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On The Accuracy Of Identification And The Design Of Identification Experiments

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ON THE ACCURACY OF IDENTIFICATION AND THE DESIGN OF IDENTIFICATION EXPERIMENTS

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ABSTRACT.

The result of an identification experiment depends on a number of items, such as the identification method, the class of models used and the input signal generation. In this report the effect of these items on the accuracy of the estimates is investigated. The models are assumed to be single-input single-output difference equations. Special attention is paid to systems operating in closed loop and the effects of the feedback on the accuracy. In particular, it is shown that closed loop experiments can give better accuracy than open loop experiments in the case of constrained output variance.

TABLE OF CONTENTS. Page

1. INTRODUCTION. 1

2. IDENTIFIABILITY PROPERTIES. 3

3. ACCURACY OF IDENTIFICATION. 10

4. INFLUENCE OF THE MODEL STRUCTURE ON THE ACCURACY. 14

5. INFLUENCE OF THE IDENTIFICATION METHOD ON THE ACCURACY. 19

6. INFLUENCE OF THE EXPERIMENTAL CONDITION ON THE ACCURACY. 27

6.1 Introduction.

6.2 First Order Systems with White Noise.

6.3 First Order Systems with Coloured Noise.

6.4 Optimization with Constrained Input Variance.

7. EXAMPLES. 42

8. CONCLUSIONS 51

9. ACKNOWLEDGEMENTS 54

10. REFERENCES 55

APPENDIX 1 - Proof of Lemma 4.1.

APPENDIX 2 - Proof of Theorem 6.1.

APPENDIX 3 - Proof of Lemma 6.1.

APPENDIX 4 - Proof of Lemma 6.2

1. INTRODUCTION.

Process identification is a valuable tool for modelling of dynamic systems. If the goal of the modelling is the design of a control law it is desirable to know the open loop characteristics, i.e. the transfer function from u to y and how the disturbance v_1 influence the output y, see Fig. 1.1, which shows a typical configuration for a system with feedback.

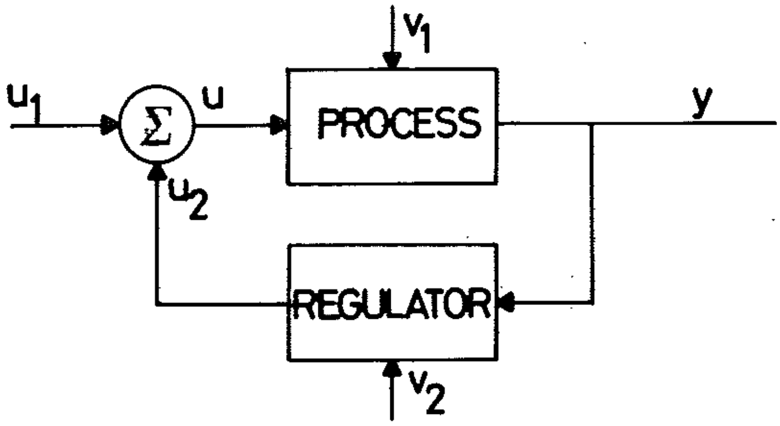

Fig. 1.1 - Block diagram of a closed loop system.
u - input signal to the process (measurable)
y - output signal (measurable)
u_1 - extra perturbation signal, also called external input signal (measurable)
u_2 - feedback signal (measureable)
v_1, v_2 - disturbances (unmeasurable)

However, in many cases identification experiments cannot be carried out in open loop, since several processes operate in closed loop. A typical reason is that the process in open loop is unstable or unsatisfactorily damped so that a regulator has to be used. Other types of processes, e.g. many biological, economical and social systems are inherently in closed loop. Thus it turns out to be very important to know under what circumstances it is possible to identify the open loop characteristics of a system operating in closed loop. This question is thoroughly treated in Gustavsson-Ljung-Söderström (1974). Some main results of that report will be restated.

Identifiability is a concept dealing with the asymptotic properties of the results of identification. A system is said to be identifiable if the open loop characteristics are obtained exactly when the number of data tends to infinity. If a parametric model is used it is not necessary that the parameter estimates are uniquely determined. The concept of identifiability is treated e.g. in Ljung- Gustavsson-Söderström (1974) and will be briefly reviewed in the next section.

A more practical aspect of identifiability is to consider the accuracy of the result of the identification. Suppose that an identifiable system is given. It is then an interesting and important question how the experimental conditions(e.g. the feedback law) will influence the accuracy of the result. In particular it is important to know how to optimize the accuracy. Note that this problem is closely related to design of optimal experiments, the study of which is one of the main objects of the report.

The report is organized as follows.

In the next section some basic concepts and identifiability results are discussed. In Section 3 different ways of measuring the accuracy of identification are considered. The next three sections deal with how the accuracy is influenced by the model structure, the identification method and the experimental configuration respectively. It is e.g. shown that it can be advantageous from the accuracy point of view to perform the identification experiments in closed loop. Finally some numerical examples are given in Section 7.

2. IDENTIFIABILITY PROPERTIES.

In this section some identifiability results will be reviewed from Ljung-Gustavsson-Söderström (1974). Consider a linear, discrete time, stochastic system S, given on the general form

$$y(t) = G_S(q^{-1})u(t) + H_S(q^{-1})e(t) \tag{2.1}$$

The output, $y(t)$, is a vector of dimension n_y and the input, $u(t)$, has dimension n_u. The variables $\{e(t)\}$ are a sequence of independent, random vectors with zero mean values and covariances $E\, e(t)e^T(t) = \Lambda$. It is assumed that $e(t)$ has the same dimension as $y(t)$, which introduces no loss of generality. $G_S(z)$ and $H_S(z)$ are matrices of appropriate dimensions with rational functions as entries and q^{-1} is the backward shift operator, i.e. $q^{-1}u(t) = u(t-1)$. It is assumed that $G_S(0) = 0$, i.e. there is a time delay in the system. It is also assumed that $H_S(0) = I$ and that $\det[H_S(z)]$ has all zeroes strictly outside the unit circle. Then $H_S^{-1}(q^{-1})$ is a well-defined, exponentially stable filter. This assumption is not very restrictive, cf. the spectral factorization theorem, see e.g. Åström (1970).

The input, $u(t)$, to the process can be determined in several ways. It can, as in open loop experiments, be chosen freely by the experiment designer. It can also be determined partly from output feedback by a regulator of a given structure, etc. On the whole, the manner in which the input is determined will be referred to as the experimental condition, denoted by X.

To determine a model of the system the functions $G(z)$ and $H(z)$ are parametrized in a suitable manner by a parameter vector $\hat{\theta}$. A model corresponding to a certain value of $\hat{\theta}$

is denoted by $M(\hat{\theta})$ and is given by

$$y(t) = G_{M(\hat{\theta})}(q^{-1})u(t) + H_{M(\hat{\theta})}(q^{-1})\varepsilon(t) \tag{2.2}$$

where $\{\varepsilon(t)\}$ is a sequence of independent, random vectors with zero mean values and covariances $\hat{\Lambda}$. When $\hat{\theta}$ is varied over the region of feasible values, eq. (2.2) represents a family of models denoted by M. This family will sometimes be referred to as the "model structure". The identification problem is to determine the parameter $\hat{\theta}$ so that $M(\hat{\theta})$ in some sense suitably describes the system S given by eq. (2.1).

Remark: The recursive expressions (2.1) and (2.2) require certain initial values to be started up. Since the analysis in this paper concerns asymptotic properties, these initial values will have no effect on the results and can, for example, be taken as zero.

In Section 4 this general model structure will be treated. In Sections 5-7 specialization will be made to single input single output systems of the form

$$y(t) = q^{-\hat{k}} \frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} u(t) + \frac{\hat{C}(q^{-1})}{\hat{A}(q^{-1})} \varepsilon(t) \tag{2.2'}$$

where

$$\hat{A}(z) = \hat{A} = 1 + \hat{a}_1 z + \ldots + \hat{a}_{\hat{n}_a} z^{\hat{n}_a}$$

$$\hat{B}(z) = \hat{B} = \hat{b}_1 z + \ldots + \hat{b}_{\hat{n}_b} z^{\hat{n}_b}$$

$$\hat{C}(z) = \hat{C} = 1 + \hat{c}_1 z + \ldots + \hat{c}_{\hat{n}_c} z^{\hat{n}_c}$$

$$\hat{k} \geq 0 \quad \hat{n}_a \geq 0 \quad \hat{n}_b \geq 1 \quad \hat{n}_c \geq 0$$

and

$$\hat{\theta} = \left[\hat{a}_1 \ldots \hat{a}_{\hat{n}_a}, \hat{b}_1 \ldots \hat{b}_{\hat{n}_b}, \hat{c}_1 \ldots \hat{c}_{\hat{n}_c}\right]^T \qquad (2.3)$$

For single input single output systems Λ is substituted with λ^2 for convenience only.

In the general case G and H can be parameterized e.g. via vector difference equations or via state space realizations, see Ljung-Gustavsson-Söderström (1974).

Let the identification method used be denoted by I; some different specific methods are discussed below. In order to be able to treat the problem in a systematic manner some useful identifiability concepts will be introduced.

Let

$$D_T(S,M) = \left\{\hat{\theta} \mid G_{M(\hat{\theta})}(z) = G_S(z) \text{ and } H_{M(\hat{\theta})}(z) = H_S(z) \text{ a.e.z.}\right\}$$

This set consists of the parameter values that give models $M(\hat{\theta})$ with the same transfer function and the same noise characteristics as the system S.

Let the parameter estimates at time N for given S, M, I and X be denoted by $\hat{\theta}(N;S,M,I,X)$.

Definition 1. The system S is said to be System Identifiable under M, I and X, SI(M,I,X), if $\hat{\theta}(N;S,M,I,X) \to$ $\to D_T(S,M)$ w.p.1 as $N \to \infty$.†

Definition 2. The system S is said to be Strongly System Identifiable under I and X, SSI(I,X), if it is SI(M,I,X) for all M such that $D_T(S,M)$ is non-empty.

Definition 3. The system S is said to be Parameter Identifiable under M, I and X, PI(M,I,X), if it is SI(M,I,X) and $D_T(S,M)$ consists of only one element.

Notice that PI(M,I,X) is always implied by SI(M,I,X) if $D_T(S,M)$ consists of only one point. This condition on $D_T(S,M)$ involves neither I nor X, and is the problem of canonical representation of transfer functions. It is convenient to treat this difficult problem separately and study the identifiability properties for different experimental conditions X by considering SI(M,I,X). Clearly, a necessary condition on M to achieve SI(M,I,X) is that $D_T(S,M)$ is non-empty. If the system is SSI(I,X), this condition is also a sufficient condition on M for SI(M,I,X). In that case the fact that the system may operate in closed loop does not add any extra difficulties when choosing appropriate model structures M. Experimental conditions that give SSI(I,X) therefore are equivalent to open loop from the viewpoint of identifiability.

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† By this it is meant that

$$\inf_{\theta \in D_T} |\hat{\theta}(N;S,M,I,X) - \theta| \to 0 \text{ w.p.1 as } N \to \infty.$$

There are many identification methods that can be used for identification of the system S. Special attention will be paid to two methods.

1. Direct identification, denoted by I_1. This method means that the signals u and y are used straightforwardly assuming no a priori knowledge about the regulator or even that the process is operating in closed loop. Moreover, it is assumed that a statistically efficient prediction error method is applied. The maximum likelihood method, see e.g. Åström-Bohlin (1965), is such a method.

2. Indirect identification, denoted by I_2. It consists of two steps. As a first step the closed loop system is identified, e.g. the transfer function from u_1 to y, if an external input signal is used. If $u_1 = 0$ the output can be modelled as a pure time series. The second step consists of algebraic calculations. In this step it is assumed that the regulator is known, and the parameter vector $\hat{\theta}$ is solved for, using the closed loop characteristics estimated in step 1. This way of identification is not applicable if there are disturbances in the regulator, i.e. when $v_2 \neq 0$.

Before reviewing the identifiability results it will be appropriate to consider the experimental condition X. It will also be valuable to make use of sets of experimental conditions. Such sets will in general be denoted by $\{X\}$. An example of such a set may be to require that $u_1 = 0$ and that the regulator is proportional. Then $\{X\}$ can be considered as a set of constants of the proportional regulator.

An extension of optimal input design is to determine the experimental condition $\mathcal{X}$ in a given set $\{\mathcal{X}\}$ that will give optimal accuracy. In this report three different sets of experimental conditions will be considered. It turns out that they often give identifiability as stated explicitly below.

1. $\{\mathcal{X}\}_1$ - The open loop configuration, i.e. $u_2 = 0$ in Figure 1.1. For such experimental configurations there is no feedback.

2. $\{\mathcal{X}\}_{2(r,n)}$ - The feedback consists of shifts between r different linear regulators of order n. No external input u_1 is used. The shifts are performed in such a way that the transients have influence a negligable part of the experiment time. During 100 γ_i percent (i = = 1,...,r) of the total experiment time the following regulator is used

$$F_i(q^{-1})u(t) = G_i(q^{-1})y(t) \qquad (2.4)$$

where

$$\begin{aligned} F_i(z) &= F_i = 1 + f_{i1}z + \dots + f_{in}z^n \\ G_i(z) &= G_i = g_{i0} + g_{i1}z + \dots + g_{in}z^n \end{aligned} \qquad (2.5)$$

The notation $\{\mathcal{X}\}_{2(r,\cdot)}$ will mean that the orders of the regulators are not specified.

3. $\{\mathcal{X}\}_{3(n)}$ - One regulator of order n and external input. The input u(t) is given by

$$F(q^{-1})u(t) = F(q^{-1})u_1(t) + G(q^{-1})y(t) \qquad (2.6)$$

where

$$F(z) = F = 1 + f_1 z + \dots + f_n z^n$$
$$G(z) = G = g_0 + g_1 z + \dots + g_n z^n \tag{2.7}$$

The notation $\{X\}_{3(\cdot)}$ will be used when the order of the regulator is not specified.

With some abuse of the notations the set of experimental configurations will in the following be denoted by X_1, $X_2(r,n)$ and $X_3(n)$.

The concepts M, I and X will have the following influence on the identifiability, cf. Gustavsson-Ljung-Söderström (1974).

M: A necessary condition for identifiability clearly is that $D_T(S,M)$ is not empty.

I: If I_1 fails, then any method will fail. If I_2 is applicable and the system is $SI(M,I_2,X)$ then it is also $SI(M,I_1,X)$.

X: The experimental conditions X_1 (i.e. open loop), $X_2(r,\cdot)$ (i.e. shift between r regulators) with $(r-1)n_y \geq n_u$ and $X_3(\cdot)$ (i.e. one regulator and external input signal) will in general imply identifiability, provided that for X_1 the input signal u and for $X_3(\cdot)$ the signal u_1 is persistently exciting (see e.g. Åström-Bohlin (1965) for a definition of this concept). This means that the system is $SSI(I,X_1)$, $SSI(I,X_2(r,\cdot))$ and $SSI(I,X_3(\cdot))$. However, the system is not $SSI(I,X_2(1,\cdot))$, i.e. experimental conditions in $X_2(1,\cdot)$ (one regulator and no external input) will sometimes give identifiability and sometimes not.

3. ACCURACY OF IDENTIFICATION.

There are several ways to analyse the accuracy of the result of an identification. Such analysis can be found in several references treating optimal input design, e.g. Aoki-Staley (1970), Mehra (1974) and Nahi-Napjus (1971). Cf. also e.g. Åström (1967).

