Modeling of Dynamic Systems Applications to Physics, Engineering, Biology, and Medicine

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Modeling of Dynamic Systems
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Abstract

This course, given at University of Pavia, Italy, Spring 2001, gives an introduction to modeling and analysis of complex dynamical systems from a broad perspective. A special feature of the course is that it draws from examples from a wide range of fields, physics, biology, medicine, engineering, and economics. The course consists of 10 lectures organized as follows:

1. Introduction
2. First Order Systems (2 lectures)
3. Linear Time Invariant Systems (2 lectures)
4. Compartment Models, Pharmacokinetics
5. Nonlinear Systems
6. Mechanical Systems, Advanced Modeling Tools
7. Electrical Systems and Mechatronics
8. Neurons and Neural Systems

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Modeling of Dynamic Systems Applications to physics, engineering, biology, and medicine.

Lectures at University of Pavia, Spring 2000
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Introduction

This course gives an introduction to modeling and analysis of complex dynamical systems from a broad perspective. A special feature of the course is that it draws from examples from a wide range of fields, physics, biology, medicine, engineering, and economics. The course consists of 10 lectures organized as follows:

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2. First Order Systems (2 lectures)
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7. Electrical Systems and Mechatronics
8. Neurons and Neural Systems

Modeling of Complex Dynamic Systems

K. J. Åström

Goals

- Appreciation of modeling in a wide range of fields
- A good understanding of modeling of systems
- A good understanding of the underlying mathematics
- Familiarity with the standard models for dynamics
- Ability to model and simulate moderately complex systems
- Awareness of computational tools for systems modeling
- Practical experience of modeling through a project

Organization

1. Introduction
2. First Order Systems (2 lectures)
3. Linear Time Invariant Systems (2 lectures)
4. Compartment Models, Pharmacokinetics
5. Nonlinear Systems
6. Mechanical Systems, Advanced Modeling Tools
7. Electrical Systems and Mechatronics
8. Neurons and Neural Systems

Team Formation

- A wide spread of students
- Formation of project teams

Lecture 1 - Introduction

1. Introduction
2. Cosmology - A Role Model
3. Impact of Computers
4. Modeling Paradigms
5. Summary

Models and Modeling

Why Models

- Compact summary of knowledge

  Reductionism

  Tycho Brahe, Kepler and Newton
- Communication
- Education
- Easier or safer to work with than the real world
- Sometimes a necessity, the systems do not exist
- The role of experiments

Caution

- A model captures only some aspects of the system
- Simplicity. Parsimony
- The purpose of modeling
- What parts are essential for the problem under investigation
- Families of models

Many Elements

- Physics
- Engineering
- Biology
- Medicine
- Economy
- Mathematics
- Statistics
- Numerical Mathematics
- Computer Science
- Software Engineering

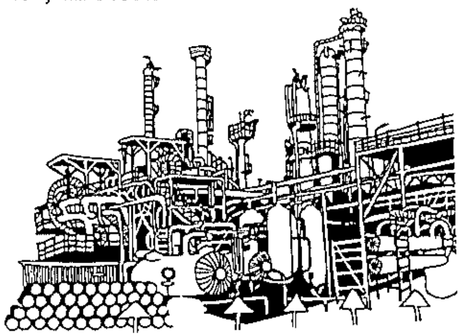
A Voice from The Past

Vannevar Bush 1927 inventor of the mechanical differential analyzer 1928-1931.

“Engineering can proceed no faster than the mathematical analysis on which it is based. Formal mathematics is frequently inadequate for numerous problems pressing for solution, and in the absence of radically new mathematics, a mechanical solution offers the most promising and powerful attack ... ”

A Voice from Industry

Ralph P Schlenker, Exxon:



Modeling and simulation technologies are keys to achieve **manufacturing excellence** and to **assess risk** in unit operations. As we make our plant more flexible to response to **business opportunities** efficient **modeling and simulation** techniques will become **commonly used tools**.

Uses of Modeling

- Insight and understanding
- Analysis
- Simulation (Virtual reality)
- Design, decision and optimization
- Diagnosis fault detection
- Control design
  The internal model principle
- Operator training
- Hardware in the loop simulation (Eg VTI, SSPA)
- Rapid prototyping

A Rich Field

- Mechanical systems
- Electrical systems
- Fluids and hydraulics
- Thermal systems
- Vehicles
- Chemical processes
- Biological systems
- The Human Body
- Ecosystems
- The Economy

How to Handle the Diversity?

- Appropriate abstractions
- Standard forms of models
- Dimension free or normalized variables
- Libraries
- Team working
- Software tools

Complexity

- Different physical domains
- Complex behavior
- Large physical dimensions
- Large number of components

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Cosmology - A Role Model

The Problem: Predict the future positions of the planets.

- The emergence of Natural Science
- Conceptual insight
- How do they move
- What causes the motion
- How can be motion be described
- Abstractions Physical Laws
- Spin offs

A prime example of a modeling effort that spanned many centuries with brillant contributors and revolutionary consequences.

Natural Science and Engineering Science

Many similarities but also many differences

Natural Phenomena	**Technical Systems**
Analysis	Synthesis
Isolation	Interaction
Fundamental Laws	System Principles

Feedback is a a good system principle!

The Emergence of Cosmology

- The heliocentric view
- Phytagoras cosmology
- Aristarchus 300 BC
- The dark ages
- Revival of helicentrism
- Copernicus 1474-1542
- G. Galilei 1564-1642
- Experiments
- Tycho Brahe 1546-1601
- Timur Lenk
- Insight from data
- J. Kepler 1571-1630
- Theory emerges
- I. Newton 1643-1727

The Origin of Ideas

The early speculations were based on crude observations, philosophy about the heaven and some mathematics.

Pythagoras had a heliocentric view.

Aristoteles was earth centered, prevailed for a long time for religious reasons.

Ptolemeios had an effective description based on circles.

Copernicus reintroduced the earth centric view.

The scientific view started with measurements by Tycho Brahe, data analysis by Kepler, who was schooled in Copernicus spirit. Newton made the final synthesis.

Early Cosmology

In Pythagoras cosmology the earth, along with the sun, moon, and planets, revolved around a body known as a central fire,... . The Pythagorean astronomer Aristachus simplified the system by putting the sun at the center. Wertheim p. 32:

Ptolemaios approx 100-165 BC Geocentric system. Moon, Mercurius, Venus and sun rotates around the earth. Motions in cycloids. Circles and spheres the perfect forms. Nationalencyclopedin:

Copernicus 1474-1542 proposed heliocentric system. Explained time for rotation, Mercury 88, Venus 225, Earth 365, Mars 687, Jupiter 4333, Saturn 10759. Planets moved in epicycles as complicated as Ptolemeios. Wertheim p. 66:

Features from Observation

Tycho Brahe was mathematician at the court of Emperor Rudolf II in Pragye, Kepler was his assistant. Brahe gave reluctantly Kepler access to data for Mars, the planet whose path deviates most from a circle.

By analysis of the data Kepler found three laws.

1. Planets move in ellipses with the sun at the center
2. Equal areas are covered in equal times
3. Time to go around the sun related to the size of the orbit
4. Keplers formula

$$M = E - e \sin EXS$$

An Example - The Giant Modelers

The Problem: Predict the future positions of the planets.

The different phases

- Observations: Tycho Brahe and Timur Lenk
- Finding features: Kepler
- Theory development: Newton
- Improved data treatment: Gauss
- Abstraction: Euler, Lagrange and Hamilton
- Further abstractions: Poincare, Birkhoff
- Recent contributions Smale, Arnold and Chaos

Concepts, Observations and Patterns

Early astronomers came up with the conceptual idea that the planets rotate around the sun in circles. Cycloids had to be introduced to fit observations.

Tycho Brahe and Timur Lenk made extensive observations of the planets and gathered a large experimental material.

Kepler analyzed Tycho Brahes data and found that the planets moved in ellipses. He also observed several regularities.

Newton used Keplers results to invent the law of gravitation, Newtons equations and calculus.

Newton a Modeling Giant

Newton investigated the motion of two planets subject to a gravitational force. He formulated the law for gravitation

$$F = k\frac{mM}{r^2}$$

and he also formulated the law of momentum balance

$$\frac{d}{dt}mv = F, \quad F = ma = m\frac{d^2x}{dt^2}$$

and the analog for angular momentum.

He also developed differential calculus to be able to manipulate the equations.

The theory that emerged covered much more than the original problem. The three-body problem defied analysis.

Abstraction 1 - Lagrange

Introduce

q generalized coordinates

p generalized momenta

Compute

Potential energy $V(q)$

Kinetic energy $T(p,q)$

Lagrangian $L = T - V$

The equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

A Nice Way to Obtain Equations of Motion

- Introduce coordinates
- Compute positions and velocities of centers of masses
- Compute angular velocities
- Compute potential energy $V = mgy_{cm}$
- Compute kinetic energy $2T = m\dot{x}^2 + J\dot{\theta}^2$
- Form Lagrange function $L = T - V$
- Lagranges equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F$$

give the equations of motion

Abstraction 2 - Hamiltonian

Introduce $\partial L(p,q)/\partial q$ and form

$$H(p,q) = p^T q - T(p,q),$$

If the system is scleronomic (only stationary constraints) we have

$$H(p,q) = V(q) + T(p,q)$$

Hamiltons equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Compare with Pontryagins Maximum Principle!

Effective Use of Observation

The story of the planet Ceres, discovered in 1781, almost circular orbit. Vanished from view. Recovered by Gauss method in 1801. K. F. Gauss Teoria Motus Corporum Coelestium 1809.

"The most probable values of the unknown parameters, are those which minimize the sum of the squares of the differences between the observed and computed values."

"The principle that the sum of the squares of the differences between observed and computed quantities must be a minimum may be considered independently of the calculus of probabilities."

"Instead of using the sum of squares (our principle) we could use sum of any even power of the errors. But of all these principles ours is the most simple."

The Three Body Problem - Poincare

Newton could solve his equations for two bodies, the sun and the earth, and obtain ellipsoidal orbits.

Efforts to solve the equations for three planets failed. Poincare gave a new view. He emphasized the qualitative aspects and started a new vigorous development.

Dynamics Differential Equations and Flows

Ordinary differential equation

$$\frac{dx}{dt} = f(x)$$

Controlled differential equation

$$\frac{dx}{dt} = f(x,u)$$

Philosophical implications:

- Determinism and free will.
- Chaos

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Impact of Computers

- Typical uses
  - Solve the equations, find trajectories
  - Fit parameters to experimental data
- Absent in the early development
- Early efforts 1850-1950
- Explosive development 1950-2000
- Moores law, doubling in 18 months (100 in 10 years)

Herman Goldstine: When things change by two orders of magnitude it is revolution, not evolution. With Moores Law a revolution every 10 years!

Commentary on Computations

- Herman Goldstine: "When things change by two orders of magnitude it is revolution not evolution."
- Important to complement computation by understanding and insight
- Hamming: “The purpose of computing is insight not numbers”
- Expect software errors! Important to check results to make sure that they are reasonable. Look at results and *Think*
  - Look for special case where you know the solution
  - Compute an auxiliary quantity to check the results

Vannevar Bush 1927

"Engineering can proceed no faster than the mathematical analysis on which it is based. Formal mathematics is frequently inadequate for numerous problems pressing for solution, and in the absence of radically new mathematics, a mechanical solution offers the most promising and powerful attack ... "

The mechanical differential analyzer 1928-1931.

Simulation and modeling.

Analog Simulation

- Kelvins Tide Predictor 1879
- The Differential Analyzer 1925
  Ball and disk integrator
  Torque amplifier
- Modeling by analogs
  Masses, springs, dash-pots
  Capacitors, inductors, resistances
- Electronic Analog Computers 1947
  The operational amplifier
  Commercial systems Philbrick, Electronic Associates, Applied Dynamics

Analog Simulation

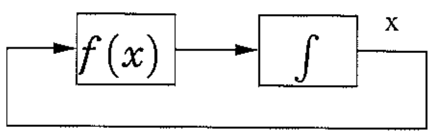
Idea: Physical analogs. Differential equations as a unifying concept description.

