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Some unexpected phenomena have been observed when a self-tuning regulator is used in a servoloop where only the error is available for feedback. Analysis is given which gives insight into the observed phenomena. The analysis shows that it is in general not feasible to use the self-tuning regulator in this way. Tools for analysing adaptive systems with deterministic inputs are obtained as byproducts.

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1. INTRODUCTION

The self-tuning regulator discussed in Åström and Wittenmark (1973) was originally developed for steady state regulation where the major disturbances can be described as stochastic processes. Some theory has been developed for this case. There is also some experience from industrial process control. See Åström et al (1977).

In this paper we will discuss the application of the self-tuning regulator (STR) in a servoproblem. The main task is thus to make a system follow deterministic command inputs. It will also be assumed that only the error i.e. the difference between the command input and the output is available for feedback. Such a system is called a single-degree-of-freedom (SDF) system by Horowitz (1963).

Simulations by Wittenmark (1973) indicated that the self-tuning regulators in a single-degree-of-freedom configuration would converge to dead-beat like regulators when subject to deterministic command signals. Experiments by Andersson (1977) also indicated that the system could in some cases oscillate with a very long period. The purpose of this work has been to gain insight into the observed phenomena.

The closed loop system obtained with a STR in a feedback system is a nonlinear system. If the command signal is deterministic the problem is to analyse a nonlinear difference equation. This problem formulation is given in Chapter 2. In Chapter 3 the standard procedure of determining the stationary solutions and their local properties is persued. A major problem is to find a suitable characterization of the command signal. A class of signals called *piece-wise deterministic* are introduced. Using this notion a convenient characterization of the stationary solutions is given. The results show that the stationary solutions do indeed correspond to dead-beat like regulators. The analysis makes it possible to conclude that it is in general not feasible to use a self-tuning regulator in a servo loop in a single-degree-of-freedom configuration. This is of course not surprising because a single-degree-of-freedom configuration is known to have certain limitations. See e.g. Horowitz (1963, p. 239-245). To investigate the local properties of the stationary solutions the nonlinear equations are linearized. This leads to linear time-varying difference equations. Such equations are dealt with in detail for specific examples in Chapter 4. In particular it is found that the experimental results obtained by Andersson (1977) can be explained. Since the difficulties encountered are due to the fact that a single-degree-of-freedom configuration is used they can be avoided by using a two-degree-of-freedom configuration. It is shown in Chapter 5 that this is indeed the case.

2. PROBLEM FORMULATION

Consider the closed loop system shown in Fig. 2.1, where the self-tuning regulator STURE described in Åström and Wittenmark (1973) is used in a simple command servo loop.

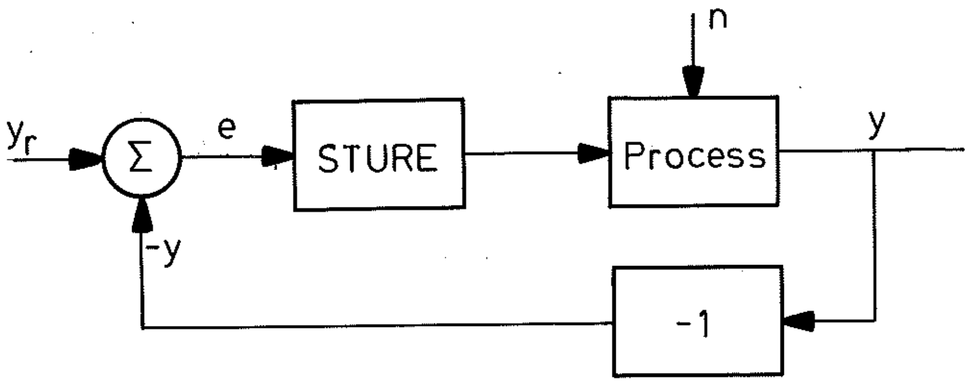

Figure 2.1. Block diagram of a self-tuning regulator in a single-degree-of-freedom configuration.

It is assumed that only the control error e is available and that y_r and y are not available separately. This case is called a single-degree-of-freedom system by Horowitz (1963). It is well known that the system shown in Fig. 2.1 is not the best way to handle command inputs. See Horowitz (1963, p. 239-245). It is, however, the only possibility if only the error e is measured. Because of its simplicity the system shown in Fig. 2.1 is also quite common even if both y and y_r can be measured.

The behaviour of the system shown in Fig. 2.1 is well understood if the reference value y_r is zero and if the process disturbance n is a stochastic process. In this report we will develop the theory for the case when the reference value y_r is a deterministic signal. The analysis also applies to the case when the process disturbance is a deterministic signal.

3. ANALYSIS

The properties of the closed loop system shown in Fig. 2.1 will now be analysed.

Preliminaries

It is assumed that the process disturbance n is zero and that the process can be described as

$$y(t) = \frac{B(q^{-1})}{A(q^{-1})} u(t-k), \tag{3.1}$$

where A and B are polynomials

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_n q^{-n},$$

$$B(q^{-1}) = b_0 + b_1 q^{-1} + \ldots + b_m q^{-m}, \qquad b_0 \neq 0$$

in the backward shift operator q^{-1}.

The k step prediction model which corresponds to (3.1) is

$$\hat{y}(t+k) = \mathcal{A}_0(q^{-1})\, y(t) + \mathcal{B}_0(q^{-1})\, u(t) \tag{3.2}$$

where the polynomials

$$\mathcal{A}_0(q^{-1}) = \alpha_{10} + \alpha_{20}\, q^{-1} + \ldots + \alpha_{r0}\, q^{-r+1} =$$

$$= - \theta_{10} - \theta_{20}\, q^{-1} - \ldots - \theta_{r0}\, q^{-r+1}$$

and

$$\mathcal{B}_0(q^{-1}) = \beta_0 + \beta_1\, q^{-1} + \ldots + \beta_s\, q^{-s} =$$

$$= \beta_0 + \theta_{r+1}\, q^{-1} + \ldots + \theta_{r+s}\, q^{-s}$$

are the solution to the polynomial equation

$$B = A\mathcal{B}_0 + q^{-k} B\mathcal{A}_0.$$

The prediction model (3.2) can also be written as

$$\hat{y}(t+k) = \beta_0 u(t) + \varphi^T(t)\ \theta_0 \tag{3.3}$$

where

$$\varphi(t) = [-y(t) \ldots -y(t-r+1) \quad u(t-1) \ldots u(t-s)]^T.$$

The self-tuning regulator STURE discussed in Åström and Wittenmark (1973) was originally derived for the case $y_r = 0$. It can be described as follows. The parameters of the prediction model (3.3) are estimated by recursive least squares. The control signal is then determined from the condition that $\hat{y}(t+k) = 0$, i.e.

$$\beta_0 u(t) = -\ \varphi^T(t)\ \theta(t) \tag{3.4}$$

where $\theta(t)$ are the estimates of the parameters at time t. For simplicity it is assumed that $\beta_0 = b_0$ is known. This is not essential. See Åström and Wittenmark (1973) and Ljung and Wittenmark (1974). The estimate $\theta(t)$ is given by

$$\theta(t+1) = \theta(t) + P(t+1)\ \varphi(t-k+1)\ \varepsilon(t+1)$$

$$\varepsilon(t+1) = y(t+1) - \hat{y}(t+1) = y(t+1) - \beta_0 u(t-k+1) - \varphi^T(t-k+1)\ \theta(t) =$$
$$= y(t+1) + \varphi^T(t-k+1)[\theta(t-k+1) - \theta(t)]$$

where the last equality follows from (3.4). The matrix P(t+1) is given by

$$P^{-1}(t+1) = \lambda P^{-1}(t) + \varphi(t-k+1)\ \varphi^T(t-k+1).$$

This implies that the matrix P must be regular which in essence is an identifiability requirement.

When the self-tuning regulator is connected as shown in Fig. 2.1 the above equations hold if y is replaced by $y - y_r = -e$. The regulator is thus described by the equations

$$\begin{cases} \varphi(t) = [e(t) \ldots e(t-r+1) \quad u(t-1) \ldots u(t-s)]^T \\ \theta(t+1) = \theta(t) + P(t+1)\, \varphi(t-k+1)\, \varepsilon(t+1) \\ \varepsilon(t+1) = -\, e(t+1) + \varphi^T(t-k+1)[\theta(t-k+1) - \theta(t)] \\ P^{-1}(t+1) = \lambda P^{-1}(t) + \varphi(t-k+1)\, \varphi^T(t-k+1) \\ \beta_0 u(t) = -\, \varphi^T(t)\, \theta(t). \end{cases} \tag{3.5}$$

It is well-known that the matrix P also satisfies the following recursive equation

$$P(t+1) = \frac{1}{\lambda} \Big\{ P(t) - $$

$$- P(t)\, \varphi(t-k+1)[\lambda + \varphi^T(t-k+1)\, P(t)\, \varphi(t-k+1)]^{-1}\, \varphi^T(t-k+1)\, P(t) \Big\}.$$

The equation (3.4) can be written as

$$\mathcal{B}(q^{-1})\, u(t) = \mathcal{A}(q^{-1})\, e(t) \tag{3.6}$$

where

$$\begin{aligned} \mathcal{A}(q^{-1}) &= -\, \theta_1(t) - \theta_2(t)\, q^{-1} - \ldots - \theta_r(t)\, q^{-r+1} \\ &\triangleq \alpha_1(t) + \alpha_2(t)\, q^{-1} + \ldots + \alpha_r(t)\, q^{-r+1} \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \mathcal{B}(q^{-1}) &= \beta_0 + \theta_{r+1}(t)\, q^{-1} + \ldots + \theta_{r+s}(t)\, q^{-s} \\ &\triangleq \beta_0 + \beta_1(t)\, q^{-1} + \ldots + \beta_s(t)\, q^{-s}. \end{aligned} \tag{3.8}$$

The definition of the control error e gives

$$e(t) = y_r(t) - y(t) = y_r(t) - \frac{B(q^{-1})}{A(q^{-1})}\, u(t-k).$$

Combining this with (3.6) we find the following equations

$$\begin{cases} A(q^{-1})\, e(t) + B(q^{-1})\, u(t-k) = A(q^{-1}) y_r(t) \\ \mathcal{A}(q^{-1})\, e(t) - \mathcal{B}(q^{-1})\, u(t) = 0 \end{cases} \tag{3.9}$$

which describes how u and e evolve if y_r and θ are known. Notice that the difference operators $\mathcal{A}(q^{-1})$ and $\mathcal{B}(q^{-1})$ are in general timevarying because they depend on the parameter estimates. Compare (3.7) and (3.8).

The closed system obtained when the self-tuning regulator is connected to the process can thus be described by the difference equations

$$\begin{cases} \theta(t+1) = \theta(t) + P(t+1)\ \varphi(t-k+1)\ \varepsilon(t+1) \\ \varepsilon(t+1) = -\ e(t+1) + \varphi^T(t-k+1)[\theta(t-k+1) - \theta(t)] \\ P^{-1}(t+1) = \lambda P^{-1}(t) + \varphi(t-k+1)\ \varphi^T(t-k+1) \\ A(q^{-1})\ e(t) + B(q^{-1})\ u(t-k) = A(q^{-1})\ y_r(t) \\ A(q^{-1})\ e(t) - \mathcal{B}(q^{-1})\ u(t) = 0. \end{cases} \tag{3.10}$$

These nonlinear difference equations are fairly complex. It is not possible to obtain analytical solutions. To get insight into the properties of the closed loop system the standard path of investigating the stationary solutions and their local properties is used.

Stationary Solutions

The state of the system (3.10) can be chosen as θ, P, the statevector of a realization of the dynamical system (3.9), and the delayed values of $e(t)$ and $u(t)$, which are necessary to determine $\varphi(t-k+1)$. The system is driven by the reference signal y_r. For a given reference signal y_r it will first be investigated if there are some stationary solutions. Since there is a time delay in the process there will always be a control error for at least k sampling periods after an unpredictable change in the reference signal. It is thus not possible to have stationary solutions in the sense that all variables are constant except in the trivial case of $y_r = 0$. It will therefore be explored if there are stationary solutions in the sense that the parameter estimates are constant. Introduce

DEFINITION 1

A solution to the equations (3.10) is called *stationary* if $\theta(t) =$ = constant. □

Let θ_0 be a constant parameter vector. If $\theta(t) = \theta_0$ the closed loop system is described by (3.9). Assume that θ_0 is such that (3.9) is stable. Let e_0 and u_0 denote the solution to (3.9) corresponding to θ_0. Similarly let φ_0 and P_0 be the functions defined by (3.10) with e_0 and u_0 substituting e and u. It follows from (3.10) that θ_0 is a stationary solution if and only if

$$P_0(t+1)\ \varphi_0(t-k+1)\ e_0(t+1) = 0, \quad \forall\ t. \tag{3.11}$$

Since $P_0(t)$ was assumed to be regular this is equivalent to

$$\varphi_0(t-k+1)\ e_0(t+1) = 0, \quad \forall\ t.$$

It follows from the definition of φ that the stationarity conditions can be written as

$$\begin{cases} e_0(t+\tau)\ e_0(t) = 0, & \forall\ t \quad \text{and} \quad \tau = k,\ldots,\ k+r-1 \\ e_0(t+\tau)\ u_0(t) = 0, & \forall\ t \quad \text{and} \quad \tau = k+1,\ldots,\ k+s. \end{cases} \tag{3.12}$$

Stationary solutions will not exist unless special conditions are imposed on the reference signal. Such conditions will now be introduced.

Piece-wise Deterministic Signals

It is convenient for analysis to use signals that can be described as solutions to linear difference equations. Such a description covers a wide range of signals. It follows, however, from the description that the signals can be predicted exactly for all future, which is not desirable for our purposes. In stochastic control theory the problem is overcome by introducing random processes which naturally can not be predicted exactly. Since we want an essentially deterministic description a different approach will be taken. It will be assumed that the signals can be described as solutions to linear difference equations over certain time intervals but that there are isolated points where the signals change in an unpredictable manner. Certain restrictions are also imposed on the changes. It is assumed that there are discontinuities only in the highest difference which appears in the difference equation.

Let $T_i(\ell)$ be a set of discrete integers

$$T_i(\ell) = \{\ldots, t_{-1}, t_0, t_1, \ldots\}$$

such that

$$\min(t_{i+1} - t_i) = \ell > 1$$

and let $T_r(\ell)$ be the complement of $T_i(\ell)$ with respect to all integers. The points in $T_i(\ell)$ are obviously isolated. Introduce

DEFINITION 2

Let $Q(q^{-1})$ be a polynomial of degree $n < \ell$ in the backward shift operator. A signal y is called a *piece-wise deterministic* signal of degree n and index ℓ if

$$Q(q^{-1})\ y(t) = 0 \quad \text{if} \quad t \in T_r(\ell) \tag{3.13}$$

and

$$Q(q^{-1})\ y(t) \neq 0 \quad \text{if} \quad t \in T_i(\ell). \tag{3.14}$$

□

The polynomial $Q(q^{-1})$ is called the *generator* of the signal. The set $T_r(\ell)$ is called the set of *regular points* and $T_i(\ell)$ is called the set of *irregular points*. The irregular points are at least ℓ units apart, where ℓ is the index of the signal. The index is thus a measure of how irregular the signal is. The smaller ℓ is, the more irregular is the signal. It follows from (3.13) that a piece-wise deterministic signal can be predicted exactly in an interval that does not contain any irregular points. This is shown in detail in the following.

