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A PRACTICAL ROBUSTNESS THEOREM FOR ADAPTIVE CONTROL

by

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ABSTRACT

In this paper, two theorems are quoted which, when applied together, provide much information about the robustness of adaptive control schemes. From these two theorems, another theorem is developed which can explain why adaptive controllers can perform robustly in certain practical situations, while possibly failing in other situations. In particular, if the bandwidth constraints on a control systems are lenient enough to allow the use of a sampling frequency which is smaller than the frequency at which unstructured uncertainty becomes significant, an adaptive controller can behave robustly. Many, if not all, of the applications of adaptive control which have been successful employ relatively slow sampling of the process. Thus, the results of this paper provide a theoretical explanation of how certain adaptive controllers are performing robustly in practice.

In addition, the final theorem is of a form which provides insight into what a priori knowledge is required to achieve robust adaptive control and how this knowledge may be used.

1. INTRODUCTION

This paper considers the problem of robust adaptive control from an input-output or transfer function viewpoint. The plant is considered in continuous-time as a combination of a parameterized nominal plant and a multiplicative perturbation [1]. The magnitude of the multiplicative perturbation is assumed to be bounded as a function of frequency. Such a perturbation is referred to as unstructured uncertainty. The output of the continuous-time plant is sampled and a discrete-time adaptive controller is applied to the resulting discrete-time system. We wish to investigate the question of whether the discrete-time adaptive control system will be stable despite the original continuous-time unstructured uncertainty.

Two theorems are quoted and applied to the problem. The first theorem, [2], addresses the question of the uncertainty present in the discrete-time system resulting from the sampling of the continuous-time system with a sampling frequency ω_s. We let the discrete-time system be represented as a parameterized nominal plant and an unstructured multiplicative perturbation in discrete-time. The theorem states that the maximum size of the multiplicative perturbation in discrete-time over all frequencies will be approximately equal to the maximum size of the continuous-time multiplicative perturbation in the frequency range between zero and $\omega_s/2$. Since the continuous-time multiplicative perturbation is generally very small at low frequencies, the practical meaning of the result is that, by sampling a continuous-time system slowly enough, a discrete-time system results which has very little unstructured uncertainty.

The second theorem quoted is due to Ortega, Praly and Landau [3] and provides an adaptive algorithm in discrete-time which is robust in the sense that, if a discrete-time plant has small enough unstructured uncertainty, the adaptive control system will be stable.

At first glance, the allowable size of unstructured uncertainty in the second theorem appears quite small indeed. However, from the first theorem we see that unstructured uncertainty in sampled data systems can be extremely small in practical situations if a low enough sampling rate is chosen. Therefore, practical robust adaptive control is possible. Such is the main point of this paper.

Finally, a third theorem and an accompanying lemma are developed which provide direct tests of the robustness of the adaptive controller. The results give theoretical foundation and quantification to a two-part design process. First, a nominal control design is carried out which is robust for the plant with no unstructured uncertainty. Then the system must be sampled slowly enough so that the discrete-time multiplicative perturbation is small.

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2. SAMPLING THE PROCESS

The process or plant is represented by the following transfer function:

$$P(s,p) = P_n(s,p) (1 + M(s)) \qquad (1)$$

The function $P_n(s,p)$ represents a parameterized nominal plant model with p a vector of parameters. Ignorance of the exact value p which best represents the plant is an indication of structured uncertainty and is the reason for using an adaptive controller. The function M(s) is a multiplicative perturbation representing unstructured uncertainty. We do not know exactly what causes a particular M(s) to be present. The causes of M(s) may be neglected high frequency dynamics, the apparent effects of non-linearities, or other effects that cannot be measured accurately enough to be acceptably modeled. The only knowledge we will assume about the multiplicative perturbation is that M(s) represents stable dynamics and we know a bound on its magnitude as a function of frequency.

$$|M(j\omega)| < m(j\omega) \qquad (2)$$

We assume no knowledge at all about the phase characteristics of M(s). In general $m(j\omega)$ will be much less than one at low frequencies and will grow as frequency increases. Notice that M(s) values larger than unity translate into completely unknown phase characteristics for the plant P(s,p).