Generate solutions to

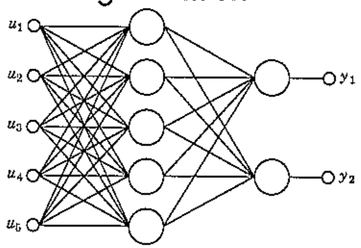
$$\frac{dx}{dt} = f(x)$$

Two components
- Integration
- Function generation

A feedback loop



Function generation



Notice combination of
- Scalar product
- Function $f : R \to R$

To Perform a Simulation

- Derive differential equations
- Convert to explicit state space form
  - Scale equations to make sure numbers are in the right range. A very useful exercise!
  - Make analog computer diagram
  - Make the connections on a patch board and connect recorders
- Simulate Initial conditions, Operate, Hold
- Change parameters - Almost instantaneous response
- Document

Digital Simulation

- Integration of ordinary differential equations
- Mixture of modeling and integration
- Need for modeling tools
- Special modeling simulation languages
  - Different paradigms
  - Mechanical systems
  - Ecological systems
  - Electronic systems
  - Control systems
- Object Oriented Modeling - Modelica
- Tools for this course

Mimic an Analog Computer

- Try to make a digital computer behave like an analog
- Man machine interfaces
- Standardization effort spear-headed by the Simulation Council (Society for Computer Simulation)
- Order out of chaos. Farsighted and long lasting.
- Good features of earlier programs, Mimic, DSL90.

  Textual representation of analog world

  Expressions ($J_\ell + n^2 J_m$)

  Sorting

  Macros
- ACSL de facto standard

  A Fortran preprocessor

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Modeling Paradigms

- Ordinary Differential Equations

  Matlab
- Block Diagram Modeling
- Physical Modeling

  Mechanical systems

  Electrical circuits

  Hydraulics

  A General Approach
- Modeling Languages
- Summary

Ordinary Differential Equations

- Derive the equations for the system
- Write them in standard form

$$\frac{dx}{dt} = f(x)$$

- Use an integration routine to simulate the equations
- Simple and straight forward
- Much tedious work
- Error prone
- Difficult to structure and reuse

Solving Differential Equations using Matlab

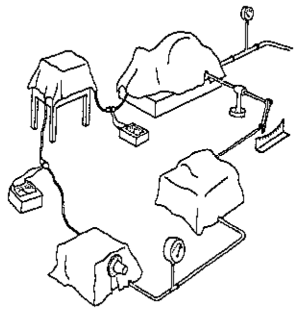
```
%Basic ODE simulator
tspan=[0,11];
x0=[0;0.5;1;1];
[t,x] = ODE45('nf',tspan,x0);
plot(x(:,1),x(:,3),'b-')

function dxdt=f(t,x)
%The right hand side for a two-body problem
r=sqrt(x(1)^2+x(3)^2);
r3=r^3;
k=1;%G=6.6720 10^(-11) Mm^2/kg^2
dxdt=[x(2);-k*x(1)/r3;x(4);-k*x(3)/r3];
```

Block Diagram Modeling

The block diagram is a nice abstraction which is an early example of information hiding.

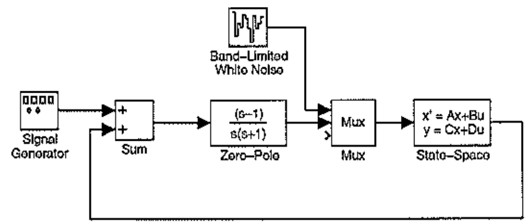
- A nice way to structure a system
- Well coupled to transfer functions
- Good simulation tools available

  SystemBuild
  Simulink
  VisSim
- Unidirectional interaction



Block Diagram Modeling Environments System Build, Simulink and VisSim

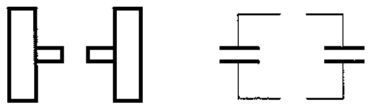
- "Virtual" Analog Computer
- Easy to use



- Granularity and Structuring
- Graphical aggregation and disaggregation
- Much manual manipulation from physics to blocks
- Libraries restricted to strictly input output form

Limitations of Block Diagram Modeling 1

- States may disappear



- Unidirectional interaction very limiting
- Much manual error prone work required to go from physics to a state space model
- Progress requires a paradigm shift!

Electrical Circuits

- Large systems
- A few component types
  - Resistors, capacitors, inductors
  - Amplifiers, transformers, gyrators
- Two-ports, four-ports and n-ports
- Network Theorems
  - Superposition (Linear)
  - Kirkhoffs voltage and current Laws
  - Telegens Theorem
  - Thevenins Therorem
  - Passivity
- A history lesson

Kirkhoffs Laws

Number nodes

Z_{ij} impedance between i and j

I_{ij} current from i to j

V_i voltage at node i

$V_{ij} = V_i - V_j$

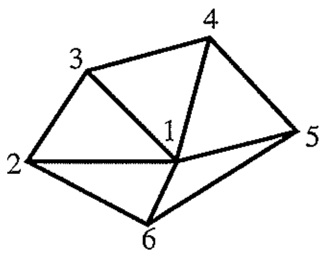
Kirchoff's current law

$$\sum_i I_{ij} = 0, \quad \sum_j I_{ij} = 0$$

Kirchoff's voltage law

$$\sum_{anyloop} V_{ij} = 0$$

Generalizations *through variables* and *across variables*!



Modeling and Simulation of Electric Circuits

- SPICE - Donald O Pedersen, Berkeley
- Write all circuit equations
- Match equations iteratively
- Mix modeling and simulation
- Very large following
  - Public domain version
  - Many commercial versions
  - Complicated transistor models

Mechanical Systems

- Rigid bodies, springs, dampers
- Tedious and error prone to write all the equations
- Many coordinate systems
- Transformation of coordinates
- A graphical description of the system is intuitively very clear
- Special integration routines may be required
- Special software
  - Adams
  - Dymola

A General Approach

- Cut a system into subsystems
- Write mass, momentum and energy balances for each subsystem
- Use object orientation to structure the system
- Let software handle book keeping and transformations
- Code generators for many different purposes
- Build component libraries

A General Approach

- Eliminate unnecessary variable
- Use graph theory to reduce equations to block diagonal form
- Solve linear blocks analytically
- Generate iterations for nonlinear blocks
- Generate code for finding equilibria
- Generate code for simulation
- Generate code for linearized models

The Modelica Effort

- Multi-domain object oriented physical modeling
- Background
- Standard language backed by international body not by an individual company
- An European effort
- Close to physics
- Multiple views
- Many different representations
  - Equations
  - Schematic pictures tailored to different domains
- Libraries public domain and commercial
- Reuse

Lecture 1 - Introduction

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Summary

- Modeling is a key activity in all sciences
- Cosmology is a nice prototype - The beginning of Natural Science

  Observations
  Conceptualization
  Mathematical models
  Abstraction
- The role of computers
- Modeling paradigms

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Lecture 2 & 3 - First Order Systems

K. J. Åström

1. Introduction
2. Examples
3. Linear systems
4. Nonlinear systems
5. Bifurcations
6. Summary

Theme: Starting to look at simple dynamics.

Introduction

- A gentle beginning
- The richness of dynamics
- Similarities betweeen different areas
- A simple way to look at basic concepts
  - Equilibria
  - Stability
  - Vector fields and flows
- Linear systems - Analytical solutions
- Nonlinear systems - Qualitative analysis
- Numerical solutions

Lecture 2 & 3 - First Order Systems

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First Order Systems

- A water tank
- RC and LC circuits
- Heat conduction
- Population dynamics
- The Logistics equation
- Michaelis-Menten kinetics
- A simple neuron
- A simple model of a national economy

A Water Tank

Let V be the volume of water in the tank. Let q_{in} be the inflow and q_{out} be the outflow. Then

$$\frac{dV}{dt} = q_{in} - q_{out}$$

Integrating this equation we find that

$$V(t) = V_0 + \int_0^t (q_{in}(\tau) - q_{out}(\tau))d\tau$$

- Notice that the solution is the sum of two terms, one depends on the initial volume, and the other depends on inflow and outflow.
- Notice role of storage variable.

The RC Circuit

The charge of the capacitor is

$$Q = CV$$

The rate of change of the charge is equal to the current, hence

$$\frac{dQ}{dt} = C\frac{dV}{dt} = I = \frac{E - V}{R}$$

Cleaning up we get

$$\frac{dV}{dt} = -\frac{V}{RC} + \frac{E}{RC}$$

- Notice similarity to water tank
- Notice the role of the storage variable, charge = $Q = CV$

The RL Circuit

The flux of the inductor is

$$\Psi = LI$$

The rate of change of the flux is equal to the voltage across the inductor, hence (Faraday's Law)

$$\frac{d\Psi}{dt} = L\frac{dI}{dt} = V = E - RI$$

Cleaning up we get

$$\frac{dI}{dt} = -\frac{R}{L}I + \frac{E}{L}$$

- Notice similarity to water tank
- Notice the role of the storage variable, flux $\Psi = LI$

Thermal Systems

Consider a steel ingot in a liquid cooling bath

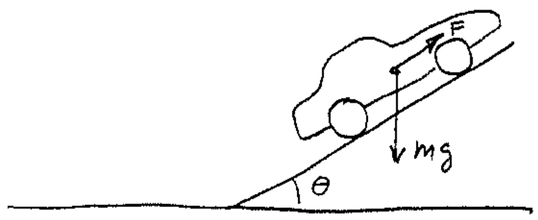
$$mC_p\frac{dT}{dt} = k(T_l - T)$$

Rapid thermal processing of silicon wafers

$$mC_p\frac{dT}{dt} = \sigma A(T_l^4 - T^4)$$

- Notice similarity to water tank
- Notice the role of the storage variable, energy mC_pT

Cruise Control



- Process input or control variable: gaspedal u
- Process output: velocity v
- Desired output or reference signal v_r
- Disturbances: slope θ

Mathematical Model

A simplified mathematical model

$$m\frac{dv}{dt} + vd = F - mg\theta$$

With reasonable parameters Audi in 3rd gear

$$\frac{dv}{dt} + 0.02v = u - 10\theta$$

where v velocity [m/s], u normalized throttle $0 \leq u \leq 1$ and θ slope in [rad/s]

- Notice similarity to water tank
- Notice the role of the storage variable, momentum mv

Population Dynamics

Let the population be N, the birth rate b and the death rate d. THe population is then governed by

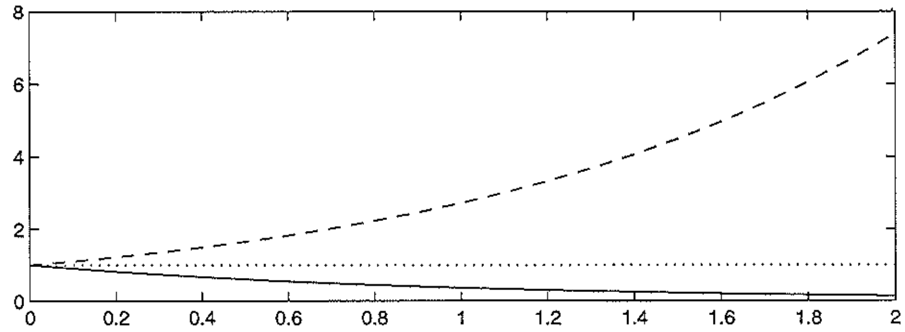
$$\frac{dN}{dt} = bN - dN$$

The solution is

$$N(t) = N_0 e^{(b-d)t}$$

Malthus 1798 (actually earlier by Euler)

The Shape of the Solutions



Notice the drastic differences between the cases

$b > d$ (dashed)

$b = d$ (dotted)

$b < d$ (solid)

Is it a Good Model?

Population of the Earth in billions

≈ *1650*	*1800*	*1918-28*	*1960*	*1974*	*1987*	*1999*
0.5	1	2	3	4	5	6

Is this compatibel with exponential growth?