The name piece-wise deterministic signal is chosen because its resemblance to the notion of a completely deterministic stochastic process. In the early literature on stochastic processes a process was called completely deterministic or singular if (3.13) holds for all t. See e.g. Wold (1954). Piece-wise constant signals and piece-wise linear signals are examples of piece-wise deterministic signals.

In analogy with the terminology for random processes the signal ν defined by

$$\nu(t) = Q(q^{-1})\, y(t), \tag{3.15}$$

is called the *innovations*.

EXAMPLE 3.1

A piece-wise constant signal has the generator

$$Q = 1 - q^{-1}.$$

The set of irregular points are all the points where the signal changes level. □

EXAMPLE 3.2

A piece-wise linear signal has the generator

$$Q = 1 - 2q^{-1} + q^{-2}.$$

The set of irregular points are all points such that the change of slope is immediately to the left of the points. See Fig. 3.1. □

Prediction of Piece-wise Deterministic Signals

A piece-wise deterministic signal can be predicted exactly in an interval which does not contain an irregular point. A k-step predictor can be constructed as follows. Let $F(q^{-1})$ and $G(q^{-1})$ be polynomials

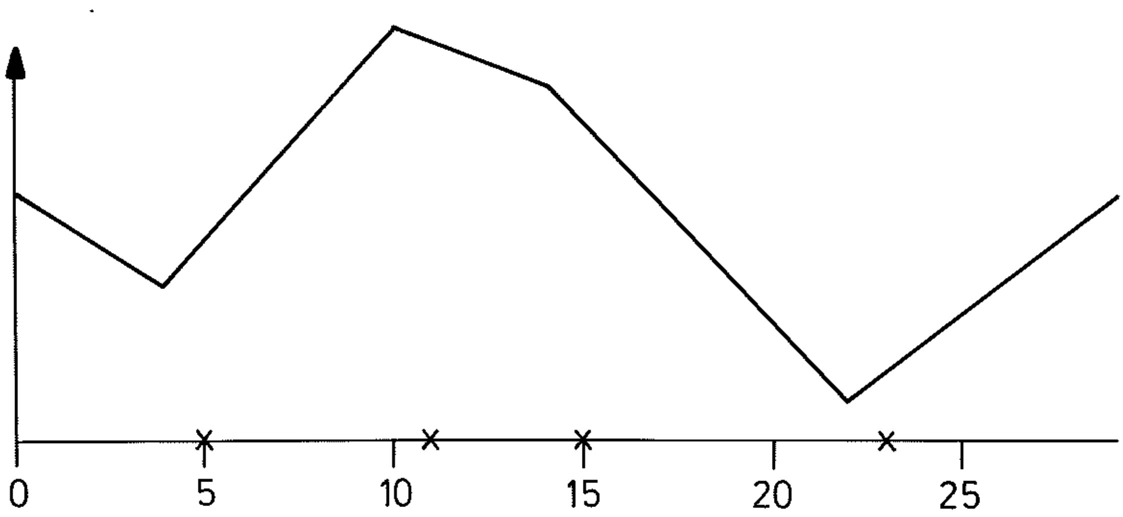

Figure 3.1. The set of irregular points for a piece-wise linear signal.

of degrees $k-1$ and deg $Q-1$ which are the unique solutions of the equation

$$1 = F(q^{-1})\ Q(q^{-1}) + q^{-k}\ G(q^{-1}). \tag{3.16}$$

If deg $Q + k < \ell$ it follows from (3.13) that

$$[1 - q^{-k}\ G(q^{-1})]\ y(t) = F(q^{-1})\ Q(q^{-1})\ y(t) = 0$$

for $t_i + k \leqslant t < t_{i+1}$, where t_i, $t_{i+1} \in T_i(\ell)$.

The k-step predictor is thus given by

$$\hat{y}(t) = G(q^{-1})\ y(t-k)$$

and the prediction error is

$$e(t) = y(t) - \hat{y}(t) = F(q^{-1})\ Q(q^{-1})\ y(t) = F(q^{-1})\ \nu(t). \tag{3.17}$$

Since $Q(q^{-1})\ y(t)$ is a sequence which is different from zero at the irregular points only, it follows that the error of the k-step predictor is different from zero at the irregular points and their $k-1$ right successors.

A periodic signal with the period p has the property

$$(1 - q^{-p})\ y(t) = 0 \quad \forall\ t.$$

This means that a periodic signal is deterministic and that it can be predicted exactly provided that a predictor with sufficiently large memory is available. Certain periodic signals can, however, be considered also as piece-wise deterministic signals. This is illustrated by the following example.

EXAMPLE 3.3

Consider a square wave with period 2p. The signal can be predicted using the predictor

$$\hat{y}(t+1|t) = y(t).$$

This predictor gives the correct prediction except of those points where the square wave changes level. The square wave can thus be regarded as a piece-wise deterministic signal with generator $Q = 1 - q^{-1}$. The square wave can, however, also be predicted exactly by the predictor

$$\hat{y}(t+1|t) = y(t) - y(t-p+1) + y(t-p)$$

which requires that $p+1$ past values of the signal are stored. This means that

$$(1 - q^{-1})\ (1 + q^{-p})\ y(t) = 0 \quad \forall\ t.$$

The square wave can thus also be regarded as a purely deterministic signal.

Characterization of Stationary Solutions

The stationary solutions were previously characterized implicitly by the equations (3.12). A more direct characterization can be given if the command signal is a piece-wise deterministic signal. To obtain the result it is necessary to assume that there are enough parameters in the regulator and that the unpredictable changes in the command signal are sufficiently far apart. To be precise the following notion is introduced.

DEFINITION 3

A self-tuning regulator is called *compatible* with the process model (3.1) and the command signal if

$$\deg \mathcal{A} \geq \deg A + \deg Q - \deg \text{lcd}\,[A,Q] - 1 \tag{3.18}$$

$$\deg \mathcal{B} \geq k + \deg B + \deg Q - \deg \text{lcd}\,[A,Q] - 1 \tag{3.19}$$

where lcd(A,Q) is the largest common divisor of A and Q and

$$\deg \mathcal{A} + \deg \mathcal{B} < \ell - 2k. \tag{3.20}$$

□

The following result can now be given.

THEOREM 1

Let the system (3.1) be controlled by a self-tuning regulator which is compatible with the system and the piece-wise deterministic command signal y_r. Assume that there is a stationary solution. Then

(i) Q divides $\mathcal{A}$, B divides G

and the stationary solution is such that

(ii) $\frac{\mathcal{A}}{\mathcal{B}} = \frac{AG}{BFQ}$

□

Proof

Let θ_0 be the stationary value of the parameter estimate. Let the signals obtained for $\theta(t) = \theta_0$ be indexed by the subscript " $_0$ ". When the parameter estimate is constant the closed loop system is characterized by (3.9). The operators $\mathcal{A}$ and $\mathcal{B}$ are time invariant because θ_0 is constant. See (3.7) and (3.8). The closed loop system is thus linear and time invariant. The characteristic equation of the closed loop system is

$$A(z^{-1})\, \mathcal{B}(z^{-1}) + z^{-k}\, \mathcal{A}(z^{-1})\, B(z^{-1}) = 0. \tag{3.21}$$

The parameter estimate θ_0 must be such that this equation has all its zeros inside the unit disc. It may happen that the polynomials $\mathcal{A}$ and $\mathcal{B}$ have common factors. Let $\mathcal{A}_1$ and $\mathcal{B}_1$ be the polynomials obtained when the largest common divisor of A and B is cancelled. Solving (3.9) for e_0 and u_0 we get

$$[A\mathcal{B}_1 + q^{-k} A_1 B]\, e_0 = A\mathcal{B}_1\, y_r \tag{3.22}$$

$$[A\mathcal{B}_1 + q^{-k} A_1 B]\, u_0 = AA_1\, y_r\ . \tag{3.23}$$

The solution associated with the possible common factors of A and $\mathcal{B}$ will vanish in steady state. Introduce the signal v_0 defined by

$$[A\mathcal{B}_1 + q^{-k} A_1 B]\, v_0 = A\, y_r. \tag{3.24}$$

Then

$$\begin{cases} e_0 = \mathcal{B}_1(q^{-1})\, v_0 \\ u_0 = A_1(q^{-1})\, v_0. \end{cases} \tag{3.25}$$

Since θ_0 is a stationary point it follows from (3.12) that

$$\begin{cases} e_0(t+\tau)\ \mathcal{B}_1(q^{-1})\ v_0(t) = 0 & \text{for } \tau = k,\dots, k+r-1 \text{ and all } t \\ e_0(t+\tau)\ A_1(q^{-1})\ v_0(t) = 0 & \text{for } \tau = k+1,\dots, k+s \text{ and all } t \end{cases} \tag{3.26}$$

where $r - 1 = \deg A$ and $s = \deg \mathcal{B}$.

The equations (3.26) can be written as

$$\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \cdots \alpha_{r_1} & & & \\ & \ddots & & & & \\ & & 0 & \alpha_1 & \alpha_2 \cdots \alpha_{r_1} & \\ & & & \ddots & & \\ & & & 0 & \alpha_1 & \alpha_2 \cdots \alpha_{r_1} \\ 1 & \beta_1 \cdots \beta_{s_1} & & & & \\ & \ddots & & & & \\ & & 1 & \beta_1 \cdots \beta_{s_1} & & \\ & & & \ddots & & \\ & & & 1 & \beta_1 \cdots \beta_{s_1} & \end{pmatrix} \begin{pmatrix} e_0(t+k)\ v_0(t) \\ e_0(t+k)\ v_0(t-1) \\ \vdots \\ \vdots \\ e_0(t+k)\ v_0(t-r_1-s+1) \end{pmatrix} = 0$$

for all t, where $r_1 - 1 = \deg A_1$ and $s_1 = \deg \mathcal{B}_1$. The parameters of $A_1(q^{-1})$ and $\mathcal{B}_1(q^{-1})$ are here denoted by α_i and β_i. Notice that

$r_1 + s = s_1 + r$. The matrix on the left is the resultant of the polynomials A_1 and B_1. See van der Waerden (1966). Since A_1 and B_1 are relatively prime the columns are linearly independent and it follows that

$$e_0(t+k)\ v_0(t-i) = 0$$

for $i = 0,\dots,\ r_1 + s - 1$ and all t. Hence

$$e_0(t+\tau)\ v_0(t) = 0 \tag{3.27}$$

for all t and $\tau = k,\ k+1,\dots,\ k+r_1+s-1$.

The signal v_0 has the property

$$v_0(t_i) \neq 0.$$

This is shown by contradiction. Assume $v_0(t_i) = 0$. Let $Q = Q_1Q_2$ and $A = A_1Q_2$, where Q_2 is the largest common divisor of A and Q. Equation (3.24) gives

$$Q_1[AB_1 + q^{-k}\ A_1B]\ v_0 = Q_1Ay_r = Q_1Q_2A_1y_r = A_1\nu.$$

The right hand side is different from zero for $t = t_i$ because the leading term of A_1 is nonzero. The polynomial operator of the left hand side has the order

$$n_s = \deg Q_1 + \max[\deg(AB_1),\ k + \deg(A_1B)].$$

It follows from the compatibility conditions (3.18) and (3.19) that $n_s \leqslant r_1 + s$. It thus follows that $v_0(t)$ can not be zero for all t in an interval of length n_s between two irregular points. It then follows from (3.27) that $e_0(t)$ is zero for $t_{i-1} + n_s + k \leqslant t \leqslant t_i$. Equation (3.22) gives

$$Q_1[AB_1 + q^{-k}\ A_1B]\ e_0(t_i) = Q_1AB_1y_r(t_i) = A_1B_1\nu(t_i).$$

The left hand side is zero, but the right hand side is not zero. A contradiction is thus obtained and it is proven that $v_0(t_i) \neq 0$. Equation (3.27) then implies that

$$e_0(t_i + \tau) = 0,\quad \tau = k,\ k+1,\dots,\ k+r_1+s-1.$$

Furthermore e_0 satisfies (3.22). Then

$$Q_1[A\mathcal{B}_1 + q^{-k} A_1 B]\, e_0 = Q_1 A\mathcal{B}_1 y_r = 0$$

for $t_i + n_s < t < t_{i+1}$. Hence

$$e_0(t_i + \tau) = 0, \; \tau = k, k+1, \ldots, t_{i+1} - t_i - 1.$$

The signal e_0 can thus be represented as

$$e_0(t) = W(q^{-1})\, \nu(t) \tag{3.28}$$

where deg $W = k - 1$ and ν is the innovation of the command signal. It follows from (3.22) that

$$e_0 = \frac{A\mathcal{B}_1}{A\mathcal{B}_1 + q^{-k} A_1 B}\, y_r = \left[1 - \frac{q^{-k} A_1 B}{A\mathcal{B}_1 + q^{-k} A_1 B}\right] y_r =$$

$$= QFy_r + q^{-k}\left[G - \frac{A_1 B}{A\mathcal{B}_1 + q^{-k} A_1 B}\right] y_r =$$

$$= F\nu + q^{-k}\left[G - \frac{A_1 B}{A\mathcal{B}_1 + q^{-k} A_1 B}\right] y_r$$

where the last equality follows from (3.16) and deg $F = k - 1$. A comparison with (3.28) shows that

$$W = F \tag{3.29}$$

and

$$G = \frac{A_1 B}{A\mathcal{B}_1 + q^{-k} A_1 B}\, .$$

Hence

$$A_1 B[1 - q^{-k} G] = A\mathcal{B}_1 G$$

or

$$A_1 BQF = AG\mathcal{B}_1. \tag{3.30}$$

Since A_1 and $\mathcal{B}_1$ are obtained by cancelling common factors in A and $\mathcal{B}$ condition (ii) now follows.

Proceeding in an analogous way it can be shown that

$$v_0(t) = V(q^{-1})\ \nu(t) \tag{3.31}$$

where deg $V = k-1$. Equation (3.23) gives after some calculations

$$v_0 = \frac{F}{\mathcal{B}_1}\nu + q^{-k}\left[G - \frac{A_1 B}{A\mathcal{B}_1 + q^{-k}A_1 B}\right] y_r.$$

If this should be compatible with (3.31) $\mathcal{B}_1$ must divide F. Since A and B are relatively prime and Q and G are also relatively prime it follows from (3.30) that B divides G and that Q divides A. This proves part (i) of the theorem. □

Remark 1

It follows from the theorem that there will not be any stationary solutions unless specific conditions are satisfied. The condition that Q divides A implies that the dynamics of process which generates the command signal must be part of the process dynamics. This can be achieved by introducing a precompensator which, however, requires a priori knowledge of the command signal. To have stationary solutions when the command signal is piece-wise constant the process must contain an integrator or else it must be cascaded with an integrator. This is an example of the "internal model principle".