The analysis in this report is based on asymptotic result. It is possible to use the covariance matrix P of the parameter estimates as a measure of the accuracy. An alternative is to use Fisher's information matrix J, see e.g. Kendall--Stuart (1961) for a definition. $\varepsilon(t)$ in the equation (2.2) can be regarded as a function of $\hat{\theta}$. Let $\varepsilon'(t)$ denote the derivative of $\varepsilon(t)$ with respect to $\hat{\theta}$. Then the information matrix can be written as

$$J = E\varepsilon'(t)^T \Lambda^{-1} \varepsilon'(t) \cdot N \tag{3.1}$$

Moreover, when the ML method is used, asymptotically

$$P = J^{-1} \tag{3.2}$$

For an arbitrary identification method the Cramér-Rao inequality, see Cramér (1946) and for this particular case also e.g. Åström (1967), can be stated as

$$P \geq J^{-1} \tag{3.3}$$

It follows from (3.2) that it is equivalent to use P or J as a measure of the accuracy. However, if two experimental conditions are to be compared, then such an approach means that two matrices, say P_1 and P_2, and not two scalars are to be compared. It may now very well happen that neither $P_1 \geq P_2$ nor $P_2 \geq P_1$ is true. Thus such an approach will not in general make it possible to judge which will be the optimal experimental condition. For this reason scalar measures of the accuracy will be used.

In this report two different measures are treated. One of these measures is det J = $(\det P)^{-1}$. This measure is interpreted as follows. The larger the value of det J is, the better the identification result is considered. However, this measure has a disadvantage. If such M, I and X are considered that the system S is SI(M,I,X) but <u>not</u> PI(M,I,X) (i.e. the set $D_T(S,M)$ contains more than one point) then det J will in general be zero. Nevertheless, the obtained model can give a good description of the true system.

The other type of measure was described in Söderström (1973). The idea is to use a scalar differentiable function $V(\hat{\theta})$ of $\hat{\theta}$ such that

$$\hat{\theta} \in D_T(S,M) \Rightarrow V(\hat{\theta}) = \inf_{\theta} V(\theta) \tag{3.4}$$

This means that all models giving a true description of the system will give the absolute minimum value of the loss function V, cf. Figure 3.1. Note that the reverse is not necessarily true.

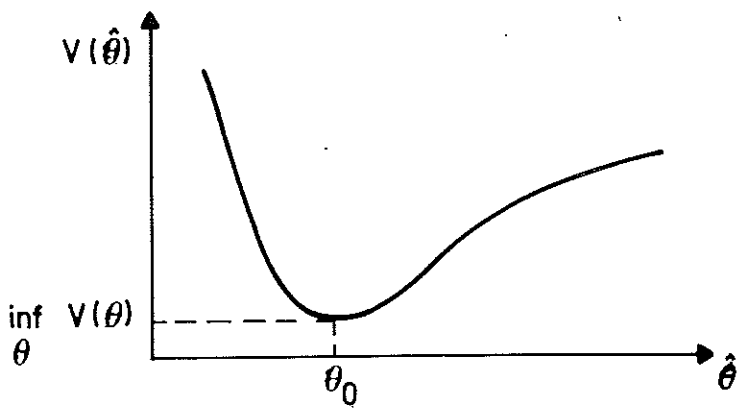

Figure 3.1 - Illustration of a function $V(\hat{\theta})$. The set $D_T(S,M)$ consists of the single point θ_0.

So far general multivariable systems have been considered. In the following examples of functions $V(\hat{\theta})$ the single-input single-output model (2.2') is chosen. It is assumed that the system has the same structure.

Example 3.1.

$$V_1(\hat{\theta}) = E\,\varepsilon^2(t) \tag{3.5}$$

The loss function $V_1(\hat{\theta})$ expresses the variance of the one step prediction errors (i.e. $\varepsilon(t)$) that will be obtained using the model.

□

Example 3.2.

$$V_2(\hat{\theta}) = E\left[\left(\frac{\hat{B}(q^{-1})q^{-\hat{k}}}{\hat{A}(q^{-1})} - \frac{B(q^{-1})q^{-k}}{A(q^{-1})}\right) u(t)\right]^2 \tag{3.6}$$

The loss function $V_2(\hat{\theta})$ expresses how well the model describes the deterministic part of the system. If in (3.6) the signal $u(t)$ is chosen as white noise then

$$V_2(\hat{\theta}) = \sum_{i=1}^{\infty} (\hat{h}_i - h_i)^2 \tag{3.7}$$

where h_i ($\hat{h}_i$) is the discrete time impulse response of the system (resp. the model).

□

Example 3.3. Assume that the model is used for construction of a minimum variance regulator, cf. Åström (1970). Suppose that the true system is controlled by this regulator and take

$$V_3(\hat{\theta}) = E\, y^2(t) \tag{3.8}$$

The regulator is given by

$$u(t) = -\frac{q^{-1}G(q^{-1})}{\hat{B}(q^{-1})F(q^{-1})}\, y(t)$$

where $F(q^{-1})$ and $G(q^{-1})$ are defined by

$$\hat{C}(q^{-1}) = \hat{A}(q^{-1})F(q^{-1}) + q^{-\hat{k}-1}G(q^{-1})$$

Thus the closed loop system will be

$$[A(q^{-1})\hat{B}(q^{-1})F(q^{-1}) + q^{\hat{k}-k}B(q^{-1})\{\hat{C}(q^{-1}) -$$

$$- \hat{A}(q^{-1})F(q^{-1})\}]\, y(t) = C(q^{-1})\hat{B}(q^{-1})F(q^{-1})\, e(t) \tag{3.9}$$

If, in particular $\hat{k} = 1$ then F reduces to 1. □

Functions satisfying (3.4) can be used for measuring the accuracy of $\hat{\theta}$ by considering the scalar

$$E\, V(\hat{\theta}_N) \tag{3.10}$$

where expectation is to be taken over $\hat{\theta}_N$. It may be possible that the expectation (3.10) does not exist. This is further discussed in the next section. Also note that in evaluation of a single realization $V(\hat{\theta}_N)$ expresses the accuracy of the identification result from this single experiment.

4. INFLUENCE OF THE MODEL STRUCTURE ON THE ACCURACY.

The model structure M can be chosen in many ways. It will influence the identifiability as well as the accuracy of the result. The purpose of this section is to study the influence on the accuracy. A criterion of the form

$$E\,V(\hat{\theta}_N) \tag{4.1}$$

as discussed in the previous section will be used. It will be generally assumed in this section that the system S is PI(M,I,X) for the present M, I and X. This limitation is made for technical reasons.

The expression (4.1) will now be approximated. Let θ_0 denote the parameter vector describing the true system. The function $V(\hat{\theta})$ is approximated by three terms of its Taylor series as

$$V(\hat{\theta}) \approx \tilde{V}(\hat{\theta}) = V(\theta_0) + V'(\theta_0)(\hat{\theta}-\theta_0) + \frac{1}{2}(\hat{\theta}-\theta_0)^T V''(\theta_0)(\hat{\theta}-\theta_0) \tag{4.2}$$

Since $V(\hat{\theta})$ has a minimum in θ_0, cf (3.4), it follows that $V'(\theta_0) = 0$. Now

$$EV(\hat{\theta}_N) \approx E\tilde{V}(\hat{\theta}_N) = V(\theta_0) + \frac{1}{2}\,E(\hat{\theta}_N-\theta_0)^T V''(\theta_0)(\hat{\theta}_N-\theta_0) \tag{4.3}$$

Note that the left hand side of (4.3) may not exist. This is the case e.g. for the function given in Example 3.3, since for that example there is a non zero probability that the closed loop system is unstable. Nevertheless the right hand side of (4.3) describes an appropriate measure of the obtained accuracy of the identification.

The expression (4.3) is easily rewritten as

$$EV(\hat{\theta}_N) \approx V(\theta_0) + \frac{1}{2} E \operatorname{tr}(\hat{\theta}_N-\theta_0)^T V''(\theta_0)(\hat{\theta}_N-\theta_0) =$$

$$= V(\theta_0) + \frac{1}{2} E \operatorname{tr} V''(\theta_0)(\hat{\theta}_N-\theta_0)(\hat{\theta}_N-\theta_0)^T =$$

$$= V(\theta_0) + \frac{1}{2} \operatorname{tr} V''(\theta_0)\operatorname{Cov}(\hat{\theta}_N) =$$

$$= V(\theta_0) + \frac{1}{2} \operatorname{tr} V''(\theta_0)J^{-1} \tag{4.4}$$

In (4.4) $V(\theta_0)$ is a constant independent of M, I and X. This means that it is of interest to study the expression

$$\operatorname{tr} V''(\theta_0)J^{-1}$$

for different model structures. Note that the dimension of the square matrices $V''(\theta_0)$ and J is dependent on the model structure. Note the similarity with a criterion of the form $\operatorname{tr} QJ^{-1}$, where Q is a weighting matrix. Such a criterion is used e.g. in Mehra (1974).

<u>Example 4.1</u>. Consider the specific choice $V_1(\theta) = E\varepsilon^2(t)$ as discussed in Example 3.3. Then

$$V_1(\theta_0) = \lambda^2 \qquad V_1''(\theta_0) = \frac{2\lambda^2}{N} J$$

so

$$E\,V_1(\hat{\theta}_N) \approx \lambda^2(1 + \frac{1}{N} \operatorname{tr} JJ^{-1}) = \lambda^2(1 + \frac{1}{N} \dim J) \tag{4.5}$$

This means that if more than the minimal number of parameters are used then dim J will be unnecessarily large and so will the criterion $EV_1(\hat{\theta}_N)$.

□

After these preliminary considerations it is possible to treat the general case. Consider two model structures M_1 and M_2. Since a model structure can be interpreted as a set of parameters the relation

$$M_1 \subset M_2 \tag{4.6}$$

is well defined. A possible interpretation of (4.6) can be that M_1 is obtained from M_2 by fixing some parameters to zero.

Let the θ-vector corresponding to M_1 be denoted by θ_1 and the θ-vector corresponding to M_2 be denoted by θ_2. Note that θ_1 and θ_2 in general have different dimensions. Then, due to (4.6) the θ_2 vector can be parameterized with θ_1 to describe also M_1, i.e. the model structure M_1 is "given" by some function

$$\theta_2 = \theta_2(\theta_1) \tag{4.7}$$

Let now $V''(\theta_0)$ and J_2 denote the matrices corresponding to the model structure M_2. To find expressions for these matrices for M_1 the relation (4.7) will be used. Consider the function $V(\theta_2(\theta_1))$. Clearly (with $d^2V/d\theta_2^2 = V''(\theta_0)$)

$$\frac{dV}{d\theta_1} = \frac{dV}{d\theta_2}\frac{d\theta_2}{d\theta_1}$$

$$\frac{d^2V}{d\theta_1^2} = \left(\frac{d\theta_2}{d\theta_1}\right)^T \frac{d^2V}{d\theta_2^2}\frac{d\theta_2}{d\theta_1} + \frac{dV}{d\theta_2}\frac{d^2\theta_2}{d\theta_1^2} \tag{4.8}$$

The last term in (4.8) is written just in a formal way. Since $dV/d\theta_2$ vanishes in the minimum point

$$\frac{d^2V}{d\theta_1^2} = \left(\frac{d\theta_2}{d\theta_1}\right)^T \frac{d^2V}{d\theta_2^2}\left(\frac{d\theta_2}{d\theta_1}\right) \tag{4.9}$$

Similarly the information matrix J_1 for the model structure M_1 becomes

$$J_1 = \left(\frac{d\theta_2}{d\theta_1}\right)^T J_2 \left(\frac{d\theta_2}{d\theta_1}\right) \tag{4.10}$$

Evidently to compare the expressions of $EV(\hat{\theta}_N)$ for M_1 and M_2 is nothing but to compare $tr(d^2V/d\theta_1^2)J_1^{-1}$ and $tr(d^2V/d\theta_2^2)J_2^{-1}$. To do this comparison the following lemma will be utilized. The proof is rather long and given in Appendix 1.

Lemma 4.1. Let A and B be two symmetric and nonnegative definite matrices of the same dimensions. Assume that the nullspace of B lies in the nullspace of A. Let R be an arbitrary matrix of proper dimensions. Then

$$tr\ AB^+ \geq tr(R^TAR)(R^TBR)^+ \tag{4.11}$$

where B^+ denotes the pseudoinverse of B, see e.g. Zadeh-Desoer (1963) for a definition*. Moreover, equality in (4.11) holds if and only if

$$AB^+ = AR(R^TBR)^+R^TBB^+ \tag{4.12}$$

□

The application of the lemma is given in the following theorem.

---

*If B^{-1} exists, then $B^+=B^{-1}$.

Theorem 4.1. Consider two model structures satisfying (4.6)

$$M_1 \subset M_2$$

Denote the matrices of second order derivatives of an arbitrary appropriate loss function by V_1'' and V_2'' respectively and the information matrices in the same way by J_1 and J_2. Then

$$\text{tr } V_1''J_1^{-1} \leq \text{tr } V_2''J_2^{-1} \tag{4.13}$$

Proof. With use of (4.9) and (4.10) the assertion follows from the lemma by taking $A = V_2''$, $B = J_2^{-1}$, $R = d\theta_2/d\theta_1$.
□

The conclusion of the theorem is that it is preferable from the accuracy point of view to use as small a model structure M as possible. The only restriction is, of course, that M must be chosen such that the system S is $PI(M,I,X)$, i.e. such that the set $D_T(S,M)$, see (2.10), contains a point.

This result is a confirmation of the popular opinion that the model structure is not to be chosen unnecessarily large. However, here a proof of this fact is presented. Note that the result is independent of the experimental conditions. It does not matter e.g. if the process is operating in open loop or in closed loop. Note also that the analysis holds for multivariable systems and for a general class of models.

5. INFLUENCE OF THE IDENTIFICATION METHOD ON THE ACCURACY.

There exist great many different identification methods. One reason for this is that the identification method has influence on the accuracy. In particular, those giving minimal variance on the parameter estimates are called statistically efficient. If it is important to obtain as good an estimate as possible such a method must be used. The maximum likelihood method is statistically efficient, see e.g. Åström-Bohlin (1965).

In this section the influence of the identification method on the accuracy will be treated for single input-single output systems operating in closed loop. The model structure (2.2') will be considered. As mentioned earlier there are essentially two ways to perform the identification (see Section 2):

- direct identification, I_1,
- indirect identification, I_2.

It has been earlier assumed that the ML method is used in I_1. To get as good an estimate as possible when I_2 is used, the closed loop system must be identified with an efficient method. It is assumed in the following that this is the case.

The following facts about the indirect identification method I_2 can be stated.

- I_2 is not always applicable (e.g. if there is noise in the regulator).
- I_2 requires special programs to solve the open loop characteristics from knowledge of the regulator and the closed loop characteristics.

- It will be shown that for some cases I_2 will give the same accuracy as direct identification I_1.

In the analysis in this section it is generally assumed that M and X are such that the system S is $PI(M,I,X)$ and that M and S have the same structure (order). The experimental condition X may be $X_2(r,\cdot)$ (i.e. shifts between r different regulators) or $X_3(\cdot)$ (i.e. one regulator and an external input signal). In order to simultaneously treat both the cases assume that

$$u(t) = u_1(t) + \frac{G_i(q^{-1})}{F_i(q^{-1})} y(t) \qquad (i = 1 \dots r) \tag{5.1}$$

in $100\gamma_i$ percent of the experiment time. The cases $u_1(t) \equiv 0$, i.e. X_2, or $G_i(q^{-1}) \equiv 0$ some i are thus allowed.

With use of (5.1) the closed loop system with regulator number i is given by

$$y(t) = y^{(i)}(t) = \frac{q^{-k}B(q^{-1})F_i(q^{-1})}{H_i(q^{-1})} u_1(t) + \frac{C(q^{-1})F_i(q^{-1})}{H_i(q^{-1})} e(t) \tag{5.2}$$

$$u(t) = u^{(i)}(t) = \frac{A(q^{-1})F_i(q^{-1})}{H_i(q^{-1})} u_1(t) + \frac{C(q^{-1})G_i(q^{-1})}{H_i(q^{-1})} e(t) \tag{5.3}$$

where

$$H_i(z) = A(z)F_i(z) - z^k B(z)G_i(z) \tag{5.4}$$

(5.2) and (5.3) are assumed to hold when the regulator is $G_i(q^{-1})/F_i(q^{-1})$ $(i = 1 \ldots r)$. Since this regulator is used in $100\gamma_i$ percent of time the covariance element $r_y(\tau)$ is to be interpreted as

$$r_y(\tau) = \sum_{i=1}^{r} \gamma_i r_y^{(i)}(\tau) \tag{5.5}$$

Similar expressions will be used for other covariance elements.