In order to create a sample data discrete-time system, filters which appropriately condition the plant's response will be allowed. Such filters are common in practice. First, the plant may be followed by an anti-aliasing filter, $F_a(s)$. Second, the plant may be preceded by a filter, $F_p(s)$, in order to remove the potential high frequency components which may be present in the output of the digital to analog converter. Finally, the effect of the digital to analog converter itself is well represented by an analog reconstruction as a train of impulses followed by a filter

$$F_2(s) = \frac{1 - e^{-sT}}{s} \qquad (3)$$

where T is the sampling period.

Let

$$G_n(s,p) = F_2(s)F_p(s)P_n(s,p)F_a(s)$$

The discrete-time system that results from condition the plant with $F_a(s)$ and $F_p(s)$ and sampling at a frequency of $\omega_s = 2\pi/T$ is given by the following equation.

$$G_d(j\omega,p) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_n(j\omega + k\omega_s,p)(1 + M(j\omega + k\omega_s)) \qquad (4)$$

We can define a nominal parameterized discrete-time plant as the discrete-time system which would result if M(s) were zero.

$$G_{dn}(j\omega,p) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_n(j\omega + k\omega_s,p) \qquad (5)$$

The discrete-time unstructured uncertainty, $M_d(j\omega,p)$ is then defined implicitly as follows:

$$G_d(j\omega,p) = G_{dn}(j\omega,p)(1 + M_d(j\omega,p)) \qquad (6)$$

Theorem 1, taken from [2], provides a bound on the magnitude of M_d.

Theorem 1:

$$|M_d(j\omega,p)| < m_d(j\omega,p) \qquad (7)$$

with

$$m_d(j\omega,p) = \frac{\sum_{k=-\infty}^{\infty} G_n(j\omega + k\omega_s,p)m(j\omega + k\omega_s)}{\sum_{k=-\infty}^{\infty} G_n(j\omega + k\omega_s,p)} \qquad (8)$$

where $m(j\omega)$ is taken from equation (2).

The proof is given in [2].

First, we remark that all the sums given in equation (4), (5) and (8) are absolutely convergent under the mild condition that the term found by setting k=0 has more poles than zeroes. More importantly, under similarly mild conditions which are always realized in practice, all these sums are completely dominated by the k=0 term for the frequency range

$$-\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \qquad (9)$$

See [4] for further illustration of this point. In addition, all discrete-time transfer functions are periodic with period ω_s so that the behavior for all frequencies is completely determined by the behavior for the frequencies given in (9). By assuming the k=0 term dominates each sum, equation (8) is reduced.

$$m_d(j\omega,p) = m(j\omega) \qquad -\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2} \qquad (10)$$

Note that, with these assumptions, the dependence of m_d on p disappears. With these results we can see that the maximum of $m_d(j\omega)$ over all frequencies is approximately equal to the maximum of $m(j\omega)$ over the frequency range given in (9). Usually $m(j\omega)$ is small at low frequencies so that if ω_s is small the resulting discrete-time system may have very little unstructured uncertainty at all frequencies. This phenomena was observed for a specific example using pole-zero plots in [5]

In summary, by sampling a system slowly enough one can arrive at a discrete-time transfer function for a process represented by the following equation:

$$G_d(z^{-1},p) = G_{dn}(z^{-1},p)(1 + M_d(z^{-1})) \qquad (11a)$$

with $\qquad |M_d(j\omega)| < m_d(j\omega) \qquad$ (11b)

where $m_d(j\omega)$ is small for all ω.

3. THE ROBUSTNESS THEOREM OF ORTEGA, PRALY AND LANDAU [3]

In this section we paraphrase a theorem from [3] which shows that if a plant can be expressed as in equation (11) and $m_d(j\omega)$ is small enough, there is an adaptive controller which remains stable despite the unstructured uncertainty.

Starting from equation 11, write

$$G_d(z^{-1},p) = \frac{z^{-d}B_0(z^{-1})}{A_0(z^{-1})} \qquad (12)$$

where A_0 and B_0 are polynomials

Let

$$\hat{\theta}[t] = [\hat{s}_0[t],\hat{s}_1[t],\ldots,\hat{s}_{n_s}[t],\hat{r}_0[t],\hat{r}_1[t]$$