Limitations to growth Verhulst 1836

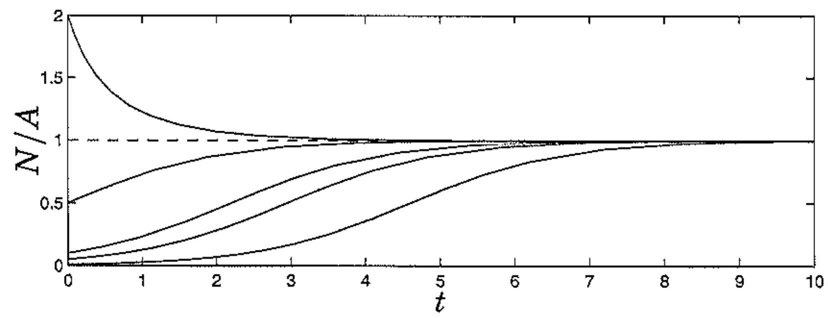
$$\frac{dN}{dt} = bN\Big(1 - \frac{N}{A}\Big)$$

Logistic growth. Steady state solution $N = 0$ or $N = A$. Parameter A is called carrying capacity. More general models

$$\frac{dN}{dt} = f(N)$$

Logistic Growth

$$\frac{dN}{dt} = bN\Big(1 - \frac{N}{A}\Big)$$



The Logistic Model

- Good agreement with lab test on bacteria and yeast under ideal conditions
- Believed to be a universal growth law Pearl 1927
- Totally different behavior found in many cases

  No steady state, oscillatory behavior
- Good review by Krebs Ecology: The Experimental Analysis of Distribution and Abundance. Harper and Row. New York, 1972.

Michaelis Menten Kinetics

Consider a substance (medicine) with concentration x i the body. The concentration can be described by the model

$$\frac{dc}{dt} = -\frac{rc}{a+c} + bu$$

where c is the concentration of the substance, u the input, i.e. drug injection and r, k, and b are parameters. The equation implies that excreation rate

$$\frac{rc}{a+c}$$

decreases with increasing rate.

A Static Model of a Neuron

The input is the membrane potential and the output is spike rate, a static model is

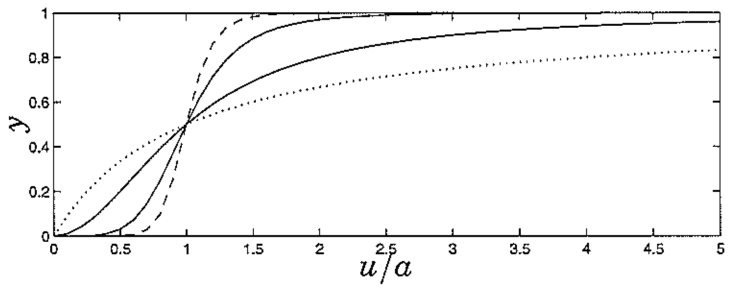
$$y(u) = \begin{cases} \dfrac{ku^n}{a^n + u^n} & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases}$$

where k is a constant which gives the maximum spike rate and a a parameter that gives the input where the spike rate is half of the maximum value.

This function $y(u)$ is called the Nake-Rushton function. Compare with Michaelis-Menten kinetics.

Naka-Rushton Function

Normalized spike rate y/b as a function of normalized potential u/a, for $n = 1$ (dotted), n=2, n=5, and n=10 (dashed)



A Dynamic Model of the Neuron

To obtain a dynamic model we must take into account that a change of the potential does not give an instantaneous change in spike rate. A simple model that captures this is

$$\frac{dy}{dt} = \frac{1}{T}(-y + f(u))$$

where f is the Naka-Rushton function. This model gives the steady state behavior $y = f(u)$ and dynamics is characterized by the time constant T which is in the range of tenth of milli seconds.

This model can be considered as a static nonlinearity followed by a first order dynamics with time constant T.

A Keynesian Economy Model

Let Y be the gross national product, C consumption and I investment. We have the following accounting relation

$$Y = C + I$$

One model for the consumers is that consumption increases with increasing GNP with with some dynamics.

$$\frac{dC}{dt} = -aC + bY$$

Combining the equations we get

$$\frac{dC}{dt} = -aC + bY = (b - a)C + bI$$

First we observe that to have a stable system we must have $b < a$, economic interpretation!

Keynes Multiplier

The model

$$\frac{dC}{dt} = (b-a)C + bI$$

is stable if $b < a$. Notice that b/a is the steady state relation between consumption and GNP! The simple model indicates that the economy runs away when consumers spend more than the filtered increase in GNP!

The steady state relation between investment and GNP is

$$Y/I = (2b-a)/(b-a) = K$$

, where K is Keynes' multiplier.

- Keynes' theory, which was verbal, not formal, was very influential to bring countries out of the depression
- Remark on economic modeling

Lecture 2 & 3 - First Order Systems

1. Introduction
2. Examples
3. Linear systems
4. Nonlinear systems
5. Bifurcations
6. Summary

Linear Systems

Standard model

$$\frac{dx}{dt} = ax + bu$$

- What do we mean by linear
- Why use standard models?

Linear systems are nice because they can be solved explicitely. Steady state solution, put $dx/dt = 0$

$$x = -\frac{bu}{a}$$

Solution

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)d\tau$$

Solving the Equation

$$\frac{dx}{dt} = ax + bu$$

The homogeneous equation, put $u = 0$

$$\frac{dx}{dt} = ax$$

has the solution

$$x(t) = Ce^{at}$$

where C is an arbitrary constant ($C = x(0)$). The inhomogeneous equation has the solution

$$x(t) = e^{at}x(0) + \int_0^t be^{a(t-\tau)}u(\tau)dt$$

Verification

Differentiation of

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)d\tau$$

gives

$$\frac{dx}{dt} = ae^{at}x(0) + ab\int_0^t e^{a(t-\tau)}u(\tau)d\tau + bu(t) = ax(t) + bu(t)$$

Furthermore the initial condition is $x(0)$.

The Input-Output Relation

If the initial value is zero we have

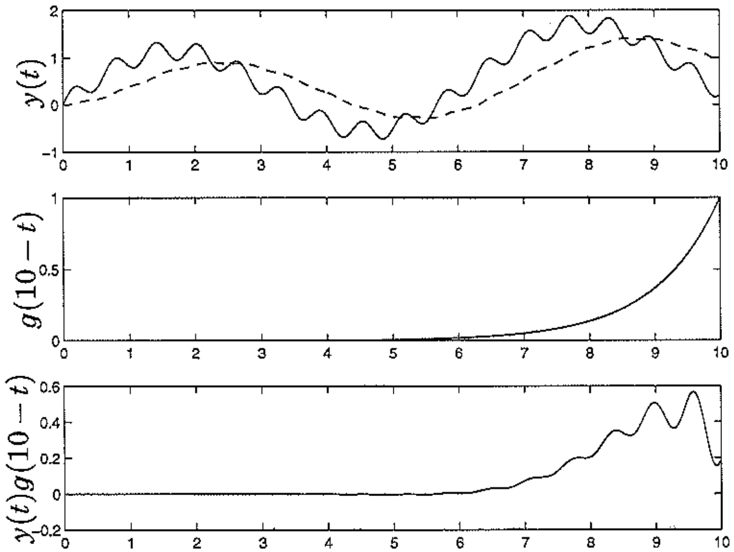
$$x(t) = \int_0^t be^{a(t-\tau)}u(\tau)dt = \int_0^t g(t-\tau)u(\tau)dt$$

to obtain the output at time t, the input at time τ is multiplied with $g(t-\tau)$ and the product is integrated from 0 to t.

The funtion $g(\tau)$ is therefore called the weighting function, another name is the impulse response.

The **state** is a number of variables that summarizes the past behavior for the purpose of predicting the future development of the system. The state is the only information about the past required for prediciton.

Graphical Illustration



Summary

The equation

$$\frac{dx}{dt} = ax + bu$$

has the solution

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)d\tau$$

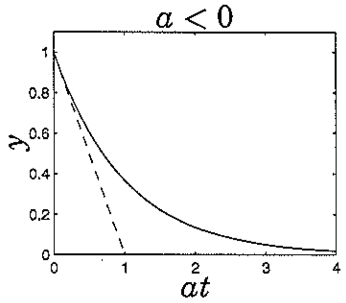
- The solution has two terms
- One depends on the initial condition
- The other depends on the control signal
- Parameter a is the root of the equation $s - a = 0$

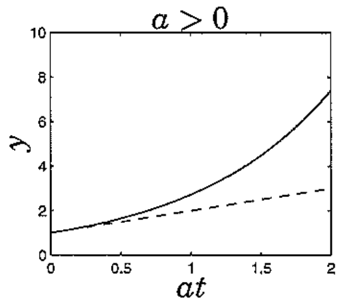
  Looks trivial but useful later

Parameter a Tells a lot!

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)dt$$

First term depends on initial conditions, second term depends on input signal





The system is stable if $a < 0$ and unstable if $a > 0$.

Example - The RC network

What is the effect of the drive voltage E

$$\frac{dV}{dt} = -\frac{1}{RC}V + \frac{1}{RC}E$$

A comparison with the solution of standard model

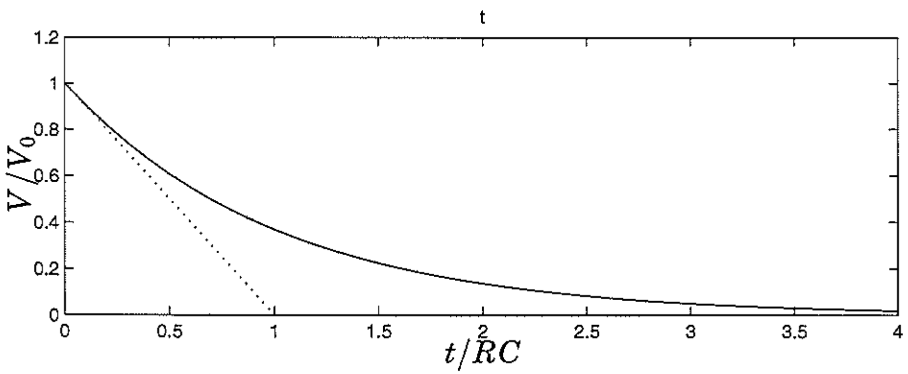
$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)dt$$

gives

$$V(t) = V_0 e^{-t/RC} + \int_0^t e^{-(t-\tau)/RC}E(\tau)d\tau$$

Notice $a = 1/RC$, where RC is the time constant

The Shape of the Solution



Notice the role of the time constant RC

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2. Examples
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4. Nonlinear systems
5. Bifurcations
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Nonlinear System

Why standard models?

$$\frac{dx}{dt} = f(x,u)$$

The steady state relation is given by

$$f(x,u) = 0$$

- Much richer behavior than linear equations
- Equation can be solved analytically only in a few special cases
- What to do?
- Qualitative analysis and numerical solutions

Equilibria and Stability

$$\frac{dx}{dt} = f(x)$$

Equilibria are given by

$$f(x) = 0$$

An equilibrium $x = a$ is stable if all solutions starting in the neighborhood of $x = a$ moves towards it. A sufficient condition is that $f'(a) < 0$.

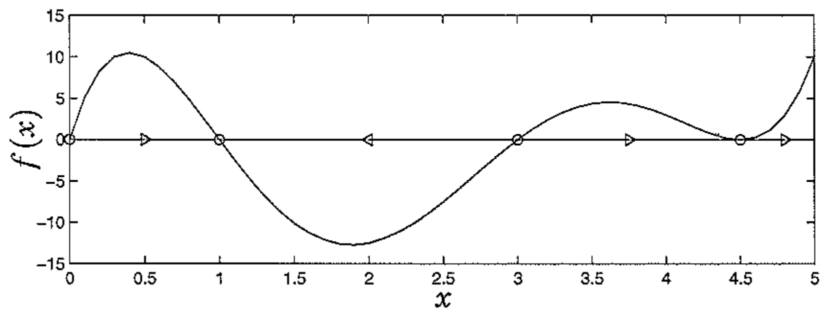
A complete characterization is obtained by plotting the function $f(x)$.

Discuss interpretation.

Example

$$\frac{dx}{dt} = f(x)$$

Interpretation as flow



How fast does it move?

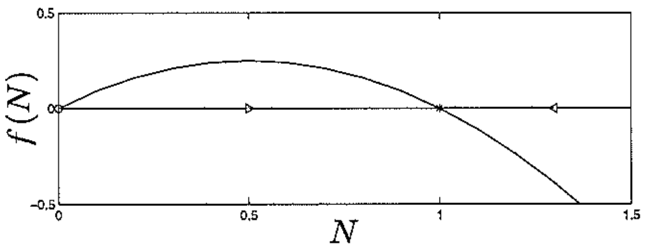
Logistic Growth

$$\frac{dN}{dt} = bN(1 - \frac{N}{A}) = bN - \frac{bN^2}{A} = f(N)$$

Equilibria $N = 0$ and $N = A$.

$$f'(N) = r - \frac{2bN}{A}$$

Hence $N = 0$ unstable and $N = A$ stable. Vector field.