Consider now the method I_1. Since it is efficient (3.1) and (3.2) imply that the covariance matrix of $\hat{\theta}$ becomes

$$P_{I_1} = \frac{\lambda^2}{N}[E\varepsilon'^T(t)\varepsilon'(t)]^{-1} \tag{5.6}$$

where

$$\varepsilon'(t) = \Big[\frac{1}{C(q^{-1})} y(t-1) \ldots \frac{1}{C(q^{-1})} y(t-n_a),$$
$$- \frac{1}{C(q^{-1})} u(t-k-1) \ldots - \frac{1}{C(q^{-1})} u(t-k-n_b),$$
$$- \frac{1}{C(q^{-1})} e(t-1) \ldots - \frac{1}{C(q^{-1})} e(t-n_c)\Big]$$

cf. Åström-Bohlin (1965).

The expression (5.6) is to be interpreted as (use (5.5))

$$P_{I_1} = \frac{\lambda^2}{N}\left[\sum_i \gamma_i E\, \varepsilon^{(i)'T} \varepsilon^{(i)'}\right] \tag{5.7}$$

where

$$\varepsilon^{(i)'}(t) = \left[\frac{1}{C(q^{-1})} y^{(i)}(t-1) \; \ldots \; \frac{1}{C(q^{-1})} y^{(i)}(t-n_a), \right.$$

$$- \frac{1}{C(q^{-1})} u^{(i)}(t-k-1) \; \ldots \; - \frac{1}{C(q^{-1})} u^{(i)}(t-k-n_b),$$

$$\left. - \frac{1}{C(q^{-1})} e(t-1) \; \ldots \; - \frac{1}{C(q^{-1})} e(t-n_c)\right]$$

Consider now the method I_2. The first step of the identification means that the polynomials

$$AF_i - z^k BG_i \triangleq A_i^*, \quad z^k BF_i \triangleq z^k B_i^*, \quad CF_i \triangleq C_i^* \quad i = 1 \ldots r$$

are estimated. The sequence of B_i^* is to be omitted if no external input is used. It is now further assumed that for each i there is no common factor to all three polynomials A_i^*, B_i^* and C_i^*. The $\hat{A}$, $\hat{B}$ and $\hat{C}$ can be computed from

$$\begin{aligned} \hat{A}F_i - z^k \hat{B}G_i &= A_i^* \\ \hat{B}F_i &= B_i^* \qquad\qquad i = 1 \ldots r \\ \hat{C}F_i &= C_i^* \end{aligned} \tag{5.8}$$

Equation (5.8) can be interpreted as a system of linear equations, namely

$$A\hat{\theta} = \begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix} \hat{\theta} = \begin{bmatrix} b_1 \\ \vdots \\ b_r \end{bmatrix} = b \qquad (5.9)$$

where

$$A_i = \left[\begin{array}{c:c:c}
\begin{matrix} 1 & & & 0 \\ f_{i1} & \ddots & & \\ \vdots & \ddots & 1 & \\ & \ddots & f_{i1} & \\ 0 & & \ddots & \vdots \end{matrix} &
\begin{matrix} -g_{i0} & & & 0 \\ -g_{i1} & \ddots & & \\ \vdots & \ddots & -g_{i0} & \\ & \ddots & -g_{i1} & \\ 0 & & \ddots & \vdots \end{matrix} &
0 \\ \hdashline
0 &
\begin{matrix} 1 & & & 0 \\ f_{i1} & \ddots & & \\ \vdots & \ddots & 1 & \\ & \ddots & f_{i1} & \\ 0 & & \ddots & \vdots \end{matrix} &
0 \\ \hdashline
0 & 0 &
\begin{matrix} 1 & & & 0 \\ f_{i1} & \ddots & & \\ \vdots & \ddots & 1 & \\ & \ddots & f_{i1} & \\ 0 & & \ddots & \vdots \end{matrix}
\end{array}\right]$$

$$b_i = \begin{bmatrix} a^*_{i1} - f_{i1} \\ a^*_{i2} - f_{i2} \\ \vdots \\ \hline b^*_{i1} \\ b^*_{i2} \\ \vdots \\ \hline c^*_{i1} - f_{i1} \\ c^*_{i2} - f_{i2} \\ \vdots \end{bmatrix}$$

Equation (5.9) is an overdetermined linear system of equations. The estimate $\hat{\theta}$ without bias and with smallest variance is the Markov estimate

$$\hat{\theta} = \left[A^T[\mathrm{Var}\ b]^{-1}A\right]^{-1}A^T[\mathrm{Var}\ b]^{-1}b \tag{5.10}$$

which gives

$$P_{I_2} = \left[A^T(\mathrm{Var}\ b)^{-1}A\right]^{-1} \tag{5.11}$$

The expression in (5.11) is easily rewritten as

$$P_{I_2} = \left\{\left[A_1^T \ \dots \ A_r^T\right]\begin{bmatrix} (\mathrm{Var}\ b_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & (\mathrm{Var}\ b_r)^{-1}\end{bmatrix}\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}\right\}^{-1} =$$

$$= \left\{\sum_{I}^{r} A_i^T(\mathrm{Var}\ b_i)^{-1}A_i\right\}^{-1} \tag{5.12}$$

However,

$$\text{Var } b_i = \frac{\lambda^2}{\gamma_i N}\left[E\varphi_i\varphi_i^T\right]^{-1} \tag{5.13}$$

where

$$\varphi_i^T = \left[\frac{1}{C_i^*(q^{-1})}\, y^{(i)}(t-1) \ldots \frac{1}{C_i^*(q^{-1})}\, y_i^{(i)}(t-n_{a_i^*}),\right.$$

$$-\frac{1}{C_i^*(q^{-1})}\, u_1(t-k-1) \;\ldots\; -\frac{1}{C_i^*(q^{-1})}\, u_1(t-k-n_{b_i}^*),$$

$$\left.-\frac{1}{C_i^*(q^{-1})}\, e(t-1) \;\ldots\; -\frac{1}{C_i^*(q^{-1})}\, e(t-n_{c_i}^*)\right]$$

Thus

$$P_{I_2} = \frac{\lambda^2}{N}\left\{\sum_1^r \gamma_i A_i^T E\varphi_i\varphi_i^T A_i\right\}^{-1} = \frac{\lambda^2}{N}\left\{\Sigma\gamma_i E(A_i^T\varphi_i)(\varphi_i^T A_i)\right\}^{-1}$$

But

$$\varphi_i^T A_i = \frac{1}{C_i^*(q^{-1})}\,[F_i(q^{-1})y^{(i)}(t-1),\ldots,F_i(q^{-1})y^{(i)}(t-n_a),$$

$$-G_i(q^{-1})y^{(i)}(t-k-1) - F_i(q^{-1})u_1(t-k-1),\ldots,$$

$$-G_i(q^{-1})y^{(i)}(t-k-n_b) - F_i(q^{-1})u_1(t-k-n_b),$$

$$-F_i(q^{-1})e(t-1),\ldots,-F_i(q^{-1})e(t-n_c)]$$

But (5.1) and (5.8) give

$$\frac{F_i}{C_i^*} = \frac{1}{\hat{C}} = \frac{1}{C}$$

$$-\frac{G_i(q^{-1})}{C_i^*(q^{-1})} y^{(i)}(t) - \frac{F_i(q^{-1})}{C_i^*(q^{-1})} u_1(t) = -\frac{F_i(q^{-1})}{C_i^*(q^{-1})} u^{(i)}(t) = -\frac{1}{C(q^{-1})} u^{(i)}(t)$$

Thus

$$\varphi_i^T A_i = \varepsilon^{(i)'}$$

which finally implies

$$P_{I_1} = P_{I_2} \tag{5.14}$$

Thus it has been proved that for appropriate assumptions the direct identification method I_1 and the indirect identification method I_2 yield the same accuracy. The basic assumption was that for

$$A_i^* = AF_i - z^k BG_i$$

$$B_i^* = BF_i$$

$$C_i^* = CF_i$$

there is no common factor to all the three polynomials. This implies that the closed loop system is parameter identifiable for every single regulator so that the matrix Var b_i is non-singular.

6. INFLUENCE OF THE EXPERIMENTAL CONDITION ON THE ACCURACY.

6.1. Introduction.

As mentioned earlier the three concepts, the model structure M, the identification method I and the experimental condition X will influence the identifiability of a system as well as the accuracy when it is identifiable. In the previous sections it has been shown that simple rules can be used for determining how M and I should be chosen in order to obtain the greatest possible accuracy.

In this section the influence of X will be discussed. In one sense the experimental condition will have a greater influence on the result. If it is found that an X does not yield the best possible accuracy then the whole identification experiment has to be repeated. In contrast, if M (or I) was not chosen in a suitable manner it is possible to try other M (or I) on the sama data.

Special attention will be paid to the choice of an optimal experimental condition. This problem can be regarded as a generalization of optimal input design, which has been studied in a great number of papers, e.g. Levin (1960), Levadi (1966), Gagliardi (1967), Aoki-Staley (1970), Nahi--Napjus (1971), Goodwin-Murdoch (1972), Keviczky-Bányász (1973), Goodwin-Payne (1973), Ljubojevič (1973) and Mehra (1974). In all these papers the only experimental condition at the designer's disposal has been the characteristics of the input signal. The possibility to change the experimental conditions by for example introducing a feedback has not been considered.

In the papers mentioned above the optimal input design problem has been discussed for both continuous and discrete time systems, in both the frequency and time domains, and with different constraints on the signals. In most of the papers the input energy or amplitude has been constrained. Only a few papers, e.g. Aoki-Staley (1970), consider the output constrained case,which seems to be the more natural choice. Most of the analysis here is carried out for this case. For the sake of completeness the case of constrained input variance is considered in Section 6.4. The obtained results have essential differences.

The work on optimal input design has given valuable insight into the more general problem of optimal experimental design. It has resulted in a number of algorithms which are valuable also in a more general setting of the problem as proposed here. Also notice that in order to solve the general problem of optimal experimental design also e.g. the choice of sampling rate should be included, cf. Goodwin-Zarrop-Payne (1974). In order to amplify the importance of feedback for optimal experimental design the influence of the sampling rate is not considered in this report.

Since the problems lead to complicated calculations only first order systems will be discussed in the sequel. Two systems will be considered. The following systems

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + e(t) \tag{6.1}$$

and

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + (1+cq^{-1})e(t) \tag{6.2}$$

are treated. In the whole section it is assumed that M is chosen minimal, i.e.

$$(1+\hat{a}q^{-1})y(t) = \hat{b}q^{-1}u(t) + \varepsilon(t) \tag{6.3}$$

and

$$(1+\hat{a}q^{-1})y(t) = \hat{b}q^{-1}u(t) + (1+\hat{c}q^{-1})\varepsilon(t) \tag{6.4}$$

respectively.

6.2. First Order Systems with White Noise.

In this section systems of the form (6.1) will be treated

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + e(t)$$

The sets X_1 (open loop), $X_2(r,n)$ (r different regulators of order n) and $X_3(n)$ (one regulator of order n and an external input signal) have elements which are the parameters of the regulator(s) and/or the covariance function of the (external) input signal. Thus it is trivial that

$$X_2(1,\cdot) \subset X_2(r,\cdot) \tag{6.5}$$

$$X_2(1,\cdot) \subset X_3(\cdot) \tag{6.6}$$

$$X_1 \subset X_3(\cdot) \tag{6.7}$$

Let $\{J(X)\}$ denote the set of possible information matrices generated by all the elements in the set X. With this notation the following theorem is formulated.

Theorem 6.1. The sets $\{J(X)\}$ for the different sets X satisfy

$$\Big\{J(X_1)\Big\} \subset \Big\{J\big(X_2(1,\cdot)\big)\Big\} = \Big\{J\big(X_2(r,\cdot)\big)\Big\} = \Big\{J\big(X_3(\cdot)\big)\Big\} \tag{6.8}$$

□

The proof is somewhat technical and is given in Appendix 2. The result is very strong. It means e.g. that given an information matrix obtained with $X_3(\cdot)$, then it is possible to find a regulator in $X_2(1,\cdot)$ giving the same information matrix. In particular this means that in searching for optimal regulators the optimization can be performed in $X_2(1,\cdot)$, $X_2(r,\cdot)$ or in X_3 without affecting the result. In the following, the optimization will be done in $X_2(1,\cdot)$. Then the optimal J will be considered. How this J can be obtained in the simpler sets of experimental conditions $X_2(2,0)$ and $X_3(0)$ will be examined in particular.

It follows from the proof of Theorem 6.1 that when $X_2(1,\cdot)$ is used J can be parameterized as follows

$$J = \frac{N}{\lambda^2}\begin{bmatrix} r_y(0) & -r_{yu}(0) \\ -r_{yu}(0) & r_u(0) \end{bmatrix} =$$

$$= N\begin{bmatrix} 1 + b^2 r & -\left(\frac{a}{b} + \eta + br(a+\rho)\right) \\ -\left(\frac{a}{b} + \eta + br(a+\rho)\right) & \frac{a^2}{b^2} + r(1+a^2+2a\rho) + 2\frac{a}{b}\eta \end{bmatrix} \qquad (6.9)$$

The variables r, ρ and η are formally defined by

$$r = Ev(t)^2$$

$$r\rho = Ev(t)v(t-1)$$

$$\eta = Ev(t)e(t)$$

where

$$v(t) = \frac{G(q^{-1}) - \frac{a}{b} F(q^{-1})}{A(q^{-1})F(q^{-1}) - bq^{-1}G(q^{-1})} e(t)$$

The variables are constrained by

$$0 \leq r \qquad \rho^2 \leq 1 - \frac{\eta^2}{r} \tag{6.10}$$

When optimizing different criteria for the accuracies some constraints must be added. In design of optimal input signals it is e.g. common to limit the values of the input or the output. Here it seems to be reasonable to require that the variance of the output is bounded, say

$$r_y(0) \leq \lambda^2(1+\delta) \tag{6.11}$$

where δ is a positive number. If $\delta = 0$ then the system must be controlled with a minimum variance strategy. Thus (6.10) will be replaced by

$$0 \leq r \leq \frac{\delta}{b^2} \qquad \rho^2 \leq 1 - \frac{\eta^2}{r} \tag{6.12}$$

Now two scalar functions of J, given by (6.9), will be optimized with respect to r, ρ and η under the constraints (6.12).

Consider first optimization of the determinant of J.

It is easy to establish that

$$\frac{1}{N^2} \det J = \left[b^2r^2(1-\rho^2) + r - 2br\eta\rho - \eta^2\right]$$

$$= \left[r(b^2r+1) - (\eta+br\rho)^2\right] \tag{6.13}$$

Clearly, det J is maximized by the following choice

$$r = \frac{\delta}{b^2} \tag{6.14}$$

$$\eta + \frac{\delta\rho}{b} = 0 \tag{6.15}$$

which imply

$$J = N \cdot \begin{bmatrix} 1+\delta & -\frac{a}{b}(1+\delta) \\ -\frac{a}{b}(1+\delta) & \frac{a^2}{b^2}(1+\delta) + \frac{\delta}{b^2} \end{bmatrix} \tag{6.16}$$

$$\det J = N^2 \frac{1}{b^2} \delta(1+\delta) \tag{6.17}$$

Note that there are several η and ρ that satisfy (6.12) and (6.15).

Consider now the loss function $V_3(\hat{\theta})$, given in Example 3.3. This function satisfies

$$E\, V_3(\hat{\theta}) \approx E\, \tilde{V}_3(\hat{\theta}) = \lambda^2 + \lambda^2 \operatorname{tr} \begin{bmatrix} 1 & -\frac{a}{b} \\ -\frac{a}{b} & \frac{a^2}{b^2} \end{bmatrix} J^{-1} \tag{6.18}$$

which after use of (6.9) and straightforward calculations reduces to

$$\lambda^2 + \frac{\lambda^2}{N} \frac{r}{r(b^2r+1)-(br\rho+\eta)^2} \tag{6.19}$$

and it is easy to see that this quantity is optimized subject to the constraints (6.12) if and only if (6.14) and

(6.15) are satisfied.

