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1. INTRODUCTION

An attractive idea when investigating large systems is to mimic the success of statistical mechanics, i e to consider a large system composed of many copies of identical subsystems. A fundamental difficulty is that it is hard to find meaningful systemtheoretic problems which can be obtained by interconnecting identical subsystems. This paper therefore considers a class of linear systems called flow systems. Although the systems are not identical many of their properties remain invariant for different interconnections. Flow systems can therefore be used as a starting point for analysing certain large systems. Flow systems have been used as models for industrial and biological processes.

2. TANK SYSTEMS AND FLOW SYSTEMS

A collection of tanks connected by pipes is called a tank system. Such systems are common in industry. They have also been extensively used as models for biological and ecological systems. Tank systems are often explored by tracer analysis. A traceable substance, which propagates through the system in the same way as the fluid, is introduced at some point of the system. The tracer concentration at another point in the system is then measured. One key problem is to analyse what properties of a tank system that can be found from such an experiment.

Assume that the flows and volumes are in equilibrium, the tracer propagation can be described as a linear time invariant dynamical system. The dynamical systems describing tracer propagation have, however, some special properties which motivates that they are given a special name flow systems. The impulse response of a flow system is non-negative which reflects the fact that the tracer concentration is never negative. Moreover, if the tanks system is open which means that all tanks are connected to an outlet (possibly indirectly through other tanks) all tracer will eventually leave the system. The corresponding open flow systems then have the property that the integral of the impulse response is unity.

Open flow systems will be investigated in this paper. They have many interesting properties which have largely been found in connection with impulse response analysis of tank systems. The results are widely scattered in literature. Important contributions are found both in engineering and medical literature. This paper is an attempt to present a unified approach.

Two simple examples corresponding to a tank with pure mixing and a tank with pure plug flow are first investigated. A formal definition of an open flow system is then given and interconnections of open flow systems are introduced. The so called Stewart--Hamilton equation which can be used to determine the total volume of an open tank system is then derived. The volume obtained is the part of the volume which participates in the flow also called the volume of distribution.

3. EXAMPLES

Two simple examples of flow systems will first be given.

EXAMPLE 1 (IDEAL MIXING)
Consider a tank with volume V and constant inflow and outflow q (volume flow). Assume that there is perfect mixing in the tank and that the fluid is not compressible. Let c_i be the concentration of a tracer in the inflow and c the tracer concentration in the tank and at the outflow. A mass balance for the tracer gives

$$V \frac{dc}{dt} = q(c_i - c)$$

The propagation of the tracer through the system can thus be described as a linear time invariant dynamical system whose input output relation is characterized by the impulse response

$$h(t) = (V/q)e^{-qT/V} \qquad (1)$$

EXAMPLE 2 (PURE TRANSPORT OR PLUG FLOW)
Consider a pipe where there is a pure material transport with uniform velocity and no mixing. Let the volume of the tube be V and

the flow q. Let c_i denote the concentration of some substance in the inlet and c the concentration of the same substance at the outlet. The concentrations are related by

$$c(t) = c_i(t-V/q)$$

and the impulse response of the system becomes

$$h(t) = \delta(t-V/q) \qquad (2)$$

where δ is the Dirac delta function. The propagation of a tracer through a tank with ideal mixing and for a pipe with pure plug flow can be described by linear time invariant dynamical systems.

In both cases the impulse responses have the properties.

$$h(t) \geq 0 \qquad (3)$$

$$\int_0^\infty h(t)dt = 1 \qquad (4)$$

and

$$\int_0^\infty th(t)dt = - V/q \qquad (5)$$

The equation (3) means that the tracer concentration is never negative and the equation (4) implies that all tracer will finally leave the system. If the impulse response is measured by injecting a tracer in the inlet and measuring the tracer concentration in the outlet the volume to flow ratio V/q can thus be determined from the equation (5) both for an ideal mixing tank and for a pipe with pure plug flow.

4. AN AXIOMATIC APPROACH

The theory of flow systems will now be developed systematically. The analysis will be carried out for systems with one inlet and one outlet. There are, however, no difficulties to extend the results to more general situations. In analogy with the simple examples the systems will be characterized by their impulse responses. Introduce

DEFINITION 1
A single-input single-output time invariant linear system is called a <u>flow system</u> if the impulse response has the property (3). It is called an <u>open flow system</u> if the impulse response also has the property (4).

It follows from the previous examples that the transportation of a substance through a tank with perfect mixing and through a pipe with pure mass transport without mixing can be described by flow systems.

