C_2}{N^2\omega_0^2 + \delta^2}, \qquad l \in Z, \quad |l| \geq 2N + 1, \tag{15}$$

$$|\hat{g}_l(s)| \leq \frac{C_1}{|\delta + s|^2}, \qquad l \in Z, \tag{16}$$

The first bound follows since $\|G_q - P_\Omega G_q P_\Omega\|_{L_2 \to L_2} \leq \|(I - P_\Omega)G_q)\|_{L_2 \to L_2} + \|G_q(I - P_\Omega)\|_{L_2 \to L_2} \leq (C_1 + C_2)/(N^2\omega_0^2 + \delta^2)$ when $\Omega = (N + 1/2)\omega_0$. The modulus of the analytic elements of the HTF of $G_q - P_\Omega G_q P_\Omega$ must be less or equal to the L_2-induced norm according to (7). As the transfer functions $\hat{g}_l(s)$, $|l| \geq 2N + 1$, are not truncated with this choice of Ω, (15) follows. The second bound follows since the modulus of the analytic functions $\hat{g}_l(s)$ must be less than the L_2-induced norm bound in (12), and then we can choose $s = q + j\Omega$.

Hence, roll-off 2 for a time-periodic system as defined in Proposition 1 implies that the transfer functions $\hat{g}_l(s)$ on the diagonals of $\hat{G}(s)$ have roll-off 2 in the classical sense (Remark 1). Now we can prove that $\hat{G}(s)$ is analytic.

PROPOSITION 2
If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is an analytic operator for $s \in J_0$. □

Proof. If we use the bounds (15)–(16) for $s = j\omega$ in (14) we show, see below, that the sum converges uniformly and there is a constant K such that

$$\|\hat{G}(j\omega)\|_\infty \leq \|\hat{G}(j\omega)\|_2 \leq K, \quad \omega \in I_0. \tag{17}$$

By the maximum modulus theorem $|\hat{g}_l(q+j\omega)| \leq |\hat{g}_l(j\omega)|$, $q > 0$. We then also show that $\|\hat{G}(s)\|_2 \leq K$, $s \in J_0$.

We now prove (17). To compute the Hilbert-Schmidt norm we shall sum over the indices k and l. By using (16) the sum in the k-direction converges for each l,

$$S(l) = \sum_{k=-\infty}^{\infty} |\hat{g}_l(j\omega + jk\omega_0)|^2 < \infty.$$

We need to show that $\sum_{l=-\infty}^{\infty} S(l) \leq K^2$. From (15)–(16) we have

$$|\hat{g}_{\pm(2N+1)}(j\omega)| \leq (C_1 + C_2) \min\left\{\frac{1}{N^2\omega_0^2}, \frac{1}{|\omega|^2}\right\}.$$

For $\omega \in I_0$ and a fixed $N > 0$ we have

$$S(\pm(2N+1)) \leq (C_1 + C_2)^2 \left(\frac{2N-1}{N^4\omega_0^4} + 2\sum_{k=N}^{\infty} \frac{1}{|\omega + k\omega_0|^4}\right) \leq \frac{C}{N^3}$$

where C is a constant independent of ω. We can derive a similar bound for $S(\pm 2(N+1))$. Hence, we have that $S(l) = O(|l|^{-3})$ as $|l| \to \infty$, and the sum $\sum_l S(l)$ converges and there exists a constant K as in (17).

Since $\hat{G}(s)$ fulfills the conditions K1 and K2, it is analytic and the proposition follows. ■

4. Trace Class Operators and Determinants

In the linear time-invariant MIMO Bode integral (1) the determinant of the transfer function matrix is used. We need to define a determinant for infinite-dimensional operators also. This can be done for so-called trace class operators, see [Gohberg and Krein, 1969; Böttcher and Silbermann, 1990]. For a trace class operator $\hat{G}$ the determinant is defined as

$$\det(I + \hat{G}) = \prod_k \left(1 + \lambda_k(\hat{G})\right), \tag{18}$$

where $\lambda_k(\hat{G})$ are the eigenvalues of $\hat{G}$. Trace class operators are compact operators and have a countable number of eigenvalues. The possibly infinite product (18) converges for trace class operators, see (21). Note that for finite matrices, (18) coincides with the regular determinant.