Remark 2

The condition that B divides G can be relaxed. By back-tracking the arguments used in the proof it can be shown that if B does not divide G then

$$e_0(t+\tau)\ u_0(t) \neq 0$$

for $t = t_i,\ t_i+1, \ldots,\ t_i+r-1$ and $\tau = k+1, \ldots, k+s$. This means that the stationarity condition (3.12) is violated. Since the signal u_0 is governed by

$$u_0 = \frac{A_1}{\mathcal{B}_1} e_0 = \frac{AG}{BFQ} e = \frac{AG}{BQ}\nu$$

it follows that if Q divides A, if B is stable, and if ℓ is sufficiently large then $u_0(t_i-\tau)$ will be arbitrarily small for $\tau = 1, 2, \ldots, s$. This means that the products $e_0(t+\tau)u_0(t)$ can be made arbitrarily small and there will not be any noticable change in the parameter estimates at the irregular points. This explains the simulations presented in Example 8.2 in Wittenmark (1973, p. 108) where $B = 1 + 0.5\, q^{-1}$ and $\ell = 25$. The magnitude of the product $e_0(t+\tau)u_0(t)$ will be of the magnitude $\nu(t_i)^2 \cdot 2^{-25} \approx 3\cdot 10^{-8}\, \nu(t_i)^2$ which is not noticable in the simulations.

Theorem 1 can be used to give further insight into the properties of possible stationary solutions. We have

THEOREM 2

Consider a system according to Fig. 2.1 with a self-tuning regulator which is compatible with the process and the command signal. If there is a stationary solution then the open loop transfer function is such that

$$G_0 = \frac{q^{-k}G}{FQ} \tag{3.32}$$

and the input - output relation of the closed loop system is

$$y(t) = G(q^{-1})\, y_r(t-k). \tag{3.33}$$

□

Proof

The open loop transfer function is

$$G_0(z^{-1}) = z^{-k}\, \frac{A_1(z^{-1})\; B(z^{-1})}{B_1(z^{-1})\; A(z^{-1})} .$$

Equation (3.32) now follows from Theorem 2 (ii). The closed loop transfer function is

$$\frac{z^{-k}\; G(z^{-1})}{F(z^{-1})\; Q(z^{-1}) + z^{-k}G(z^{-1})} = z^{-k}G(z^{-1}).$$

The last equality follows from (3.16). The result (3.33) is thus also proven. □

Remark 1

Theorem 2 explains the empirical observation that a self-tuning regulator connected as in Fig. 2.1 gives a dead-beat like response if the parameter estimates converge. Compare e.g. with the simulations in Examples 8.1 and 8.2 in Wittenmark (1973).

Remark 2

Notice that stationary solutions correspond to closed loop systems whose properties are determined by the command signal only. A few examples are given below.

EXAMPLE 3.3

A piece-wise constant command signal has the generator

$$Q(q^{-1}) = 1 - q^{-1}.$$

Compare with Example 3.1. It follows from (3.13) that

$$F(q^{-1}) = 1 + q^{-1} + \ldots + q^{-k+1}$$

$$G(q^{-1}) = 1.$$

The possible stationary solutions for piece-wise constant command signals are thus such that the closed loop system has the property

$$y(t) = y_r(t-k)$$

□

EXAMPLE 3.4

A piece-wise linear command signal has the generator

$$Q(q^{-1}) = 1 - 2q^{-1} + q^{-2}.$$

Compare with Example 3.2. It follows from (3.13) that

$$F(q^{-1}) = 1 + 2q^{-1} + 3q^{-2} + \ldots + kq^{-k+1}$$

$$G(q^{-1}) = (k+1) - kq^{-1}.$$

The possible stationary solutions for piece-wise linear command signals are thus such that the closed loop system has the property

$$y(t) = y_r(t-k) + k\,[y_r(t-k) - y_r(t-k-1)]$$

□

Remark 3

Notice that it follows from the proof of Theorem 2 that if θ_0 is a stationary solution then the corresponding regulator is such that the poles and zeros of the process are cancelled by the regulator. It can thus be expected that the regulator will not work well unless the process is stable and minimum phase.

Local Properties of Stationary Solutions

Having obtained parameters θ_0 which correspond to stationary solutions it is natural to investigate the local properties by linearizing the equations around the stationary solutions. The linearized equations can be written as

$$\begin{cases} \delta\theta(t+1) = \delta\theta(t) + P_0(t+1)\varepsilon_0(t+1)\delta\varphi(t-k-1) + P_0(t+1)\varphi_0(t-k+1)\delta\varepsilon(t+1) \\ \delta\varepsilon(t+1) = -\ \delta e(t+1) + \varphi_0^T(t-k+1)[\delta\theta(t-k+1) - \delta\theta(t)] + \\ \qquad\qquad + [\theta^T(t-k+1) - \theta^T(t)]\delta\varphi(t-k+1) \\ A(q^{-1})\delta e(t) + B(q^{-1})\delta u(t-k) = 0 \\ A_0(q^{-1})\delta e(t) - \mathcal{B}_0(q^{-1})\delta u(t) + \delta A(q^{-1})e_0(t) - \delta\mathcal{B}(q^{-1})u_0(t) = 0 \\ \delta[P^{-1}(t+1)] = \lambda\delta[P^{-1}(t)] + \varphi_0^T(t-k+1)\delta\varphi_0(t-k+1) + \delta\varphi_0^T(t-k+1)\varphi_0(t-k+1) \end{cases} \tag{3.34}$$

where δe denotes the deviations from the stationary solution i.e.

$$\delta e = e - e_0.$$

Notice that the equations for $\delta\theta$, δe, and δu are decoupled from the equation for δP. This means that the stability of the stationary solution can be investigated without considering the perturbation equation for P. The linearized equations will be investigated further for specific examples in Chapter 4.

4. EXAMPLES

An experimental study of μ-processor self-tuners was carried out by Andersson (1977). The purpose of this study was to find simple and robust algorithms which could be used in a variety of industrial applications. Andersson found that a self-tuning regulator in a single-degree-of-freedom configuration as in Fig. 2.1 could give unsatisfactory behaviour in some cases. The motivation for the work done in this report was partly to explain the empirical results found by Andersson. The examples presented in this section correspond to some of the simple cases investigated by Andersson. The examples were simulated in SIMNON using a special package for adaptive control developed by Gustavsson (1978).

EXAMPLE 4.1 *(A first order process with square-wave command)*

Consider a process described by (3.1) with

$$A(q^{-1}) = 1 - 0.75\ q^{-1}$$

$$B(q^{-1}) = 1$$

$$k = 1.$$

Assume that it is desired to obtain a regulator which works well for piece-wise constant command signals. Since the process does not contain an integrator it would be desirable to have a regulator with integral action. A simple regulator structure which admits this is given by

$$u(t) = -\ \theta_1\ e(t) - \theta_2\ e(t-1) - \theta_3\ u(t-1)$$

Hence r=2 and s=1. To have integral action the parameter θ_3 should have the value $\theta_3=-1$. There are two choices of values of the parameters θ_1 and θ_2 which give dead-beat like responses, $\theta_1=-1.75$, $\theta_2=0.75$ and $\theta_1=-1$, $\theta_2=0.75$. This example was originally investigated in order to find out if a self-tuning regulator would tune in on any of these parameter values. An interesting problem was for example if the self-tuning regulator in the configuration in Fig. 2.1 could introduce integral action when desired. In Fig. 4.1 through Fig. 4.5 are shown what happens when the self-tuner in the configuration of Fig. 2.1 is used. The behaviour shown is very unsatisfactory. The parameters do not converge and the response is poor. The amplitudes of the fluctuations in the estimates will depend on the forgetting factor and the period of the command signal.

The command signal is piece-wise constant and has the generator

$$Q(q^{-1}) = 1 - q^{-1}.$$

Since Q does not divide A it follows from Theorem 1 that there will not be a stationary solution in the sense that the parameters are constant. The analysis of Chapter 3 thus allows us to conclude immediately that the self-tuning regulator in the configuration in Fig. 2.1 does not have the desirable properties.

It is not straightforward to show that (3.9) has a periodic solution if the command signal is periodic. Knowing that the solution is periodic the amplitude of the parameter fluctuations can, however, be estimated crudely. A phase plane of the parameters θ_1 and θ_2 for a typical simulation is shown in Fig. 4.6 to illustrate what happens if $\lambda<1$. If $\lambda=1$ the corrections of the parameter estimates will tend to zero. In Fig. 4.7 a phase plane of the parameters θ_2 and θ_3 is given to examplify how the parameter estimates evolve for the case $\lambda=1$. Notice that without the analysis made in Chapter 3 this simulation might have been interpreted as if the parameters converge to the point $\theta_1=-0.971$, $\theta_2=0.695$ and $\theta_3=-0.936$. The period length influences on the position of this point. As the period length increases the point tends to $\theta_1=-1$, $\theta_2=0.75$ and $\theta_3=-1$.

□

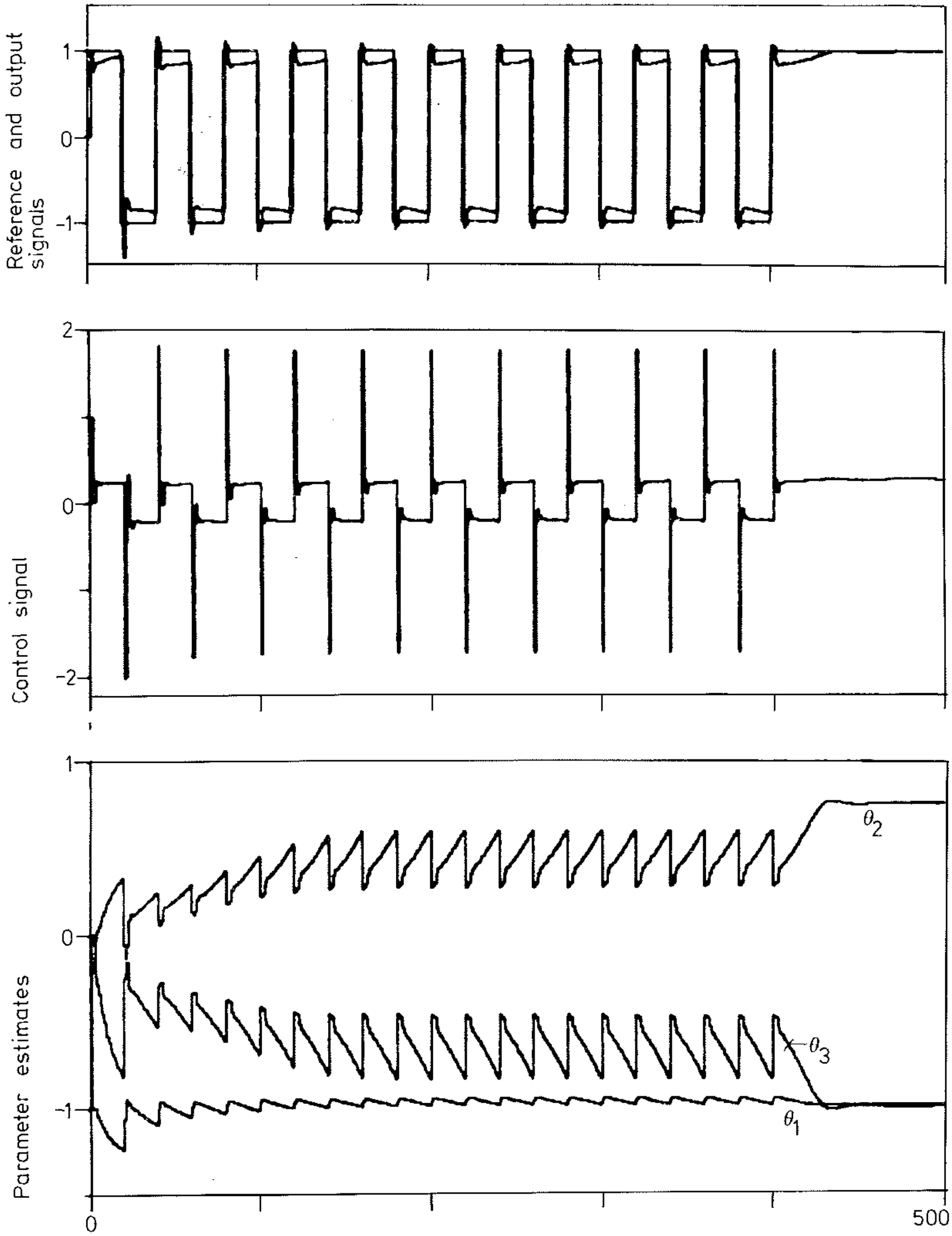

Figure 4.1. Results obtained when using a self-tuning regulator with $r = 2$ and $s = 1$ on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 40 and the forgetting factor is 0.98.

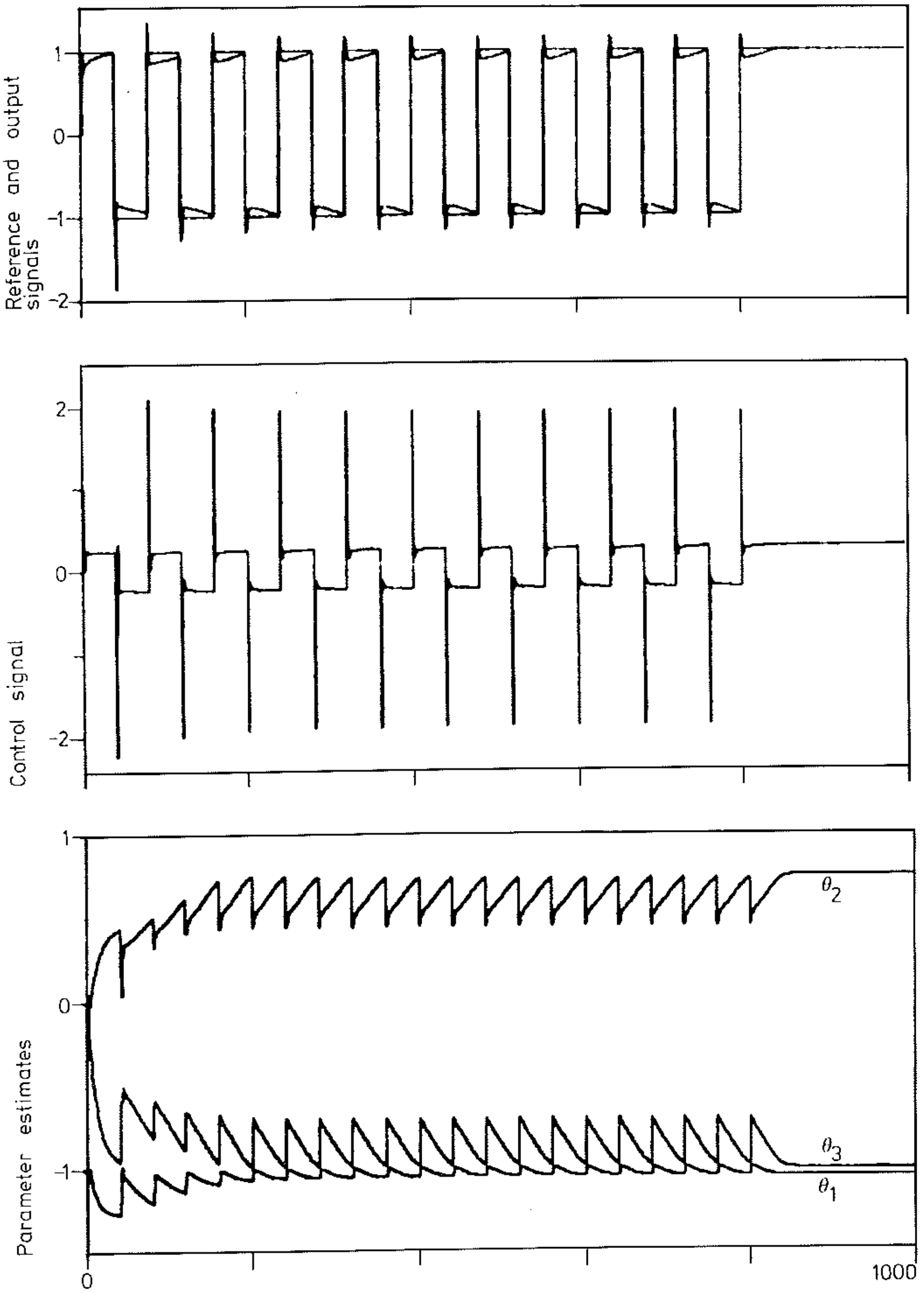

Figure 4.2. Results obtained when using a self-tuning regulator with r = 2 and s = 1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 80 and the forgetting factor is 0.98.