Notice that the quantity

$$\int_{t_1}^{t_2} h(t)dt$$

can be interpreted as the probability that a particle entering the system at time 0 will exit in the interval (t_1,t_2). The impulse response of a flow system can thus be interpreted as a probability density. It is, therefore, also called the residence time distribution or more correctly the density of the residence time distribution. The properties (3) and (4) are far reaching. A flow system is e g always input-output stable. To explore the properties further we analyse the transfer function H defined by

$$H(s) = \int_0^\infty e^{-st} h(t)dt \tag{6}$$

The equation (4) implies that

$$H(0) = \int_0^\infty h(t)dt = 1 \tag{7}$$

For Re $s \geq 0$ we have

$$|H(s)| = \left|\int_0^\infty e^{-st} h(t)dt\right| \leq \int_0^\infty |e^{-st}| h(t)dt \leq$$

$$\leq \int_0^\infty h(t)dt = 1 \qquad \text{Re } s \geq 0$$

The magnitude of the transfer function of a flow system is thus less than or equal to one in the closed right half plane.

Let ω_i be arbitrary real numbers and x_i arbitrary complex numbers. Then

$$\sum_k \sum_\ell x_k \bar{x}_\ell H(i\omega_k - i\omega_\ell) = \int_0^\infty \sum_k \sum_\ell x_k \bar{x}_\ell e^{i\omega_k t} e^{-i\omega_\ell t} h(t)dt =$$

$$= \int_0^\infty (\Sigma x_k e^{i\omega_k t})(\Sigma x_\ell e^{i\omega_\ell t}) h(t)dt =$$

$$= \int_0^\infty |\Sigma x_k e^{i\omega_k t}|^2 h(t)dt \geq 0 \tag{8}$$

It follows from a famous theorem of Bochner (1932) that the conditions (7) and (8) also imply (4) and (3).

An open flow system can thus also be defined as a linear time invariant system whose transfer function satisfies (7) and (8). This is not done because the conditions (3) and (4) are much more appealing to physical intuition.

4. INTERCONNECTION OF FLOW SYSTEMS

There are several ways to interconnect flow systems. They can e g be connected in series, parallel or in feedback connections in the same way as ordinary linear systems are interconnected. More interesting and more useful results are, however, obtained if the interconnection is done in a different way. Since flow systems are used to describe the propagation of a tracer in a tank system we will first consider different ways to connect tanks together. Interconnection of flow systems will then be defined by considering the flow systems which describe the propagation of a tracer in the interconnected tanks.

Tanks can be connected in many different ways. The outflow of one tank can be sent to another tank (series connection). A flow can be split up in different parts which are sent through tanks and again continued (parallel connection). Part of the outflow of a tank can be mixed with the inflow and sent to the tank again (feedback connection).

It seems intuitively clear that if the tracer propagation in two tanks is described by flow systems in the sense of Definition 1, then the propagation of a tracer in the interconnected tanks is also a flow system. It will now be shown formally that this is indeed the case.

By a series connection of two tanks we mean the system obtained by letting the outlet of one tank be connected to the inlet of the other tank as illustrated in Fig 1.

Assume that the tracer propagation in S_1 and S_2 can be described by flow systems with the transfer functions H_1 and H_2. Let c_i, c_1 and c denote the tracer concentrations at the inlet of S_1, the outlet of S_1 and the outlet of S_2 respectively. Then

$$C_1(s) = H_1(s)C_i(s)$$

$$C(s) = H_2(s)C_1(s)$$

Elimination of C_1 gives

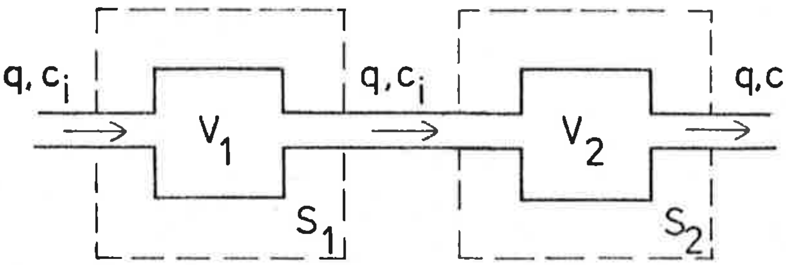

Fig 1. Series connection of the flow systems S_1 and S_2.

$$C(s) = H_2(s)H_1(s)C_i(s)$$

and we thus find that the propagation of a tracer in a series connection of two tanks can be described by a linear system with the transfer function

$$H_s(s) = H_2(s)H_1(s) \qquad (9)$$

To show that the transfer function H_s corresponds to a flow system we introduce the corresponding impulse responses, i e

$$h_s(t) = \int_0^\infty h_2(t-s)h_1(s)ds$$

It is clear that if h_1 and h_2 are non-negative then h_s is also non-negative. Furthermore it follows from (8) that

$$H_s(0) = H_2(0)H_1(0) = 1$$

Taking (9) as the definition of a series connecting of two flow systems it has thus been shown that the series connection of two flow systems is a flow system.