The remaining part of this subsection is devoted to "solutions" of (6.16), i.e. to an examination of what regulators and input signals give this optimal information matrix.

It follows from the proof of Theorem 6.1 that to "solve" (6.16) is equivalent to require that

$$r_y(0) = \lambda^2(1+\delta) \tag{6.20}$$

$$r_{yu}(0) = \lambda^2 \frac{a}{b}(1+\delta) \tag{6.21}$$

Consider first the set $X_2(2,0)$, i.e. shift between two proportional regulators. With use of equation (5.2) it is found that (6.20) means that

$$E\left[\frac{1}{1+(a-bg_{i0})q^{-1}}\, e(t)\right]^2 = \frac{\lambda^2}{1-(a-bg_{i0})^2} = \lambda^2(1+\delta)$$

or

$$g_{i0} = \frac{a \pm \sqrt{\frac{\delta}{1+\delta}}}{b} \qquad i = 1,2 \tag{6.22}$$

Thus the two regulators are entirely specified from (6.22).

The remaining equation, (6.21), will determine what relative influence the two regulators G_1 and G_2 will have, i.e. determine the two numbers γ_1 and γ_2. (These numbers satisfy by definition $\gamma_1 + \gamma_2 = 1$.) With use of (5.3)

$$\lambda^2 \frac{a}{b}(1+\delta) = \lambda^2 \gamma_1 \frac{g_{10}}{1-(a-bg_{10})^2} + \lambda^2 \gamma_2 \frac{g_{20}}{1-(a-bg_{20})^2} \quad (6.23)$$

which can be rewritten as

$$\lambda^2 \frac{a}{b}(1+\delta) = \gamma_1 \lambda^2 \frac{(1+\delta)}{b}\left[a + \sqrt{\frac{\delta}{1+\delta}}\right] +$$

$$+ \gamma_2 \lambda^2 \frac{(1+\delta)}{b}\left[a - \sqrt{\frac{\delta}{1+\delta}}\right]$$

or simplified

$$\gamma_1 - \gamma_2 = 0$$

Thus

$$\gamma_1 = \gamma_2 = 0.5 \quad (6.24)$$

The resulting experimental condition means that two regulators are each used 50% of the total time and each gives precisely the maximal allowed variance of the output signal. This experimental configuration is illustrated by the Figure 6.1.

Consider now the set $X_3(0)$. The choice

$$G = \frac{a}{b}$$

$u_1(t)$ white noise of variance $\frac{\lambda^2 \delta}{b^2}$

can easily be seen to give an information matrix (6.16). A thorough analysis would show that the experimental condition above is one of several possible choices to satisfy

(6.16). It is e.g. not necessary to use the minimum variance regulator $G = a/b$, cf Example 7.3.

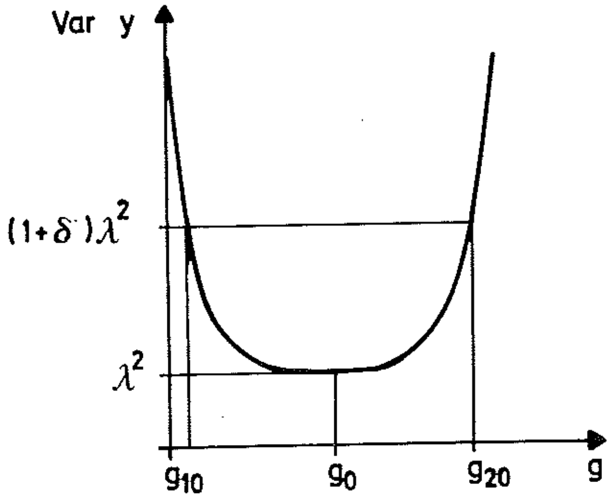

Fig. 6.1 - Illustration of optimal experimental configuration in $X_2(r,\cdot)$. g denotes the regulator parameter. g_0 is the minimum variance regulator. The optimal configuration is obtained by taking g_{10} 50% of the time and g_{20} in the remaining 50%.

Finally, in order to compare the result with optimal input design the set X_1 will be considered. Since X_1 means open loop system there is no regulator and the constraint on the variance of y(t) implies that

$$\frac{\lambda^2}{1 - a^2} \leqslant \lambda^2(1+\delta) \qquad (6.25)$$

or

$$\delta \geqslant \frac{a^2}{1 - a^2}$$

is necessary in order to meet this condition at all. Introduce

$$R_0 = E\left[\frac{1}{1 + aq^{-1}} u(t)\right]^2$$

$$R_1 = E\left[\frac{1}{1 + aq^{-1}} u(t)\right]\left[\frac{q^{-1}}{1 + aq^{-1}} u(t)\right]$$

Then (6.20) and (6.21) can be written as

$$b^2 R_0 + \frac{\lambda^2}{1 - a^2} = \lambda^2 (1+\delta) \tag{6.26}$$

$$b(R_0 a + R_1) = \frac{a}{b} \lambda^2 (1+\delta) \tag{6.27}$$

Straightforward calculations give that

$$R_0 = \frac{\lambda^2}{b^2(1-a^2)} [\delta - a^2 - a^2\delta] \tag{6.28}$$

$$R_1 = \frac{a\lambda^2}{b^2(1-a^2)} \tag{6.29}$$

If (6.28), (6.29) correspond to a physical solution it is necessary to require that $R_0 \geq 0$ (which is (6.25)) and

$$|R_1| \leq R_0 \tag{6.30}$$

Clearly, (6.30) is a stronger condition than (6.25). It can be simplified to

$$|a| \leq \delta - a^2 - a^2\delta$$

or

$$\delta \geq \frac{|a|}{1 - |a|} \qquad (6.31)$$

The inequalities (6.25) and (6.31) have the following sense. In order to meet the constraint (6.25) on $r_y(0)$ it is necessary that $\delta \geq a^2/(1-a^2)$. In order to achieve optimal accuracy, δ must be larger, namely $\delta \geq |a|/(1-|a|)$. Cf. also Figure 6.2. The result is striking. It means that although open loop experiments may be possible with respect to the constraints it can be more advantageous to use closed loop experiments!

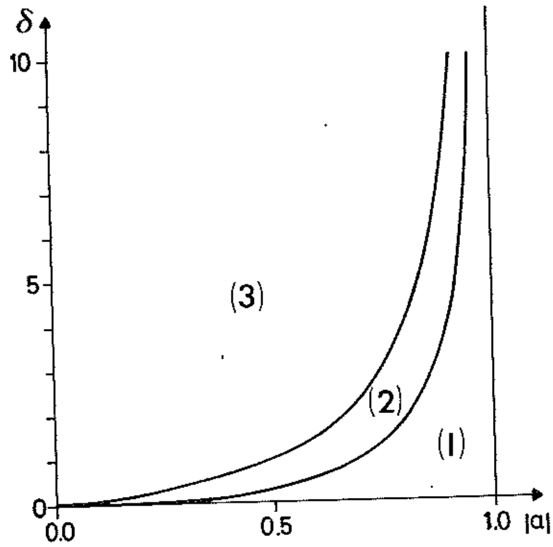

Fig. 6.2 - Illustration of the inequalities (6.25) and (6.31).
In region (1) it is impossible to use open loop experiments.
In region (2) it is unfavourable to use open loop experiments.
In region (3) open loop experiments can give optimal accuracy.

6.3. First Order Systems with Coloured Noise.

In this section systems of the form (6.2)

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + (1+cq^{-1})e(t)$$

are treated.

Restriction is made to the experimental conditions X_1, $X_2(2,0)$ and $X_3(0)$. This is done mainly for technical reasons. However, in view of the results of Section 6.2 it is natural if this choice of X can give optimal accuracy.

Lemma 6.1. Consider the criterion det J under the constraint $Ey^2(t) \leq \lambda^2(1+\delta)$. Then

$$\frac{1}{N^3} \sup_{X_1} \det J \leq \frac{\delta^2}{b^2(1-c^2)^3} \tag{6.32}$$

$$\frac{1}{N^3} \sup_{X_2(2,0)} \det J = \frac{\delta^2}{b^2(1-c^2)^3} \tag{6.33}$$

$$\frac{1}{N^3} \sup_{X_3(0)} \det J = \frac{\delta^2}{b^2(1-c^2)^3} \tag{6.34}$$

Equality in (6.32) is obtained if and only if the system fulfils a = c (a very hard requirement). Moreover, if a = c, then the optimum is reached when the signal

$$v(t) = \frac{1}{(1+cq^{-1})^2} u(t)$$

fulfils

$$Ev(t)^2 = \frac{\lambda^2\delta(1+c^2)}{b^2(1-c^2)^2} \qquad Ev(t)v(t-1) = -\frac{2c\lambda^2\delta}{b^2(1-c^2)^2} \tag{6.35}$$

Optimum in (6.33) is reached by two proportional regulators, both giving $r_y(0) = \lambda^2(1+\delta)$ and both used 50% of the total experiment time. To be precise the regulators are given by

$$g_{i0} = \frac{a}{b}\left[1 - \frac{c}{1+\delta} \pm \frac{\sqrt{\delta(1-c^2+\delta)}}{1+\delta}\right] \qquad i = 1,2 \tag{6.36}$$

Optimum in (6.34) is obtained if and only if $g_0 = (c-a)/b$ (i.e. the regulator is a minimum variance controller) and the signal

$$v(t) = \frac{1}{(1+cq^{-1})^2}\,u_1(t)$$

fulfils (6.35).

□

The proof consists of long calculations and is given in Appendix 3.

Remark. Note that (6.35) is in particular satisfied if $u(t)$ [or $u_1(t)$ respectively] is white noise with variance $\lambda^2\delta(1-c^2)/b^2$.

The inequality (6.32) implies a difference between the systems (6.1) and (6.2). For systems with a c-parameter not equal to a, it is never advantageous to use open loop experiments. There is no analogy to the inequality (6.31). As long as an upper bound on $r_y(0)$ is given the best accuracy is always obtained by closed loop experiments. Conversely, suppose that some given accuracy (given as det J) is required. Then every open loop experiment will require a larger variance of the output signal than optimal closed loop experiments.

6.4. Optimization with Constrained Input Variance.

As mentioned earlier it is the authors' opinion that it is more natural to require that the variance of the output is bounded than that the variance of the input is so. Nevertheless it is instructive to see how this change of the constraint will influence the result.

Consider the system (6.1)

$$(1+aq^{-1})y(t) = bq^{-1}u(t) + e(t)$$

and the criterion det J under the constraint

$$r_u(0) \leq \lambda^2\delta/b^2 \tag{6.37}$$

where δ is a positive number.

Now the following result can be stated.

Lemma 6.2. Consider the system (6.1), the criterion det J and the constraint (6.37). Then

$$\sup_{X_1} \det J = \sup_{X_2(1,\cdot)} \det J = \sup_{X_2(r,\cdot)} \det J =$$

$$= \sup_{X_3(\cdot)} \det J = \sup_{X_2(2,0)} \det J = N^2 \frac{\delta(\delta+1-a^2)}{b^2(1-a^2)^2} \tag{6.38}$$

□

The proof consists of long calculations and is given in Appendix 4.

Remark. Note that in contrast to the results in Section 6.2 the lemma implies that it is not disadvantageous to use open loop operation (provided of course that the system is stable). This means in particular that for the present case there is no analog to the regions (1) and (2) in Figure 6.2.

7. EXAMPLES

In this section some numerical examples related to the discussed problem will be presented. The intention is not to give extensive simulations illustrating all the results given in the previous sections. Instead only a few points will be illuminated here.

Example 7.1.

The simulations presented in this example are taken from the Master Thesis, Elvgren - Krantz (1974). The system

$$y(t) + 0.8y(t-1) = u(t-1) + e(t) \tag{7.1}$$

has been simulated with different choices of the input signal. Open loop experiments as well as closed loop experiments using different types of regulators were performed. The noise e(t) was a sequence of normally distributed random numbers. The parameters $\hat{a}$ and $\hat{b}$ of the model

$$y(t) + \hat{a}y(t-1) = \hat{b}u(t-1) + \varepsilon(t) \tag{7.2}$$

have been fitted to the data using the maximum likelihood method, which in this case is identical to the least squares method. In Figure 7.1 the determinant of the estimated information matrix is plotted versus the output variance for a number of different input signals.

The intention with this example is to study if there are any systematic differences between different experimental conditions that all yield identifiability. Normally, the reason for feedback control is to decrease the variance of the output. This will of course also decrease the information contents of the measured data and give less accurate estimates. However, if different experimental conditions are compared for the same output variance the previous analysis on simple examples has shown that there is no systematic difference for the experimental conditions considered. In this example also non-linear regulators have been used. From Figure 7.1 it is seen that this does not violate the conclusion above. The accuracy is more a question of how the experimental condition is chosen in the set of experimental conditions considered than which set is chosen. In the same way as a poor choice of the input signal gives inaccurate estimates in an open loop experiment, unsuitable regulator parameters will result in poor estimates.

Example 7.2.

Also in this example the system (7.1) will be used. The system is simulated in open loop, and in closed loop with an additional input signal and a regulator,

$$u(t) = 0.6y(t) + u_1(t) \tag{7.3}$$

The amplitude of the input signal (PRBS) for the open loop experiment was chosen so that the output variance became 5.11. Direct identification with the maximum likelihood method gave the following estimates for 1000 samples,

$\hat{a}$: 0.771 ± 0.017

$\hat{b}$: 1.048 ± 0.036

$\hat{c}$: −0.022 ± 0.036

The estimated standard deviations have also been given.

For the closed loop experiment the additional input signal was the same PRBS with the amplitude chosen so that the output variance became approximately the same as in the open loop experiment (the actual output variance was 5.06). The data were used both for direct and indirect

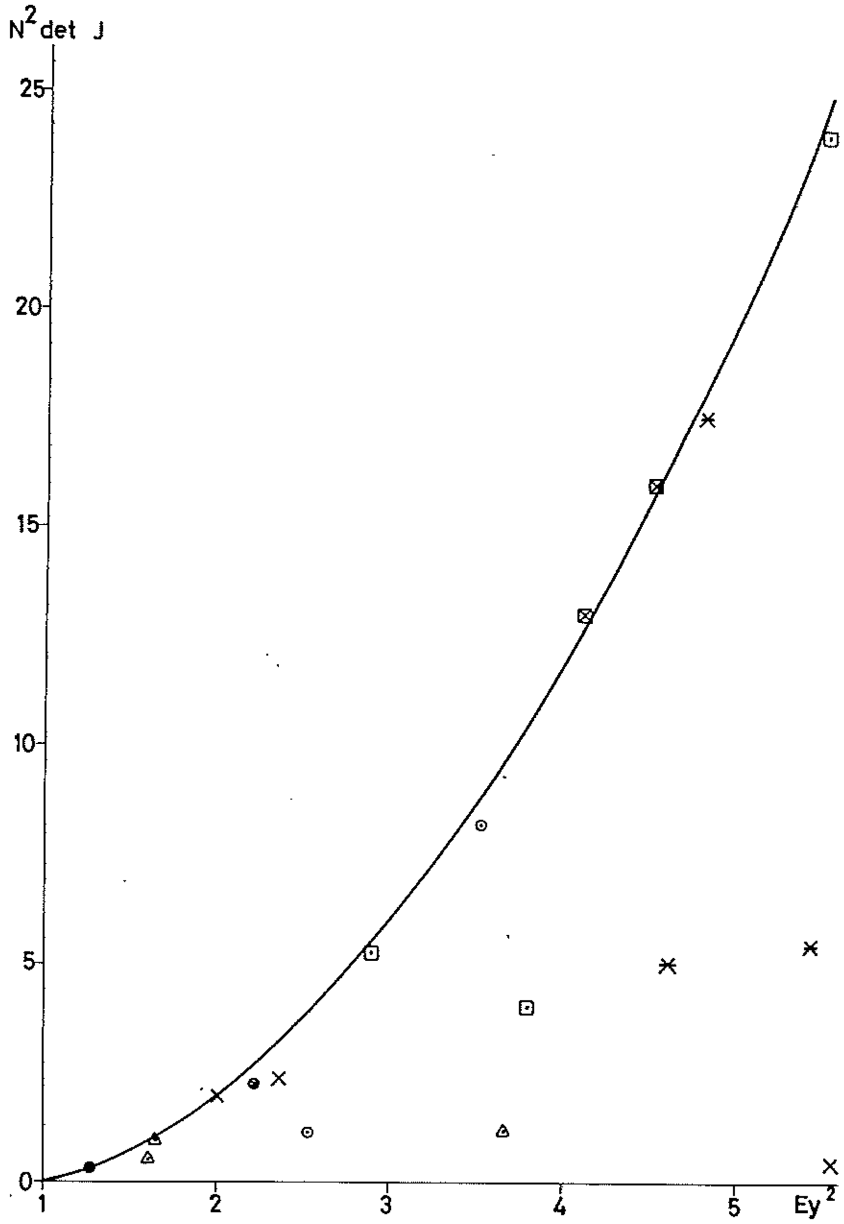

Fig. 7.1. The determinant of the information matrix as a function of the output variance obtained from simulations of the system (7.1) with various input signals. The number of data used in each simulation has been N=2000.