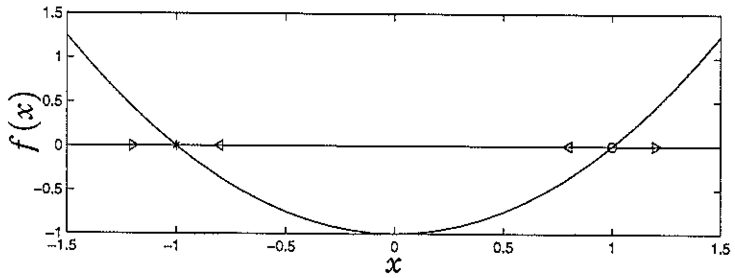
Another Example

$$\frac{dx}{dt} = x^2 - 1$$

Equilibria $x = 1$ and $x = -1$.

$$f'(x) = 2x$$

Hence $x = -1$ stable $x = 1$ unstable. Vector field.



Potentials

Consider the differential equation

$$\frac{dx}{dt} = f(x)$$

The potential $V(x)$ associated with the function $f(x)$ is defined by

$$f(x) = -\frac{dV}{dx}$$

We have

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = -f^2(x)$$

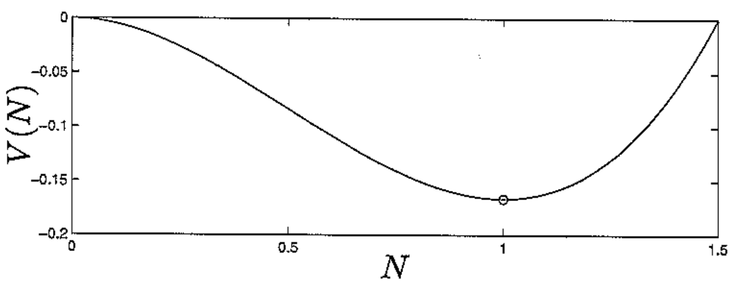
Since $f^2 \geq 0$ the potential is never increasing along a trajectory of the differential equation. The function $V(x(t))$ will thus decrease unlesss $x(t)$ happens to be an equilibrium where it remains constant.

Example - The Logistic Equation

$$\frac{dN}{dt} = bN(1 - \frac{N}{A}) = bN - \frac{bN^2}{K} = f(N)$$

The potential is

$$V(N) = -\frac{bN^2}{2} + \frac{bN^3}{3K} + C$$



Numerical Solutions

- The role of computers
- H. Goldstine: When things change by 2 orders of magnitude it is revolution not evolution. Moores law implies revolution every 10 years.
- The dangers of computers
- Numerical solutions are useful both for linear and nonlinear systems
- What do we mean by a solution to a problemXS
- Tools are available: Matlab, SCilab, Octave, SysQuake,
- Computer algebra tools: Mathematica, Maple

Commentary on Computations

- Herman Goldstine: "When things change by two orders of magnitude it is revolution not evolution."
- Important to complement computation by understanding and insight
- Hamming: "The purpose of computing is insight not numbers"
- Expect software errors! Important to check results to make sure that they are reasonable. Look at results and *Think*
  - Look for special case where you know the solution
  - Compute an auxiliary quantity to check the results

ODE Solvers

```
ODE23  Solve non-stiff differential equations.
  [T,Y] = ODE23('F',TSPAN,Y0) with TSPAN = [T0 TFINAL]
  integrates the differential equations y' = F(t,y)
  from time T0 to TFINAL with initial conditions Y0.
  'F' is a string containing the name of an ODE file.
  Function F(T,Y) must return a column vector. Each
  row in solution array Y corresponds to a time
  returned in column vector T. To obtain solutions at
  specific times T0, T1, ..., TFINAL (all increasing
  or all decreasing), use TSPAN = [T0 T1 ... TFINAL].

See also ODEFILE and other ODE solvers: ODE45, ...
  options handling: ODESET, ODEGET
  output functions: ODEPLOT, ODEPHAS2, ODEPHAS3, ...
  odefile examples: ORBITODE, ORBT2ODE, RIGIDODE, ...
```

Simulation Code

```
%Simulation of logistic growth
tspan=[0,10];
%---------------------
N0=0.1;
[t,N] = ODE23('nl',tspan,N0);
subplot(2,1,1)
plot(t,N)

function dNdt=f(t,N)
%The right hand side for logistic growth
b=1;
A=1;
dNdt=b*N*(1-N/A);
```

Lecture 2 & 3 - First Order Systems

1. Introduction
2. Examples
3. Linear systems
4. Nonlinear systems
5. Bifurcations
6. Summary

Bifurcations

- First order systems are simple
- Can be completely understood by plooting $f(x)$
- Are there interesting questions?
- Parameter dependencies
- Can changes of parameters cause qualitative changes?
- Bifurcations
- Catastrophe theory

Consider the equation

$$\frac{dx}{dt} = f(x, a)$$

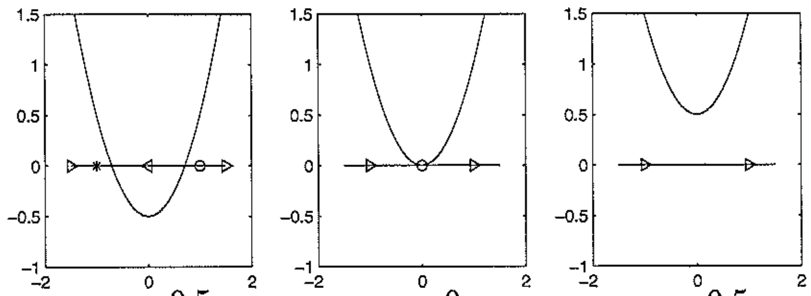
where a is a parameter. How does the behavior change with a?

Example

Consider the equation

$$\frac{dx}{dt} = a + x^2$$

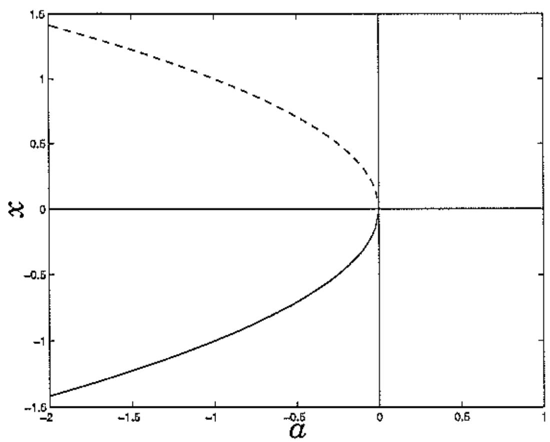
Equilibria changes with the parameter, different qualitative behavior



$a = -0.5$ $\quad a = 0$ $\quad a = 0.5$

Representation of Bifurcations

A diagram which shows the equilibria as a function of the state. The dashed line indicates unstable equilibra and solid lines represents stable equilibria



Terminology

- Bifurcation theory is filled with bewildering names
- The bifurcation we have studied has been named: fold bifurcation, saddle-node bifurcation, turning point bifurcation, blue sky bifurcation (equilibria appears out of the clear blue sky)
- Other types of bifurcations are

  Transcritical bifurcation

  Pitchfork bifurcation

  Hopf bifurcation

  Flip bifurcation

The Audience is Thinking

Sketch the function $f(x)$ for the system

$$\frac{dx}{dt} = ax - x^2$$

- Determine equilibria and their stability
- Sketch the bifurcation diagram
- Interprete the bifurcation diagram

Transcritical Bifurcation

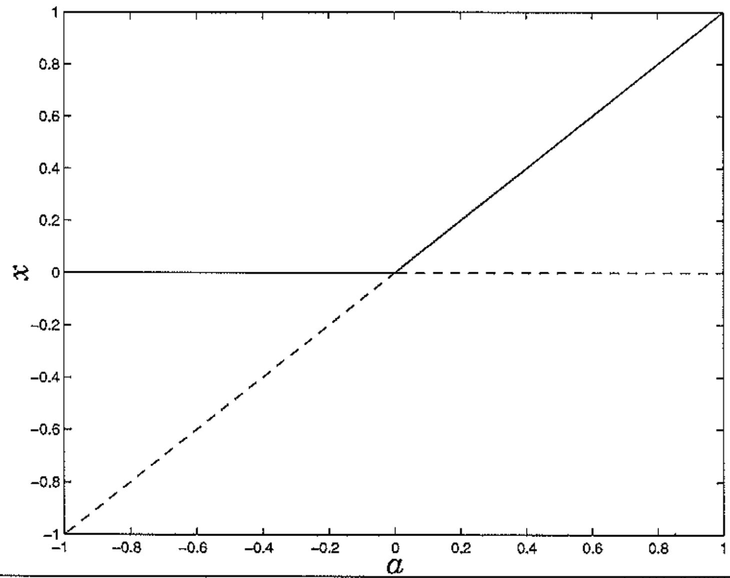
The system

$$\frac{dx}{dt} = f(x) = ax - x^2$$

has the equilibria $x = 0$ and $x = a$, the stability of the equilibria depends on the parameter a.

- Equilibrium $x = 0$, $f'(0) = a$, stable if $x < 0$ unstable if $x > 0$
- Equilibrium $x = a$, $f'(0) = -a$, stable if stable if $a > 0$ unstable if $a < 0$

Bifurcation Diagram



The Audience is Thinking

Sketch the function $f(x)$ for the system

$$\frac{dx}{dt} = f(x) = ax - x^3$$

- Determine equilibria and their stability
- Sketch the bifurcation diagram
- Interprete the bifurcation diagram

Pitchfork Bifurcation

Consider the system

$$\frac{dx}{dt} = ax - x^3$$

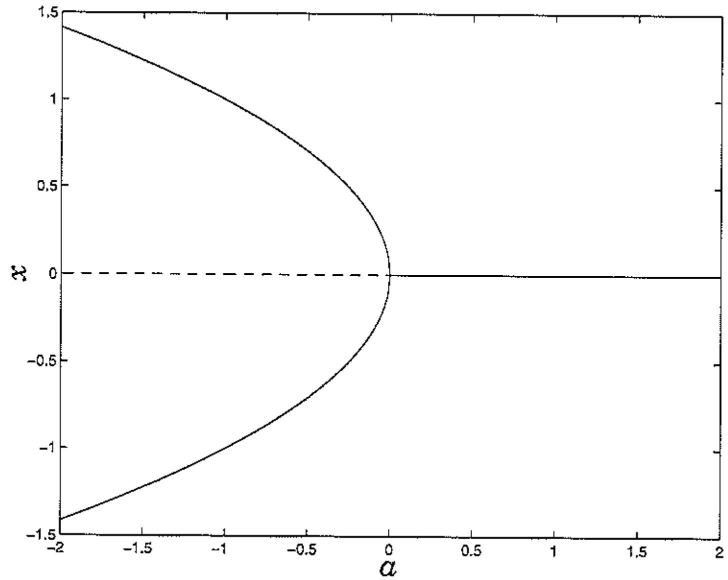
The system has one equilibria for $a \le 0$ and three equilibria for $a > 0$: $x = -\sqrt{a}$, $x = 0$ and $x = \sqrt{a}$. the stability of the equilibria depends on the parameter a.

Equilibrium $x = -\sqrt{a}$, $f'(0) = -2a$, unstable if $a > 0$

Equilibrium $x = 0$, $f'(0) = a$, stable if $a < 0$ unstable if $a > 0$

Equilibrium $x = \sqrt{a}$, $f'(0) = -2a$, stable if $a > 0$

Pitchfork Bifurcation



Several Parameters

Consider the system

$$\frac{dx}{dt} = f(x) = b + ax - x^3$$

which has two parameters. For b=0 we have a pitchfork bifurcation and the function $f(x)$ is symmetric in x.

The fixed points are given by

$$b + ax - x^3 = 0$$

This equation has one, two or three roots. If $a < 0$ there is always one equilibria. If $a > 0$ there are one, two or three equilibria.

The Number of Equilibria

To determine when there are three equilibria we first determine the extrema of the function when $a > 0$

$$y = b + ax - x^3$$

We have

$$y' = a - 3x^2$$

The derivative vanishes for $x = \pm x_0 = \pm\sqrt{a/3}$ we have

$$y(x_0) = b - \frac{2a}{3}\sqrt{\frac{a}{3}}$$

The Number of Equilibria ...