We will now proceed to other ways of connecting flow systems. A parallel connection of two tanks is obtained by splitting the inflow q into two flows $\alpha_1 q$ and $\alpha_2 q$ where $0 \le \alpha_1 \le 1$ and $\alpha_1 + \alpha_2 = 1$. These flows are then taken as inflows to the tanks S_1 and S_2 whose outflows are then combined assuming perfect mixing. The parallel connection is illustrated in Fig 2.

To analyse the propagation of a tracer through two tanks S_1 and S_2 in parallel it is assumed that the tracer propagation through S_1 and S_2 can be described by flow systems with the transfer functions H_1 and H_2. Let c_i denote the tracer concentration at the inlet and c_1 and c_2 the tracer concentrations at the outlets of the

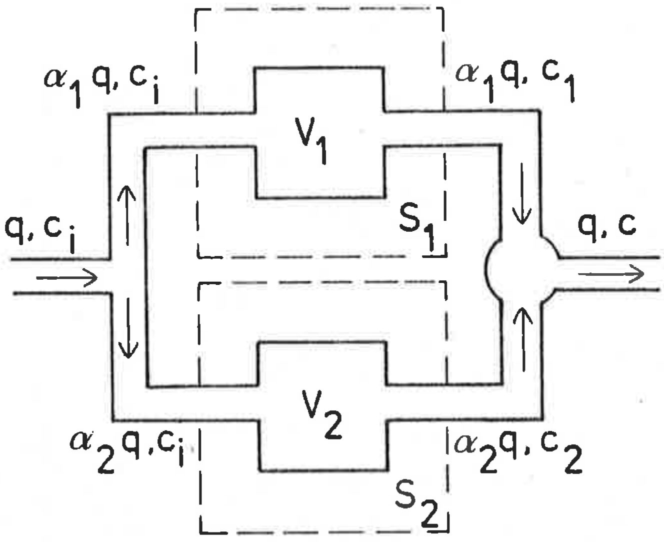

Fig 2. Parallel connection of the flow systems S_1 and S_2.

tanks. Then

$$C_1(s) = H_1(s)C_i(s)$$

$$C_2(s) = H_2(s)C_i(s)$$

Since the output flow is obtained by ideal mixing of the flows $\alpha_1 q$ and $\alpha_2 q$, with tracer concentrations c_1 and c_2, the concentration at the outlet becomes

$$C(s) = \alpha_1 C_1(s) + \alpha_2 C_2(s) = [\alpha_1 H_1(s) + \alpha_2 H_2(s)]\ C_i(s)$$

The propagation of a tracer through a parallel connection of two tanks can thus be described by a linear system with the transfer function

$$H_p(s) = \alpha_1 H_1(s) + \alpha_2 H_2(s) \qquad 0 \le \alpha_1,\ \alpha_2 \le 1,\ \alpha_1 + \alpha_2 = 1 \tag{10}$$

To verify that this is a transfer function of a flow system the impulse responses are introduced. Hence

$$h_p(t) = \alpha_1 h_1(t) + \alpha_2 h_2(t)$$

It is clear that if h_1 and h_2 satisfy (3) and (4), then h_p will also satisfy the same equations.

The feedback connection S_f of two tanks or two flow systems S_1 and S_2 is illustrated in Fig 3. Let the inflow to S_3 be q and the tracer concentration c_i. Furthermore let the proportion α of the outflow of S_1 be the inflow to S_2. It is assumed that the outflow of S_2 is perfectly mixed with the system inflow.

