Steady-state heat loss from insulated pipes

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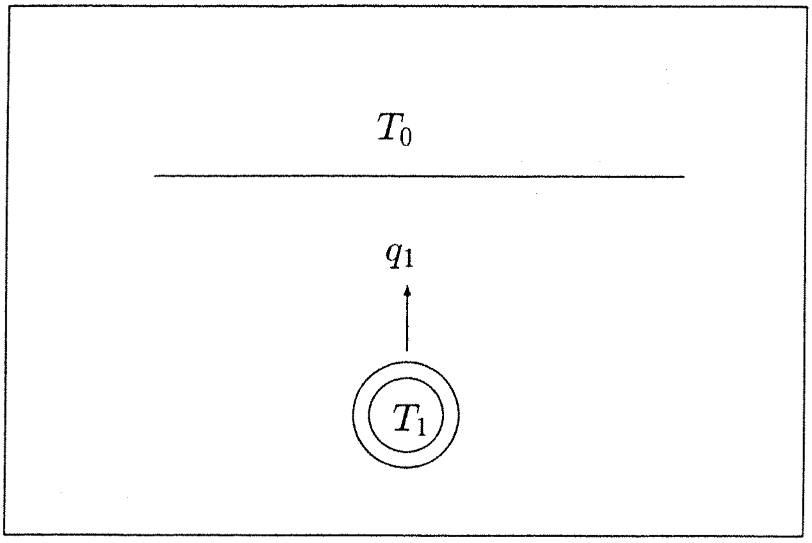
Petter Wallentén



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STEADY-STATE HEAT LOSS FROM INSULATED PIPES

Petter Wallentén

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Preface

This report deals with the steady-state heat loss from insulated pipes. The objective has been to present explicit formulae for the heat loss from the pipes.

The report is divided into four parts: Summary of new formulae, Part A,B and C. The first part is a summary of the results from A,B and C. Part A,B and C are separate reports that previously have been published.

Part A deals with the heat loss from one or two pipes in the ground, Part B deals with the heat loss from two pipes imbedded in a circular insulation and Part C deals with the heat loss from two pipes in the ground imbedded in a circular insulation. Part A,B, and C are in the summary referred to as [A],[B] and [C].

This work has been initiated by Dr. Johan Claesson at the Department of building Physics in Lund. He invented the method that makes the new formulae possible. I want to express my deep gratitude for his support and constructive criticism.

Lund, May 1991
Petter Wallentén

SUMMARY OF NEW FORMULAE

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Nomenclature

Symbol	Defining equation	Definition, (dimension)
h_s	(3.3,4.1,5.2)	Heat loss factor for the symmetrical problem, (-)
h_a	(3.4,4.2,5.3)	Heat loss factor for the anti-symmetrical problem, (-)
h_1	(2.3)	Heat loss factor for one pipe in the ground, (-)
H	Figure 2.1	Depth between ground surface and center of pipe, (m)
H_{eff}	(2.10)	Effective depth when surface resistance is used, (m)
d	(2.9)	Extra depth when surface resistance is used, (m)
d_e	(4.10,5.12)	Parameter used in old formulae, (m)
D	Figure 3.1	Half the distance between center of pipes, (m)
R_1	(4.11,14,5.13,17)	Thermal resistance used in old formulae, (mK/W)
R_2	(4.12,15,5.14,18)	Thermal resistance used in old formulae, (mK/W)
R_3	(5.15,19)	Thermal resistance used in old formulae, (mK/W)
r_c	Figure 4.1	Outer radius of the large circumscribing pipe, (m)
r_e	(4.9,5.11)	Equivalent radius used in old formulae, (m)
r_o	Figure 2.1	Outer radius of the pipe, (m)
r_i	Figure 2.1	Inner radius of the pipe, (m)
λ_g		Thermal conductivity of the ground, (W/mK)
λ_i		Thermal conductivity of the insulation, (W/mK)
$1/\alpha_a$	(2.8)	Surface resistance from ground to air, (m^2K/W)
T_c	Figure 4.1	Temperature at the large circumscribing pipe, (°C)
T_s	(3.1)	Temperature in the symmetrical problem, (°C)
T_a	(3.2)	Temperature in the anti-symmetrical problem, (°C)
T_0	Figure 2.1	Temperature at the ground surface, ($°C$)
T_1	Figure 3.1	Temperature in pipe 1, (°C)
T_2	Figure 3.1	Temperature in pipe 2, (°C)
q_s	(3.3,4.1,5.2)	Heat loss in the symmetrical problem, (W/m)
q_a	(3.4,4.2,5.3)	Heat loss in the anti-symmetrical problem, (W/m)
q_1	Figure 3.1	Heat loss from pipe 1, (W/m)
q_2	Figure 3.1	Heat loss from pipe 2, (W/m)
β	(2.2)	Parameter describing the insulation of the pipe, (-)
σ	(5.1)	Parameter describing a relation between λ_i and λ_g, (-)
γ	(5.10)	Parameter used in formula, (-)

1 Introduction

This is summary of the results from Part [A], [B] and [C]. They all deal with the problem of finding explicit formulae for the steady-state heat loss from insulated pipes. Four different problems are dealt with: one pipe in the ground, two pipes in the ground, two pipes imbedded in a circular insulation and two pipes imbedded in a circular insulation in the ground. Part [A] deals with the problem of one pipe in the ground and two pipes in the ground, Part [B] deals with two pipes imbedded in a circular insulation and Part [C] deals with two pipes imbedded in a circular insulation in the ground. In Part [A] also the effect of a surface resistance is investigated.

