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ALGEBRAIC SYSTEM THEORY AS A TOOL FOR REGULATOR DESIGN

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Algebraic system theory as a tool for regulator design

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When the characteristics of a system, its environment and the specifications are given in terms of external models it is very natural to use algebraic system theory to design control systems. Different problems of this type are discussed in the paper. It is shown that many design problems can be solved in a uniform, simple and direct way using algebraic system theory. Relations to other methods of solving the design problem are also discussed. The importance of Blomberg's contributions to the pole-zero cancellation problem is also discussed.

Referat skrivet av
Author

Förslag till ytterligare nyckelord
Observers. Pole-zero placement. Linear quadratic control.

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1. INTRODUCTION

My first contacts with algebraic system theory were in basic courses
on Heaviside operator calculus, Laplace transform theory and linear
control theory. I believe that my reactions were fairly typical for
a generation who learned automatic control in a similar way. Initially
it was very pleasing to see how simple it was to reduce analysis and
design of control systems to pure algebra. It was also interesting
to see how simple the algebraic manipulations were and how easy it was
to get insight into properties of control systems using the algebraic
tools. As more familiarity about the algebraic methods were obtained
there was an increasing dissatisfaction about certain matters that
remained obscure. The problem of cancellation of poles and zeros was
the heart of the difficulties. It seemed so strange that pure algebra
had to be augmented either by analysis or by rules which at that time
seemed rather arbitrary in order to get a full picture of linear
control systems. Intellectually it was also very dissatisfying to
resort to nonalgebraic methods to obtain a sensible theory. Gradually
it became clear that Laplace transform algebra was not the right
framework for linear systems problems.

State space theory, particularly the notions of reachability and
observability and Kalman's decomposition theorem [1] and [2], give a
good insight into the nature of the pole-zero cancellation problem.
It is, however, also possible to understand the problem in other ways.
I remember my first scientific contacts with Hans Blomberg in the
early sixties. He had an unusually clear picture of the difficulties
of pole-zero cancellation and also a very good feel for how the
Laplace-transform algebra should be modified to obtain the desired
results. Intuitively the idea is to describe linear systems by poly-
nomials but not to allow division. An appropriate mathematical struc-
ture is to interpret the signal space algebraically as a module over
the ring of polynomial operators. The germ of the idea is expressed
in [3]. It has been worked out in fine detail in many of Blomberg's

later works, [4], [5], [6]. A slight modification is to allow division
of polynomials whose factors correspond to well damped modes [7].

The purpose of this paper is to show how several design problems for
control systems can be solved using algebraic system theory. The
approach given here is probably the most natural one if the system
to be controlled is characterized as a rational transfer function. My
own first work along these lines was the development of the minimum
variance control law [8]. In that case a model of a system and its
environment were given in transfer function form. A model of this
form was actually obtained from system identification. To determine a
feedback a realization of the transfer function was first introduced
and the solution was then obtained by applying standard state space
theory. It was interesting to see that the problem could be solved in
a much simpler way by direct application of algebraic methods to the
input output model. In this paper it is shown how several design
problems can be formulated and solved using pure algebraical polynomial
manipulation. The discussion includes design of observers, servos and
regulators. It is also shown how the algebraic methods relate to other
design techniques.

2. PRELIMINARIES

The notations used will now be discussed together with the basic assumptions. Lower case letters u, x, y, z denote signals i.e. real time functions. Time can be either the real numbers (the continuous time problem) or the integers (the discrete time problem). Upper case letters A, B, C etc. denote polynomials.

Dynamical systems are represented as

$$Ay = Bu \qquad (2.1)$$

where the independent variable of the polynomials is a differential operator for continuous time problems and a forward shift operator for discrete time problems. For discrete time systems it is natural to require that

$$\deg B \leq \deg A \qquad (2.2)$$

which means that the discrete time system is causal. For continuous time systems the inequality (2.2) means that the input output relation does not contain any pure derivatives. Since there are problems where it is meaningful to have systems which are differentiators the condition (2.2) can not always be imposed on continuous time systems.

Let Z be a region of the complex plane which corresponds to sufficiently well damped modes. For continuous time systems Z can be chosen as a sector in the left half plane. For discrete time systems Z can be chosen as a region well inside the unit disc. Introduce a ring R of rational functions of the form B/A where the denominator polynomial has all its zeros in Z. The signal space is taken as a module M over R. This means for example that if $y \in M$ and

$$Ay = 0 \qquad (2.3)$$

then y = 0 if the polynomial A has all its zeros in Z.