Summarizing we find that

- One equilbrium if $a < 0$ or $a > 0$ and

$$b < -\frac{2a}{3}\sqrt{\frac{a}{3}}, \quad \text{or}\, b > \frac{2a}{3}\sqrt{\frac{a}{3}}$$

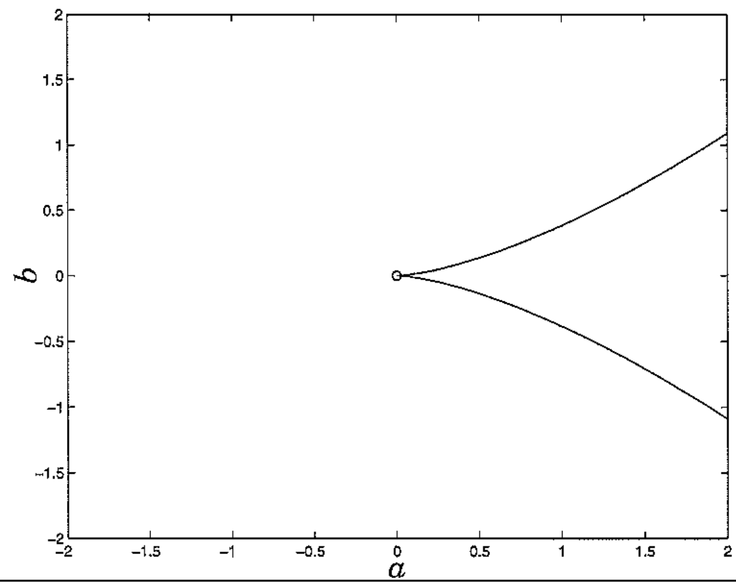
- Two equilibria if $a > 0$

$$b = -\frac{2a}{3}\sqrt{\frac{a}{3}}, \quad \text{or}\, b = \frac{2a}{3}\sqrt{\frac{a}{3}}$$

- Three equilibria if $a > 0$

$$-\frac{2a}{3}\sqrt{\frac{a}{3}} < b < \frac{2a}{3}\sqrt{\frac{a}{3}} \tag{1}$$

Stability Diagram



Spruce Budworm Breakout

Biological background: The insect called spruce budworm is a serous pest in eastern Canada, which attacks fir trees. The behavior is characterized by sudden unexpected outbreaks where the trees are defoliated and killed in about four years.

A very elegant model has been proposed in

Ludwig, D, Jones D. D. and Hol ling C. S. (1978) Qualitative analysis of insect outbreak systems: the spruce budworm and forest. J. Anim Ecol. 47 (1978) 315–332.

A characteristic feature is the difference in timescales of worm and wood growth. The budworm can increase by a factor of five in a year. The trees grow slowly and can replace their foliage in about 10 years. Treat tree foliage as a parameter.

Model for Budworm Population

$$\frac{dN}{dt} = BN\Big(1 - \frac{N}{A}\Big) - p(N)$$

where $p(N)$ is predation by birds. This is assumed to follow a Michaelis-Menten relation

$$p(N) = \frac{CN^2}{D^2 + N^2}$$

Hence

$$\frac{dN}{dt} = BN\Big(1 - \frac{N}{A}\Big) - \frac{CN^2}{D^2 + N^2}$$

This model has four parameters: carrying capacity A, birthrate B, maximum predation C and 50 % predation D

Scaling

$$\frac{dN}{dt} = BN\Big(1-\frac{N}{A}\Big) - \frac{CN^2}{D^2+N^2}$$

Introduce the scaled population $x = N/D$ and divide by C

$$\frac{D}{C}\frac{dx}{dt} = \frac{B}{C}x\Big(1-\frac{Dx}{A}\Big) - \frac{x^2}{1+x^2}$$

Introduce a new time scale and parameters a and b

$$\tau = \frac{Ct}{D}, \quad b = \frac{B}{C}, \quad a = \frac{A}{D}$$

The model then becomes

$$\frac{dx}{dt} = bx\Big(1-\frac{x}{a}\Big) - \frac{x^2}{1+x^2}$$

This model only has two parameters a and b

Equilibria

$$\frac{dx}{dt} = bx\Big(1-\frac{x}{a}\Big) - \frac{x^2}{1+x^2}$$

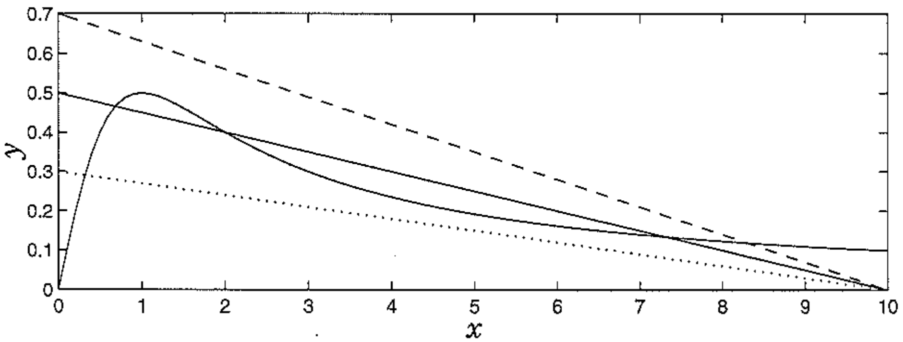
the equilibria are given by

$$bx\Big(1-\frac{x}{a}\Big) - \frac{x^2}{1+x^2} = 0$$

A fourth order equation, $x = 0$ is always a solution, the other can be visualized graphically by plotting $y = bx(1-x/a)$ and $y = x^2/(1+x^2)$.

Equilibria

Plot the functions $y = b(1-x/a)$ and $y = x/(1+x^2)$.



There are thus one, two, three or four equilibria depending on the particular values of the parameters. Recall that $x = 0$ is always an equilibrium.

Coinciding Equilibria

$$f(x) = bx\Big(1-\frac{x}{a}\Big) - \frac{x^2}{1+x^2} = 0$$

$$f'(x) = b\Big(1-\frac{2x}{a}\Big) - \frac{2x}{(1+x^2)^2} = 0$$

Dividing these equations we get

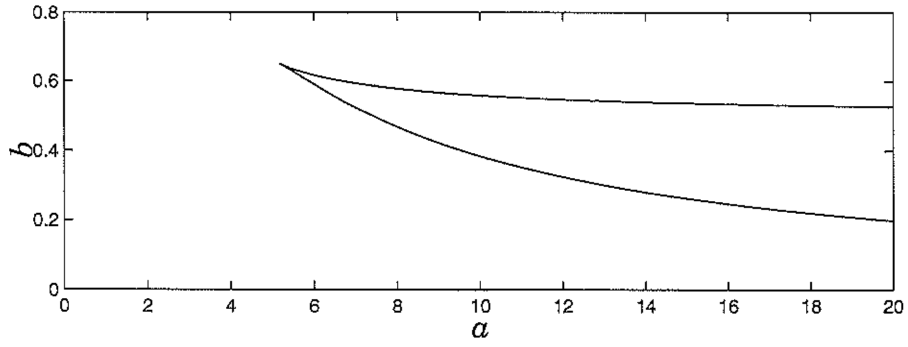
$$2\Big(1-\frac{x}{a}\Big) = \Big(1-\frac{2x}{a}\Big)(1+x^2)$$

Hence

$$a(x) = \frac{2x^3}{x^2-1}, \qquad b(x) = \frac{2x^3}{(1+x^2)^2}$$

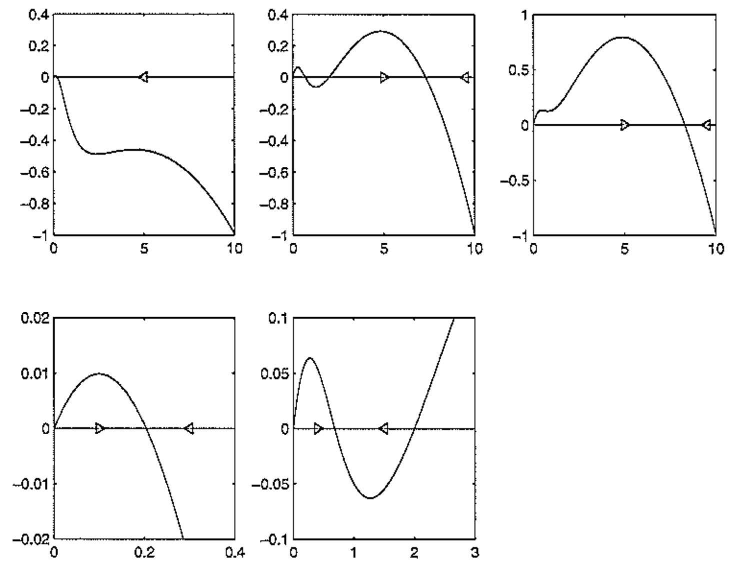
The smallest value of $b(x)$ is obtained for $x = \sqrt{3}$ and we have $a(\sqrt{3}) = 3\sqrt{3} = 5.1962$ and $b(\sqrt{3}) = 3\sqrt{3}/8 = 0.6495$.

Number of Equilibria



When a is sufficiently small there are two equilibria, $x = 0$ and another close to the origin $x \approx b$. When parameters are inside the wedge-shaped region there are four equilibria and at the borders of the region there are three equilibria.

Stability of Equilibria



Lecture 2 & 3 - First Order Systems

1. Introduction
2. Examples
3. Linear systems
4. Nonlinear systems
5. Bifurcations
6. Summary

Summary

- Many examples from widely different areas
- First order dynamics
- Linear systems - explicit solutions
- Nonlinear systems - qualitative thery
  - Equilibria
  - Stability
  - Flow
  - Potential
- Easy to get complete understanding
- Parameter dependencies, bifurcations, catastrophes
- But first order systems do not exhibit oscillations

References

John Maynard Keynes (1936) The General Theory of Employment Interest and Money. MacMillan, London

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Lecture 4 & 5 - Linear Time-Invariant Systems

K. J. Åström

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Summary
6. References

Theme: Increasing Dimensions

Introduction

- First order systems easy to deal with
- Standard form and analytical solutions
- Linear systems of high order
- Approximate nonlinear systems close to equilibria
- Standard form
- Numerical solutions

First Order Systems

Recall linear systems of first order

$$\frac{dx}{dt} = ax + bu$$

which has the solution

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)dt$$

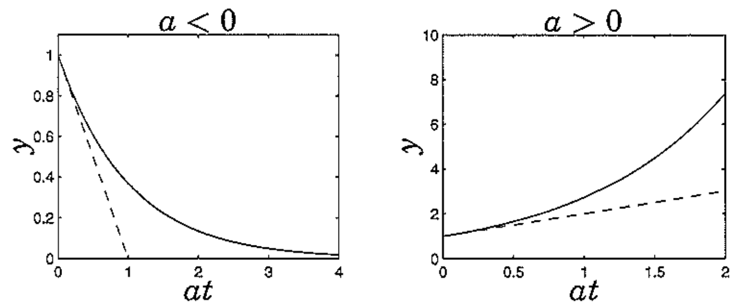
- The solution has two terms
- One depends on the initial condition
- The other depends on the control signal

How can this be generalized to systems of high order?

Parameter a Gives Much Insight

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)dt$$

First term depends on initial conditions, second term depends on input signal



The system is stable if $a < 0$ and unstable if $a > 0$.

Lecture 4 & 5 - Linear Time-Invariant Systems

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Input-output relations
6. Summary
7. References

Examples

- Spring mass system
- RLC circuit
- Simple compartment model
- Electric motor
- A compartment model
- A simple model of a macro econonmy

Can we find a suitable standard form

Spring Mass Damper System

A spring mass damper system can be described by the equations

$$m\frac{d^2y}{dt^2} + d\frac{dy}{dt} + ky = ky_r$$

This is a second order differential equation. We will write it as a system of first order equations. Introducing $x_1 = y$ and $x_2 = dy/dt$ the system can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= x_2\\ \frac{dx_2}{dt} &= -\frac{k}{m}x_1 - \frac{d}{m}x_2 + \frac{k}{m}y_r\end{aligned}$$

An RLC Network

There is energy stored in the capacitor and the inductor. The system can be described by the equations

$$\begin{aligned}E &= RI + L\frac{dI}{dt} + V\\ I &= C\frac{dV}{dt}\end{aligned}$$

These equations can be written as

$$\begin{aligned}\frac{dV}{dt} &= \frac{1}{C}I\\ \frac{dI}{dt} &= -\frac{1}{L}V - \frac{R}{L}I + \frac{1}{L}E\end{aligned}$$

An RLC Network ...

$$E = RI + L\frac{dI}{dt} + V$$
$$I = C\frac{dV}{dt}$$

By differentiating the first equation and multiplying it by C and subtracting the second equation we find that the model can also be written as

$$LC\frac{d^2I}{dt^2} + RC\frac{dI}{dt} + I = C\frac{dE}{dt}$$

By instead eliminating I we find that the system can also be written as

$$LC\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V = E$$

An Electric Motor

Energy stored in capacitor, inductor and rotor.