If αq_1 is the flow through S_2, a flow balance then gives

$$(\alpha q_1 + q) = q_1$$

Hence

$$q_1 = \frac{q}{1 - \alpha}$$

Let c_2 denote the concentration at the outlet of S_2 then

$$C_2(s) = H_2(s)C(s)$$

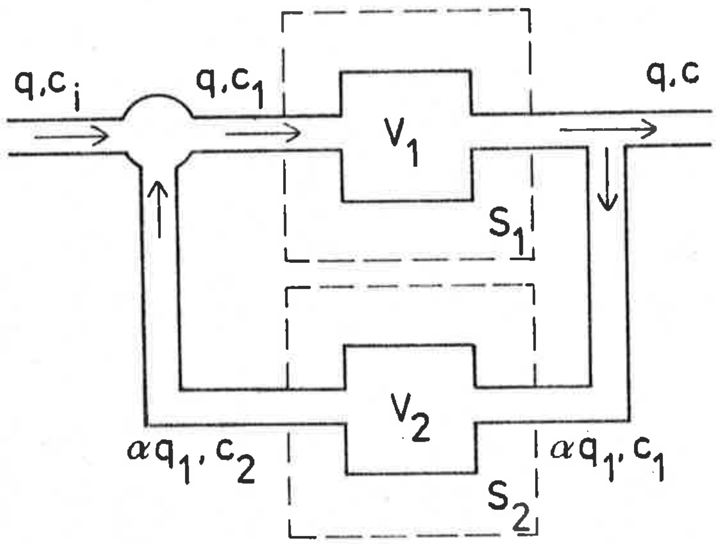

Fig 3. Feedback connection of the flow systems S_1 and S_2. The inflow q is perfectly mixed with the outflow of S_2, and the mixture is fed to S_1. The outflow of S_1 is split into two streams, one of which goes to S_2 and the other part is the outflow of S_f.

The input to S_1 is a mix of two flows q and $\alpha q/(1-\alpha)$, having concentrations c_i and c_2 respectively. The concentration c_1 at the inlet of S_1 is thus

$$C_1(s) = (1-\alpha)C_i(s) + \alpha C_2(s)$$

Furthermore

$$C(s) = H_1(s)C_1(s) = (1-\alpha)H_1(s)C_i(s) + \alpha H_1(s)H_2(s)C(s)$$

which gives

$$C(s) = \frac{(1-\alpha)H_1(s)}{1 - \alpha H_1(s)H_2(s)} C_i(s)$$

The tracer propagation through a feedback connection of two tanks can thus be described by a linear system with the transfer function

$$H_f(s) = \frac{(1-\alpha)H_1(s)}{1 - \alpha H_1(s)H_2(s)} \qquad 0 \le \alpha < 1 \qquad (11)$$

Assuming that H_1 and H_2 are transfer functions of flow systems it will now be shown that H_f is also such a transfer function. We have

$$H_f(0) = \frac{(1-\alpha)H_1(0)}{1 - \alpha H_1(0)H_2(0)} = \frac{1-\alpha}{1-\alpha} = 1$$

Furthermore introduce $H = H_1H_2$. Since S_1 and S_2 are flow systems, it follows from the equation (7) that

$$|H(s)| \le 1 \quad \text{for} \quad \text{Re } s \ge 0$$

The series expansion

$$H_f(s) = (1-\alpha)H_1(s)[1 + \alpha H(s) + \alpha^2H^2(s) + \ldots]$$

thus converges uniformly for $\alpha \le \alpha_0 < 1$ and Re $s \ge 0$. The corresponding impulse response then satisfies

$$h_f = (1-\alpha)h_1*[1 + \alpha h + \alpha^2 h*h + \ldots]$$

where * denotes convolution. Since S_1 and S_2 are flow systems, we have $h_1(t) \ge 0$ and $h_2(t) \ge 0$, and we find $h_f(t) \ge 0$.

Summing up we get

THEOREM 1
Let S_1 and S_2 be open flow systems with the transfer functions H_1 and H_2. The series S_s, parallel S_p and feedback S_f connections of S_1 and S_2 whose transfer functions are defined by

$$H_s = H_2H_1 \tag{9}$$

$$H_p = \alpha_1H_1 + \alpha_2H_2 \qquad 0 \le \alpha_1,\ \alpha_2 \le 1,\ \alpha_1 + \alpha_2 = 1 \tag{10}$$

$$H_f = \frac{(1-\alpha)H_1}{1 - \alpha H_1H_2} \qquad 0 \le \alpha < 1 \tag{11}$$

are then also open flow systems.

Remark. Notice that the series connection of two flow systems is identical to the series connection of two linear systems. The parallel and feedback connections of flow systems are, however, not the same as the parallel and series connection of linear systems.