Equation (2.3) represents a differential equation in the continuous time case and a difference equation in the discrete time case. In both cases the solution is a sum of decaying exponentials. The rate

of decay is determined by the choice of the region Z. The chosen
signal space is a convenient way to formalize the statement that
solutions of differential equations that decay sufficiently fast are
regarded as zero or equivalently that it is possible to cancel factors
which correspond to sufficiently well damped modes. The module intro-
duced here is a slight generalization of Blomberg's modules. A formal
treatment is given in [7].

3. OBSERVERS

Consider a system with one input signal u and two output signals y and z. The input signal u is known and the output signal y is measured. It is desired to determine the output signal z. Let the signals be related through

$$Az = Bu \tag{3.1}$$

$$Cy = Dz \tag{3.2}$$

where A, B, C and D are polynomials in the differential operator p or the shift operator q. The pairs (A,B) and (C,D) are assumed to be coprime. It is assumed that the polynomials A and D do not have any common factors. If A and D have common factors the signal z is not observable from y and the stated problem can not be solved. Strictly speaking it is sufficient to assume that A and D are coprime in R. This means that z is detectable from y.

Since all signals are scalars we have

$$ACy = BDu. \tag{3.3}$$

It is assumed that

$$\deg B \leqslant \deg A \tag{3.4}$$

and

$$\deg BD \leqslant \deg AC. \tag{3.5}$$

To determine the signal z from the signals u and y the following estimate is formed

$$A\hat{z} = Bu + G^{-1} H (Cy - D\hat{z})$$

where G and H are polynomials. This equation is the result obtained if $\hat{z}$ is determined from the model (3.1) with feedback from the output signal. The equation can be written as

$$(GA + HD) \hat{z} = GBu + HCy. \tag{3.6}$$

Notice that the same result is obtained if the signal z is predicted from (3.2) with a feedback from (3.1) i.e.

$$D\hat{z} = Cy + H^{-1} G(Bu - A\hat{z}).$$

Equation (3.6) is called an observer for z. It is a dynamical system with u and y as inputs. The output $\hat{z}$ will be close to z as is seen in the following.

Analysis

Having obtained an observer its properties will now be analysed. Equations (3.1), (3.2) and (3.6) give

$$(GA + HD) \, \hat{z} = GAz + HDz = (GA + HD) \, z.$$

Hence

$$(GA + HD) \, (z - \hat{z}) = 0.$$

This shows that if the polynomial

$$F = GA + HD \tag{3.7}$$

is stable then the observer (3.6) will give an estimate $\hat{z}$ that converges to z as time goes to infinity. Moreover if $z = \hat{z}$ over a time interval then $\hat{z}$ will equal z for all times and all input signals.

Design of observers

If A and D are coprime there always exists a solution of equation (3.7). This solution is unique if it is required that

$$\deg H < \deg A. \tag{3.8}$$

It is reasonable to require that

$$\deg GB \leq \deg (GA + HD) \tag{3.9}$$

and

$$\deg HC \leq \deg (GA + HD). \tag{3.10}$$

For discrete time systems this means that the observer (3.6) is causal and for continuous time systems it means that the observer does not use derivatives. Equation (3.9) always holds because of (3.4).

It follows from (3.8) that (3.10) is always satisfied if

$$\deg F = \deg AC - 1. \tag{3.11}$$

This means that the degree of the observer polynomial F is chosen as the degree of the order of the system (3.3) minus one. For specific A and C it may actually be possible to choose an observer polynomial of lower degree.

The design of the observer (3.6) is straightforward. The observer polynomial is first chosen subject to the constraint (3.11). Equation (3.7) is then solved for G and H subject to (3.8).

If only detectability is assumed A and D have a common factor A_1 in Z. The problem can still be solved in this case provided that the observer polynomial is chosen so that it is divisible by A_1.

4. POLE AND ZERO PLACEMENT

The problem of designing a servo with a given closed loop transfer function will now be described and solved.