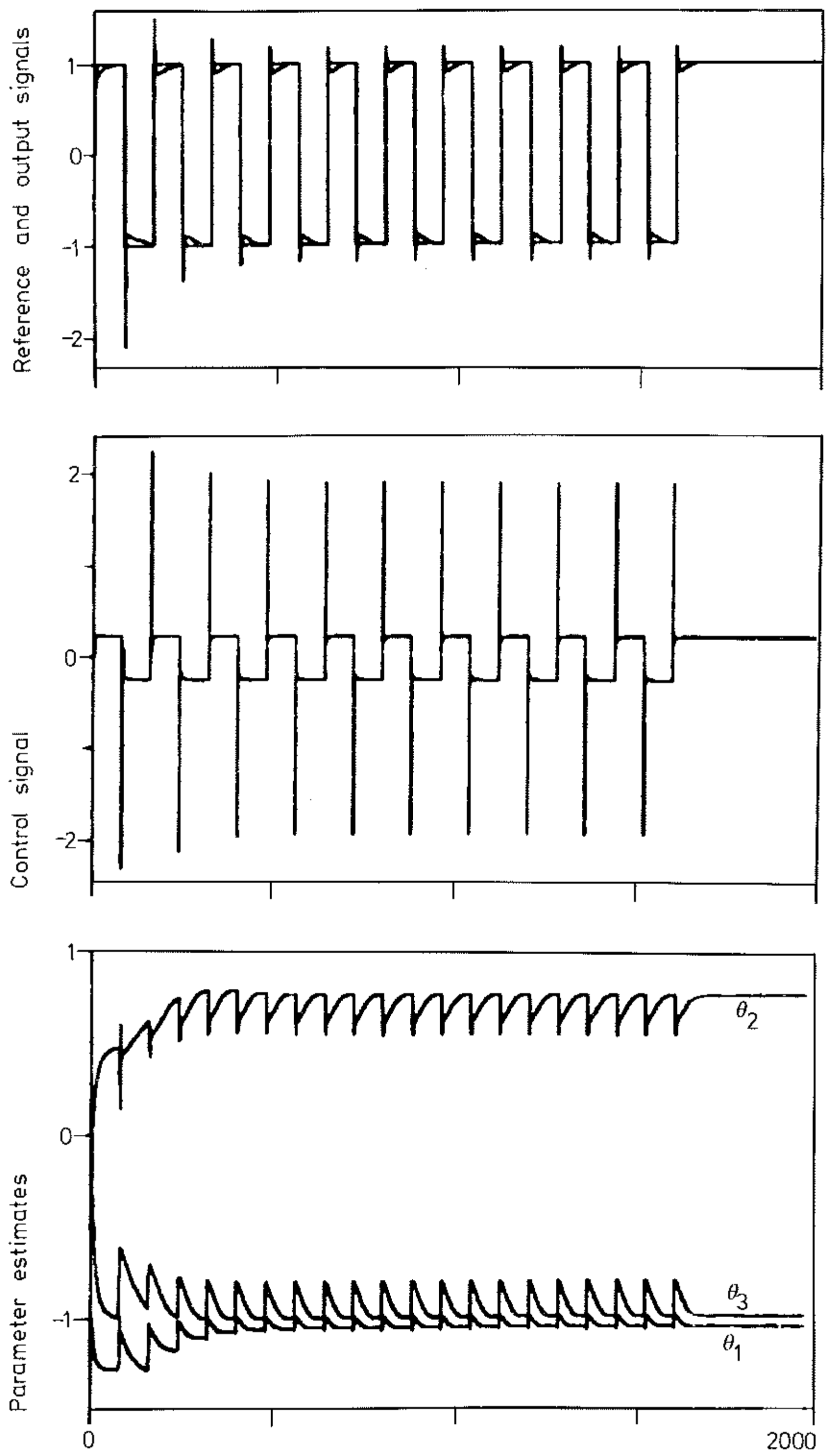

Figure 4.3. Results obtained when using a self-tuning regulator with r = 2 and s = 1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 160 and the forgetting factor is 0.98.

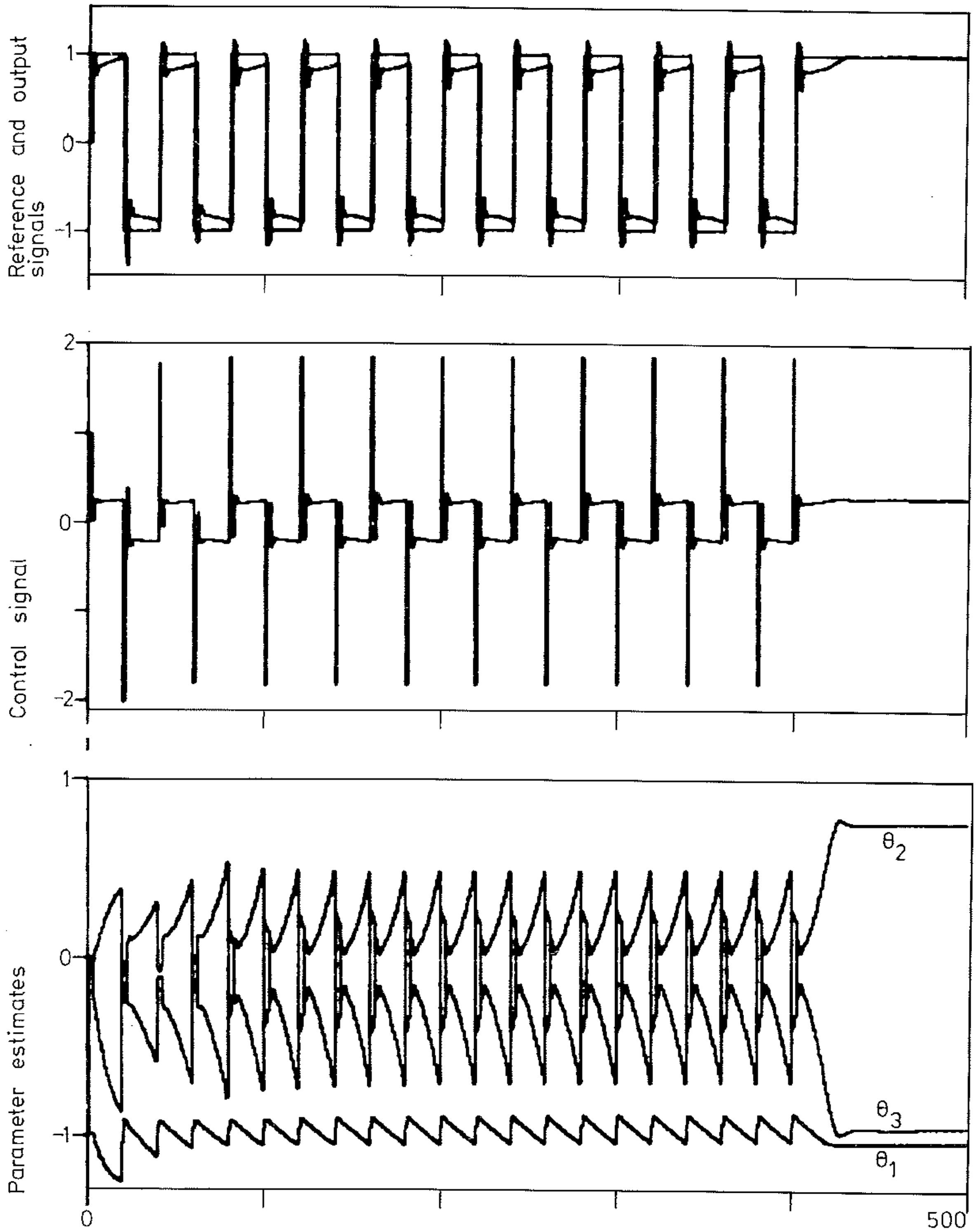

Figure 4.4. Results obtained when using a self-tuning regulator with r = 2 and s = 1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 40 and the forgetting factor is 0.95.

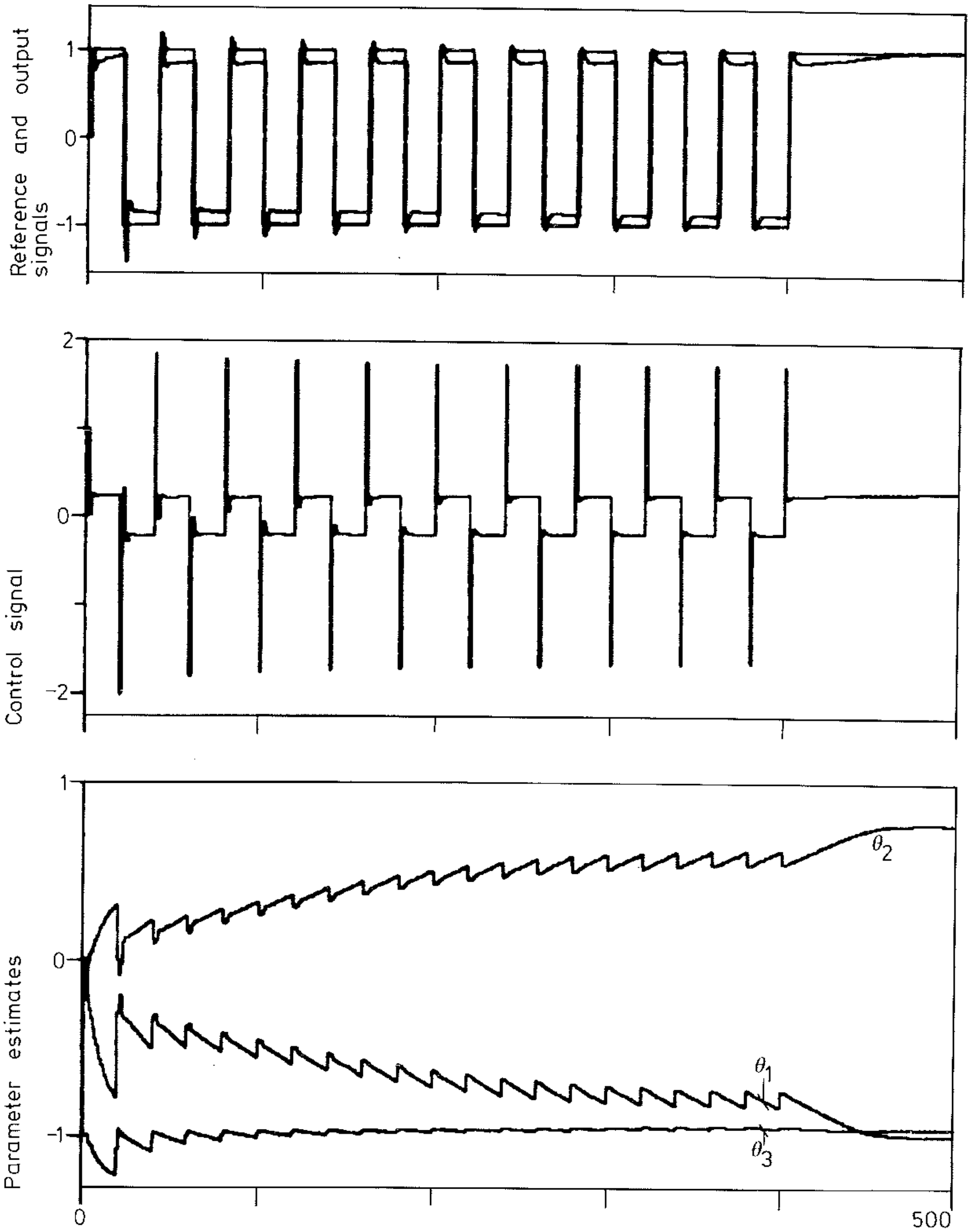

Figure 4.5. Results obtained when using a self-tuning regulator with r = 2 and s = 1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 40 and the forgetting factor is 0.995.

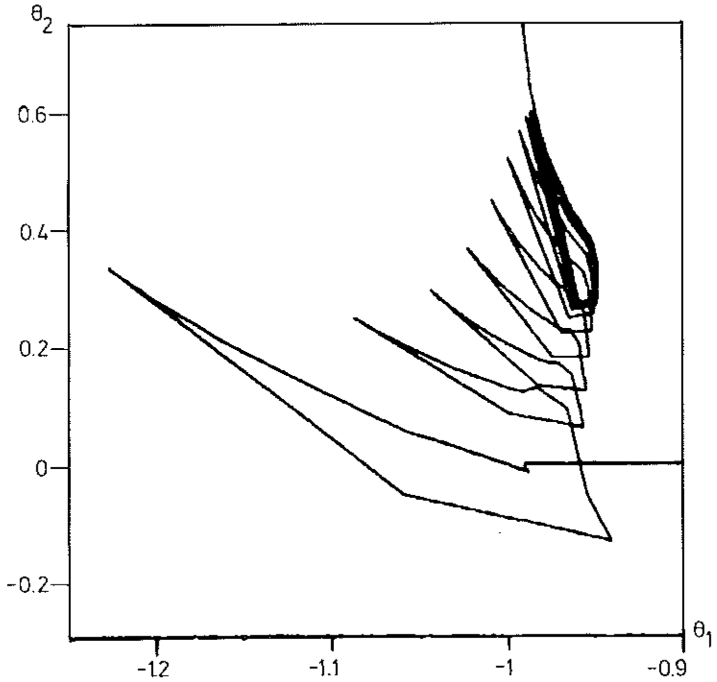

Figure 4.6. Phase plane corresponding to Fig. 4.1.

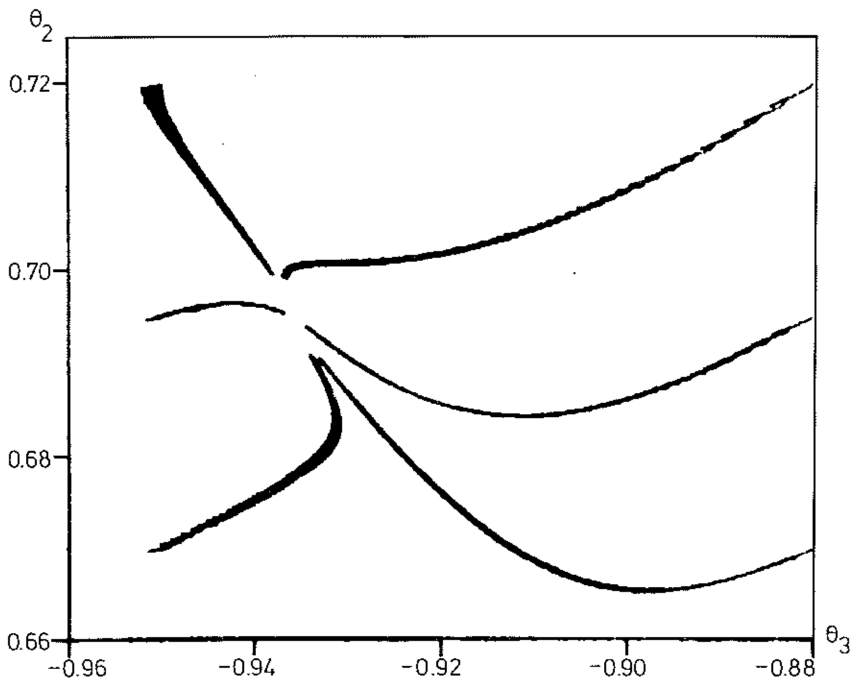

Figure 4.7. Phase plane of the parameter estimates when using a self-tuning regulator with r=2 and s=1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 80 and the forgetting factor is 1. Notice that each curve corresponds to approximately 20 000 time steps.