Momentum balance

$$J\frac{d\omega}{dt} + D\omega = kI$$

Kirchoffs laws for the electric circuit

$$E = RI + L\frac{dI}{dt} + V - k\frac{d\omega}{dt}$$
$$I = C\frac{dV}{dt}$$

The equation can be written as

$$\begin{aligned}\frac{d\omega}{dt} &= -\frac{D}{J}\omega + \frac{k}{J}I\\ \frac{dV}{dt} &= \frac{1}{C}I\\ \frac{dI}{dt} &= -\frac{kD}{JL}\omega - \frac{1}{L}V + \Big(\frac{k^2}{JL} - \frac{R}{L}\Big)I + \frac{1}{L}E\end{aligned}$$

A Compartment Model

$$V_1\frac{dc_1}{dt} = k_1(c_2 - c_1) + bu$$
$$V_2\frac{dc_1}{dt} = k_1(c_1 - c_2)$$

This equation can be written as

$$\begin{aligned}\frac{dc_1}{dt} &= -\frac{k_1}{V_1}c_1 + \frac{k_1}{V_1}c_2 + \frac{b}{V_1}u\\ \frac{dc_1}{dt} &= \frac{k_1}{V_2}c_1 - \frac{k_1}{V_2}c_2\end{aligned}$$

A National Economy

Let Y be the gross national product, C consumption, I investment, and G government expenditure. We have

$$Y = C + I + G$$

Assume that consumers react to the general economic climate with first order dynamics

$$\frac{dC}{dt} = -aC + bY$$

Moreover assume that investment is based on the filtered rate of change of consumption, i.e.

$$\frac{dI}{dt} = -cI + d\frac{dC}{dt}$$

A National Economy ...

$$\frac{dC}{dt} = -aC + bY$$
$$\frac{dI}{dt} = -cI + d\frac{dC}{dt}$$

can be written as

$$\frac{dC}{dt} = (b-a)C + bI + bG$$
$$\frac{dI}{dt} = d(b-a)C + (bd-c)I + bdG$$

The static input-output relation is

$$K = \frac{Y}{G} = \frac{2b-a}{b-a} \geq 1$$

Keynes multiplier! Same as for the simple model

A National Economy ...

The system

$$\frac{dC}{dt} = (b-a)C + bI + bG$$
$$\frac{dI}{dt} = d(b-a)C + (bd-c)I + bdG$$

has the characteristic polynomial

$$\lambda^2 + (a-b-bd+c)\lambda + (a-b)c$$

The system is stable if

$$(a-b)c > 0$$
$$a-b-bd+c > 0$$

Standard Forms

The examples show that systems can be written in at least two different ways

- As a high order differential equation

$$LC\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V = E$$

- As a system of first order differential equations

$$\frac{dV}{dt} = \frac{1}{C}I$$
$$\frac{dI}{dt} = -\frac{1}{L}V - \frac{R}{L}I + \frac{1}{L}E$$

Which one should we choose to develop the theory?

Lecture 4 & 5 - Linear Time-Invariant Systems

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Input-output relations
6. Summary
7. References

Standard Form for Second Order Systems

A system of second order can be written as

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + b_1u\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + b_2u\end{aligned}$$

or with matrix notation

$$\frac{dx}{dt} = Ax + Bu$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

Looks like a first order system! Can be extended to arbitrary order!

Matrices

- Matrices are a very convenient compact notation (a good language)
- Many problems can be expressed in a convenient form
- Matrices are useful for qualitative reasoning
- There are many powerful concepts such as eigenvalues and eigenvectors
- There are powerful computational tools (Matlab) for computing with matrices. Using this you can easily obtain solutions to specific problems numerically
- There is a beautiful theory for matrices

Simple Examples

Consider the following matrices and their transpose

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

- Which matrices can be added?
- Which matrices can be multiplied?
- The inverse of A is a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

.

Determinants, and Inverses

Consider the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The determinant is

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

The trace is

$$\operatorname{tr} A = a_{11} + a_{22}$$

If the determinant is different from zero the matrix has an inverse which is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Eigenvalues and Eigenvectors

Consider an n vector v and an $n \times n$ matrix A. The product Av is also an n vector. Matrix multiplication can thus be interpreted as a transformation of a vector. Are there some vectors that are transformed in a special way.

The vectors v and Av are parallel if $Av = \lambda v$, which implies

$$\lambda v - Av = (\lambda I - A)v = 0$$

This equation has a solution different from zero if

$$\det(\lambda I - A) = 0$$

The value λ is called an eigenvalue and the corresponding vectors v is called an eigenvector.

2×2 Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The characteristic polynomial of A is

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} \\ &= \lambda^2 - \operatorname{tr} A\lambda + \det A \end{aligned}$$

This is a second order polynomial which has two zeros. Notice that the eigenvalues may be real or complex numbers.

Diagonalization - Distinct Eigenvalues

Consider an 2×2 matrix A with distinct eigenvalues λ_1 and λ_2 and corresponding eigenvectors v^1 and v^2. We have

$$\begin{aligned} A\,(v^1 \quad v^2) &= (\lambda_1 v^1 \quad \lambda_2 v^2) \\ &= (v^1 \quad v^2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = T^{-1}\Lambda \end{aligned}$$

For a matrix with distinct eigenvalues we find that

$$AT^{-1} = T^{-1}\Lambda$$

or $A = T^{-1}\Lambda T$ and $\Lambda = TAT^{-1}$, where Λ is a diagonal matrix with the eigenvalues in the diagonal and the columns of the matrix T^{-1} consists of the right eigenvectors of A.

Matlab - A Matrix Calculator

Matlab is designed for matrix calculations

```
Arithmetic operators.
   plus       - Plus                              +
   minus      - Minus                             -
   mtimes     - Matrix multiply                   *
   times      - Array multiply                   .*
   mpower     - Matrix power                      ^
   power      - Array power                      .^
   mldivide   - Backslash or left matrix divide   \
   mrdivide   - Slash or right matrix divide      /
   ldivide    - Left array divide                .\
   rdivide    - Right array divide               ./
```

Eigenvalues and Eigenvectors in Matlab

```
EIG    Eigenvalues and eigenvectors.
   E = EIG(X) is a vector containing the eigenvalues
       of a square matrix X.

   [V,D] = EIG(X) produces a diagonal matrix D of
           eigenvalues and a full matrix V whose
           columns are the corresponding eigenvectors
           so that X*V = V*D.
```

Notice $X = VDV^{-1}$ comparing with our previous calculation we find that matlab gives $V = T^{-1}$.

Second Order Homogeneous Equations

A system of second order can be written as

$$\frac{dx}{dt} = Ax = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let v be an eigenvector of the matrix A and consider the solution $x(t)$ with initial condition $x(0) = v$.

$$x(t) = ve^{\lambda t}$$

Proof: $\quad \dfrac{dx}{dt} = \lambda ve^{\lambda t}, \quad Ax = Ave^{\lambda t} = \lambda ve^{\lambda t}$

The general solution is thus

$$x(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$$

where C and D are arbitrary vectors

Eigenvalues Give Much Insight

The equation

$$\frac{dx}{dt} = Ax$$

where A is a 2×2 matrix has the solution

$$x(t) = Ce^{\lambda_1 t} + De^{\lambda_2 t}$$

where λ_1 and λ_2 are the eigenvectors of A and C and D are arbitrary vectors.

Discuss the form of the solution for

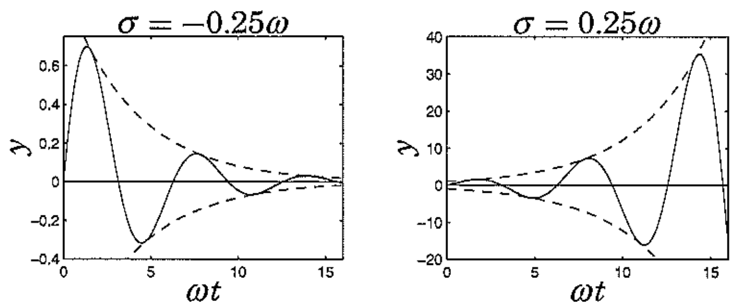
- Real eigenvalues
- Complex eigenvalues

Roots of Characteristic Equation give Insight!

A real eigenvalue λ corresponds to the time function $e^{\lambda t}$.

A complex eigenvalue $\lambda = \sigma \pm i\omega$ corresponds to the time functions.

$$e^{\sigma t} \sin \omega t, \qquad e^{\sigma t} \cos \omega t$$



Example RLC Circuit

$$\frac{dV}{dt} = \frac{1}{C}I$$
$$\frac{dI}{dt} = -\frac{1}{L}V - \frac{R}{L}I + \frac{1}{L}E$$

Introducing $x_1 = V$ and $x_2 = I$ and assume that $E = 0$, hence

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1/C \\ -1/L & -R/L \end{pmatrix} x = Ax$$

Characteristic polynomial

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1/C \\ 1/L & \lambda + R/L \end{pmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

Compare with the other way of writing the equations

$$\frac{d^2V}{dt^2} + \frac{R}{L}\frac{dV}{dt} + \frac{1}{LC}V = 0$$

Example RLC Circuit ...

Characteristic polynomial

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1/C \\ 1/L & \lambda + R/L \end{pmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

Introduce the numerical values $C = 10^{-6}$, $L = 0.0025$ and $R = 10$. The characteristic polynomial then becomes

$$\lambda^2 + 4000\lambda + 4 \times 10^8$$

This equation has the roots $\lambda = -200 \pm 19900i$. The solution to the differential equation is of the form

$$C_1 e^{-2000t} \sin 19900t + C_2 e^{-200t} \cos 19900t$$

A Compartment Model

- Consider the body as a collection of compartments with constant concentration in each compartment
- Rate of transport between two compartment is proportional to the difference in concentration (C) or activity. For a system with two compartments we have

$$\frac{dx_1}{dt} = k(C_2 - C_1) = k\Big(\frac{x_2}{V_2} - \frac{x_1}{V_1}\Big) = k_2 x_2 - k_1 x_1$$
$$\frac{dx_2}{dt} = k(C_1 - C_2) = k\Big(\frac{x_1}{V_1} - \frac{x_2}{V_2}\Big) = k_1 x_1 - k_2 x_2$$

where x_k is the number of molecules in compartment k, C_k the concentration and V_k the volume in compartment k

A Compartment Model ...