⋇ : Open loop experiments.

× : Proportional regulator and extra input signal.

⊠ : Regulator shifting between two linear feedback laws; ten shifts.

⊡ : As above, but just one shift.

△ : Linear constant regulator of order one.

⊙ : Nonlinear regulator: Proportional with u(t)<C.

⊗ : As above, but |u(t)|<C.

The curve corresponds to the optimum value, that can be achieved, cf (6.17).

identification. The direct identification gave the following result,

$\hat{a}$: 0.779 ± 0.019

$\hat{b}$: 1.022 ± 0.017

$\hat{c}$: -0.034 ± 0.036

Indirect identification gave the following estimates of the parameters of the closed loop system,

$\widehat{a-bf}$: 0.166 ± 0.016

$\hat{b}$: 1.022 ± 0.017

$\hat{c}$: -0.034 ± 0.036

Inserting f=0.6 (the regulator parameter) gives

$\hat{a}$: 0.779 ± 0.019

It is immediately clear that direct and indirect identification for this example give identical estimates.

It is also seen that the only essential difference between the open loop and closed loop experiments is that the parameter $\hat{b}$ seems to be more accurately estimated in the closed loop situation. The reason in this case is the straightforward choice of input signal in the open loop experiment which obviously was not the optimal choice. The additional input signal in the closed loop experiment was the same as in the open loop experiment, except that the amplitude was different.

Example 7.3.

The system

$$y(t) + ay(t-1) = bu(t-1) + e(t) \tag{7.4}$$

under the experimental condition

$$u(t) = u_1(t) - ky(t) \tag{7.5}$$

will be considered.

It was shown in page 34 that the determinant of the information matrix was maximized when $u_1(t)$ was white noise and $k=-a/b$. It will now be investigated how sensitive the optimum is for the choice of $u_1(t)$ and k. Two cases will be considered:

a) $u_1(t)$ is supposed to be white noise. Its variance is chosen for any given k so that det J is maximized.

b) No a priori restriction on $u_1(t)$. For any given k the covariance function of $u_1(t)$ is chosen so that det J is maximized.

In both the cases the result will be a relation between the value of k and the value of det J.

Let

$$\alpha = a + bk \tag{7.6}$$

Then

$$(1+\alpha q^{-1})y(t) = bu_1(t-1) + e(t)$$

$$(1+\alpha q^{-1})u(t) = (1+aq^{-1})u_1(t) - ke(t)$$

Let

$$v(t) = \frac{1}{1+\alpha q^{-1}} u_1(t)$$

$$R = E\, v(t)^2$$

$$R\rho = E\, v(t)v(t-1)$$

$$V = \lambda^4 N^{-2} \det J$$

Then it follows easily

$$r_y(0) = b^2 R + \frac{\lambda^2}{1-\alpha^2} \qquad (7.7)$$

$$r_{yu}(0) = bR(\rho+a) - \frac{k\lambda^2}{1-\alpha^2}$$

$$r_u(0) = R(1+a^2+2a\rho) + \frac{k^2\lambda^2}{1-\alpha^2}$$

Then

$$V = r_y(0)r_u(0) - r_{yu}(0)^2 =$$

$$= b^2R^2(1-\rho^2) + \frac{R\lambda^2}{1-\alpha^2}(1+\alpha^2+2\alpha\rho) \qquad (7.8)$$

Let the following constraint hold

$$r_y(0) = \lambda^2(1+\delta) \qquad (7.9)$$

The results to follow will not change if an inequality sign in (7.9) is chosen.

Then from (7.7) and (7.9)

$$b^2 R = \frac{\lambda^2}{1-\alpha^2}[\delta(1-\alpha^2)-\alpha^2]$$

Inserting this in (7.8) gives

$$V = \frac{\lambda^4[\delta(1-\alpha^2)-\alpha^2]}{b^2(1-\alpha^2)^2}[\delta(1-\alpha^2)(1-\rho^2) + (1+\alpha\rho)^2] \qquad (7.10)$$

Case a) u_1 is white noise. Then

$$\rho = -\alpha$$

Inserting this in (7.10) gives

$$V(\alpha) = \lambda^4 b^{-2}[\delta-(1+\delta)\alpha^2](1+\delta) \qquad (7.11)$$

<u>Case b)</u> u_1 is not assumed to be white noise. First maximize V with respect to ρ without considering the constraint

$$|\rho| \leq 1$$

It follows

$$\frac{\partial V}{\partial \rho} \sim -2\rho\delta(1-\alpha^2) + 2\alpha(1+\alpha\rho) = 0$$

and

$$\rho = \frac{\alpha}{\delta(1-\alpha^2)-\alpha^2}$$

Inserting this value of ρ in (7.10) gives

$$V(\alpha) = \lambda^4 b^{-2}\delta(1+\delta) \qquad (7.12)$$

This solution is realizable if and only if $|\rho|\leq 1$, i.e.

$$|\alpha| \leq \delta(1-\alpha^2)-\alpha^2$$

i.e. when

$$\left|\,|\alpha| + \frac{1}{2(1+\delta)}\right| \leq \frac{1+2\delta}{2(1+\delta)}$$

or

$$|\alpha| \in (-1,\frac{\delta}{1+\delta})$$

or since $|\alpha|\geq 0$

$$|\alpha| \in (0,\frac{\delta}{1+\delta})$$

However, also $|\alpha|\in(\frac{\delta}{1+\delta},1)$ will give a stable closed loop system.

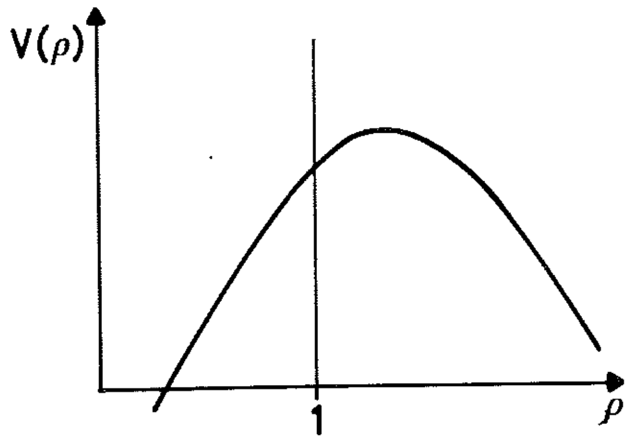

Fig.7.2. V as a function of ρ for the case $\alpha \in (\frac{\delta}{1+\delta},1)$

Then it can be seen, Fig. 7.2, that subject to the constraint $|\rho|\leq 1$

$$\rho = \mathrm{sgn}(\alpha)$$

will give the largest value to V, i.e.

$$V(\alpha) = \frac{\lambda^4[\delta(1-\alpha^2)-\alpha^2]}{b^2(1-|\alpha|)^2} \qquad (7.13)$$

To summarize, for case a) the function V is given by (7.11). For case b) V is given by (7.12) for $|\alpha|\leq\frac{\delta}{1+\delta}$ and by (7.13) for $\frac{\delta}{1+\delta}<|\alpha|<1$. The function V is plotted against α in Figure 7.3 for the cases a) and b).

This example thus illustrates that for the system (7.4) with input according to (7.5) it is possible to use different combinations of the value of the regulator parameter k and the characteristics of the additional input signal u_1.

Notice that this result is in contrast to Lemma 6.1 which holds for the system

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$

with the input signal generated by

$$u(t) = u_1(t) - ky(t)$$

In this case only the minimum variance strategy combined with u_1 as white noise will give the optimal value of the determinant.

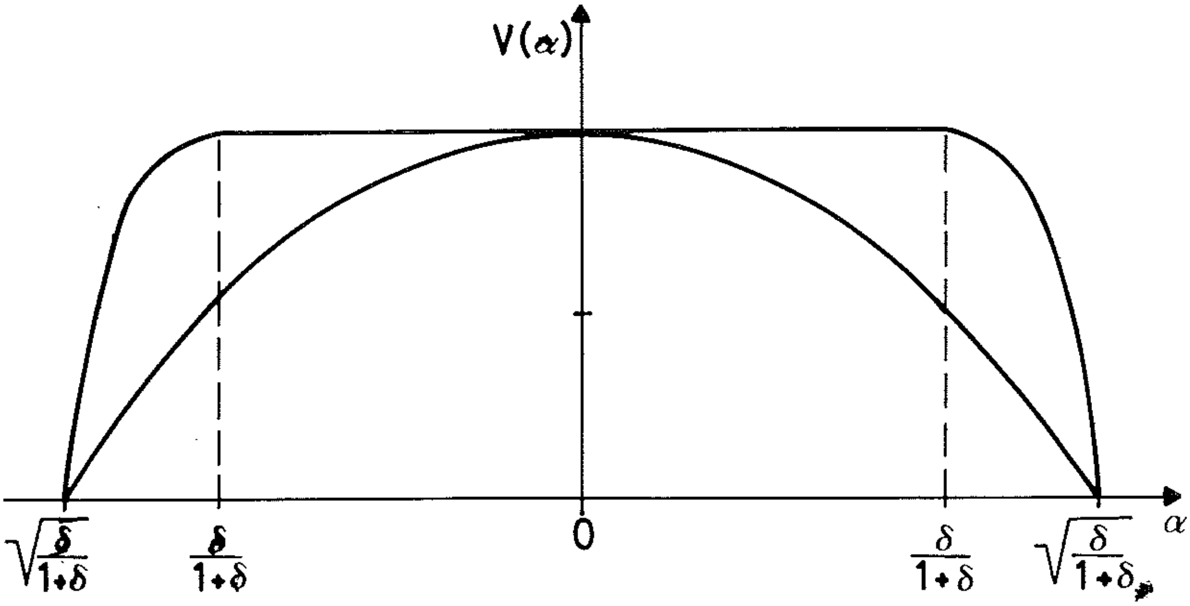

Fig. 7.3. V as a function of $\alpha = a+bk$ for the system (7.4) with input according to (7.5)

8. CONCLUSIONS.

It is commonly held, that closed loop identification experiments should be avoided. One reason may be that the identifiability properties for such experimental conditions are largely unknown. In Gustavsson-Ljung-Söderström (1974) and Ljung-Gustavsson-Söderström (1974) a concept, Strong System Identifiability (SSI) has been introduced, which extends the identifiability properties of open loop identification experiments to other classes of experimental conditions. It has also been shown that SSI is obtained also for closed loop experiments, when e.g. there is an extra perturbation signal added to the input or the regulator alternates between several linear feedback laws. Hence, from this identifiability point of view there is no reason to avoid closed loop experiments.

From a practical viewpoint, however, it is more important to consider the "degree of identifiability", i.e. accuracy aspects for the identification experiments. This has been the objective of this report.

The result of an identification experiment clearly depends on a number of items, such as the identification method used, I, the class of models, M, among which an appropriate model is fitted to the data, and the experimental conditions, X,like input signal generation etc. The influence on the accuracy of the obtained models has been investigated for each of these three items. The analysis of the influence of M is carried out for a general form of the model structure. The influence of the other two items is considered essentially for single-input single-output models on difference equation form.

It has been shown in Section 4 that the set of models should be chosen as small as possible under the obvious constraint that it must include a true description of the system. Inclusion of additional parameters decreases the accuracy of the results obtained. This result, quite expected, holds for all kinds of experimental conditions and all accuracy criteria discussed in Section 4.

The choice of identification method, I, is most often dictated by accuracy considerations. For open loop identification there are several different methods that yield a high accuracy and which have been compared in various contexts, see e.g. Åström-Eykhoff (1971), Gustavsson (1972). For system operating in closed loop there is an additional interesting problem; namely to compare a straightforwardly applied identification method (Direct Identification) with schemes that are specially designed for closed loop data (Indirect Identification).

In Section 5 it has been shown that Direct Identification using the maximum likelihood method is never inferior to Indirect Identification. In some circumstances they are equivalent. This implies that there is no need for the more complex indirect identification methods.

The most interesting question is probably the one concerning the influence of different experimental conditions. It is sometimes claimed that even if identification of a process operating in closed loop may be theoretically possible, it gives such poor accuracy that the results are questionable. In Section 6 the influence of feedback in the system on the accuracy of the estimates has been analysed for some simple examples, and in Section 7 the results of some simulations have been presented.

It is immediately clear, that since the objective of a regulator is to decrease the output variance, this tends to decrease the information contents of the measured signals and hence causes more inaccurate estimates. But, taking this effect into account, the important question then asked is: Does the mere presence of feedback deteriorate the identification results, even if the output variance is the same? The numerical results in Section 7 gives an answer to this question: No. On the contrary, the accuracy for a closed loop experiment can be better than for an open loop experiment even for an optimal input signal! For the simple examples considered in Section 6 this is the case when the output variance is constrained.

This result has a most important implication: The extensively discussed problem of optimal input design, see e.g. Aoki-Staley (1970), Nahi-Napjus (1971), Mehra (1974), is not quite well posed (under a given constraint on the output variance). By including feedback terms in the input signal which by no means poses technical problems the accuracy can be improved compared to the "optimal" non-feedback input.

The design of optimal feedback input suffers from the same dilemma as the design of an optimal non-feedback input: The true system must be known. For the feedback input (when the output variance is constrained) the following rule of thumb may be given:Design a feedback regulator that decreases the output variance as much as possible. This makes use of the available a priori information about the system. Then add as much extra input to the feedback signal (e.g. a white noise signal) as the constraint on the output variance allows. This scheme can also be applied on-line by letting the feedback regulator be adjusted to the current knowledge of the system. Notice, however, a white noise signal may not give the optimal accuracy.

9. ACKNOWLEDGEMENTS

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APPENDIX 1 - Proof of Lemma 4.1.

In order to prove Lemma 4.1 another lemma will be used.

Lemma A.1. Let A and B be two symmetric non-negative definite matrices. Then

$$\text{tr } AB \geq 0 \tag{A1.1}$$

where equality holds if and only if

$$AB = 0 \tag{A1.2}$$

□

Proof. There exists an orthogonal matrix U such that*

$$U^T BU = D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where D_1 is a positive definite diagonal matrix. (The trivial case B = 0 is not considered). Introduce

$$U^T AU = \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{12}^T & \bar{A}_{22} \end{bmatrix}$$

Thus

$$\text{tr } AB = \text{tr } U^T AUU^T BU = \text{tr } \bar{A}D = \text{tr } \bar{A}_{11}D_1 =$$

$$= \Sigma(\bar{A}_{11})_{ii}(D_1)_{ii}$$

But $\bar{A}_{11}$ is non-negative definite and thus $(\bar{A}_{11})_{ii} \geq 0$ and (A1.1) follows.