EXAMPLE 4.2 (An integrator with square-wave command)

Consider a process described by (3.1) with

$$A(q^{-1}) = 1 - q^{-1}$$

$$B(q^{-1}) = 1$$

$$k = 1.$$

Assume that the main task is to make the system follow a piece-wise constant command signal. Since the process has integration a proportional regulator will suffice, i.e. r=1 and s=0. It is shown in Fig. 4.8 what happens when such a self-tuning regulator is used. The behaviour of the system is excellent. The parameter estimate converges quickly to the correct value and the output then follows the command signal with one unit time-delay.

The analysis presented in Chapter 3 can be applied to get insight into the problem. Since the command signal has the generator $Q(q^{-1}) = 1 - q^{-1}$ it follows from (3.16) that

$$F(q^{-1}) = 1$$

$$G(q^{-1}) = 1$$

Since Q divides A and B divides G it follows from Theorem 1 that there is a stationary solution characterized by

$$\frac{A}{B} = \frac{AG}{BFQ} = 1$$

The parameter value associated with the stationary solution is thus

$$\theta_1 = -1.$$

To investigate the local properties of the system around the stationary solution the linearized equations (3.34) will be explored. In this particular case the closed loop system is described by

$$\theta(t+1) = \theta(t) + P(t+1)\ \varphi(t)\ \varepsilon(t+1)$$

$$\varepsilon(t+1) = -\ e(t+1) = y_r(t+1) - y(t+1)$$

$$P^{-1}(t+1) = \lambda\ P^{-1}(t) + \varphi(t)\ \varphi^T(t)$$

$$(1 - q^{-1})\ e(t) + u(t-1) = (1 - q^{-1})\ y_r(t)$$

$$\theta_1(t)\ e(t) + u(t) = 0$$

$$\varphi(t) = e(t)$$

Elimination of u(t), y(t) and φ(t) from these equations gives

$$
\begin{aligned}
e(t+1) &= [1 + \theta_1(t)]\, e(t) + [y_r(t+1) - y_r(t)] \\
\theta(t+1) &= \theta(t) - P(t+1)\, e(t)\, e(t+1) \\
P^{-1}(t+1) &= \lambda\, P^{-1}(t) + e^2(t)
\end{aligned}
\tag{4.1}
$$

The variables $\theta(t)$, $P^{-1}(t)$ and $e(t)$ can thus be chosen as the state variables of the closed loop system. If the command signal is periodic with period 2ℓ then the set of irregular points is $T_i = \{\ldots\ t_0-\ell,\ t_0,\ t_0+\ell, \ldots\}$. The stationary solution is

$$
\begin{aligned}
\theta_0(t) &= -1 \\
e_0(t) &= \nu(t) = \begin{cases} 0 & t \neq t_i \\ \pm\Delta & t = t_i \end{cases} \\
P_0^{-1}(t_i+1) &= \frac{e_0^2(t_i)}{1 - \lambda^{\ell}} \\
P_0^{-1}(t_i+k) &= \lambda^{k-1} P_0^{-1}(t_i+1) \qquad 2 \leq k \leq \ell
\end{aligned}
$$

Linearization of (4.1) around the stationary solution gives

$$
\begin{bmatrix} \delta e(t+1) \\ \delta\theta(t+1) \\ \delta P^{-1}(t+1) \end{bmatrix} = \begin{bmatrix} 0 & e_0(t) & 0 \\ -P_0(t+1)\, e_0(t+1) & 1 - P_0(t+1)\, e_0^2(t) & 0 \\ 2e_0(t) & 0 & \lambda \end{bmatrix} \begin{bmatrix} \delta e(t) \\ \delta\theta(t) \\ \delta P^{-1}(t) \end{bmatrix} \tag{4.2}
$$

The transition matrix of the equation (4.2) is denoted by A(t). Introduce the notations

$$
\begin{aligned}
\Delta &= e_0(t_i) \\
P_a &= P_0(t_i) = \frac{1 - \lambda^{\ell}}{\lambda^{\ell-1}\Delta^2} \\
P_b &= P_0(t_i+1) = \frac{1 - \lambda^{\ell}}{\Delta^2}
\end{aligned}
$$

Then

$$
A(t_i-1) = \begin{bmatrix} 0 & 0 & 0 \\ -P_a\Delta & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}
$$

$$
A(t_i) = \begin{bmatrix} 0 & \Delta & 0 \\ 0 & 1-P_b\Delta^2 & 0 \\ 2\Delta & 0 & \lambda \end{bmatrix}
$$

$$A(t_i+k) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \text{for } 1\leq k\leq \ell-2 \text{ and } \ell+1\leq k\leq 2\ell-2$$

$$A(t_i+\ell-1) = \begin{bmatrix} 0 & 0 & 0 \\ P_a\Delta & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A(t_i+\ell) = \begin{bmatrix} 0 & -\Delta & 0 \\ 0 & 1-P_b\Delta^2 & 0 \\ -2\Delta & 0 & \lambda \end{bmatrix}$$

The linearized system (4.2) is a periodic system with the period 2ℓ. In order to study the stability the time invariant system

$$\begin{bmatrix} \delta e(t_{i+2}) \\ \delta\theta(t_{i+2}) \\ \delta P^{-1}(t_{i+2}) \end{bmatrix} = \begin{bmatrix} \delta e(t_i+2\ell) \\ \delta\theta(t_i+2\ell) \\ \delta P^{-1}(t_i+2\ell) \end{bmatrix} = A \begin{bmatrix} \delta e(t_i) \\ \delta\theta(t_i) \\ \delta P^{-1}(t_i) \end{bmatrix}$$

with

$$A = \prod_{k=0}^{2\ell-1} A(t_i+k)$$

is formed. Simple calculations give

$$A = \begin{bmatrix} 0 & 0 & 0 \\ P_a\Delta(1-P_b\Delta^2)^2 & (1-P_b\Delta^2)^2 & 0 \\ 0 & 0 & \lambda^{2\ell} \end{bmatrix}$$

The stability is determined by the eigenvalues of A. Inserting the values of P_a and P_b we find that the eigenvalues are 0, $\lambda^{2\ell}$ and $\lambda^{2\ell}$. For $\lambda<1$ the solution $\theta_0(t)=-1$ is thus locally stable. If there is a perturbation from the stationary solution the parameter estimate will return to the stationary value. Notice that one element of the matrix A depends on the sign of Δ. This means that a disturbance in e(t) will influence on $\theta(t)$ differently depending on the sign of the disturbance.

In this example the equation for δP^{-1} is always stable if $\lambda<1$. Also notice that the matrices A(t) have the structure

$$A(t) = \begin{bmatrix} x & x & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

which means that the equations for $\delta\theta$ and δe do not involve δP^{-1}. This is generally true as was discussed in Chapter 3.

□

It was shown in Example 4.1 that the self-tuning regulator in the configuration of Fig. 2.1 could not introduce an integrator in the loop. In the next example it will be shown that the regulator could be made to work satisfactory if an integrator is introduced into the system.

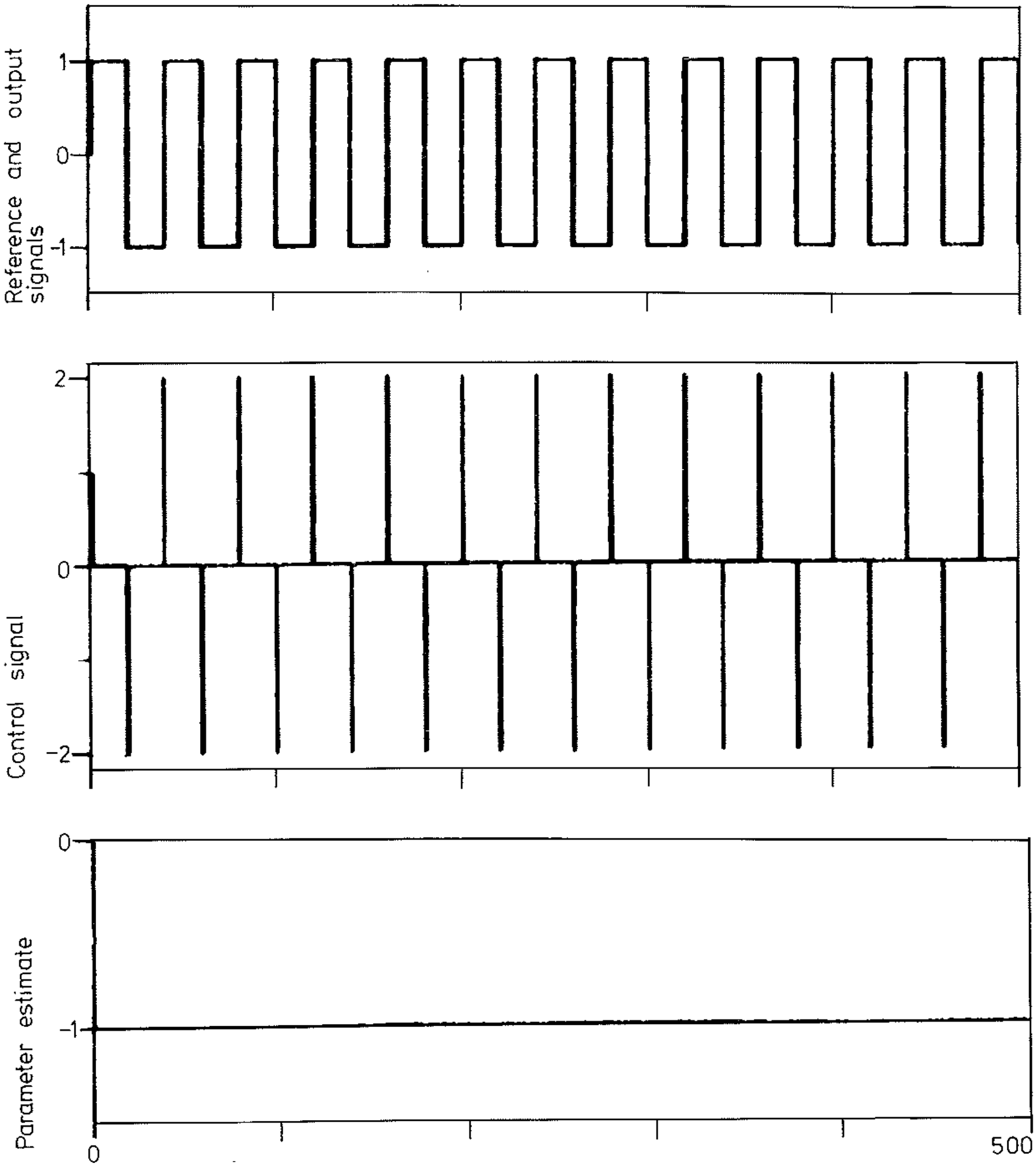

Figure 4.8. Results obtained when using a self-tuning regulator with $r = 1$ and $s = 0$ on a process with the pulse transfer function $H(z) = \frac{1}{z - 1}$. The period of the command signal is 40 and the forgetting factor is 0.98.

EXAMPLE 4.3 (Same as Example 4.1 but the process is cascaded with an integrator)

Consider the process in Example 4.1. Assume that an integrator is added to the plant. The process is then described by

$$A(q^{-1}) = 1 - (a+1)\, q^{-1} + a\, q^{-2}$$

$$B(q^{-1}) = 1$$

$$k = 1.$$

Let the command signal be a piece-wise constant signal with the generator

$$Q(q^{-1}) = 1 - q^{-1}.$$

The simulation results are shown in Fig. 4.9 and Fig. 4.10. Solving the equation (3.16) for F and G gives

$$F(q^{-1}) = 1$$

$$G(q^{-1}) = 1.$$

Since Q divides A and B divides G it follows from Theorem 1 that there is a stationary solution given by

$$\frac{A}{B} = \frac{AG}{BFQ} = 1 - a\, q^{-1}$$

Thus r=2 and s=0 and there will be two parameters to estimate with the stationary values

$$\theta_1 = -1$$

$$\theta_2 = a.$$

The following difference equations describe the closed loop system

$$\theta(t+1) = \theta(t) + P(t+1)\, \varphi(t)\, \varepsilon(t+1)$$

$$\varepsilon(t+1) = - e(t+1) = y_r(t+1) - y(t+1)$$

$$P^{-1}(t+1) = \lambda\, P^{-1}(t) + \varphi(t)\, \varphi^T(t)$$

$$A(q^{-1})\, e(t) + u(t-1) = A(q^{-1})\, y_r(t)$$

$$\theta_1\, e(t) + \theta_2\, e(t-1) + u(t) = 0$$

$$\varphi(t) = [e(t),\, e(t-1)]^T.$$

Elimination of $u(t)$, $y(t)$ and $\varphi(t)$ from the equations gives

$$e(t+1) = [\theta_1(t) + 1 + a]\, e(t) + [\theta_2(t) - a]\, e(t-1) + v(t+1)$$

$$\theta_1(t+1) = \theta_1(t) - P_{11}(t+1)\, e(t)\, e(t+1) - P_{12}(t+1)\, e(t-1)\, e(t+1)$$

$$\theta_2(t+1) = \theta_2(t) - P_{21}(t+1)\, e(t)\, e(t+1) - P_{22}(t+1)\, e(t-1)\, e(t+1)$$

$$P^{-1}(t+1) = \lambda\, P^{-1}(t) + \varphi(t)\, \varphi^T(t)$$

where

$$v(t) = y_r(t) - y_r(t-1) - a[y_r(t-1) - y_r(t-2)].$$

For a square-wave command signal $v(t) \neq 0$ only for $t=t_i$ and $t=t_i+1$. As discussed in Chapter 3 the δP^{-1} does not influence on $\delta e(t)$ and $\delta\theta(t)$. The δP^{-1}-equation will be stable if $\lambda<1$. To study the stability only the equations for $\delta e(t)$ and $\delta\theta(t)$ thus have to be analysed. Let $e_1(t)= e(t)$ and $e_2(t)=e(t-1)$. The stationary solution will be

$$\theta_{10}(t) = -1$$

$$\theta_{20}(t) = a$$

$$e_0(t) = \begin{cases} \pm\Delta & t=t_i \\ 0 & \text{elsewhere} \end{cases}$$

$$\bar{P}(t_i) = P_0(t_i) = \frac{1-\lambda^\ell}{\lambda^\ell \Delta^2} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{bmatrix}$$

$$\bar{P}(t_i+1) = P_0(t_i+1) = \frac{1-\lambda^\ell}{\lambda^\ell \Delta^2} \begin{bmatrix} \lambda^\ell & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\bar{P}(t_i+2) = P_0(t_i+2) = \frac{1-\lambda^\ell}{\lambda^\ell \Delta^2} \begin{bmatrix} \lambda^{\ell-1} & 0 \\ 0 & \lambda^\ell \end{bmatrix}$$