For the definition of a trace class operator we need the s-numbers (or singular numbers) of an operator $\hat{G}$,

$$s_k(\hat{G}) = \inf\{\|\hat{G} - \hat{G}_k\|_\infty : \operatorname{rank} \hat{G}_k \leq k\}.$$

The numbers s_k tell how well $\hat{G}$ may be approximated by a finite-rank operator. If $\hat{G}$ is compact we have that $s_k \to 0$ as $k \to \infty$ and $s_0 = \|\hat{G}\|_\infty$. The trace class operators are those operators for which

$$\|\hat{G}\|_1 = \sum_{k=0}^{\infty} s_k < \infty. \tag{19}$$

With the norm $\|\cdot\|_1$ the trace class operators form a complete normed space, see [Gohberg and Krein, 1969]. We have that

$$\operatorname{trace} \hat{G} = \sum_k \lambda_k(\hat{G}) \leq \|\hat{G}\|_1, \tag{20}$$

$$|\det(I + \hat{G})| \leq \exp(\|\hat{G}\|_1). \tag{21}$$

Next we will see that under the assumptions of Proposition 1, the HTF $\hat{G}(s)$ is in fact a trace class operator for all $s \in J_0$.

The HTF of $G_q P_\Omega$, with $\Omega = (N + 1/2)\omega_0$, has elements equal to zero everywhere except for its $2N + 1$ middle columns which are identical to the $2N + 1$ middle columns of $\hat{G}(s)$ defined by (6). Hence, the truncated HTF has at most rank $2N + 1$. We know that $G_q P_\Omega$ converges to G_q as $O(\Omega^{-2}) = O(N^{-2})$ from (12). Using the norm formula (7) and the continuity of $\hat{G}_q(j\omega)$ ($\hat{G}(s)$ is analytic), we conclude that for each $q + j\omega \in J_0$ we have that

$$\begin{aligned} s_{2N+1}(\hat{G}_q(j\omega)) &\leq \|\hat{G}_q(j\omega)(I - \hat{P}_\Omega(j\omega))\|_\infty \\ &\leq \|G_q(I - P_\Omega)\|_{L_2 \to L_2} \leq \frac{C_1}{|\delta + q + j\Omega|^2} \\ &\leq \frac{C_1}{(\delta + q)^2 + N^2\omega_0^2}. \end{aligned} \tag{22}$$

For each fixed s the singular numbers $s_k(\hat{G}(s))$ decay as $O(k^{-2})$ for systems with roll-off 2. We are now ready to state the proposition of this section.

PROPOSITION 3
If the periodic system G fulfills the assumptions of Proposition 1, then its harmonic transfer function $\hat{G}(s)$ is a bounded trace class operator in J_0,

$$\|\hat{G}(q + j\omega)\|_1 \leq \frac{K_1}{K_2 + q}, \quad q + j\omega \in J_0,$$

for some positive constants K_1, K_2. □

Proof. We have that

$$s_0(\hat{G}(q+j\omega)) = \|\hat{G}(q+j\omega)\|_\infty \leq \|G_q\|_{L_2\to L_2} \leq \frac{C_1}{(\delta+q)^2},$$

and for $N = 0, 1, 2, \dots$ we have that $s_{2N+1}(\hat{G}(q+j\omega))$ is bounded as in (22). The singular numbers form a decreasing sequence and hence we can make the upper estimate

$$s_{2N+2}(\hat{G}(q+j\omega)) \leq s_{2N+1}(\hat{G}(q+j\omega)).$$

Now we can use these estimates to bound the trace norm (19),

$$\|\hat{G}(q+j\omega)\|_1 \leq \sum_{k=0}^{\infty} \frac{2C_1}{(\delta+q)^2 + \omega_0^2 k^2} \leq \frac{K_1}{K_2+q},$$

for some constants K_1, K_2. ■

Before stating the main result, we need the following lemma.

LEMMA 1—[GOHBERG AND KREIN, 1969]
If Ω is an open set in $\boldsymbol{C}$ and if $\hat{G}(s)$ is an analytic trace-class-operator-valued function for $s \in \Omega$, then $\det(I + \hat{G}(\cdot)) : \Omega \to \boldsymbol{C}$ is an analytic function. □

5. Main Result

Using Propositions 1–3 and Lemma 1 we are finally ready to state the analogue of Bode's sensitivity integral, applicable to time-periodic systems.