The standard form of the model is

$$\frac{dx}{dt} = \begin{pmatrix} -k_1 & k_2 \\ k_1 & -k_2 \end{pmatrix} x = Ax$$

The matrix A has the characteristic polynomial

$$\det \lambda I - A = \det\begin{pmatrix} s + k_1 & -k_2 \\ -k_1 & s + k_2 \end{pmatrix} = \lambda^2 + (k_1 + k_2)\lambda$$

The characteristic equation thus have two roots $\lambda_1 = 0$ and $\lambda_2 = -k_1 - k_2$. The solution is thus of the form

$$x_i(t) = A_i + B_i e^{-(k_1+k_2)t}$$

Another Compartment Model

Consider the compartment model

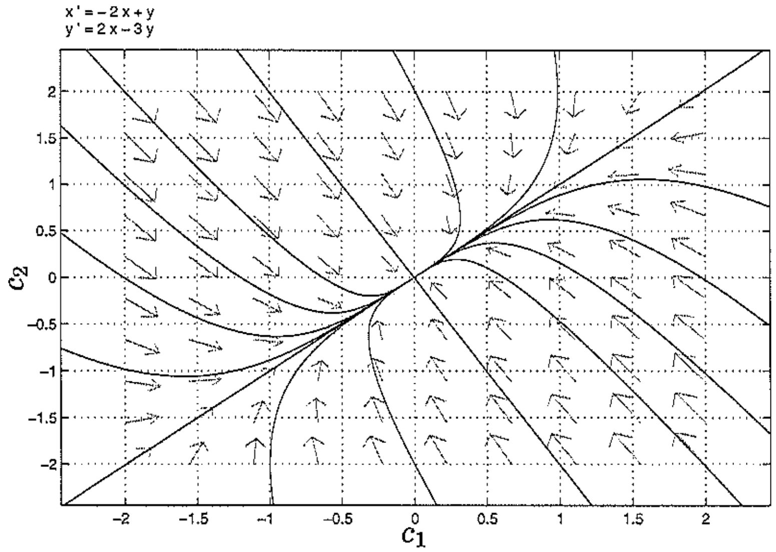
$$\frac{dx}{dt} = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix} x$$

This model has extra leakage from the second compartment. The system matrix A has the eigenvalues $\lambda_1 = -1.382$ and $\lambda_2 = -3.618$. The corresponding eigenvalues are

$$v^1 = \begin{pmatrix} 0.8507 \\ -0.5257 \end{pmatrix}, \quad v^1 = \begin{pmatrix} 0.5257 \\ 0.8507 \end{pmatrix}$$

The solution is thus composed of the exponential function $e^{-1.382t}$ and $e^{-3.618t}$. The vector v^1 is the slow eigenvector and all solutions except those who start on the fast eigenvecor will approach v^1 for large t.

Graphical Illustration



Diagonalization

A 2×2 matrix with distinct eigenvalues can be written as

$$A = T^{-1} \Lambda T$$

Consider the system

$$\frac{dx}{dt} = Ax + Bu$$

Introduce new coordinates $z = Tx$, $x = T^{-1}z$, then

$$\frac{dz}{dt} = T\frac{dx}{dt} = T(Ax + Bu) = TAT^{-1}z + TBu = \Lambda z + \beta u$$

Diagonalization ...

The equation $dz/dt = \Lambda z + \beta u$ can be written as

$$\frac{dz_1}{dt} = \lambda_1 z_1 + \beta_1 u$$
$$\frac{dz_2}{dt} = \lambda_1 z_2 + \beta_2 u$$

This is two uncoupled first order systems and we can use the results for first order systems to obtain the solution

$$z_1(t) = e^{\lambda_1 t} z_1(0) + \beta_1 \int_0^t e^{\lambda_1 (t-\tau)} u(\tau) d\tau$$
$$z_1(t) = e^{\lambda_1 t} z_2(0) + \beta_2 \int_0^t e^{\lambda_1 (t-\tau)} u(\tau) d\tau$$

The solution in the original coordinates is then obtained as $x = T^{-1}z$

Lecture 4 & 5 - Linear Time-Invariant Systems

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Input-output relations
6. Summary
7. References

Systems of Arbitrary Order

The first order system

$$\frac{dx}{dt} = ax + bu$$

has the solutions

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-\tau)}u(\tau)d\tau$$

We have generalized to second order systems and we will now generalize even further to systems of arbitrary high order

$$\frac{dx}{dt} = Ax + Bu$$

Linear Dynamical Systems - The State Model

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$

- Variables denote deviations from equilibrium
- Think scalar and interpret as vectors
- x state vector
- u control variable, input
- y measured variables, output

All information about the system in the matrices A, B, C and D and the initial condition.

Vector and Matrix Notations

- Very compact and practical notation
- Numerical calculations supported by nice software
- Learn to formulate and interpret
- Essentially the same as for scalar equations
- BUT remember that $AB \neq BA$! for matrices
- Use scalar results to guess the results for vectors and matrices

An Educated Guess

Consider the equation

$$\frac{dx}{dt} = Ax + Bu$$

If A and B are real numbers the solution is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Can this be the solution also when A is a matrix?
- How to define funtion of matrices?

Matrix Functions

Let $f(x)$ be a polynomial in real variables

$$f(x) = a_0 + a_1x + \ldots + a_nx^n$$

Let A be a square matrix, the matrix function $f(A)$ can then be defined as

$$f(A) = a_0I + a_1A + \ldots + a_nA^n$$

If $f(x)$ has a converging series expansion

$$f(A) = a_0I + a_1A + \ldots + a_nA^n + \ldots$$

The matrix exponential

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \ldots + \frac{1}{n!}A^nt^n + \ldots$$

Calculating with the Matrix Exponential

The matrix exponential is defined as

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{3!}(At)^3 \ldots + \frac{1}{n!}(At)^n + \ldots$$

Differentiate!

$$\frac{d}{dt}e^{At} = A + At + \frac{1}{2}(At)^2 \ldots + \frac{1}{(n-1)!}(At)^{n-1} + \ldots = Ae^{At}$$

"Solving" Linear Equations

The equation

$$\frac{dx}{dt} = Ax + Bu$$

has the solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Check: differentiation gives

$$\begin{aligned}\frac{dx}{dt} &= Ae^{At}x(0) + A\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)\\ &= Ax(t) + Bu(t)\end{aligned}$$

An Example

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

We have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence

$$e^{At} = I + At = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad e^{At}B = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

and we get

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x(0) + \int_0^t \begin{pmatrix} t-\tau \\ 1 \end{pmatrix} u(\tau) d\tau$$

Diagonalization - Distinct Eigenvalues

Consider an $n \times n$ matrix A with distinct eigenvalues λ_i and corresponding eigenvectors v^i. We have

$$A \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix} = \begin{pmatrix} \lambda_1 v^1 & \lambda_2 v^2 & \dots & \lambda_n v^n \end{pmatrix}$$
$$= \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} = T^{-1}\Lambda$$

For a matrix with distinct eigenvalues we find that

$$AT^{-1} = T^{-1}\Lambda$$

or

$$A = T^{-1}\Lambda T, \quad \text{and } \Lambda = TAT^{-1}$$

Diagonalization ...

$$\frac{dx}{dt} = Ax + Bu$$

Assume that there exist a matrix T such that $\Lambda = TAT^{-1}$. Introduce new coordinates $z = Tx$, then

$$\frac{dz}{dt} = T(Ax + Bu) = TAT^{-1}z + TB = \Lambda z + \beta u$$

Componentwise

$$\frac{dz_1}{dt} = \lambda_1 z_1 + \beta_1 u$$
$$\vdots$$
$$\frac{dz_n}{dt} = \lambda_n z_n + \beta_n u$$

Diagonalization ...

The equation

$$\frac{dz_1}{dt} = \lambda_1 z_1 + \beta_1 u$$
$$\vdots$$
$$\frac{dz_n}{dt} = \lambda_n z_n + \beta_n u$$

has the solution

$$z_1(t) = e^{\lambda_1 t} z_1(0) + \beta_1 \int_0^t e^{\lambda_1 (t-\tau)} u(\tau) d\tau$$
$$\vdots$$
$$z_n(t) = e^{\lambda_n t} z_n(0) + \beta_n \int_0^t e^{\lambda_n (t-\tau)} u(\tau) d\tau$$

The Cayley-Hamilton Theorem

Let the $n \times n$ matrix A have the characteristic equation

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} \ldots + a_n = 0$$

then it follows that

$$\det(\lambda I - A) = A^n + a_1 A^{n-1} + a_2 A^{n-2} \ldots + a_n I = 0$$

A matrix satisfies its characteristic equation.

Proof for Distinct Eigenvalues

If a matrix has distinct eigenvalues it can be diagonalized and we have $A = T^{-1}\Lambda T$. This implies that

$$A^2 =^{-1} T\Lambda T T^{-1}\Lambda T = T^{-1}\Lambda^2 T$$
$$A^3 =^{-1} T\Lambda T A^2 = T^{-1}\Lambda T T^{-1}\Lambda^2 T = T^{-1}\Lambda^3 T$$

and that $A^n = T^{-1}\Lambda^n T$. Since λ_i is an eigenvalue it follows that

$$\lambda_i^n + a_1\lambda_i^{n-1} + a_2\lambda_i^{n-2} \ldots + a_n = 0$$

Hence

$$\Lambda_i^n + a_1\Lambda_i^{n-1} + a_2\Lambda_i^{n-2} \ldots + a_n I = 0$$

Multiplying by T^{-1} from the left and T from the right and using the relation $A^k = T^{-1}\Lambda^k T$ now gives

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} \ldots + a_n I = 0$$

A Consequence for Matrix Functions

Let A be an $n \times n$ matrix. A matrix function of A can always be written as

$$f(A) = a_0 I + a_1 A + \ldots + a_{n-1} A^{n-1}$$

For matrices having distinct eigenvalues the coefficients can be determined from the equations

$$f(\lambda_i) = a_0 I + a_1 \lambda + \ldots + a_{n-1}\lambda^{n-1}$$

A Sharper Result

The minimal polynomial of a matrix is the polynomial of lowest degree such that $g(A) = 0$. The characteristic polynomial is generically the minimal polynomial. For matrices with common eigenvalues the minimal polynomila may, however, be different from the characteristic polynomial. The matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

have the minimal polynomials

$$g_1(\lambda) = \lambda - 1, \quad g_2(\lambda) = (\lambda - 1)^2$$

Lecture 4 & 5 - Linear Time-Invariant Systems

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Input-output relations
6. Summary
7. References

Two Views on Dynamics

State Models - White Boxes

- A detailed description of the inner workings of the system
- The heritage from mechanics
- The notion of state and stability
- States describe storage of mass, energy and momentum

Input-Output Models - Black Boxes

- A description of the input output behavior
- The Table
- The heritage of electrical engineering
- The notion of weighting function

Input and Output Relations

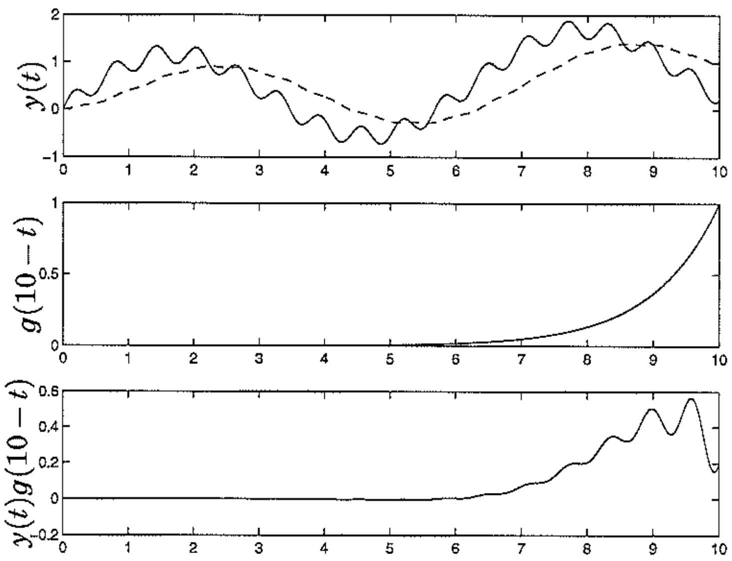
$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Input-output relation

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$
$$= Ce^{At}x(0) + \int_0^t g(t-\tau)u(\tau)d\tau$$

- The same formula as for first order systems!
- Notice two terms, initial conditions and input
- Weighting function or impulse response $g(t)$

The Impulse Response



Input and Output Relations

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

$$\begin{aligned}y &= Cx\\ \frac{dy}{dt} &= C\frac{dx}{dt} = CAx + CBu\\ \frac{d^2y}{dt^2} &= CA\frac{dx}{dt} + CB\frac{du}{dt} = CA^2x + CBu + CB\frac{du}{dt}\\ &\vdots\\ \frac{d^ny}{dt^n} &= CA^{n-1}\frac{dx}{dt} + CAB\frac{du}{dt} + CA^2B\frac{d^2u}{dt^2} + \ldots + CA^{n-1}B\frac{d^{n-1}u}{dt^{n-1}}\\ &= CA^nx + CBu + CAB\frac{du}{dt} + CA^2B\frac{d^2u}{dt^2} + \ldots + CA^{n-1}B\frac{d^{n-1}u}{dt^{n-1}}\end{aligned}$$

Input and Output Relations

$$\begin{aligned}y &= Cx\\ \frac{dy}{dt} &= C\frac{dx}{dt} = CAx + CBu\\ &\vdots\\ \frac{d^ny}{dt^n} &= CA^nx + CBu + CAB\frac{du}{dt} + CA^2B\frac{d^2u}{dt^2} + \ldots + CA^{n-1}B\frac{d^{n-1}u}{dt^{n-1}}\end{aligned}$$

Let the characteristic polynomial of the matrix A be

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$$

Multiply the first equation by a_n the second by a_{n-1} etc and add

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_ny = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_nu$$

The Transfer Function

The system

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_ny = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_nu$$

is characterized by two polynomials

$$\begin{aligned}A(s) &= s^n + a_1s^{n-1} + \ldots + a_{n-1}s + a_n\\ B(s) &= b_1s^{n-1} + b_2s^{n-2} + \ldots + b_{n-1}s + b_n\end{aligned}$$

The rational function

$$G(s) = \frac{B(s)}{A(s)}$$

is called the transfer funtion of the system

Summary

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu\\ y &= Cx\end{aligned}$$

has the input-output relation

$$\frac{d^ny}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_ny = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_nu$$

a_i coefficients of the characteristic polynomial and

$$\begin{aligned}b_1 &= CB\\ b_2 &= CAB + a_1CB\\ &\vdots\\ b_n &= CA^{n-1}B + a_1CA^{n-2}B + \ldots + a_{n-1}CB\end{aligned}$$

Linear Time Invariant Systems

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Variables now denote deviations from steady state. Solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}Bu(s)ds$$
$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-s)}Bu(s)ds + Du(t)$$

First terms depend on initial condition the second on the input.