Linearization around the stationary solution gives

$$\begin{bmatrix} \delta e_1(t+1) \\ \delta e_2(t+1) \\ \delta\theta_1(t+1) \\ \delta\theta_2(t+1) \end{bmatrix} = \begin{bmatrix} a & 0 & e(t) & e(t-1) \\ 1 & 0 & 0 & 0 \\ A_{31} & 0 & A_{33} & 0 \\ A_{41} & A_{42} & 0 & A_{44} \end{bmatrix} \begin{bmatrix} \delta e_1(t) \\ \delta e_2(t) \\ \delta\theta_1(t) \\ \delta\theta_2(t) \end{bmatrix}$$

with

$$A_{31} = -\bar{P}_{11}(t+1)[e(t+1) + a\, e(t)]$$

$$A_{33} = 1 - \bar{P}_{11}(t+1)\, e^2(t)$$

$$A_{41} = -\bar{P}_{22}(t+1)\, a\, e(t-1)$$

$$A_{42} = -\bar{P}_{22}(t+1)\, e(t+1)$$

$$A_{44} = 1 - \bar{P}_{22}(t+1)\, e^2(t-1)$$

The transition matrix of this system is denoted by A(t). Then

$$A(t_i-1) = \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -\overline{P}_{11}(t_i)\Delta & 0 & 1 & 0 \\ 0 & -\overline{P}_{22}(t_i)\Delta & 0 & 1 \end{bmatrix}$$

$$A(t_i) = \begin{bmatrix} a & 0 & \Delta & 0 \\ 1 & 0 & 0 & 0 \\ -a\overline{P}_{11}(t_i+1)\Delta & 0 & 1-\overline{P}_{11}(t_i+1)\Delta^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A(t_i+1) = \begin{bmatrix} a & 0 & 0 & \Delta \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a\overline{P}_{22}(t_i+2)\Delta & 0 & 0 & 1-\overline{P}_{22}(t_i+2)\Delta^2 \end{bmatrix}$$

$$A(t_i+k) = \begin{bmatrix} a & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad k=2,\dots,\ell-2$$

Thus

$$\prod_{k=2}^{\ell-2} A(t_i+k) = \begin{bmatrix} a^{\ell-3} & 0 & 0 & 0 \\ a^{\ell-4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For $k=\ell-1,\dots,2\ell-2$ we have

$$A(t_i+k) = A(t_i+k-\ell)$$

with Δ replaces by $-\Delta$.

As in Example 4.2 a time invariant system can be obtained for the perturbed equations by looking at what happens between points that are 2ℓ steps apart. The system matrix for this time invariant system will be A, where

$$A = \prod_{k=0}^{2\ell-1} A(t_i+k)$$

In this case it is not easy to get analytical expressions for the eigenvalues. It is easy to see, however, that one eigenvalue is zero. The other three eigenvalues must be computed numerically. As an example the eigenvalues were computed for different values of λ when a=-0.75. The results are given in Table 4.1.

λ	Eigenvalues			
0.75	6.715	0.266	$5.63\cdot10^{-6}$	$2.36\cdot10^{-7}$
0.76	3.548	0.304	$9.31\cdot10^{-6}$	$-2.91\cdot10^{-8}$
0.77	1.653	0.398	$1.53\cdot10^{-5}$	$4.59\cdot10^{-8}$
0.78	$0.513\pm0.376i$		$2.49\cdot10^{-5}$	$1.02\cdot10^{-8}$

Table 4.1

A more detailed examination shows that the system is locally stable if $\lambda>0.7744$. Simulations verify these computations. A typical behaviour in the unstable region with $\lambda=0.75$ is shown in Fig. 4.11. The parameters and the covariances have been initialized to the expected asymptotical values. A small perturbation has then been introduced in the initial value of one of the parameters. But if λ is chosen for example as 0.78 the parameters converge to the expected values. The simulations also show that the region where local stability holds is quite small close to the computed boundary $\lambda=0.7744$, i.e. the perturbation must be small to obtain convergence to the stationary point. But if λ is chosen larger, e.g. $\lambda\geq0.82$, then the system is stable in a large region including for example the initial value $\theta_1(0)=\theta_2(0)=0$. □

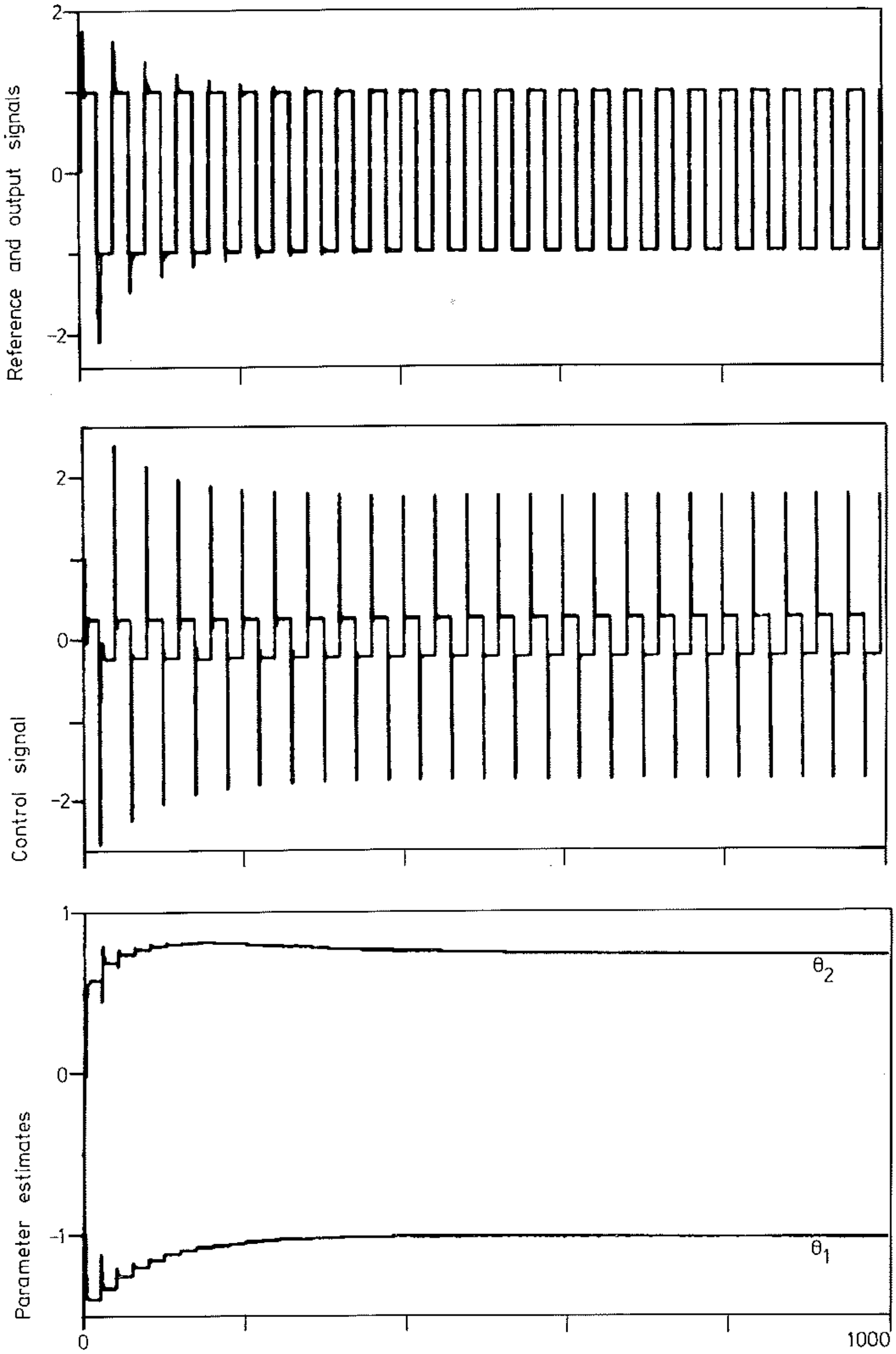

Figure 4.9. Results obtained when using a self-tuning regulator with $r = 2$ and $s = 0$ on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$ cascaded with an integrator. The square wave command signal has the period 40 and the forgetting factor is 0.99.

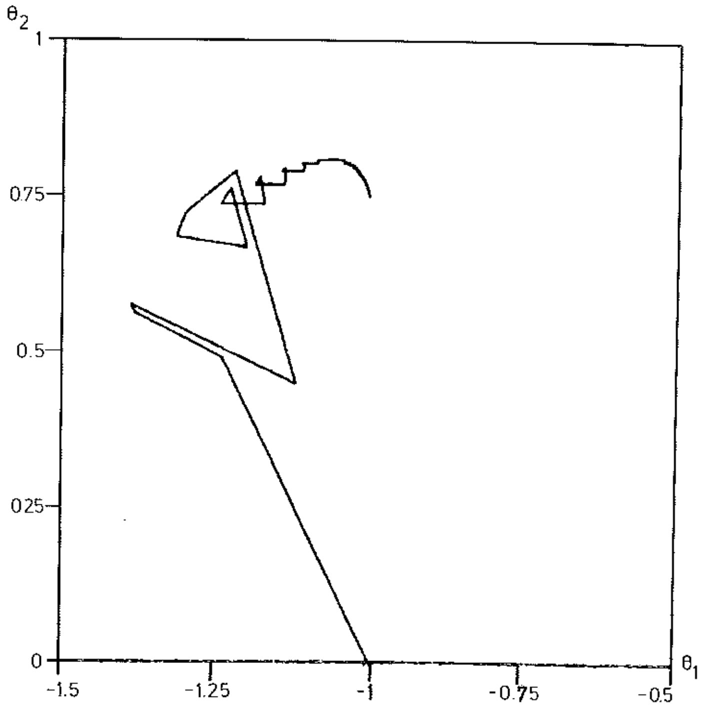

Figure 4.10. Phase plane corresponding to Fig. 4.9.

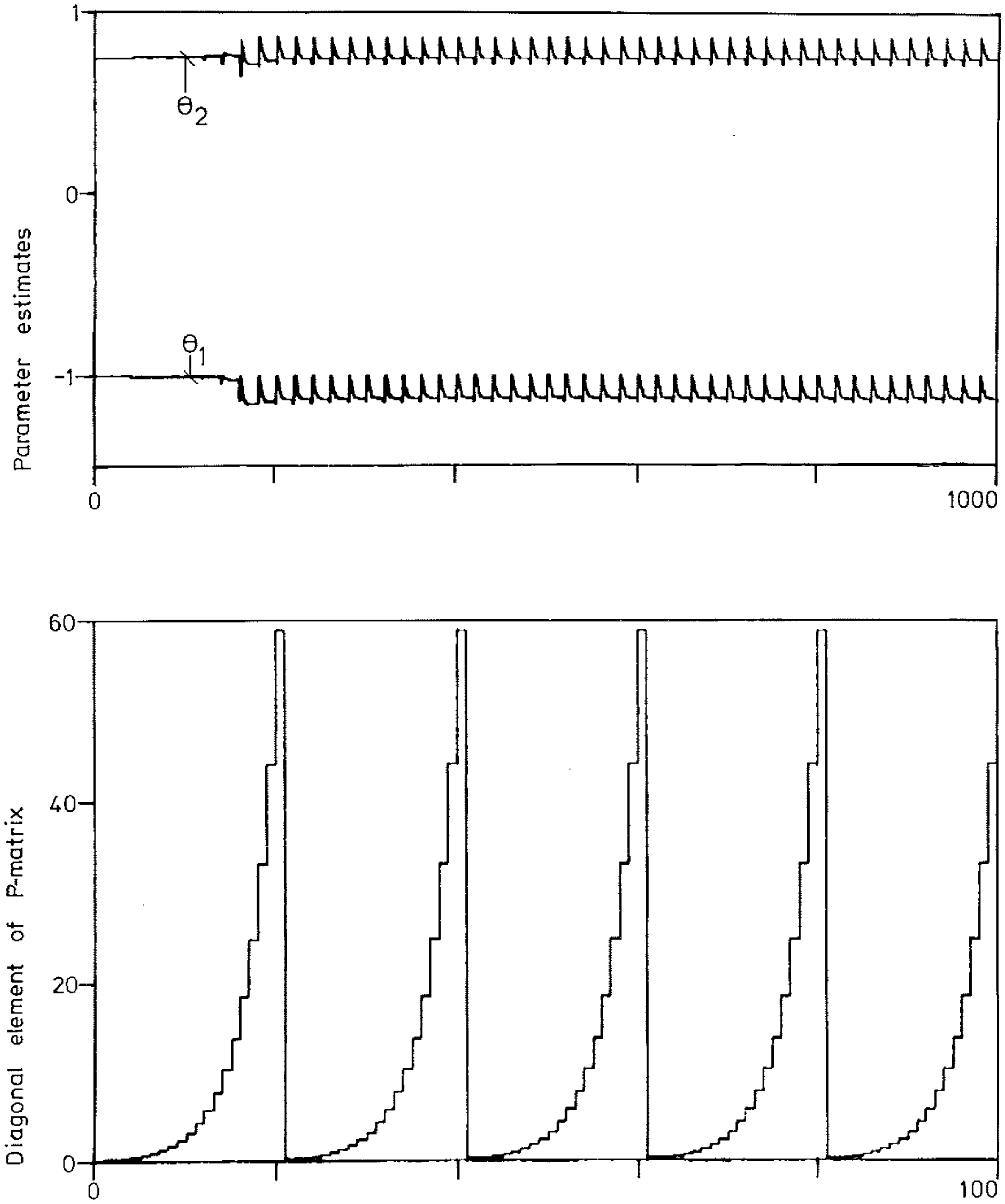

Figure 4.11. Results obtained when using a self-tuning regulator with r = 2 and s = 0 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$ cascaded with an integrator. The square wave command signal has the period 40 and the forgetting factor is 0.75. Notice the different time scales.

Example 4.1 illustrates that very poor results may be obtained when tracking a piece-wise constant command signal if the process does not contain an integrator. It was also shown that the self-tuning regulator was unable to supply the desired regulator. In Example 4.2 it was shown that good results could indeed be obtained if there was an integrator in the process. An example which illustrates what may happen if the process contains too many integrators is given in the following example.

EXAMPLE 4.4 *(A double integrator plant)*

Consider a process described by

$$A(q^{-1}) = (1 - q^{-1})^2$$

$$B(q^{-1}) = 1$$

$$k = 1.$$

Solving (3.16) for F and G gives

$$F(q^{-1}) = 1$$

$$G(q^{-1}) = 1.$$

Since Q divides A and B divides G it follows from Theorem 1 that there is a stationary solution. This solution corresponds to a regulator with the transfer function

$$\frac{\mathcal{A}}{\mathcal{B}} = \frac{AG}{BFQ} = 1 - q^{-1}.$$

This corresponds to a regulator with r=2 and s=0. Notice that this regulator has a zero at z=1 which cancels one of the process poles. This means that the stationary solution corresponds to a closed loop system where the mode z=1 is cancelled. The analysis in Example 4.3 is still valid. Numerical calculations indicate that the matrix A is unstable for all choices of λ. Simulation results are shown in Fig. 4.12 through Fig. 4.14.