THEOREM 1—SENSITIVITY INTEGRAL
Assume that a stable linear time-periodic system G has roll-off 2 in the sense of Proposition 1. Assume furthermore that the sensitivity operator $S = (I+G)^{-1}$ is stable, i.e., there is an ε such that

$$|\det(I + \hat{G}(s))| \geq \varepsilon > 0, \quad s \in J_0. \tag{23}$$

Then

$$\int_0^{\omega_0/2} \log|\det(I + \hat{G}(j\omega))^{-1}|\, d\omega = 0. \tag{24}$$

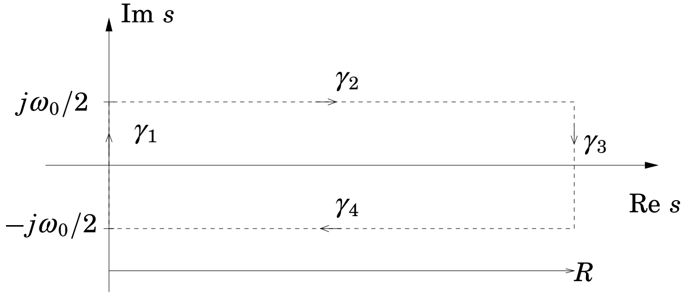

Figure 2. The integration path Γ_R.

□

Proof. We have that $\det(I+\hat{G}(s))^{-1} = 1/\det(I+\hat{G}(s))$, see [Böttcher and Silbermann, 1990]. From Proposition 3 we know that $\|\hat{G}(s)\|_1 \leq K_1/K_2$. Using (21) and (23) we then have that

$$\frac{1}{\exp(K_1/K_2)} \leq |\det(I+\hat{G}(s))^{-1}| \leq \frac{1}{\varepsilon}$$

and hence $\det(I+\hat{G}(s))^{-1}$ is a bounded function which does not become zero for $s \in J_0$.

Since $\det(I+\hat{G}(s))^{-1}$ is nonzero in J_0, we can define a complex logarithm there. Now,

$$\log\det(I+\hat{G}(s))^{-1} = -\log\det(I+\hat{G}(s)).$$

From Propositions 1–3 and Lemma 1 we know that $\det(I+\hat{G}(s))$ is an analytic function in J_0. Then for any simply closed curve $\Gamma \subset J_0$,

$$\int_\Gamma \log\det(I+\hat{G}(s))^{-1} ds = 0, \tag{25}$$

by Cauchy's integral formula. To prove the theorem we choose the curve Γ_R shown in Figure 2 and let $R \to \infty$.

First we evaluate the integral (25) along γ_2 and γ_4. Notice that

$$\int_0^R \log\det(I+\hat{G}(q+j\omega_0/2))^{-1} dq + \int_R^0 \log\det(I+\hat{G}(q-j\omega_0/2))^{-1} dq = 0$$

for all R. The cancellation is because

$$\det(I+\hat{G}(q-j\omega_0/2)) = \det(I+\hat{G}(q+j\omega_0/2))$$

for all q. This follows by the structure (6) of the HTF and the definition of the determinant.

Next we evaluate the integral along γ_3. The complex logarithm is defined as

$$\log\det(I+\hat{G}(s)) = \log|\det(I+\hat{G}(s))| + j\arg\det(I+\hat{G}(s)).$$

When the impulse response $g(t,\tau)$ is real we have that $\hat{g}_l(s) = \overline{\hat{g}_{-l}(\bar{s})}$ and by the structure (6) and the definition of the determinant that

$$\begin{aligned} \arg\det(I+\hat{G}(s)) &= -\arg\det(I+\hat{G}(\bar{s})) \\ |\det(I+\hat{G}(s))| &= |\det(I+\hat{G}(\bar{s}))|. \end{aligned} \tag{26}$$

The argument is an anti-symmetric function, so when we integrate over the symmetric interval γ_3 it disappears,

$$\begin{aligned} &\left|\int_{\omega_0/2}^{-\omega_0/2} \log\det(I+\hat{G}(R+j\omega))d(j\omega)\right| \\ &= \left|\int_{\omega_0/2}^{-\omega_0/2} \log|\det(I+\hat{G}(R+j\omega))|d(j\omega)\right| \\ &\leq \int_{\omega_0/2}^{-\omega_0/2} \|\hat{G}(R+j\omega)\|_1\, d\omega, \end{aligned}$$

for each fixed R. The last bound follows by (21). Now $\|\hat{G}(R+j\omega)\|_1$ converges uniformly to zero as $R\to\infty$ according to Proposition 3. The integral along γ_3 then goes to zero as $R\to\infty$. The only term remaining of (25) is the integral along γ_1,

$$\int_{-\omega_0/2}^{\omega_0/2} \log\det(I+\hat{G}(j\omega))^{-1} d(j\omega) = 0.$$