Transfer function: $G(s) = C(sI - A)^{-1} + D$

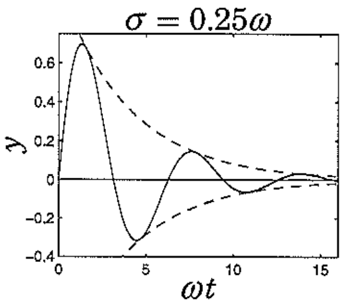
Impulse response: $h(t) = Ce^{At}B + D\delta(t)$

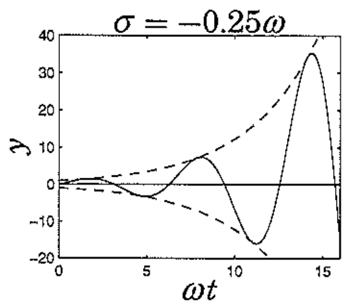
Roots of Characteristic Equation give Insight!

A real root $s = \alpha$ to the characteristic equation corresponds to the time function $e^{\alpha t}$.

Complex roots $s = \sigma \pm i\omega$ corresponds to the time functions.

$$e^{\sigma t}\sin\omega t, \qquad e^{\sigma t}\cos\omega t$$





Coordinate Changes

Coordinate changes are often useful

$$\frac{dx}{dt} = Ax + Bu \qquad z = Tx \qquad \frac{dz}{dt} = \tilde{A}z + \tilde{B}u$$
$$y = Cx + Du \qquad x = T^{-1}z \qquad y = \tilde{C}z + \tilde{D}u$$

Transformed system has the same form but the matrices are different

$$\tilde{A} = TAT^{-1}, \quad \tilde{B} = TB, \quad \tilde{C} = CT^{-1}, \quad \tilde{D} = D$$

The impulse response is an invariant with coordinate transformations.

$$\tilde{g}(t) = \tilde{C}e^{\tilde{A}t}\tilde{B} = CT^{-1}e^{TAT^{-1}t}TB = Ce^{At}B = g(t)$$

and

$$\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = CT^{-1}(sI - TAT^{-1})^{-1}TB$$
$$= C(sI - A)^{-1}B = G(s)$$

Diagonal Form

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} z + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} u$$
$$y = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix} z + Du$$

Transfer function

$$G(s) = \sum_{i=1}^{n} \frac{\beta_i\gamma_i}{s - \lambda_i} + D$$

Notice appearance of eigenvalues of matrix A

Controllable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \end{pmatrix} z + Du$$

Input-output relation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_n u$$

Transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n} + D$$

Observable Canonical Form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \dots & 0 \end{pmatrix} z + Du$$

Input-output relation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_n u$$

Transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n} + D$$

Working with Linear Systems

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

- Compute the eigenvalues of the matrix A, gives the components of the solution
- Determine the input output relation

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_n u$$

- Plot the impulse response using Matlab

$$g(t) = Ce^{At}B$$

Using Matlab

```
IMPULSE(SYS) plots the impulse response of the LTI
   model SYS (created with either TF, ZPK, or SS).

IMPULSE(SYS,TFINAL) simulates the impulse response
   from t=0 to the final time t=TFINAL.

IMPULSE(SYS,T) uses the user-supplied time vector T
   for simulation.

STEP(SYS,T) uses the user-supplied time vector T
   for simulation.

LSIM(SYS,U,T) plots the time response of the model
   SYS to the input signal described by U and T.
```

Lecture 4 & 5 - Linear Time-Invariant Systems

1. Introduction
2. Examples
3. Second Order Systems
4. High Order Systems
5. Input-output relations
6. Summary
7. References

Summary

- Computational tools
- Combine qualitative and quantitative techniques
- The standard input-output models
- Impulse response representations

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t g(t-\tau)u(\tau)d\tau$$

- High order differential equations

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = b_1\frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_n u$$

- Relations between representations

$$G(s) = \int_0^\infty e^{-st}g(t)dt = C(sI-A)^{-1}B + D$$

Summary

- Linear systems are very common
- The role of standar models
- State models and input output models
- The concepts of **state**, **input** and **output**
- The state variables describe storage of mass, momentum and energy
- The standard state model for linear time invariant systems

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du$$

- Vector and matrix notations, the matrix exponential e^{At}

References

John Maynard Keynes (1936) The General Theory of Employment Interest and Money. MacMillan, London

S. H Strogatz (1998) Nonlinear Dynamics and Chaos. With applications to Physics, Biology, Chemistry and Engineering. Perseus Books, Cambridge MA.

Lecture 6 - Compartment Models

K. J. Åström

1. Introduction
2. Compartment Models
3. Flow Systems
4. Measurement of Volumes and Flows
5. Summary
6. References

Theme: Compartment Models and Flow Systems

Introduction

- Early work by Widmark on propagation of alcohol in the body 1920
- Teorell coined the term compartment model around 1937
- Extensive application in pharmacokinetics Dost 1953

  Models required for FDA approval of new drugs

  Sheppard and Householder 1953
- Pulse testing

  Standard technique in ecological systems

  Measurement of blood volume and blood flow in vessels

  Extensive use in industry

A Practical View

Doctors need simple models for the daily work. Key questions:

- How much drug should be administered?
- How should it be taken: inhalation, intravenously, intramuscularly, orally?
- How quickly will it act?
- How long will it act?

These questions all relate to the dynamics of propagation of drugs in the body.

A Single Compartment Model

Consider a single compartment with volume V and flow q. Assume that the amount m is injected into the volume. The concentration is then given by

$$\frac{dc}{dt} = -\frac{q}{V}c$$

where the initial concentration is $c(0) = m/V$. Solving the differential equation we find that the concentration decays exponentially.

$$c(t) = \frac{m}{V}e^{-qt/V}$$

The dynamics can be captured by two quantities

- Volume of distribution V $[m^3]$
- Clearance q $[m^3/s]$

The ratio q/V $[s^{-1}]$ is called the elimination constant.

Finding the Parameters

The concentration is given by

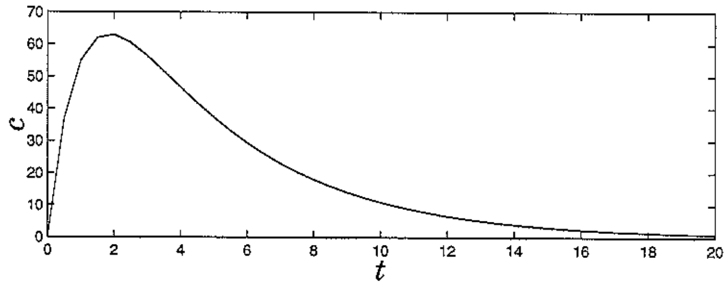
$$c(t) = \frac{m}{V} e^{-qt/V}$$

If the clearance and the volume of distribution are known it is easy to determine the concentration as a function of time t and dose m. The parameters q and V can be determined experimentally in the following way

- The volume of distribution can be determined from the dose m and the initial concentration $c(0)$ using the formula $V = m/c(0)$
- The elimination constant q/V can be determined by observing that $\log c(t) = -qt/V$. Plotting $\log c(t)$ versus time gives a straight line with slope t.

Back to Reality

Unfortunately the simple model does not agree with experiments. The concentration curve typically looks like this



which indicates that the first order model is too simplistic. In practice it is still used by neglecting the initial part of the curve and approximating the tail by a single exponential.

Volume of Distribution

Typical values

- Extracellular 0.5 l/kg
- Plasma 0.1 l/kg
- High tissue concentration $20 - 30$ l/kg

Make a detailed example for a two compartment model.

Lecture 6 - Compartment Models

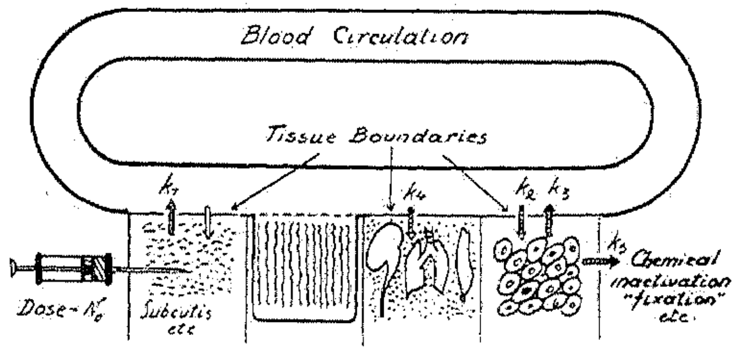
1. Introduction
2. Compartment Models
3. Flow Systems
4. Measurement of Volumes and Flows
5. Summary
6. References

Compartment Models

- Widmark 1920 Effect of alcohol on the human
- Teorell 1937 How drug injection propagates in the body
- Pharmacokinetics
- A special class of linear dynamical systems
- Nowadays required for FDA registration of medicines
- Advanced dosage (2 aspirin 3 times per day)
- Indirect measurements of volume and flow
- Use of radioactive tracers

Propagation of Drugs in the Body

T. Teorell (1937) Kinetics of Distribution of Substances Administered to the Body I and II. Arch. Int. de Pharm. et de Therapies 57(205-225), 57(226-240).



A Mathematical Model

- Consider the body as a collection of compartments.
- Rate of transport between two compartment is proportional to the difference in concentration. For a system with two compartments we have

$$\frac{dx_1}{dt} = k(c_2 - c_1) = k\Big(\frac{x_2}{V_2} - \frac{x_1}{V_1}\Big) = k_2 x_2 - k_1 x_1$$
$$\frac{dx_2}{dt} = k(c_1 - c_2) = k\Big(\frac{x_1}{V_1} - \frac{x_2}{V_2}\Big) = k_1 x_1 - k_2 x_2$$

  where x_1 is the number of molecules in compartment 1, and V_1 its volume, and c_1 the concentration in compartment 1, etc.
- Nonlinear transfer mechanism can also be used

Two Ways to Write the Models

In terms of the extensive variable mass x as

$$\frac{dx_1}{dt} = -\frac{k}{V_1}x_1 + \frac{k}{V_2}x_2$$
$$\frac{dx_2}{dt} = \frac{k}{V_1}x_1 - \frac{k}{V_2}x_2$$

or in terms of the intensive variable concentration c

$$\frac{dc_1}{dt} = -\frac{k}{V_1}c_1 + \frac{k}{V_1}c_2 = k_1(c_1 - c_2)$$
$$\frac{dc_2}{dt} = \frac{k}{V_2}c_1 - \frac{k}{V_2}c_2 = k_2(c_2 - c_1)$$

Notice structure of equations and coefficients

$$\frac{dx}{dt} = \begin{pmatrix} -k/V_1 & k/V_2 \\ k/V_1 & -k/V_2 \end{pmatrix} x, \quad \frac{dc}{dt} = \begin{pmatrix} -k/V_1 & k/V_1 \\ k/V_2 & -k/V_2 \end{pmatrix} c$$