□

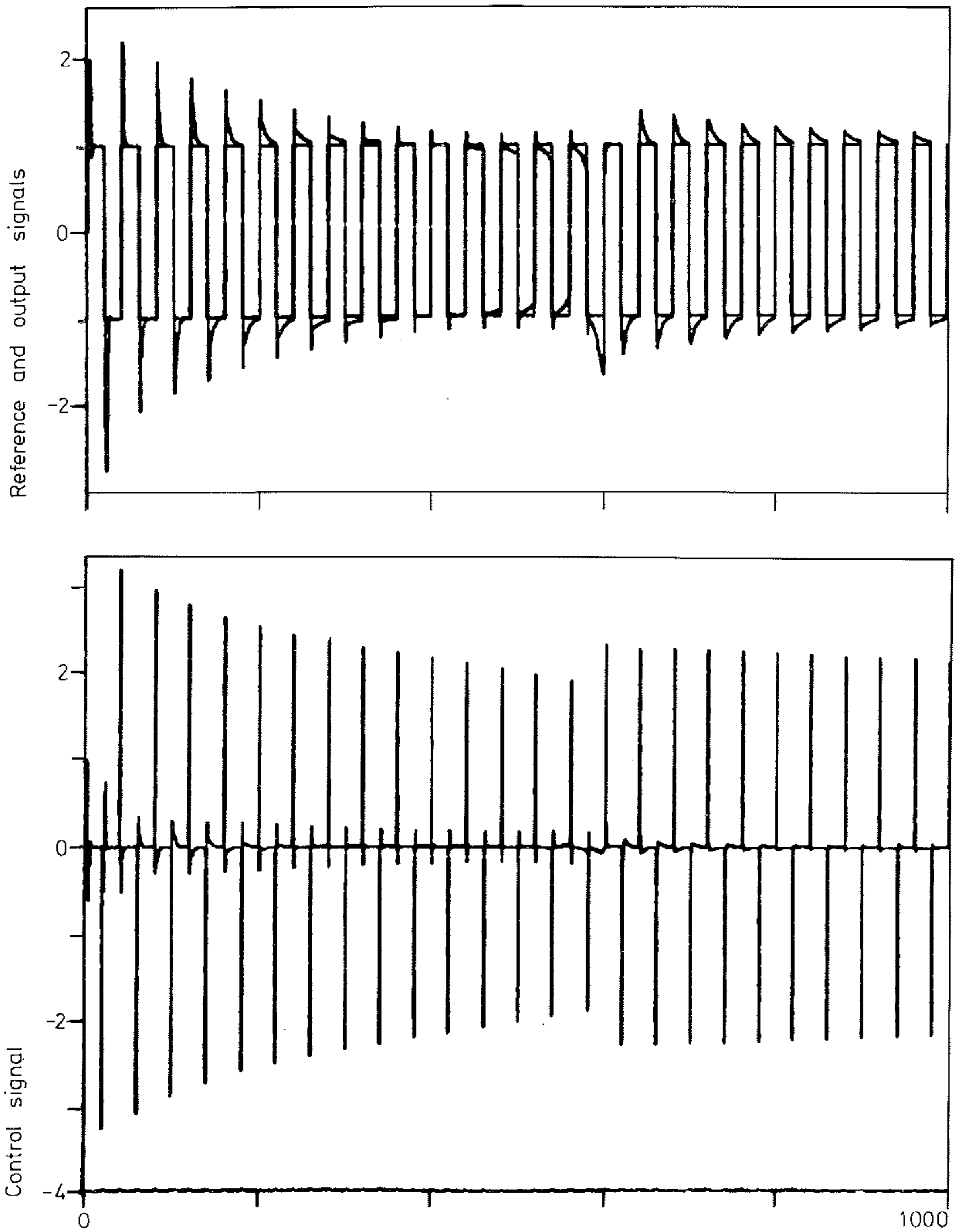

Figure 4.12. Results obtained when using a self-tuning regulator with r = 2 and s = 0 on a process with the pulse transfer function $H(z) = \frac{1}{z - 1}$ cascaded with an integrator. The square wave command signal has the period 40 and the forgetting factor is 0.99.

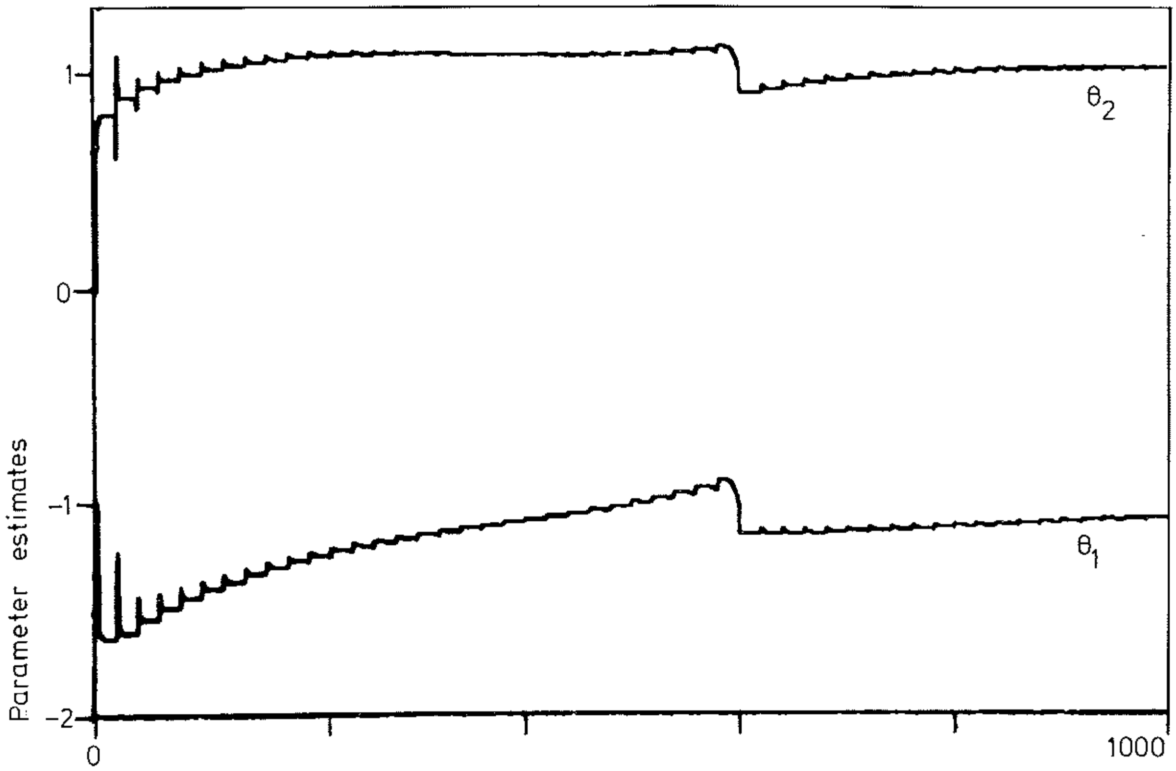

Figure 4.12 continued.

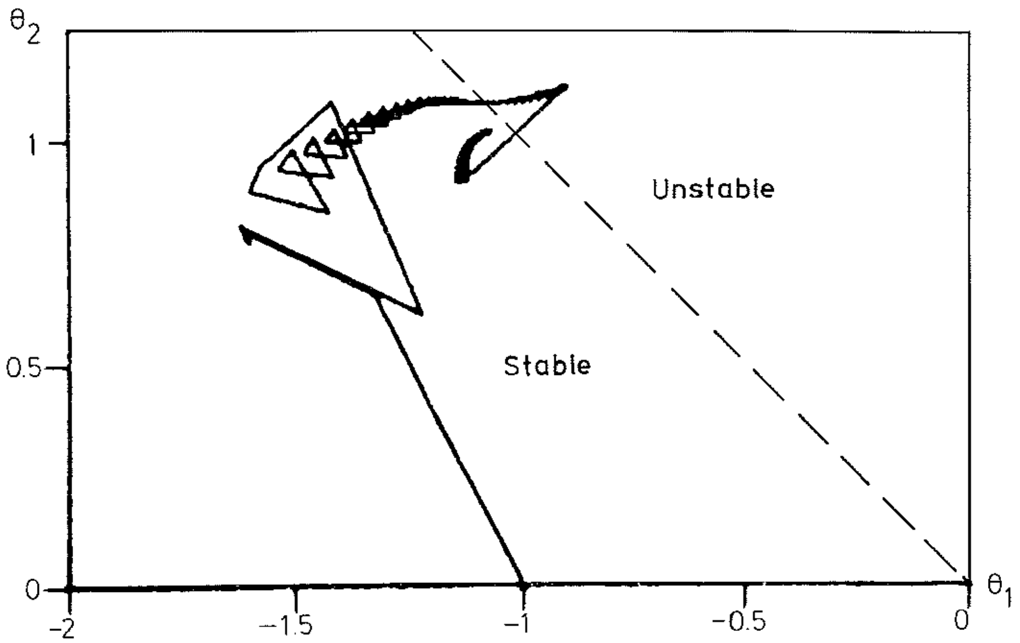

Figure 4.13. Phase plane corresponding to Fig. 4.12. The dashed line shows the boundary between the stable and unstable regions of the closed loop system.

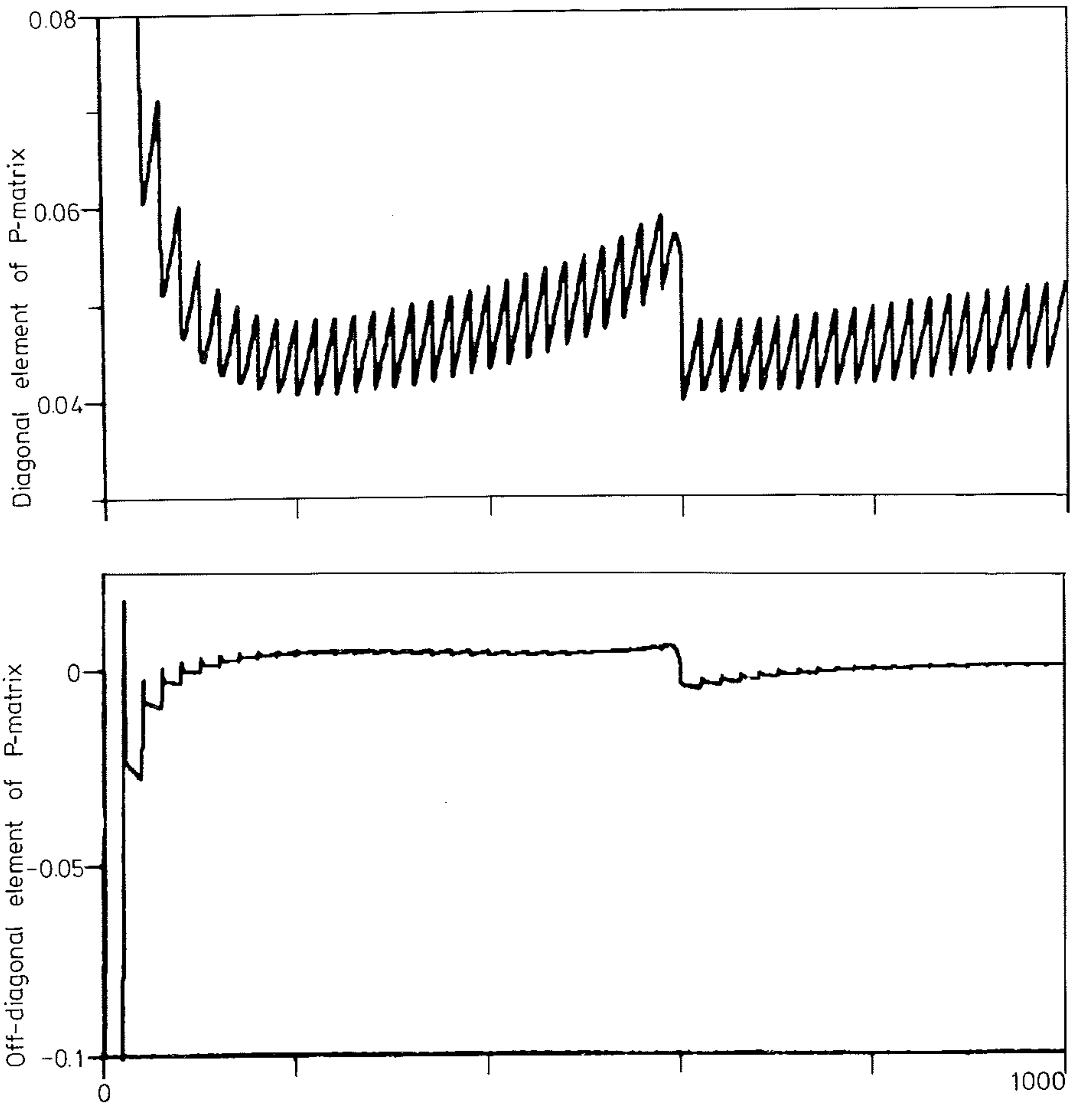

Figure 4.14. Elements P(1,1) and P(1,2) of the P-matrix for the simulation shown in Fig. 4.12.

5. BETTER REGULATOR STRUCTURES

It is well known that single-degree-of-freedom configurations are often unsatisfactory for the servo problem. It is therefore not surprising that the self-tuning regulator in the configuration of Fig. 2.1 also works poorly. The difficulties found in the previous sections can in fact be avoided simply by using a two-degree-of-freedom structure as the basis for the design. This means that both the command signal y_r and the output signal y must be separately available. It will be shown by a few examples that the difficulties can indeed be avoided by such structures.

STRUCTURE 1 (Clarke and Gawthrop)

Clarke and Gawthrop (1975) suggested that the model structure used in the estimation part of the regulator should be

$$\begin{aligned} y(t) - y_r(t) = & -\alpha_1 y(t-k) - \dots - \alpha_r y(t-k-r+1) + \\ & + \beta_0[u(t-k) + \beta_1 u(t-k-1) + \dots + \beta_s u(t-k-s)] - \\ & - \gamma_0 y_r(t) - \gamma_1 y_r(t-1) - \dots - \gamma_p y_r(t-p) + \delta, \end{aligned}$$

where δ is an estimate of a level. The control law is then given by

$$\begin{aligned} u(t) = & \frac{1}{\beta_0}[\alpha_1 y(t) + \dots + \alpha_r y(t-r+1)] - \beta_1 u(t-1) - \dots - \beta_s u(t-s) + \\ & + \frac{1}{\beta_0}[\gamma_0 y_r(t+k) + \gamma_1 y_r(t+k-1) + \dots + \gamma_p y_r(t+k-p) - \delta]. \end{aligned}$$

□

STRUCTURE 2 (Wittenmark)

Wittenmark (1975) suggested a somewhat different structure,

$$\begin{aligned} y(t) - y_r(t) = & -\alpha_1[y(t-k) - y_r(t-k)] - \dots - \alpha_r[y(t-k-r+1) - \\ & - y_r(t-k-r+1)] + \beta_0[u(t-k) + \beta_1 u(t-k-1) + \dots + \\ & + \beta_s u(t-k-s)] - \gamma_0 y_r(t) - \gamma_1 y_r(t-1) - \dots - \gamma_p y_r(t-p), \end{aligned}$$

with the control law given by

$$\begin{aligned} u(t) = & \frac{1}{\beta_0}\left\{\alpha_1[y(t) - y_r(t)] + \dots + \alpha_r[y(t-r+1) - y_r(t-r+1)]\right\} - \\ & - \beta_1 u(t-1) - \dots - \beta_s u(t-s) + \frac{1}{\beta_0}\left\{\gamma_0 y_r(t+k) + \gamma_1 y_r(t+k-1) + \right. \\ & \left. + \dots + \gamma_p y_r(t+k-p)\right\} \end{aligned}$$

□

Remark 1

The essential difference between the two structures is that the optimal controller will in general contain less parameters if Structure 1 is used (p will be smaller).