---

* where the zeroes are block matrices of proper dimensions

Assume now that tr AB = 0. Since $(D_1)_{ii} > 0$ (all i) it is necessary that $(\bar{A}_{11})_{ii} = 0$ (all i), which implies that $\bar{A}_{11} = 0$ ($\bar{A}_{11}$ is non-negative definite). Since also the whole matrix $\bar{A}$ is non-negative definite it is concluded that $\bar{A}_{12} = 0$. Thus

$$AB = U\begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{22} \end{bmatrix} U^T U \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix} U^T = 0$$

which is (A1.2).

It is trivial to see that (A1.2) implies (A1.1).

□

Proof of Lemma 4.1. There exists an orthogonal matrix U such that

$$U^T BU = D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where D_1 is a positive definite diagonal matrix. Introduce

$$U^T AU = \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{12}^T & \bar{A}_{22} \end{bmatrix}$$

The assumption concerning the null spaces implies that

$$\bar{A}_{12} = 0, \quad \bar{A}_{22} = 0 \tag{A1.3}$$

Introduce now

$$\bar{R} = \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} D_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix} U^T R = D^{1/2} U^T R$$

With use of the new variables straightforward calculations give

$$\text{tr } AB^+ = \text{tr } U\bar{A}U^T U D^+ U^T = \text{tr} \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \text{tr } \bar{A}_{11} D_1^{-1} \tag{A1.4}$$

$$\begin{aligned} \text{tr}(R^T A R)(R^T B R)^+ &= \text{tr } AR(R^T B R)^+ R^T = \\ &= \text{tr } U\bar{A}U^T R(R^T U D U^T R)^+ R^T = \\ &= \text{tr } \bar{A}U^T R(\bar{R}^T \bar{R})^+ R^T U \end{aligned} \tag{A1.5}$$

Due to (A1.3) it can be concluded that

$$\bar{A} = D^{1/2} D^{1/2^+} \bar{A} D^{1/2^+} D^{1/2}$$

Thus

$$\begin{aligned} \text{tr}(R^T A R)(R^T B R)^+ &= \text{tr}(D^{1/2^+} \bar{A} D^{1/2^+}) \bar{R}(\bar{R}^T \bar{R}) \bar{R}^T = \\ &= \text{tr} \begin{bmatrix} D_1^{-1/2} \bar{A}_{11} D_1^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{R}_1 \\ 0 \end{bmatrix} (\bar{R}^T \bar{R})^T [\bar{R}_1 \quad 0]^T = \\ &= \text{tr } D_1^{-1/2} \bar{A}_{11} D_1^{-1/2} \bar{R}_1 (\bar{R}_1^T \bar{R}_1)^+ \bar{R}_1 \end{aligned} \tag{A1.6}$$

Finally

$$\text{tr } AB^+ - \text{tr}(R^T A R)(R^T B R)^+ = \text{tr}(D_1^{-1/2} \bar{A}_{11} D_1^{-1/2}) \cdot \left[I - \bar{R}_1 (\bar{R}_1^T \bar{R}_1)^+ \bar{R}_1^T \right] \tag{A1.7}$$

Observing that $I - \bar{R}_1(\bar{R}_1^T\bar{R}_1)^+\bar{R}_1^T$ is an orthogonal projection and thus non-negative definite Lemma A.1 can be applied giving the desired inequality (4.11). Moreover it follows from Lemma A.1 that equality holds in (4.11) if and only if

$$D_1^{-1/2}\bar{A}_{11}D_1^{-1/2}\left[I - \bar{R}_1(\bar{R}_1^T\bar{R}_1)^+\bar{R}_1^T\right] = 0 \qquad \text{(A1.8)}$$

This equation can be rewritten as

$$\bar{A}_{11}\left[D_1^{-1} - D_1^{-1/2}\bar{R}_1(\bar{R}_1^T\bar{R}_1)^+\bar{R}_1^T D_1^{-1/2}\right] = 0$$

which can be extended to

$$\bar{A}\left[D^+ - D^{1/2^+}D^{1/2}U^T R(R^T UDU^T R)^+R^T UD^{1/2}D^{1/2^+}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right] = 0$$

or

$$U^T AU[U^T B^+ U - U^T R(R^T BR)^+ R^T U(U^T BU)(U^T BU)^+] = 0$$

$$AB^+ - AR(R^T BR)^+ R^T BB^+ = 0$$

which is (4.9).

□

APPENDIX 2 - Proof of Theorem 6.1.

Since trivially

$$X_1 \subset X_3(\cdot), \qquad X_2(1,\cdot) = X_2(r,\cdot) \cap X_3(\cdot)$$

it is easy to see that

$$J(X_1) \subset J(X_3(\cdot))$$

$$J(X_2(1,\cdot)) \subset J(X_2(r,\cdot))$$

$$J(X_2(1,\cdot)) \subset J(X_3(\cdot))$$

Thus it is sufficient to show that for any X in either $X_2(r,\cdot)$ or $X_3(\cdot)$ giving an information matrix J there exists an experimental condition in $X_2(1,\cdot)$ giving the same information matrix. To prove this it will be necessary to parameterize the elements of J in a new way.

Consider first the experimental condition $X_2(r,\cdot)$. The notation of Section 5 will be used. Introduce new variables through

$$\tilde{G}_i(z) = -\frac{a}{b} F_i(z) + G_i(z) \tag{A2.1}$$

Thus

$$H_i(z) = F_i(z) + azF_i(z) - bz\left\{\tilde{G}_i(z) + \frac{a}{b} F_i(z)\right\} =$$

$$= F_i(z) - bz\tilde{G}_i(z) \tag{A2.2}$$

Introduce now the variables r_i, ρ_i, η_i, r^*, ρ^*, η^*, through

$$r_i = \frac{1}{\lambda^2} E\left[\frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})} e(t)\right]^2 \quad \text{(A2.3)}$$

$$r_i \rho_i = \frac{1}{\lambda^2} E\left[\frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})} e(t)\right]\left[\frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})} e(t-1)\right] \quad \text{(A2.4)}$$

$$\eta_i = \frac{1}{\lambda^2} E\left[\frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})} e(t)\right][e(t)] = \tilde{G}_i(0) \quad \text{(A2.5)}$$

$$r^* = \Sigma\gamma_i r_i \quad \text{(A2.6)}$$

$$r^*\rho^* = \Sigma\gamma_i r_i \rho_i \quad \text{(A2.7)}$$

$$\eta^* = \Sigma\gamma_i \eta_i \quad \text{(A2.8)}$$

The information matrix is for the system under considera-tion

$$J = \frac{N}{\lambda^2} E\begin{bmatrix} y(t) \\ -u(t) \end{bmatrix}[\, y(t) - u(t)\,] = \frac{N}{\lambda^2}\begin{bmatrix} r_y(0) & -r_{yu}(0) \\ -r_{yu}(0) & r_u(0) \end{bmatrix} \quad \text{(A2.9)}$$

It is easy to establish that (note that $G_i(z) = \frac{a}{b} H_i(z) + A(z)\tilde{G}(z)$)

$$r_y(0) = \Sigma\gamma_i E\left[\frac{F_i(q^{-1}) - bq^{-1}\tilde{G}_i(q^{-1}) + bq^{-1}\tilde{G}_i(q^{-1})}{F_i(q^{-1}) - bq^{-1}\tilde{G}_i(q^{-1})} e(t)\right]^2 =$$

$$= \lambda^2(1+b^2 r^*) \quad \text{(A2.10)}$$

$$r_{yu}(0) = \Sigma\gamma_i E\left[e(t) + b\,\frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})}\,e(t-1)\right]^2 \cdot$$

$$\cdot \left[\frac{a}{b}\,e(t) + \frac{A(q^{-1})\tilde{G}_i(q^{-1})}{H_i(q^{-1})}\,e(t)\right] =$$

$$= \lambda^2\Sigma\gamma_i\left[\frac{a}{b} + \eta_i + br_i(\rho_i+a)\right] =$$

$$= \lambda^2\left[\frac{a}{b} + \eta^* + br^*(\rho^*+a)\right] \qquad \text{(A2.11)}$$

$$r_u(0) = \lambda^2\Sigma\gamma_i E\left[\frac{a}{b}\,e(t) + \frac{A(q^{-1})\tilde{G}_i(q^{-1})}{H_i(q^{-1})}\,e(t)\right]^2 =$$

$$= \lambda^2\Sigma\gamma_i\left[\frac{a^2}{b^2} + (1+a^2+2a\rho_i)r_i + 2\,\frac{a}{b}\,\eta_i\right] =$$

$$= \lambda^2\left[\frac{a^2}{b^2} + r^*(1+a^2+2a\rho^*) + 2\,\frac{a}{b}\,\eta^*\right] \qquad \text{(A2.12)}$$

Now, via (A2.9) - (A2.12) the information matrix J is expressed with use of the new variables r_i, ρ_i and η_i. In order to proceed, the values these variables will have when the coefficients of the polynomials $F_i(q^{-1})$ and $G_i(q^{-1})$ are varied must be examined.

Assertion A2.1. The variables r_i, ρ_i and η_i fulfil

$$r_i \geq 0 \qquad \eta_i^2 \leq r_i(1-\rho_i^2) \qquad \text{(A2.13)}$$

Conversely, if (r_i, ρ_i, η_i) are chosen to fulfil (A2.13) then it is possible to find a corresponding pair of polynomials $\left(F_i(z),\ G_i(z)\right)$.

□

Proof.

a) (Necessity) By definition $r_i \geq 0$ must hold.
Introduce the filter

$$L(q^{-1}) = \sum_{j=0}^{\infty} \ell_j q^{-j} = \frac{\tilde{G}_i(q^{-1})}{H_i(q^{-1})} \qquad \text{(A2.14)}$$

Then from (A2.3) - (A2.5)

$$r_i = \sum_{j=0}^{\infty} \ell_j^2, \quad r_i\rho_i = \sum_{j=0}^{\infty} \ell_j \ell_{j+1}, \quad \eta_i = \ell_0$$

With use of Cauchy-Schwarz inequality

$$(r_i\rho_i)^2 = \left(\sum_{j=0}^{\infty} \ell_j \ell_{j+1}\right)^2 \leq \left(\sum_{j=0}^{\infty} \ell_j^2\right)\left(\sum_{j=0}^{\infty} \ell_{j+1}^2\right) =$$

$$= r_i(r_i - \eta_i^2)$$

from which the second part of (A2.13) follows immediately.

b) (Sufficiency) Assume now that r_i, ρ_i and η_i are given and that (A2.13) is fulfilled.

Take first the case that $r_i = 0$. Then take $\tilde{G}_i(q^{-1}) \equiv 0$, i.e. $G(q^{-1})/F(q^{-1}) = a/b$ = the minimum variance regulator.

Assume thus in the following that $r_i > 0$. As a first step a first order filter $L(q^{-1})$ fulfilling (A2.3) - - (A2.5), (A2.14) will be constructed. Put

$$L(q^{-1}) = \frac{\tilde{g}_{i0} + \tilde{g}_{i1}q^{-1}}{1 + h_{i1}q^{-1}}$$

Then the equations (A2.3) - (A2.5) become

$$\begin{cases} r_i = \dfrac{\tilde{g}_{i0}^2 + \tilde{g}_{i1}^2 - 2\tilde{g}_{i0}\tilde{g}_{i1}h_{i1}}{1 - \tilde{h}_{i1}^2} \\[2ex] r_i\rho_i = \dfrac{(\tilde{g}_{i0}-\tilde{g}_{i1}h_{i1})(\tilde{g}_{i1}-\tilde{g}_{i0}h_{i1})}{1 - \tilde{h}_{i1}^2} \\[2ex] \eta_i = \tilde{g}_{i0} \end{cases}$$

After multiplying the first equation with h_{i1} and adding the second, the following systems of equations will be obtained from simple calculations

$$\begin{cases} \tilde{g}_{i0} = \eta_i \\[1ex] \eta_i\tilde{g}_{i1} - h_{i1}r_i = r_i\rho_i \\[1ex] \tilde{g}_{i1}^2 - 2\eta_i\tilde{g}_{i1}h_{i1} + r_ih_{i1}^2 = r_i - \eta_i^2 \end{cases}$$

From the second equation

$$h_{i1} = \frac{\eta_i}{r_i}\tilde{g}_{i1} - \rho_i$$

which after some manipulations transforms the third equation into

$$\tilde{g}_{i1}^2(r_i-\eta_i^2) = r_i\left[r_i(1-\rho_i^2) - \eta_i^2\right]$$

According to (A2.13) both sides are positive.

If $r_i - \eta_i^2 = 0$ it follows from (A2.13) that $\rho_i = 0$. This implies that $\tilde{g}_{i1}$ can be chosen quite freely.

On the other hand, if $r_i - \eta_i^2 > 0$ one obtains

$$\tilde{g}_{i1} = \pm\sqrt{r_i \frac{r_i(1-\rho_i^2) - \eta_i^2}{r_i - \eta_i^2}}$$

The calculations so far have given the first order filter $L(q^{-1})$. Finally a corresponding regulator has to be found by considering (A2.1) and (A2.2)

$$\begin{cases} \tilde{G}_i(z) = -\frac{a}{b}F_i(z) + G_i(z) \\ H_i(z) = F_i(z) - bz\tilde{G}_i(z) \end{cases}$$

These equations give, after some calculations, the regulator

$$\frac{G_i(q^{-1})}{F_i(q^{-1})} = \frac{(1+aq^{-1})\tilde{G}_i(q^{-1}) + a/b\ H_i(q^{-1})}{H_i(q^{-1}) + bq^{-1}\tilde{G}_i(q^{-1})} =$$

$$= \frac{(\tilde{g}_{i0} + a/b) + (a\tilde{g}_{i0}+\tilde{g}_{i1}+a/b\ h_{i1})q^{-1} + a\tilde{g}_{i1}q^{-2}}{1 + (h_{i1}+b\tilde{g}_{i0})q^{-1} + b\tilde{g}_{i1}q^{-2}}$$

□

Consider now the experimental condition $X_3(\cdot)$. Introduce the variables

$$R = \frac{1}{\lambda^2} E\left[\frac{F(q^{-1})}{A(q^{-1})F(q^{-1}) - B(q^{-1})G(q^{-1})} u(t)\right]^2 \quad (A2.15)$$

$$R\rho_R = \frac{1}{\lambda^2} E\left[\frac{F(q^{-1})}{A(q^{-1})F(q^{-1}) - B(q^{-1})G(q^{-1})} u(t)\right] \cdot$$

$$\cdot \left[\frac{F(q^{-1})}{A(q^{-1})F(q^{-1}) - B(q^{-1})G(q^{-1})} u(t-1)\right] \quad (A2.16)$$

The variables r, ρ, η are defined quite analogously with r_i, ρ_i and η_i and they will satisfy the condition

$$\eta^2 \leq r(1-\rho^2) \quad (A2.17)$$

With use of these variables it is easy to establish that for $X_3(\cdot)$,

$$r_y(0) = [b^2R + 1 + b^2r]\lambda^2 \quad (A2.18)$$

$$r_{yu}(0) = \left[bR(a+\rho_R) + \frac{a}{b} + \eta + br(\rho+a)\right]\lambda^2 \quad (A2.19)$$

$$r_u(0) = \left[R(1+a^2+2a\rho_R) + \frac{a^2}{b^2} + 2\frac{a}{b}\eta + r(1+a^2+2a\rho)\right]\lambda^2 \quad (A2.20)$$

Consider finally the experimental condition $X_2(1,\cdot)$. In analogy with the previous cases the variables $\underline{r}$, $\underline{r\rho}$ and $\underline{\eta}$ satisfying

$$\underline{\eta}^2 \leq \underline{r}(1-\underline{\rho}^2) \quad (A2.21)$$

are used as parameters giving

$$r_y(0) = [1 + b^2\underline{r}]\lambda^2 \tag{A2.22}$$

$$r_{yu}(0) = \left[\frac{a}{b} + \underline{\eta} + b\underline{r}(\underline{\rho}+a)\right]\lambda^2 \tag{A2.23}$$

$$r_u(0) = \left[\frac{a^2}{b^2} + 2\,\frac{a}{b}\,\underline{\eta} + \underline{r}(1+a^2+2a\underline{\rho})\right]\lambda^2 \tag{A2.24}$$

Assertion A2.2. Given an experimental condition in $X_2(r,\cdot)$ then there exists another experimental condition in $X_2(1,\cdot)$ giving the same information matrix.