Using (26) on the interval $[-\omega_0/2, \omega_0/2]$ we obtain

$$\int_0^{\omega_0/2} \log|\det(I+\hat{G}(j\omega))^{-1}|\, d\omega = 0,$$

and the result is shown. ■

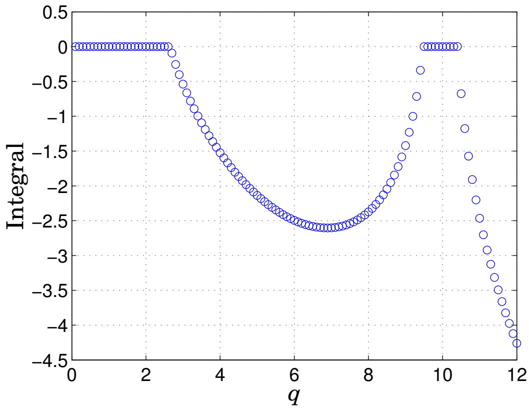

Figure 3. The values of the integral (24) for different values of q in the Mathieu equation (27). By Theorem 1, the integral must equal zero for stable closed-loop systems. It can be verified by, for instance, Floquet analysis that the system indeed is stable for $q \in [0, 2.6] \cup [9.4, 10.4]$.

REMARK 2—TIME-INVARIANT SYSTEMS
The integral in (1) is over the interval $[0, \infty)$ whereas the integral in (24) is over $[0, \omega_0/2]$. This might seem strange, but notice that for a time-invariant system with transfer function $\hat{g}(s)$, the HTF is given by

$$\hat{G}(s) = \text{diag}\,\{\dots, \hat{g}(s + j\omega_0), \hat{g}(s), \hat{g}(s - j\omega_0), \dots\},$$

for any $\omega_0 > 0$, and we see that (1) and (24) are identical if we use that $\hat{g}(\bar{s}) = \overline{\hat{g}(s)}$. □

6. Example: The Mathieu Equation

Now, we verify the main result on an example. We choose an open-loop system G with dynamics given by

$$\ddot{y}(t) + 0.4\dot{y}(t) + 2y(t) = q\cos(2t)w(t), \tag{27}$$

where q is a parameter and $w(t)$ the input. The impulse response is given by

$$g(t, \tau) = \frac{q}{1.4} e^{-0.2(t-\tau)} \sin(1.4(t-\tau))\cos(2\tau).$$

Clearly the system has roll-off 2 in the sense of Proposition 1, and it is exponentially stable. To obtain the closed-loop system in Figure 1, the

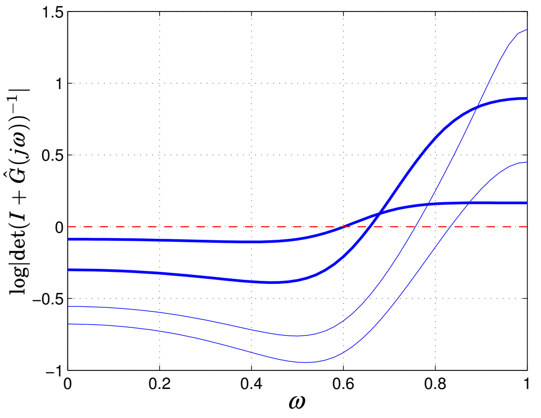

Figure 4. The logarithm of the sensitivity function for the Mathieu equation (27) is plotted for $q = 1.0$, 2.0, 3.0, and 3.5. For the bold curves (the stable systems) the conservation law in Theorem 1 applies. When q increases the sensitivity decreases for low frequencies. The sensitivity must then increase for high frequencies to keep the areas below and above the zero level equal. This is the waterbed effect.

feedback $w(t) = -(y(t) + u(t))$ is applied. Notice that when $u(t) = 0$ the dynamics of the closed-loop system is given by a damped Mathieu equation, see, for example [Wereley, 1991].

Next we compute the HTF of G using (4)–(6). Here $\omega_0 = 2$. After this we may compute the integral (24) for different values of q. For $q \in [0, 2.6] \cup [9.4, 10.4]$, the closed loop is stable. This can be shown by, for instance, Floquet analysis. According to Theorem 1 the integral should then equal zero. In Figure 3 this is verified. It is also seen that when the closed loop is unstable, the integral is strictly less than zero.

Furthermore, we can visualize the waterbed effect for periodic systems. This is done in Figure 4. When the sensitivity decreases for some frequencies, it must increase for other frequencies.

7. Conclusion

We have seen that there are fundamental limitations for feedback control of linear time-periodic systems. The modulus of the determinant of the harmonic transfer function $\hat{S}(j\omega) = (I + \hat{G}(j\omega))^{-1}$ cannot be made small for all frequencies ω. The result is a direct generalization of Bode's sensitivity integral. To prove the result we have defined roll-off 2 for a time-periodic system, and used some of the theory for analytic operators and trace class operators.

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