Remark 2

The structures above are the original suggestions. They differ also in the approach to steady state errors. Wittenmark proposed that the process should be cascaded with an integrator (using differences of the input signal instead of absolute values). Clarke and Gawthrop on the other hand have included a level parameter to be estimated. This difference is not fundamental since both approaches to steady state errors can be used with any one of the structures.

Remark 3

In the basic self-tuning algorithm used earlier in this report there is a choice whether β_0 should be estimated or it should be given a fixed value. An estimation of β_0 may result in identifiability problems and also in problems with the control if the estimate is close to zero. On the other hand the guessed fixed value must be sufficiently large to guarantee convergence, see e.g. Åström and Wittenmark (1973). In the algorithms above there is a choice between using a fixed β_0 or a fixed γ_0. It has been decided to fix γ_0 to 1 and estimate β_0 in case the a priori guess is uncertain.

EXAMPLE 5.1 (A first order process)

Consider the system in Example 4.1 with

$$A(q^{-1}) = 1 - 0.75\ q^{-1}$$

$$B(q^{-1}) = 1$$

$$k = 1.$$

Fig. 5.1 shows what happens when Structure 1 with r=1, s=0, and p=0 is used and the command signal is a square wave. δ was not estimated. In order to handle a more general situation, e.g. if β_0 is unknown and must be estimated or if there are level disturbances in the input or output signals, the structure with r=1, s=1, p=0, and an estimate of δ should be used instead. The behaviour of this regulator is also quite excellent.

Fig. 5.2 shows the results for the same system when Structure 2 with r=1, s=0, and p=1 is used. Again the behaviour is very good. As above, however, β_0 must be estimated and an integrator cascaded with the process or the number of parameters increased to handle the different cases mentioned above. Notice that the basic structures used gives one more parameter for Structure 2.

□

These regulator structures have also been used for the process in Example 4.2, a pure integrator. The behaviour of the closed loop system was very good.

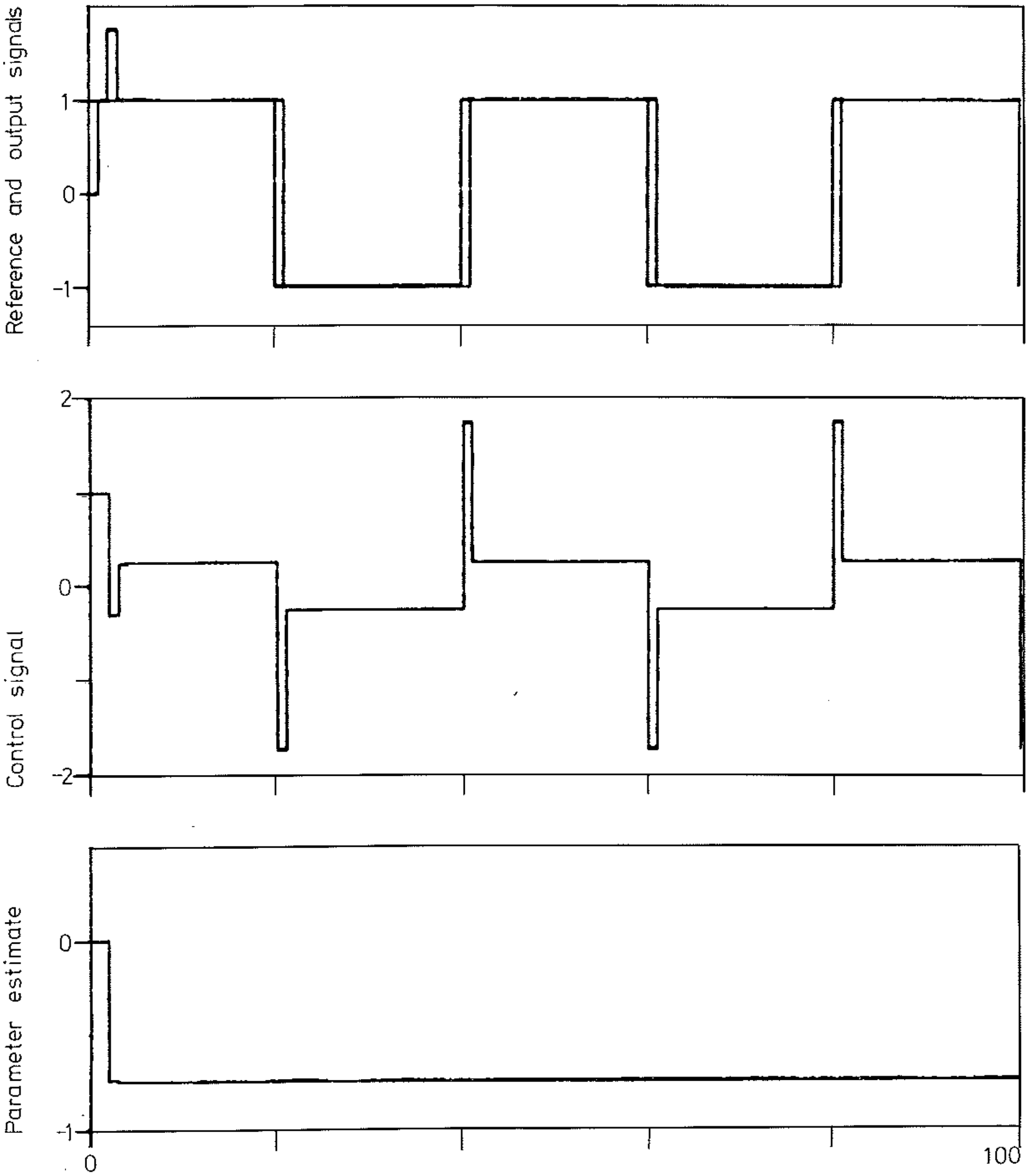

Figure 5.1. Results obtained when using a self-tuning regulator with Structure 1 and with r = 1, s = 0, and p = 0 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 40 and the forgetting factor is 0.95.

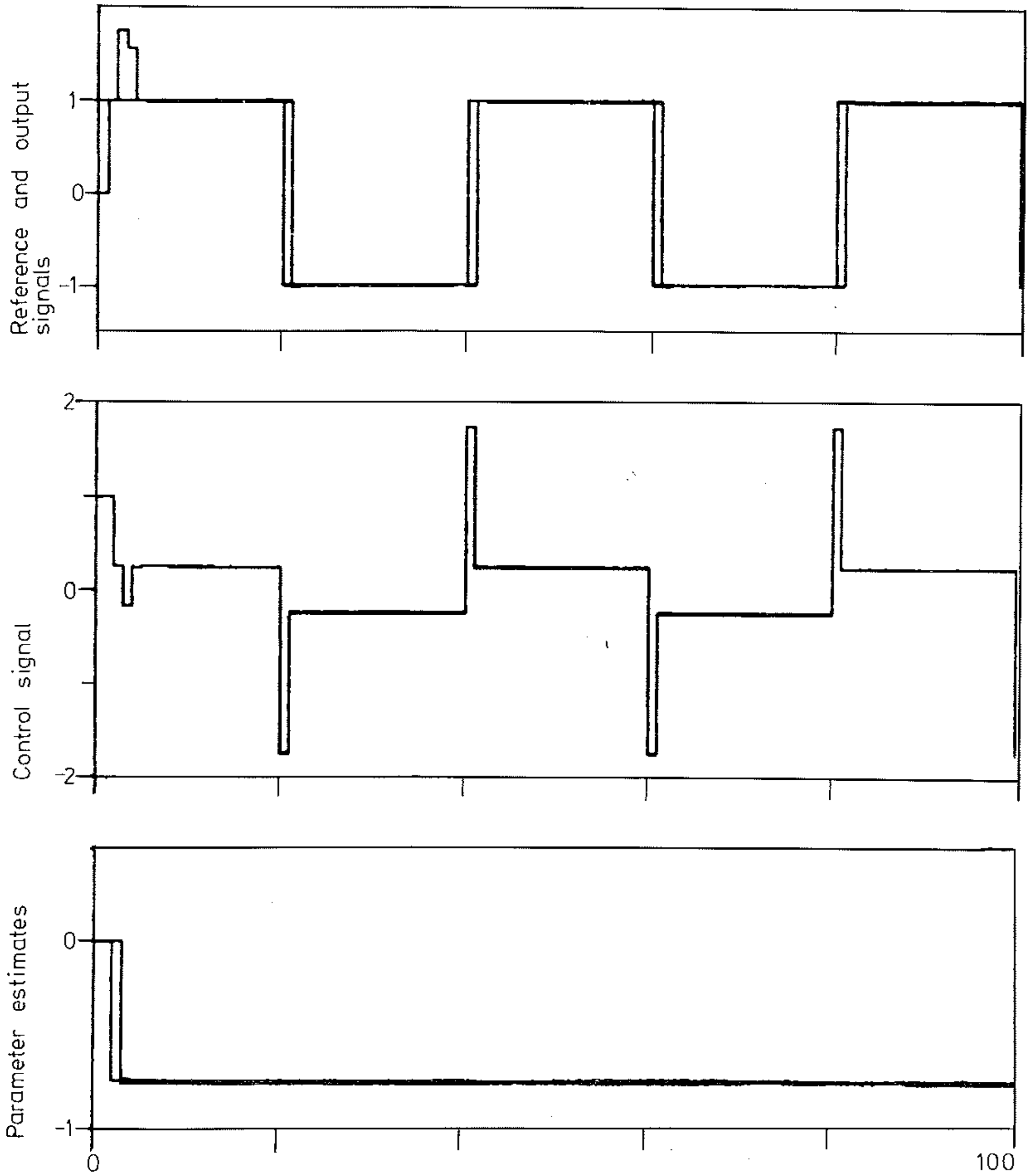

Figure 5.2. Results obtained when using a self-tuning regulator with Structure 2 and with r = 1, s = 0, and p = 1 on a process with the pulse transfer function $H(z) = \frac{1}{z - 0.75}$. The square wave command signal has the period 40 and the forgetting factor is 0.95.

6. ACKNOWLEDGEMENTS

We have benefitted substantially by discussions with L Andersson about the examples. These discussions were indeed the starting point for this work. An early version of this manuscript was read by Dr J Sternby whose suggestions led to several improvements.

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APPENDIX

The examples in Sections 4 and 5 were simulated using a program package for simulation of self-tuning regulators, Gustavsson (1978). This program package is found on disc No. 9. All the necessary commands for each simulated example are listed in this appendix.

EXAMPLE 4.1

Case a) With parameters giving the result of Fig. 4.1

External systems required: REF, SCON3

Macro's required: GLOBL, EX41

GLOBL

SYST SYS1 REG REF SCON3

EX41

Case b) For simulation of Fig. 4.6

External systems required: REF, SCON3

Macro's required: GLOBL, EX41A

GLOBL

SYST SYS1 REG REF SCON3

EX41A

EXAMPLE 4.2

External systems required: REF, SCON3

Macro's required: GLOBL, EX42

GLOBL

SYST SYS1 REG REF SCON3

EX42

EXAMPLE 4.3

Case a) For simulation of Fig. 4.9

External systems required: REF, INT, SCON1

Macro's required: GLOBL, EX43

GLOBL

SYST SYS1 REG REF INT SCON1

EX43

Case b) For simulation of Fig. 4.12

External systems required: REF, INT, SCON1
Macro's required: GLOBL, EX43A
GLOBL
SYST SYS1 REG REF INT SCON1
EX43A

EXAMPLE 4.4

External systems required: REF, INT, SCON1
Macro's required: GLOBL, EX44
GLOBL
SYST SYS1 REG REF INT SCON1
EX44

EXAMPLE 5.1
Case a) Structure 1

External systems required: REF, SCON4
Macro's required: GLOBL, EX51A
GLOBL
LET ISA.=4
SYST SYS1 REG REF SCON4
EX51

EXAMPLE 5.2
Case b) Structure 2

External systems required: REF, SCON3
Macro's required: GLOBL, EX51B
GLOBL
SYST SYS1 REG REF SCON3
EX52

External Systems

```
DISCRETE SYSTEM REF

TIME T
OUTPUT Y
TSAMP TS

OUTPUT
Y=IF MOD(T,PER)<(0.5*PER-EPS) THEN NIV1 ELSE NIV2

DYNAMICS
TS=T+DT

PER:40
NIV1:1.
NIV2:-1.
EPS:0.00001

DT:1

END
```

```
DISCRETE SYSTEM INT

TIME T
INPUT DU
OUTPUT Y1 Y2
STATE UOLD
NEW UNEW
TSAMP TS

OUTPUT
D=UOLD+DU
F=IF D>B THEN B ELSE IF D<A THEN A ELSE D
Y1=F
Y2=F-UOLD

DYNAMICS
UNEW=F

TS=T+DT

DT:1
A:-10
B:10

END
```

```
CONNECTING SYSTEM SCON1
TIME T
U1[REG]=Y[SYS1]-Y[REF]
U3[REG]=Y[REF]
DU[INT]=UR[REG]
U[SYS1]=Y1[INT]+ULEV
U2[REG]=Y2[INT]
ULEV:0
END
```

```
CONNECTING SYSTEM SCON3
TIME T
U1[REG]=Y[SYS1]-Y[REF]
U3[REG]=Y[REF]
U[SYS1]=UR[REG]+ULEV
U2[REG]=UR[REG]
ULEV:0
END
```

```
CONNECTING SYSTEM SCON4
TIME T
U1[REG]=Y[SYS1]-Y[REF]
U3[REG]=Y[REF]
U4[REG]=1.
U[SYS1]=UR[REG]+ULEV
U2[REG]=UR[REG]
ULEV:0
END
```

Macro's

```
MACRO GLOBL
LET ISA.=3
LET IVR.=6
LET ISB.=6
LET IVS.=3
END

MACRO EX41
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:2
PAR N2:1
PAR WTI:0.98
END

MACRO EX41A
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:2
PAR N2:1
PAR TH01:-0.97
PAR TH02:0.72
PAR TH03:-0.95
PAR P01:0.005
PAR P02:0.005
PAR P03:0.005
PAR PER:80
END

MACRO EX42
PAR NSA:1
PAR NSB:1
PAR A1:-1
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:1
PAR WTI:0.98
END

MACRO EX43
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:2
PAR WTI:0.99
END

MACRO EX43A
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:2
PAR WTI:0.75
PAR TH01:-1
PAR TH02:0.7499
PAR P01:0.1869
PAR P02:0.1402
PAR NIV1:0
PAR NIV2:2
END

MACRO EX44
PAR NSA:1
PAR NSB:1
PAR A1:-1
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR N1:2
PAR WTI:0.99
END

MACRO EX51A
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR REF:2
PAR N1:1
PAR K3:1
PAR WTI:0.95
END

MACRO EX51B
PAR NSA:1
PAR NSB:1
PAR A1:-0.75
PAR B1:1
PAR LAMB:0
PAR REG:1
PAR REF:1
PAR N1:1
PAR N3:1
PAR K3:1
PAR WTI:0.95
END
```