$$,\ldots,\hat{r}_{n_R}[t]]^T \qquad (13)$$

$$\phi[t] = [u[t],u[t-1],\ldots,u[t-n_s],y[t],y[t-1]$$

$$,\ldots,y[t-n_R] \qquad (14)$$

$w[t]$ be the reference input

Choose $u[t]$ to solve

$$w[t+d] = \hat{\theta}^T[t]\ \phi[t] \qquad (15)$$

The choice of $u[t]$ can be represented by the equation

$$\hat{S}_t[q^{-1}]u[t] = w_{t+d} - \hat{R}_t[q^{-1}]y[t] \qquad (16)$$

For each constant value, θ^*, there is an entity to be referred to as a nominally controlled plant, $H^*(z^{-1})$, given by

$$H^*(z^{-1}) = \frac{z^{-d}B_0(z^{-1})}{C_0(z^{-1})} \qquad (17)$$

where $C_0(z^{-1}) = S^*(z^{-1})A_0(z^{-1}) + z^{-d}R^*(z^{-1})B_0(z^{-1})$ (18)

The nominally controlled plant is simply the closed-loop system that results from applying the constant feedback represented by equation (15) or (16) with a constant θ^* to the plant of equation (12). It is standard practice in the case where there is no unstructured uncertainty in the plant ($m_d(j\omega) \equiv 0$) to pick the degrees of S and R large enough to be able to achieve pole placement and plant zero cancellation in the closed-loop. In that case, there is some θ^{**} so that

$$H^{**}(z^{-1}) = \frac{z^{-d}}{C_R(z^{-1})} \text{ when } m_d(j\omega) \equiv 0 \qquad (19)$$

where $C_R(z^{-1})$ is a polynomial whose roots are desired closed-loop poles.

The adaptation algorithm consists of the following equations:

$$e[t] = C_R[q^{-1}]\ y[t] - w[t] \qquad (20)$$

$$\hat{\theta}[t] = \hat{\theta}[t-1] + \frac{f\phi[t-d]e[t]}{\rho[t]} \qquad (21)$$

$$\rho[t] = \mu\rho[t-1] + \max(|\phi[t-d]|^2,\rho_0) \qquad (22)$$
$$\rho_0 > 0,\ 0 < \mu < 1$$

This algorithm has constant adaptation gain, f, with a normalizing factor represented by ρ.

For this algorithm, Theorem 2 may be proved from the arguments of [3]. But first the following definition is needed:

A transfer function $H(z^{-1})$ is μ stable if there are no values of z outside a circle of radius μ which cause $H(z^{-1})$ to be singular.

Theorem 2 If there exists a θ^* such that

1. $z^d\ C_R(z^{-1})H^*(z^{-1})$ is μ-stable

and

2. $\sup\limits_{z\ =\ \mu^{1/2}e^{j\omega}} |z^d\ C_R(z^{-1})H^*(z^{-1}) - 1/f| < \frac{1}{f}$

Then the following are true:

1. If $w[t] \in \ell_\infty$ and $e^*[t] \in \ell_2$
   where

   $$e^*[t] = [q^d\ C_R[q^{-1}]H^*[q^{-1}] - 1]\ w[t]$$

   then $e[t] \in \ell_2$ and $\phi[t] \in \ell_\infty$

2. If $w[t] \in \ell_\infty$, then there exists a ρ_0 from equation (22) so that $e[t] \in \ell_\infty$ and $\phi[t] \in \ell_\infty$

Remark 1: The first results say that if a θ^* which provides asymptotic tracking produces a nominally controlled plant which meets the sufficient robustness conditions, the system will asymptotically track the input.

Remark 2: Similar results are obtained in [3] where bounded output disturbances are allowed. Similar results are also obtained in [3] when a normalized Lease Squares algorithm is used.

In interpreting the second theorem, it is useful to examine the conditions of the theorem in the case of no unstructured uncertainty. When there is no unstructured uncertainty the order of the adaptive controller may be chosen large enough so that equation (19) applied. Equation (19) implies the following condition on the quantities of interest in the conditions of Theorem 2.

$$z^d\ C_R(z^{-1})H^{**}(z^{-1}) = 1 \qquad (23)$$

With equation (23) being true condition one of Theorem 2 is automatically satisfied and condition two is satisfied if $0 < f < 2$ as expected from standard adaptive control theory. In general, condition two states that the image of a circle of radius $\mu^{1/2}$ in the z-plane under the mapping $z^d C_R(z^{-1})H^*(z^{-1})$ lies in a circle whose center is $1/f$ and radius is $1/f$. This condition is always more strict than requiring $z^d C_R(z^{-1})H^*(z-01)$ to be strictly positive real. Thus, any multiplicative perturbation in a plant which reflects itself as a multiplicative perturbation in H^* with magnitude greater than one implies that the phase of H^* is indeterminable at some frequency and condition 2 cannot be satisfied. Thus, small multiplicative perturbations in discrete-time are needed for robustness. Theorem 1 shows that while nature usually dictates large high frequency multiplicative perturbations in continuous-time models, small perturbations in discrete-time are possible through slow sampling.