Formulation

Consider a process characterized by the rational operator

$$G_p = \frac{B}{A} \tag{4.1}$$

where A and B are polynomials. It is assumed that A and B are coprime and that

$$\deg B < \deg A.$$

It is desired to find a controller such that the closed loop is stable and that the transfer function from the command input u_c to the output is given by

$$G_M = \frac{Q}{P} \tag{4.3}$$

where P and Q are coprime and

$$\deg P - \deg Q \geqslant \deg A - \deg B. \tag{4.4}$$

Design procedure

A general linear regulator can be described by

$$Ru = Tu_c - Sy. \tag{4.5}$$

The regulator can be thought of as a combination of a feedback having the transfer function $-S/R$ with a feedforward with the transfer function T/R. The closed loop system is then characterized by the operator

$$G = \frac{TB}{AR + BS}.$$

Since G should equal the desired closed loop response G_M given by (4.3) we get

$$\frac{TB}{AR+BS} = \frac{Q}{P} \ . \tag{4.6}$$

The design problem is thus equivalent to the algebraic problem of finding polynomials R, S and T such that (4.6) holds. It follows from (4.6) that factors of B which are not also factors of Q must be factors of R. Since factors of B correspond to open loop zeros it means that open loop zeros which are not desired closed loop zeros must be canceled. Factor B as

$$B = B^+B^-, \tag{4.7}$$

where all the zeros of B^+ are in Z and all zeros of B^- outside Z. This means that all zeros of B^+ correspond to well damped modes and all zeros of B^- correspond to unstable or poorly damped modes.

A necessary condition for solvability of the servo problem is thus that the specifications are such that B divides Q in $\mathcal{R}$ i.e.

$$Q = Q_1 B^- . \tag{4.8}$$

Since deg P is normally less than deg(AR+BS) it is clear that there are factors in (4.6) which cancel. In state space theory it can be shown that the regulator (4.5) corresponds to a combination of an observer and a state feedback. See [9]. It is natural to assume that the observer is designed in such a way that changes in command signals do not generate errors in the observer. This means that the factor which cancels in the right hand side of (4.6) can be interpreted as the observer polynomial F.

The design procedure can be formulated as follows.

Data: Given the desired response specified by the polynomials P and Q, subject to deg P = deg A and the conditions (4.4), (4.8), and the desired observer polynomial F.

Step 1:   Solve the equation

$$AR_1 + B^-S = PF \tag{4.9}$$

with respect to R_1 and S.

Step 2:   The regulator which gives the desired closed loop response
           is then given by (4.5) with

$$R = R_1 B^+ \qquad (4.10)$$

           and

$$T = FQ_1. \qquad (4.11)$$

The equation (4.9) can always be solved because it was assumed that A
and B were relatively prime. This implies of course that A and B^- are
also relatively prime. Equation (4.9) has infinitely many solutions.
The unique solution determined by

$$\deg S < \deg A \qquad (4.12)$$

is chosen. Since deg P = deg A it then follows that deg R_1 = deg F. If
the observer polynomial is chosen in such a way that

$$\deg F = \deg A - \deg B^+ - 1 \qquad (4.13)$$

then

$$\deg R = \deg A - 1 \geq \deg S$$

$$\deg T = \deg A - 1 = \deg R. \qquad (4.14)$$

This means in the continuous time case that the regulator does not
include any pure derivatives and in the discrete time case that the
regulator is causal. Notice that in special cases the regulator may
still have the property (4.14) even if (4.13) does not hold. Also
notice that the choice (4.13) corresponds to a Luenberger observer in
the state space interpretation. Further discussions including examples
are found in [9].

Analysis

A direct calculation gives

$$\frac{TB}{AR + BS} = \frac{FQ_1 B^+ B^-}{B^+(AR_1 + B^- S)} = \frac{FQ_1 B^-}{PF} = \frac{Q}{P}$$

which shows that the regulator gives the desired closed loop response.
Notice that in this calculation we have divided with the factors B^+

and F. This is permitted since it was assumed that all their zeros are well damped.

A direct calculation shows that the closed loop system has the characteristic polynomial B^+FP. The polynomial B^+ has all its zeros in Z by definition. Since the observer polynomial F and the polynomial P were chosen to have all their zeros in Z it follows that the closed loop system has all its poles in Z.