□

Proof. It is to be shown that

$$\begin{cases} 1 + b^2\underline{r} = 1 + b^2 r^* \\ \frac{a}{b} + \underline{\eta} + b\underline{r}(\underline{\rho}+a) = \frac{a}{b} + \eta^* + br^*(\rho^*+a) \\ \frac{a^2}{b^2} + 2\,\frac{a}{b}\,\underline{\eta} + \underline{r}(1+a^2+2a\underline{\rho}) = \frac{a^2}{b^2} + 2\,\frac{a}{b}\,\eta^* + r^*(1+a^2+2a\rho^*) \end{cases} \tag{A2.25}$$

always has a solution with respect to $\underline{r}$, $\underline{\rho}$ and $\underline{\eta}$. Take now

$$\underline{r} = r^*, \quad \underline{\rho} = \rho^*, \quad \underline{\eta} = \eta^* \tag{A2.26}$$

which clearly satisfy (A2.25). It remains to be proved that the choice (A2.26) always can be made subject to the constraints (A2.13) and (A2.21). Clearly the choice $\underline{r} = r^*$ is always possible. Moreover

$$|\rho^*| = \left|\frac{\Sigma\gamma_i r_i \rho_i}{\Sigma\gamma_i r_i}\right| \leq \frac{\Sigma\gamma_i r_i |\rho_i|}{\Sigma\gamma_i r_i} \leq 1$$

which implies that it is sufficient to consider the last asserted choice $\underline{\eta} = \eta^*$. This choice is possible if (by (A2.21))

$$\sqrt{\underline{r}(1-\underline{\rho}^2)} \geq \eta^* \qquad \text{(A2.27)}$$

However, (A2.27) can always be satisfied if

$$\sqrt{\underline{r}(1-\underline{\rho}^2)} \geq \sup_{\text{constraints}} \eta^* = \sup \Sigma\gamma_i\eta_i^* = \Sigma\gamma_i\sqrt{r_i(1-\rho_i^2)} \qquad \text{(A2.28)}$$

which can be evaluated as (quadrate both sides and multiply with $\underline{r} = r^* = \Sigma\gamma_i r_i$)

$$(\Sigma\gamma_i r_i)^2 - (\Sigma\gamma_i r_i\rho_i)^2 \geq \Sigma\gamma_i r_i\left[\Sigma\gamma_i\sqrt{r_i(1-\rho_i^2)}\right]^2$$

This is satisfied if (use Cauchy-Schwarz inequality)

$$(\Sigma\gamma_i r_i)^2 - (\Sigma\gamma_i r_i\rho_i)^2 \geq \Sigma\gamma_i r_i\left[\Sigma\gamma_i\Sigma\gamma_i r_i(1-\rho_i^2)\right]$$

which is easily rewritten (note that $\Sigma\gamma_i = 1$)

$$(\Sigma\gamma_i r_i\rho_i)^2 \leq \Sigma\gamma_i r_i\Sigma\gamma_i r_i\rho_i^2$$

which is always true due to Cauchy-Schwarz inequality. □

<u>Assertion A2.3</u>. Given an experimental condition in $X_3(\cdot)$ then there exists another experimental condition in $X_2(1,\cdot)$ giving the same information matrix. □

Proof. It is to be shown that

$$\begin{cases} 1 + b^2\underline{r} = b^2R + 1 + b^2r \\ \frac{a}{b} + \underline{\eta} + b\underline{r}(\underline{\rho}+a) = bR(\rho_R+a) + \frac{a}{b} + \eta + br(\rho+a) \\ \frac{a^2}{b^2} + 2\,\frac{a}{b}\,\underline{\eta} + \underline{r}(1+a^2+2a\underline{\rho}) = R(1+a^2+2a\rho_R) + \\ \qquad + \frac{a^2}{b^2} + 2\,\frac{a}{b}\,\eta + r(1+a^2+2a\rho) \end{cases} \tag{A2.29}$$

always has a solution with respect to $\underline{r}$, $\underline{\rho}$ and $\underline{\eta}$. Try now the choice

$$\underline{r} = R + r, \qquad \underline{\rho} = \frac{R\rho_R + r\rho}{R + r}, \qquad \underline{\eta} = \eta \tag{A2.30}$$

It is easy to see that this choice satisfies (A2.29). It remains to be shown that (A2.21) is satisfied. This can always be the case if (it is easy to prove that the tried expression for $\underline{\rho}$ implies $|\underline{\rho}| \leq 1$)

$$\sqrt{\underline{r}(1-\underline{\rho}^2)} \geq \eta \tag{A2.31}$$

The condition (A2.31) can always be satisfied if

$$\sqrt{\underline{r}(1-\underline{\rho}^2)} \geq \sup|\eta|$$

or equivalent

$$(R+r) - \frac{(R\rho_R+r\rho)^2}{R + r} \geq r - r\rho^2$$

or rewritten

$$R^2(1-\rho_R^2) + Rr(1-2\rho_R\rho+\rho^2) \geq 0 \qquad (A2.32)$$

However, both terms in (A2.32) are positive. This observation completes the proof. □

With the Assertions A2.2 and A2.3 the proof of Theorem 6.1 is finished. □

APPENDIX 3 - Proof of Lemma 6.1.

The proof of the lemma is organized as proofs of three assertions.

Assertion A3.1

$$\frac{1}{N^3} \sup_{X_2(2,0)} \det J = \frac{\delta^2}{b^2(1-c^2)^3}$$

□

Proof. The notations of Section 5 will be used. For this case $F_i = 1$, $i = 1,2$. Put

$$G_i = \frac{a - \alpha_i}{b} = k_i \qquad (i = 1,\ 2)$$

Thus

$$H_i(z) = 1 + \alpha_i z$$

Moreover, for the treated system the information matrix J can be calculated to satisfy

$$J = N \sum_{i=1}^{2} \gamma_i \begin{bmatrix} \frac{1}{1-\alpha_i^2} & -\frac{k_i}{1-\alpha_i^2} & -\frac{1}{1-\alpha_i c} \\ \frac{-k_i}{1-\alpha_i^2} & \frac{k_i^2}{1-\alpha_i^2} & \frac{k_i}{1-\alpha_i c} \\ -\frac{1}{1-\alpha_i c} & \frac{k_i}{1-\alpha_i c} & \frac{1}{1-c^2} \end{bmatrix} \qquad (A3.1)$$

from which

$$\frac{1}{N^3} \det J = \frac{\gamma_1 \gamma_2 (\alpha_1 - \alpha_2)^2}{b^2 (1-\alpha_1^2)(1-\alpha_2^2)(1-c^2)} \cdot$$

$$\cdot \left[\gamma_1 \frac{(\alpha_1 - c)^2}{(1-\alpha_1 c)^2} + \gamma_2 \frac{(\alpha_2 - c)^2}{(1-\alpha_2 c)^2} \right] \quad \text{(A3.2)}$$

can be computed in a straightforward way.

Consider now the constraint

$$\frac{1}{\lambda^2} r_y(0) = 1 + \frac{(\alpha_i - c)^2}{1 - \alpha_i^2} \leq 1 + \delta \quad \text{(A3.3)}$$

or rewritten

$$\alpha_i^2 (1+\delta) - 2\alpha_i c + (c^2 - \delta) \leq 0$$

Thus the constraint (A3.3) means that the variables α_i ($i = 1, 2$) are restricted as

$$\left| \alpha_i - \frac{c}{1 + \delta} \right| \leq \frac{\sqrt{\delta(1-c^2+\delta)}}{1 + \delta} \quad \text{(A3.4)}$$

Consider now maximization of

$$f_1(\alpha_i) = \left(\frac{\alpha_i - c}{1 - \alpha_i c} \right)^2 \quad \text{(A3.5)}$$

subject to the constraint (A3.4). Since $f_1'(\alpha_i) = 0$ implies $\alpha_i = c$, which gives a minimum of $f_1(\alpha_i)$ it is necessary that the constraint (A3.4) is active. Straightforward calculations show that

$$\max_{\alpha_i} f_1(\alpha_i) = f_1\left(\frac{c}{1+\delta} \pm \frac{\sqrt{\delta(1-c^2+\delta)}}{1+\delta}\right) = \frac{\delta}{1-c^2+\delta} \qquad \text{(A3.6)}$$

Both signs in (A3.6) are possible.

Consider now maximization of

$$f_2(\alpha_1,\alpha_2) = \frac{(\alpha_1-\alpha_2)^2}{(1-\alpha_1^2)(1-\alpha_2^2)} \qquad \text{(A3.7)}$$

subject to the constraint (A3.4). It is straightforward to show that

$$f_2'(\alpha_1,\alpha_2) = \left[\frac{2(\alpha_1-\alpha_2)(1-\alpha_1\alpha_2)}{(1-\alpha_1^2)^2(1-\alpha_2^2)} \quad \frac{2(\alpha_2-\alpha_1)(1-\alpha_1\alpha_2)}{(1-\alpha_1^2)(1-\alpha_2^2)^2}\right]$$

Thus $f_2(\alpha_1,\alpha_2)$ must be maximized in the "corners"

$$(\alpha_1,\alpha_2) = \left(\frac{c}{1+\delta} + \frac{\sqrt{\delta(1+\delta-c^2)}}{1+\delta}, \quad \frac{c}{1+\delta} - \frac{\sqrt{\delta(1+\delta-c^2)}}{1+\delta}\right) \qquad \text{(A3.8)}$$

and

$$(\alpha_1,\alpha_2) = \left(\frac{c}{1+\delta} - \frac{\sqrt{\delta(1+\delta-c^2)}}{1+\delta}, \quad \frac{c}{1+\delta} + \frac{\sqrt{\delta(1+\delta-c^2)}}{1+\delta}\right) \qquad \text{(A3.9)}$$

Both (A3.8) and (A3.9) will give

$$f_2(\alpha_1,\alpha_2) = \frac{4\delta(1-c^2+\delta)}{(1-c^2)^2} \qquad \text{(A3.10)}$$

Since $f_1(\alpha_i)$ as well as $f_2(\alpha_1,\alpha_2)$ are maximized by the same value it is concluded that (A3.8) and (A3.9) will also give the maximum of det J, since

$$\frac{1}{N^3} \det J = \frac{\gamma_1\gamma_2}{b^2} \frac{f_2(\alpha_1,\alpha_2)}{1-c^2}\left[\gamma_1 f_1(\alpha_1) + \gamma_2 f_1(\alpha_2)\right] \qquad \text{(A3.11)}$$

The non-uniqueness consisting of the signs depends on the fact that, of course, α_1 and α_2 can shift meaning.

Inserting (A3.6) and (A3.10) into (A3.11)

$$\frac{1}{N^3} \det J = \frac{\gamma_1\gamma_2}{b^2} \frac{4\delta(1-c^2+\delta)}{(1-c^2)^3} \frac{\delta}{1-c^2+\delta} (\gamma_1+\gamma_2)$$

$$= \frac{4\gamma_1\gamma_2}{b^2} \frac{\delta^2}{(1-c^2)^3}$$

Finally maximization with respect to the γ-parameters will give

$$\gamma_1 = \gamma_2 = 0.5$$

and

$$\frac{1}{N^3} \sup_{X_2(2,0)} \det J = \frac{\delta^2}{b^2(1-c^2)^3}$$

□

Assertion A3.2

$$\frac{1}{N^3} \sup_{X_3(0)} \det J = \frac{\delta^2}{b^2(1-c^2)^3}$$

□

Proof. Introduce the notations

$$F = 1, \quad G = \frac{a - \alpha}{b} = k \tag{A3.12}$$

$$v(t) = \frac{1}{(1+cq^{-1})(1+\alpha q^{-1})} u(t) \tag{A3.13}$$

$$R = \frac{1}{\lambda^2} Ev(t)^2 \tag{A3.14}$$

$$R\rho = \frac{1}{\lambda^2} Ev(t)v(t-1) \tag{A3.15}$$

For this case the information matrix J satisfies

$$NJ = \begin{bmatrix} b^2R & -bR(a+\rho) & 0 \\ -bR(a+\rho) & R(1+a^2+2a\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$+ \begin{bmatrix} \frac{1}{1-\alpha^2} & -\frac{k}{1-\alpha^2} & -\frac{1}{1-\alpha c} \\ -\frac{k}{1-\alpha^2} & \frac{k^2}{1-\alpha^2} & \frac{k}{1-\alpha c} \\ -\frac{1}{1-\alpha c} & \frac{k}{1-\alpha c} & \frac{1}{1-c^2} \end{bmatrix} \tag{A3.16}$$

from which

$$\frac{1}{N^3} \det J = \frac{b^2R^2(1-\rho^2)}{1 - c^2} + R(1+\alpha^2+2\alpha\rho) \frac{(\alpha-c)^2}{(1-\alpha^2)(1-c^2)(1-\alpha c)^2} \tag{A3.17}$$

can be computed in a straightforward way.

The constraint $Ey^2(t) \leq \lambda^2(1+\delta)$ becomes rewritten

$$b^2R(1+c^2+2c\rho) + \frac{(c-\alpha)^2}{1-\alpha^2} \leq \delta \qquad (A3.18)$$

Moreover

$$R \geq 0, \quad |\rho| \leq 1 \qquad (A3.19)$$

must be satisfied.

It is easy to see that maximization with respect to R is obtained for

$$R = \frac{1}{b^2(1+c^2+2c\rho)}\left[\delta - \frac{(c-\alpha)^2}{1-\alpha^2}\right] \qquad (A3.20)$$

Introduce the notation x through

$$\delta = \frac{(c-\alpha)^2}{1-\alpha^2}(1+x) \qquad (A3.21)$$

In view of (A3.18) it is seen that $x \geq 0$ must always be fulfilled.

Inserting (A3.20) and (A3.21) into (A3.17) the following expression can be obtained after straightforward, but tedious, calculations

$$\frac{1}{N^3}\det J = \frac{\delta^2}{b^2(1-c^2)^3} + \frac{(c-\alpha)^4}{b^2(1+c^2+2c\rho)^2(1-c^2)(1-\alpha^2)^2} \cdot$$

$$\cdot [-Ax^2 + Bx - C] \qquad (A3.22)$$

where

$$A = \left[\frac{\rho(1+c^2) + 2c}{1 - c^2}\right]^2 \tag{A3.23}$$

$$B = -2\left[\frac{\rho(1+c^2) + 2c}{1 - c^2}\right]^2 + \left[\frac{\rho(1+\alpha c) + \alpha + c}{1 - \alpha c}\right]^2 - 1 + \rho^2 \tag{A3.24}$$

$$C = \left[\frac{\rho(1+c^2) + 2c}{1 - c^2}\right]^2 + (1-\rho^2) \tag{A3.25}$$

Consider the polynomial

$$p(x) = -Ax^2 + Bx - C \tag{A3.26}$$

Clearly $A \geq 0$, $C > 0$. If $B \leq 0$ then $p(x)$ is always negative and the assertion will be proved. Unfortunately $B \leq 0$ is not always true so some more calculations are needed.

Since only positive values of x are of interest it will be sufficient to consider the polynomial

$$\tilde{p}(x) = \sup_{|\alpha| \leq 1} p(x) = -Ax^2 + \tilde{B}x - C \tag{A3.27}$$

where

$$\tilde{B} = \sup_{|\alpha| \leq 1} B \tag{A3.28}$$

It can be seen that the supremum is obtained for either $\alpha = 1$ or $\alpha = -1$. It is chosen to consider the case $\alpha = 1$ in the following (the case $\alpha = -1$ can be treated analogously).