4. GUIDELINES FOR ROBUST DESIGN

Unfortunately, Theorem 2 does not provide direct condition which the plant multiplicative perturbation should satisfy to guarantee robustness. Instead, it provides a condition on the nominally controlled plant. Thus, the satisfaction of the condition of Theorem 2 depends not only upon the size of plant's multiplicative perturbation but also upon the gains given in θ^* and the nominal plant itself. In this section, we create a new theorem which displays the dependence of robustness on a priori knowledge and provides guidelines for robust adaptive control design. In order to more fully understand how the plant perturbation translates into a perturbation on the nominally controlled system, we make the following definitions.

$$H^*_{m=0}(z^{-1}) = \frac{G_{dn}(z^{-1},p)}{S^*(z^{-1}) + R^*(z^{-1})G_{dn}(z^{-1},p)} \quad (24)$$

where $G_{dn}(z^{-1},p)$ from equation (11) is the discrete-time plant with no unstructured uncertainty, and $R^*(z^{-1})$ and $S^*(z^{-1})$ represent the nominal controller from equation (16). $H^*_{m=0}(z^{-1})$ is the nominal closed-loop system that would result from using the feedback parameters θ^* on the plant if there were no unstructured uncertainty. Define $M^*(z^{-1})$ implicitly by

$$H^*(z^{-1}) = H^*_{m=0}(z^{-1}) (1 + M^*(z^{-1})) \quad (25)$$

We note also that

$$H^*(z^{-1}) = \frac{G_{dn}(z^{-1},p)(1 + M_d(z^{-1})}{S^*(z^{-1}) + R^*(z^{-1})G_{dn}(z^{-1},p)(1 + M_d(z^{-1}))} \quad (26)$$

From equations (24) to (26), we derive

$$M^*(z^{-1}) = \frac{M_d(z^{-1})}{1 + \dfrac{R^*(z^{-1})G_{dn}(z^{-1},p) + R^*(z^{-1})G_{dn}(z^{-1},p)M_d(z^{-1})}{S^*(z^{-1})} \quad \dfrac{}{S^*(z^{-1})}} \quad (27)$$

Define a bound on the magnitude of M^*.

$$\sup_{\omega} |M^*(\mu^{1/2}e^{j\omega})| < m_\mu^* \quad (28)$$

Conditions which imply the conditions of Theorem 2 can now be stated in terms of $H^*_{m=0}$, $M_d(z^{-1})$, and m_μ^*.

Theorem 3. If there exists a θ^* such that

1. $H^*_{m=0}(z^{-1})$ is μ-stable

2. $\left|1 + \dfrac{R^*(\mu^{1/2}e^{j\omega})G_{dn}(\mu^{1/2}e^{j\omega},p))^{-1}}{S^*(\mu^{1/2}e^{j\omega})}\right| > |M_d(\mu^{1/2}e^{j\omega})|$

and   for all ω

3. $\sup_{z = \mu^{1/2}e^{j\omega}} |z^d C_R(z^{-1})H^*_{m=0}(z^{-1})| <$

$\dfrac{1}{f} - \sup_{z = \mu^{1/2}e^{j\omega}} |z^d C_R(z^{-1})| m_\mu^*$

Then the conditions of Theorem 2 are met and the stability results of Theorem 2 apply.

Proof: Conditions 1 and 2 of Theorem 3 are simply conditions providing for μ-stability of $H^*(z^{-1})$ through equation (26). Condition 3 of Theorem 3 follows as a sufficient condition for condition 2 of Theorem 2.

Let

$$L^*(z^{-1}) = \frac{R^*(z^{-1})G_{dn}(z^{-1},k)}{S^*(z^{-1})} \quad (28)$$

The quantity L^* is simply the loop gain of the nominal system with no modelling errors using the controller parameters θ^*. The quantity $1+L^*$ is the return difference operator of the same loop. We note here that, in most cases, any quantity evaluated at $\mu^{1/2}e^{j\omega}$ is only marginally different from the quantity evaluated at $e^{j\omega}$. Clearly, this statement becomes less valid as μ decrease from one. The size of m_μ^* is an important quantity. Let us investigate the quantity through equation (27) which we rewrite using equation (28).

$$M^*(z^{-1}) = \frac{M_d(z^{-1})}{1 + L^*(z^{-1})(1 + M_d(z^{-1}))} \quad (29)$$

Two factors are needed to keep the magnitude of $M^*(z^{-1})$ small:

1. $M_d(z^{-1})$ must be small. This is possible through Theorem 1.