Interpretation as model following

The results obtained will now be used to give an interpretation of the regulator (4.5). It follows from (4.9), (4.10) and (4.11) that

$$\frac{T}{R} = \frac{FQ_1}{B^+R_1} = \frac{(AR_1 + B^-S)Q_1}{PB^+R_1} = \frac{AQ_1}{B^+P} + \frac{SB^-Q_1}{B^+R_1P} = \frac{A}{B} \cdot \frac{Q}{P} + \frac{S}{R} \cdot \frac{Q}{P}.$$

The feedback law (4.5) can thus be written as

$$u = \frac{A}{B} y_c + \frac{S}{R} (y_c - y) \tag{4.15}$$

where

$$y_c = \frac{Q}{P} u_c. \tag{4.16}$$

The signal y_c can be interpreted as the output obtained when the command signal u_c is applied to the model Q/P. When the regulator (4.5) is re-written as (4.15) it is clear that it can be thought of as composed of two parts, one feedforward term $(A/B)y_c = (A/B)(Q/P)u_c$ and one feedback term $(S/R)(y_c-y)$. The feedforward is a combination of the ideal model and an inverse of the process model. The feedback term is obtained by feeding the error y_c-y through a dynamical system characterized by the operator S/R. It is thus clear that the regulator can be interpreted as a model following servo. Notice, however, that the system A/B is not realizable although the combination $AQ/(BP)$ is.

The MISO regulator

So far it has been assumed that there is only one measured output signal. In practice it often happens that there are additional signals which are measured and that the regulator can be simplified and improved by using these additional signals. This will in general require a complete multivariable theory. It will be shown, however, that the problem can also be handled by marginal extensions to the single output theory. Let it be assumed that a regulator has been determined for the case of one measured output y and that the feedback operator S/R has been determined. Furthermore assume that one additional measurement y_1 is available. Let this signal be related to y as

$$y_1 = \frac{D_1}{B_1} y. \tag{4.17}$$

where D_1 and B_1 are coprime. It is easy to show that B_1 must divide B in (4.1). To find a feedback from y and y_1 which is identical to the signal $(S/R)y$ the polynomial R is first factored as

$$R = R_1 R_2. \tag{4.18}$$

It is then attempted to find two polynomials S_1 and S_2 such that

$$\frac{S}{R} y = \frac{S_1}{R_1} y + \frac{S_2}{R_2} y_1 = \left(\frac{S_1}{R_1} + \frac{S_2 D_1}{R_2 B_1} \right) y.$$

If this is possible B_1 must divide S_2. Hence

$$S_2 = B_1 S_3 \tag{4.19}$$

and S_1 and S_3 satisfies

$$S = S_1 R_2 + S_3 D_1 R_1. \tag{4.20}$$

This equation can be solved with respect to S_1 and S_3 if $D_1 R_1$ and R_2 are relatively prime. Since the factoring (4.18) may be done in many ways there may be several solutions. The choice among the different possibilities can be made with respect to simplicity, disturbance sensitivity, and causality.

It is possible to proceed recursively in a similar way if there are several measured signals. The regulator obtained when this procedure is used is of the form

$$y = \frac{T}{R} u_c - \frac{S_1}{R_1} y_1 - \frac{S_2}{R_2} y_2 - \frac{S_3}{R_3} y_3 + \ldots \qquad (4.21)$$

where

$$R = R_1 R_2 R_3 \ldots$$

This regulator is called the MISO regulator because it has many inputs, u_c, y_1, y_2, ..., and a single output u. An implementation of the MISO regulator with a highly interactive operator communication is described in [10].

5. LINEAR QUADRATIC CONTROL

In the problems discussed in Section 4 the desired closed loop
polynomial P and desired observer polynomial F are considered as given.
There are many ways to determine P and F. They can e.g. be regarded as
tuning parameters which are adjusted to obtain a suitable performance
or the polynomial P can be chosen from tables of standard forms.
Another possibility is to formulate the control problem as an optimi-
zation problem. The polynomials P and F can then be determined from
the optimization criterion. This problem will be briefly discussed in
this section.