Using Theorem 1 the propagation of a tracer through a tank system can be studied in the same way as signal propagation is analysed in an ordinary linear system.

5. THE STEWART-HAMILTON EQUATION

The analysis of the simple tank systems corresponding to a tank with ideal mixing in Example 1 and to a tank with pure plug flow in Example 2 shows that the following equation

$$\int_0^\infty th(t)dt = V/q \tag{12}$$

holds in both cases. Compare with the equation (5). Recalling the probabilistic interpretation of the impulse response h as the residence time distribution the equation (12) simply says that for a tank system with one inlet and one outlet the ratio of volume to flow equals the mean residence time. The equation (12) was first used by the physiologists Stewart (1897) and Hamilton (1932) who developed methods to determine the blood volume of the heart. The equation (12) will therefore be called the Stewart-Hamilton equation. The equation has been widely used both in biology, physiology and engineering. It has also been misinterpreted and therefore the cause of much controversy.

The equation (12) can be derived by the following heuristic argument. Consider an open tank system with inflow q. The fraction h(t)dt of the particles which enter the system at time zero will exit in the interval (t,t+dt). These particles have traversed the volume dv = t·q. Integrating over all particles now gives (12). The validity of the equation (12) can also be shown formally in many cases. We have the following result:

THEOREM 2
Let S_1 and S_2 be tank systems with one inlet and one outlet and volumes V_1 and V_2. Let the tank system S_3 be a series, parallel or feedback connection of S_1 and S_2. Assume that the Stewart-Hamilton equation holds for S_1 and S_2 then it also holds for S_3.

Proof. Let H_1 and H_2 be the transfer functions which characterize the tracer propagation in S_1 and S_2. The different ways to interconnect the systems will be discussed separately.

First consider a series connection. It follows from Theorem 1 that the tracer propagation in S_3 then is characterized by the transfer function $H_3 = H_1H_2$. The mean residence time of S_3 is then given by

$$\int_0^\infty th_3(t) = - H_3'(0) = - H_1'(0)H_2(0) - H_1(0)H_2'(0) = (V_1+V_2)/q =$$
$$= V_3/q$$

The third equality follows from the fact that the flows through S_1 and S_2 are the same in a series connection.

Now consider a parallel connection. See Fig 2. Since the flow through S_1 is $\alpha_1 q$ and that through S_2 is $\alpha_2 q$, we get

$$- H_1'(0) = V_1/(\alpha_1 q) \quad \text{and} \quad -H_2'(0) = V_2/(\alpha_2 q)$$

The mean residence time of S_3 is given by

$$\int_0^\infty th_3(t)dt = - H_3'(0) = - \alpha_1 H_1'(0) - \alpha_2 H_2'(0) = (V_1+V_2)/q =$$
$$= V_3/q$$

and the result is thus established also for a parallel connection.

For a feedback connection, Fig 3, the flow through S_1 is $q_1 = q/(1-\alpha)$ and the flow through S_2 is $\alpha q_1 = \alpha q/(1-\alpha)$. Hence

$$- H_1'(0) = V_1/q_1 = (1-\alpha)V_1/q$$

$$- H_2'(0) = V_2/(\alpha q) = (1-\alpha)V_2/(\alpha q)$$

The equation (10) gives

$$H_3' = \frac{(1-\alpha)H_1'}{1 - \alpha H_1 H_2} + \frac{(1-\alpha)H_1(\alpha H_1' H_2 + \alpha H_1 H_2)}{(1-\alpha H_1 H_2)^2} = \frac{(1-\alpha)(H_1' + \alpha H_1^2 H_2')}{(1-\alpha H_1 H_2)^2}$$

The mean residence time is then given by

$$\int_0^\infty t h_3(t)dt = - H_3'(0) = - \frac{1}{1-\alpha} H_1' - \frac{\alpha}{1-\alpha} H_2' = (V_1+V_2)/q = V_3/q$$

and the proof is now complete.

Remark 1. Combining Theorem 2 with the results of Example 1 and Example 2, it is thus found that the Stewart-Hamilton equation holds for systems which are obtained by series, parallel or feedback connections of simple flow systems with pure transport or with ideal mixing.

Remark 2. The Stewart-Hamilton equation has been derived only for systems which are open flow system. Internal recirculations are allowed provided that only a fraction of the flow is recirculated ($\alpha < 1$ in Theorem 1). All fluid particles must, however, sooner or later leave the system, or formally the equation (4) must hold. This will not be the case if all the flow is recirculated.

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