2. $(1 + L^*(z^{-1}))$ must be large. This is provided by a robust nominal control design when there is no unstructured uncertainty.

In the following lemma, we make some assumptions on $|M_d|$ and $|1 + L^*|$ in order to bound m_μ^* and assure that condition 2 of Theorem 3 is met.

Lemma 1. Assume that

1. $\sup_{\omega} |M_d(\mu^{1/2}e^{j\omega})| < \sigma_m$

and   2. $\inf_{\omega} |1 + L^*(\mu^{1/2}e^{j\omega})| > \sigma_e$

then $m_\mu^* < \dfrac{\sigma_m}{\sigma_e(1 - \sigma_m) - \sigma_m}$

If, in addition,

$$\sigma_m < \frac{\sigma_e}{1 + \sigma_e}$$

Condition 2 of Theorem 3 is met.

The proof is obtained by manipulating inequalities.

In the following we show that from the theorems stated in this paper, robust adaptive control is at least feasible for some cases. In order to show feasibility, we pick arbitrarily what we feel are plausible values for σ_e and σ_m. Assume that the set of controller gains θ^{**} achieve the nominal design objective so that equation (19) is satisfied and the controller is robust enough on the nominal system with no unstructured uncertainties so that condition 2 of Lemma 1 is satisfied for some μ with $\sigma_e = -5$. Also, assume that the sampling frequency of the system is chosen so that condition one of Lemma 1 is met with $\sigma_m = 0.1$. This assumption is easily met in applications where the required bandwidth of the system is well below the frequencies where unstructured uncertainties become significant. Lemma 1 then shows that condition 2 of Theorem 3 is met and $m_\mu^* < .33$.

Applying equation (19) to condition 3 of Theorem 3 yields the following sufficient condition for stability of the adaptive control scheme in the presence of modelling error.

$$| 1 - \frac{1}{f} | < \frac{1}{f} - \sup_{z=\mu^{1/2}e^{j\omega}} |z^d C_R(z^{-1})| m_\mu^* \quad (30)$$

or, for $f < 1$

$$m_\mu^* < \inf_{z=\mu^{1/2}e^{j\omega}} \left| \frac{z^{-d}}{C_R(z^{-1})} \right|$$

Remember that $z^{-d}/C_R(z^{-1})$ is the desired nominal design objective. If the design is based upon minimum variance of control $C_R(z^{-1}) = 1$ and we have

$$m_\mu^* < \mu^{-d/2} \quad (31)$$

as the condition that must be satisfied for robustness. With $m_\mu^* = .33$ we can easily pick an appropriate μ and obtain robust adaptive control.

5. CONCLUSIONS

This paper has presented theorems which show that robust adaptive control is possible under certain practical conditions. In addition, guidelines of how to obtain a robust adaptive controller become evident through the theorems. The lessons of the theorems are in a form which is pleasing from an intuitive sense in that they suggest requirements which are similar to the requirements that one might expect from an intuitive viewpoint.

The theorems give quantitative substance to two conditions which are required for robust adaptive control. The first requirement is for a controller design which would meet reasonable robustness constraints if there were no unstructured uncertainty in the plant. This requirement is stated in terms of a familiar measure of robustness, the return difference operator.

The second requirement is that the unstructured uncertainty be small. Due to the potential high gain of the adaptive loop [6], the unstructured uncertainty must be much smaller than in nonadaptive robust control. Indeed, a multiplicative perturbation must always have magnitude less than unity. The key here is that Theorem 1 shows that discrete-time systems which are sampled slowly may result in a multiplication perturbation whose magnitude is much less than unity for all frequencies.

The practical situations for which robust adaptive control seems feasible are those systems whose bandwidth requirements are lenient enough to allow the system to be sampled at a frequency that is smaller than the frequency at which unstructured uncertainties become significant. It is pleasing that it is exactly on systems of this nature that adaptive controllers are performing successfully.

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