Preliminaries

It is first assumed that the process to be controlled is governed by

$$Ay = Bu \tag{5.1}$$

and that the control problem is formulated as to minimize the quadratic
criterion

$$J = \int_0^\infty [y^2(t) + \rho u^2(t)] \, dt. \tag{5.2}$$

It is well known [11] that the solution of this optimization problem
gives a closed loop system whose characteristic polynomial P is given
by

$$PP_* = \rho AA_* + BB_* \tag{5.3}$$

where A_* denotes the conjugate of the polynomial A. If

$$A(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$$

then the conjugate polynomial is defined by

$$A_*(x) = \begin{cases} a_0 + a_1 x + \ldots + a_n x^n & \text{discrete time systems} \\ A(-x) & \text{continuous time systems.} \end{cases}$$

The design of a regulator which minimizes (5.2) is thus reduced to the following algebraic problems:

- Find a polynomial P which satisfies (5.3). This is called spectral factorization problem.

- Apply the algebraic design procedure given in Section 4.

Notice that in the second step it is necessary to choose the observer polynomial and to choose an appropriate solution of the equation (4.9). It will next be shown how the observer polynomial can be determined if mathematical models of the disturbances are included.

Deterministic control

Assume that the process to be controlled is described by

$$Ay = Bu + Ce \tag{5.4}$$

where e is an impulse disturbance and C a stable polynomial. Let the criterion be to minimize (5.2).

Representing the signals as infinite power series and invoking Parseval's theorem the criterion (5.2) can be written as

$$J = \frac{1}{2\pi i} \oint (yy_* + \rho uu_*) \frac{dz}{z} .$$

Using (5.1) the integrand can be written as

$$I = \frac{(Bu + Ce)(Bu + Ce)_*}{AA_*} + \rho uu_* = \frac{PP_* uu_* + (CB_* + C_* B)ue_* + CC_* ee_*}{AA_*}$$

where P is stable and given by (5.3). To complete the squares the polynomial S defined by

$$PS_* + P_* S = CB_* + C_* B \tag{5.5}$$

is introduced. Then

$$I = \frac{PP_* uu_* + PS_* ue_* + P_* Su_* e + CC_* ee_*}{AA_*} = \frac{(Pu + Se)(Pu + Se)_* + (CC_* - SS_*)ee_*}{AA_*} .$$

The loss function (5.2) is thus minimized for the control signal

$$u = -\frac{S}{P} e. \tag{5.6}$$

This is an open loop control or a control program. To find the corresponding control law introduce this signal into (5.4). Hence

$$APy = (CP - BS)e.$$

The polynomial CP - BS is divisible by A. Hence

$$AR = CP - BS$$

or

$$AR + BS = CP. \tag{5.7}$$

This implies

$$y = \frac{R}{P} e.$$

The feedback law is thus

$$u = -\frac{S}{R} y. \tag{5.8}$$

It is easy to find all solutions to (5.7). The particular solution which also satisfies (5.5) is given by

$$\deg S = \deg B + \deg C - \deg P. \tag{5.9}$$

The solution of the factorization problem is not unique, if P_1 is a stable solution then $z^k P_1$ is also a solution. The integer k can then e.g. be chosen as the smallest integer which gives a causal feedback or as the feedback which does not include any derivatives.

Notice that the solution can also be interpreted as follows. Choose the observer polynomial C and the polynomial P as the solution to the factorization problem. Then apply the design procedure of the previous section with $B^+ = 1$, where the appropriate solution to equation (4.9) is determined by condition (5.9).

Stochastic control

In this case it is assumed that the process to be controlled is described by (5.1) where e is a white noise stochastic process and that C is stable. The purpose of control is to minimize

$$E(y^2 + \rho u^2) \tag{5.10}$$

in steady state. It is straightforward to show that the control law (5.8) minimizes (5.10).

For discrete time minimum variance control we have $\rho = 0$ and

$$P = z^{\deg A - \deg B^-} B_*^- B^+.$$

See [12].

6. CONCLUSIONS

It has been shown that algebraic system theory is a simple and
convenient tool for solving several control system design problems.
Using the algebraic tools the design problems are simply reduced to
polynomial manipulation. In particular it was demonstrated that design
of observers and regulators and servos could be reduced to the
solution of the diophantine equation

$$Ax + By = C \qquad\qquad (6.1)$$

and that linear quadratic optimal control problems can be reduced to
solving the factorization problem

$$PP_* = \rho AA_* + BB_*. \qquad\qquad (6.2)$$

Relations between the algebraic methods and state space theory has
also been given. For further discussions see [6], [7], [9], [13], [14],
[15], [16] and [17].

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