Thus

$$\tilde{p}(x) = - A\left(x - \frac{\tilde{B}}{2A}\right)^2 + \frac{\tilde{B}^2 - 4AC}{4A} \tag{A3.29}$$

But

$$\tilde{B}^2 - 4AC = \left[- 2\left\{\frac{\rho(1+c^2) + 2c}{1 - c^2}\right\}^2 + \left\{\frac{(\rho+1)(1+c)}{1-c}\right\}^2 - 1 + \rho^2\right]^2 -$$

$$- 4\left[\frac{\rho(1+c^2) + 2c}{1-c^2}\right]^2\left[\left\{\frac{\rho(1+c^2) + 2c}{1 - c^2}\right\}^2 + 1 - \rho^2\right] =$$

$$= \left[\frac{(\rho+1)^2(1+c)^2}{(1-c)^2} + (\rho+1)(\rho-1)\right]^2 -$$

$$- 4\,\frac{[\rho(1+c^2) + 2c]^2}{(1-c^2)^2}\,\frac{(\rho+1)^2(1+c)^2}{(1-c)^2} =$$

$$= \frac{(\rho+1)^2}{(1-c)^4}\left[\left\{(\rho+1)(1+c)^2 + (\rho-1)(1-c)^2\right\}^2 -\right.$$

$$\left. - 4\left\{\rho(1+c^2) + 2c\right\}^2\right] =$$

$$= \frac{(\rho+1)^2}{(1-c)^4}\left[\left\{2\rho(1+c^2) + 4c\right\}^2 - 4\left\{\rho(1+c^2) + 2c\right\}^2\right] = 0$$

To summarize it has been shown that

$$\frac{1}{N^3}\det J = \frac{\delta^2}{b^2(1-c^2)^3} + \frac{(c-\alpha)^4}{b^2(1+c^2+2c\rho)(1-c^2)(1-\alpha^2)^2}\,p(x) \leq$$

$$\leq \frac{\delta^2}{b^2(1-c^2)^3} + \frac{(c-\alpha)^4}{b^2(1+c^2+2c\rho)(1-c^2)(1-\alpha^2)^2}\,\tilde{p}(x)$$

where

$p(x) \leq \tilde{p}(x) \leq 0 \quad$ for all $x \geq 0$

Thus optimum of $1/N^3$ det J requires either $c = \alpha$ or $p(x) = 0$. However, the latter case can be eliminated since it implies $B = \tilde{B}$, cf. (A3.28). However, since the supremum in (A3.28) is obtained with α on the stability bound this case is without interest in the analysis.

It remains to consider the case of $\alpha = c$. This means that the regulator is given by a minimal variance strategy. The analysis given above unfortunately cannot be utilized directly for this case because it corresponds to $x = \infty$. However, from (A3.20)

$$R = \frac{\delta}{b^2(1+c^2+2c\rho)} \tag{A3.30}$$

which inserted along with $\alpha = c$ in (A3.17) gives

$$\frac{1}{N^3} \det J = \frac{\delta^2(1-\rho^2)}{b^2(1-c^2)(1+c^2+2c\rho)^2} \tag{A3.31}$$

Straightforward optimization of the right hand side with respect to ρ will give

$$\rho = - \frac{2c}{1 + c^2} \tag{A3.32}$$

which is found to fulfil the constraint $|\rho| \leq 1$.

Then finally from (A3.30) and (A3.32)

$$R = \frac{\delta(1+c^2)}{b^2(1-c^2)^2} \tag{A3.33}$$

$$R\rho = -\frac{2c\delta}{b^2(1-c^2)^2} \tag{A3.34}$$

Thus the assertion has been proved. □

Assertion A3.3.

$$\frac{1}{N^3}\sup_{X_1} \det J \leq \frac{\delta^2}{b^2(1-c^2)^3}$$

with equality if and only if a = c. □

Proof. Two observations will be utilized

i) $X_1 \subset X_3(0)$

ii) the optimum experimental condition in $X_3(0)$ giving

$$\frac{1}{N^3}\det J = \frac{\delta^2}{b^2(1-c^2)^3}$$

is uniquely defined.

Thus the ≤ sign is proved. Moreover, equality is obtained if and only if (due to the uniqueness) the open loop case corresponds to a minimum variance regulator. This is the case exactly when a = c. □

APPENDIX 4 - Proof of Lemma 6.2.

The second and third equalities in (6.38) are simple consequences of Theorem 6.1.

In order to prove the remaining equalities it will be advantageous to introduce a new parameterization of J. Consider for simplicity only cases with r = 1.

Introduce

$$v(t) = \frac{1}{1+aq^{-1}} u(t) \tag{A4.1}$$

$$z(t) = \frac{1}{1+aq^{-1}} e(t) \tag{A4.2}$$

These two signals may be correlated if feedback is present. It is easily seen that

$$y(t) = bq^{-1}v(t) + z(t)$$

$$u(t) = v(t) + aq^{-1}v(t)$$

and thus

$$r_y(0) = b^2 r_v(0) + r_z(0) + 2br_{zv}(1) \tag{A4.3}$$

$$r_{yu}(0) = br_v(1) + abr_v(0) + r_{zv}(0) + ar_{zv}(1) \tag{A4.4}$$

$$r_u(0) = (1+a^2)r_v(0) + 2ar_v(1) \tag{A4.5}$$

Now

$$r_z(0) = \frac{\lambda^2}{1-a^2} \tag{A4.6}$$

and

$$r_{zv}(1) + ar_{zv}(0) = E[z(t+1)+az(t)]v(t) = Ee(t+1)v(t) = 0$$

independent of an eventual feedback, so

$$r_{zv}(1) = -ar_{zv}(0) \quad \text{(A4.7)}$$

Introduce now the variables R, ρ and r by

$$r_v(0) = \lambda^2 R \quad \text{(A4.8)}$$

$$r_v(1) = \lambda^2 R\rho \quad \text{(A4.9)}$$

$$r_{zv}(0) = r\sqrt{r_v(0)r_z(0)} = r\lambda^2\sqrt{R/(1-a^2)} \quad \text{(A4.10)}$$

The variables are constrained in several ways, e.g.

$R \geq 0$; $|\rho| \leq 1$; $|r| \leq 1$

There are, however, more constraints. Suppose r=1. Then $v(t)=kz(t)$ and thus $\rho=-a$.

With use of (A4.8)-(A4.10) it is easy to get

$$r_y(0) = \lambda^2(b^2R + \frac{1}{1-a^2} - 2abr\sqrt{\frac{R}{1-a^2}})$$

$$r_{yu}(0) = \lambda^2[b(a+\rho)R + r\sqrt{R(1-a^2)}]$$

$$r_u(0) = \lambda^2 R(1+a^2+2a\rho)$$

Finally, by simple calculations,

$$\det J = N^2\lambda^{-4}[r_y(0)r_u(0)-r_{yu}(0)^2] =$$

$$= b^2R^2(1-\rho^2) - 2brR^{3/2}\frac{2a+\rho+a^2\rho}{(1-a^2)^{1/2}} + \frac{R}{1-a^2}[1+a^2+2a\rho-r^2(1-a^2)^2]$$

(A4.11)

The constraint (6.37) can now be written as

$$b^2R(1+a^2+2a\rho) \le \delta \tag{A4.12}$$

Consider now open loop operation (i.e. the experimental condition X_1). Then r=0 and

$$N^{-2}\det J = b^2R^2(1-\rho^2) + \frac{R}{1-a^2}(1+a^2+2a\rho) \tag{A4.13}$$

Thus equality in (A4.12) must be chosen in the optimization. Insertion of (A4.12) into (A4.13) gives

$$N^{-2}\det J = \frac{\delta^2}{b^2}\frac{1-\rho^2}{(1+a^2+2a\rho)^2} + \frac{\delta}{b^2(1-a^2)}$$

Require now that the derivative of this expression with respect to ρ must vanish. Straightforward calculations give

$$\rho = -\frac{2a}{1+a^2} \tag{A4.14}$$

which always fulfils the constraint $|\rho|<1$. Moreover (A4.14) implies

$$N^{-2}\sup_{X_1} \det J = \frac{\delta^2}{b^2(1-a^2)^2} + \frac{\delta}{b^2(1-a^2)} \tag{A4.15}$$

Note that u(t) white noise will <u>not</u> fulfil (A4.14) unless a=0. The optimal value is e.g. obtained if

$$u(t) = \frac{1}{1+aq^{-1}} w(t)$$

where w(t) is white noise with variance $\lambda^2\delta(1-a^2)/b^2$.

The calculations made so far imply that the first equality in (6.38) is fulfilled if

$$\sup_{R,\rho,r} N^{-2}\det J = \frac{\delta(\delta+1-a^2)}{b^2(1-a^2)^2} \tag{A4.16}$$

This condition is implied by the following, where

$\sup_r \det J$ is computed without taking constraints on r into account.

$$\sup_{R,\rho} [b^2R^2(1-\rho^2) + \frac{R}{1-a^2}(1+a^2+2a\rho) + \frac{b^2R^2(2a+\rho+a^2\rho)^2}{(1-a^2)^2}] =$$

$$= \frac{\delta(\delta+1-a^2)}{b^2(1-a^2)^2} \tag{A4.17}$$

By taking sup over R (which means equality in (A4.12)) it is seen after simple calculations that this is equivalent to

$$\sup_{\rho} [\frac{\delta^2(1-\rho^2)}{b^2(1+a^2+2a\rho)^2} + \frac{\delta}{b^2(1-a^2)} + \frac{\delta^2(2a+\rho+a^2\rho)^2}{b^2(1+a^2+2a\rho)^2(1-a^2)^2}] =$$

$$= \frac{\delta(\delta+1-a^2)}{b^2(1-a^2)^2}$$

The left hand side can be rewritten as

$$\sup_{\rho} [\frac{\delta^2}{b^2(1+a^2+2a\rho)^2(1-a^2)^2}\{(1-\rho^2)(1-a^2)^2+(2a+\rho+a^2\rho)^2\} +$$

$$+ \frac{\delta}{b^2(1-a^2)}] = \sup_{\rho} [\frac{\delta^2}{b^2(1+a^2+2a\rho)^2(1-a^2)^2}\{(1+a^2)^2 +$$

$$+ 4a(1+a^2)\rho \quad + 4a^2\rho^2\} + \frac{\delta}{b^2(1-a^2)}] = \sup_{\rho} [\frac{\delta^2}{b^2(1-a^2)^2} +$$

$$+ \frac{\delta}{b^2(1-a^2)}] = \frac{\delta(\delta+1-a^2)}{b^2(1-a^2)^2}$$

Thus the first equality in (6.38) has been proved.

It follows from the calculations above that

$$R = \frac{\delta}{b^2(1+a^2+2a\rho)} \tag{A4.18}$$

$$r = -\frac{bR^{3/2}(2a+\rho+a^2\rho)}{(1-a^2)^{3/2}R} = -\frac{(2a+\rho+a^2\rho)\delta^{1/2}}{(1-a^2)^{3/2}(1+a^2+2a\rho)^{1/2}} \tag{A4.19}$$

when det J is optimized. When (A4.18) and (A4.19) are used in the general expressions for $r_y(0)$, $r_{yu}(0)$ and $r_u(0)$ straightforward calculations give

$$r_y(0) = \lambda^2[b^2R + \frac{1}{1-a^2} + 2ab^2(2a+\rho+a^2\rho)\frac{R}{(1-a^2)^2}] =$$

$$= \lambda^2[\frac{1}{1-a^2} + \frac{b^2R}{(1-a^2)^2}\{(1-a^2)^2 + 2a(2a+\rho+a^2\rho)\}] =$$

$$= \lambda^2[\frac{1}{1-a^2} + \frac{b^2R}{(1-a^2)^2}(1+a^2)(1+a^2+2a\rho)] =$$

$$= \frac{\lambda^2}{(1-a^2)^2}[1-a^2+\delta(1+a^2)]$$

$$r_{yu}(0) = \lambda^2[bR(a+\rho) - \frac{bR}{1-a^2}(2a+\rho+a^2\rho)] =$$

$$= \frac{bR\lambda^2}{1-a^2}[(a+\rho)(1-a^2) - (2a+\rho+a^2\rho)] =$$

$$= \frac{bR\lambda^2}{1-a^2}(-a)(1+a^2+2a\rho) = -\frac{a}{b}\frac{\lambda^2\delta}{(1-a^2)}$$

$$r_u(0) = \frac{\lambda^2\delta}{b^2}$$

Thus the optimal J fulfils

$$J = N\cdot\begin{bmatrix} \frac{1}{1-a^2} + \delta\frac{1+a^2}{(1-a^2)^2} & \frac{a\delta}{b(1-a^2)} \\ \frac{a\delta}{b(1-a^2)} & \frac{\delta}{b^2} \end{bmatrix} \qquad (A4.20)$$

To complete the proof it is just necessary to find an experimental condition in $X_2(2,0)$ such that (A4.20) is fulfilled.

Let the proportional regulators be given by

$$u(t) = -k_i y(t) \qquad i=1,2$$

and introduce

$$\alpha_i = a + bk_i$$

Then

$$y(t) = \frac{1}{1+\alpha_i q^{-1}} e(t)$$

$$u(t) = \frac{-k_i}{1+\alpha_i q^{-1}} e(t)$$

Let now the constants k_i be chosen so that

$$r_u(0) = \frac{\lambda^2 \delta}{b^2}$$

i.e.

$$\frac{(\alpha_i - a)^2}{1-\alpha_i^2} = \delta$$

or simplified

$$\alpha_i = \frac{a}{1+\delta} \pm \frac{\sqrt{\delta(1+\delta-a^2)}}{1+\delta} \qquad i=1,2$$

Thus

$$\alpha_1 + \alpha_2 = \frac{2a}{1+\delta}$$

$$\alpha_1 \alpha_2 = \frac{a^2-\delta}{1+\delta}$$

$$\alpha_1^2 + \alpha_2^2 = 2\,\frac{\delta+\delta^2+a^2-a^2\delta}{(1+\delta)^2}$$

$$(1-\alpha_1^2)(1-\alpha_2^2) = \frac{(1+\delta)^2 - 2(\delta+\delta^2+a^2-a^2\delta) + a^4 + \delta^2 + 2a^2\delta}{(1+\delta)^2} =$$

$$= \frac{(1-a^2)^2}{(1+\delta)^2}$$

Let now both regulators act on the system during 50% of the total experiment time. Then

$$r_y(0) = \frac{\lambda^2}{2}\left[\frac{1}{1-\alpha_1^2} + \frac{1}{1-\alpha_2^2}\right] = \frac{\lambda^2}{2}\,\frac{2-(\alpha_1^2+\alpha_2^2)}{(1-\alpha_1^2)(1-\alpha_2^2)} =$$

$$= \lambda^2\,\frac{(1+\delta)^2-\delta(1+\delta)-a^2(1-\delta)}{(1-a^2)^2} = \frac{\lambda^2}{(1-a^2)^2}[1-a^2+\delta(1+a^2)]\,.$$

$$r_{yu}(0) = \frac{\lambda^2}{2}\left[\frac{-k_1}{1-\alpha_1^2} + \frac{-k_2}{1-\alpha_2^2}\right] = \frac{\lambda^2}{2b}\,\frac{(a-\alpha_1)(1-\alpha_2^2)+(a-\alpha_2)(1-\alpha_1^2)}{(1-\alpha_1^2)(1-\alpha_2^2)} =$$

$$= \frac{\lambda^2(1+\delta)^2}{2b(1-a^2)^2}[2a-a(\alpha_1^2+\alpha_2^2)-(\alpha_1+\alpha_2)(1-\alpha_1\alpha_2)] =$$

$$= \frac{\lambda^2}{2b(1-a^2)^2}[2a(1+\delta)^2-2a(\delta+\delta^2+a^2-a^2\delta)-2a(1+\delta-a^2+\delta)] =$$

$$= \frac{a\lambda^2}{b(1-a^2)^2}(1+2\delta+\delta^2-\delta-\delta^2-a^2+a^2\delta-1-2\delta+a^2) = -\frac{a\lambda^2\delta}{b(1-a^2)}$$

Thus the optimal J of (A4.20) is obtained. With this observation the proof is completed.

□