The formulae are mainly derived for district heating pipes. They can be used on any problem with the same boundary conditions, but the listed errors of the formulae are valid for dimensions usual for district heating pipes in the ground. For the typical case in this report is the thermal conductivity in the ground 2 W/mK and the thermal conductivity in the insulation 0.04 W/mK.

The presented formulae are all derived with the use of the *multipole method* by Claesson et al [1],[2]. The *multipole method* can solve two-dimensional steady-state heat flow problems with circular boundaries and has been used to calculate the error of the new formulae. Hellström [3] has derived similar formulae, also based on the *multipole method*, to be used in ground heat storage problems.

To distinguish between the new formulae derived in [A],[B] and [C] and already existing formulae are the existing formulae simply called old formulae. Some old formulae have been investigated.

2 One pipe in the ground

The results in this chapter are presented in [A]. There is one pipe in the ground, see Figure 2.1. The distance between the center of the pipe and the ground surface is H. There is an insulation between the radii r_i and r_o. The temperature in the pipe is T_1 and the temperature at the ground surface is T_0. The thermal conductivity in the insulation is λ_i. The thermal conductivity in the ground is λ_g. The problem is to determine the steady-state heat loss q_1 per unit length from the pipe. The temperature $T(x,y)$ in a vertical cross-section of the ground satisfies the steady-state heat conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.1}$$

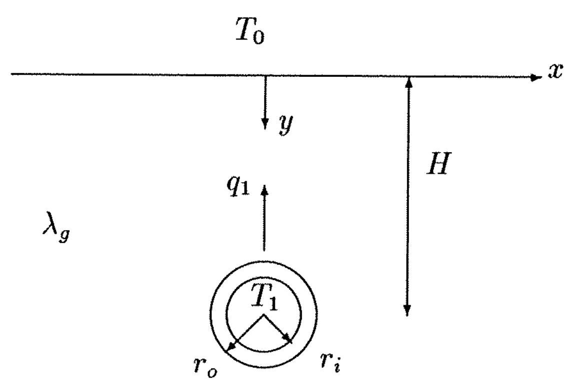

Figure 2.1. One pipe in the ground.

The dimensionless parameter β will be used in the following:

$$\beta = \frac{\lambda_g}{\lambda_i} \ln\left(\frac{r_o}{r_i}\right) \tag{2.2}$$

The heat loss q_1 is proportional to the temperature difference $T_1 - T_0$ and the thermal conductivity in the ground λ_g. We may write:

$$q_1 = 2\pi\lambda_g(T_1 - T_0) \cdot h_1(H/r_o, \beta) \tag{2.3}$$

Here $h_1(H/r_o, \beta)$ is a dimensionless heat loss factor. Note that h_1 only depends on H/r_o and β.

2.1 Approximate formulae

The zero-order multipole formula for the heat loss is:

$$h_1^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta \tag{2.4}$$

The first-order multipole formula for the heat loss factor is:

$$h_1^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta + \frac{1}{1 - (\frac{2H}{r_o})^2 \frac{1+\beta}{1-\beta}} \tag{2.5}$$

The second-order multipole formula for the heat loss factor is:

$$h_1^{-1} = \ln(\frac{2H}{r_o}) + \beta + \tag{2.6}$$

$$\left[1 + \frac{1}{2}\frac{(1+\beta)}{(1-\beta)}\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2 - \frac{3(1-2\beta)}{2(1+2\beta)}(\frac{r_o}{2H})^4\right] \cdot$$

$$\left[1 - \left((\frac{2H}{r_o})^2 - 3\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2\right)\frac{(1+\beta)}{(1-\beta)} - \frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^4\right]^{-1}$$

An old formula based on line sources investigated in [A] is:

$$h_1^{-1} = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta \tag{2.7}$$

2.2 Errors of the formulae

The relative error in the heat loss, when the old formula is used, is for district heating pipes typically less than 1 %. The error, when the zero-order formula is used, is typically less than 0.5 %. The error, when the first-order formula is used, is typically less than 0.05 %. The error, when the second-order formula is used, is typically less than 0.01 %.

2.3 Approximation of the insulation

It is standard practise to replace the thermal insulation between r_o and r_i by a surface resistance described by the dimensionless thermal resistance parameter β. The error of this approximation is typically less than 0.02 % for the ratio $\lambda_i/\lambda_g < 0.1$.

2.4 Approximation of the surface resistance

A surface resistance $1/\alpha_a$ is introduced between the ground surface and the air. The boundary condition at the ground surface then becomes:

$$T(x,y) - \frac{\lambda_g}{\alpha_a}\frac{\partial T(x,y)}{\partial y} = T_0 \qquad y = 0 \tag{2.8}$$

This resistance may be approximated with an equivalent layer of soil:

$$d = \frac{\lambda_g}{\alpha_a} \tag{2.9}$$

$$H_{eff} = H + d \tag{2.10}$$

The depth H_{eff} is used instead of H in the formulae. The error in the temperature field, when this approximation is used, is typically less than 0.01%.

3 Two pipes in the ground

The results in this chapter are presented in [A]. There are two pipes in the ground at the depth H, see Figure 3.1. The distance between the center of the pipes is $2 \cdot D$. The radius and insulation are identical for the two pipes. The temperatures in the pipes are T_1 and T_2. The temperature at the ground surface is T_0. The problem is to determine the steady-state heat losses q_1 and q_2 per unit length from the pipes.

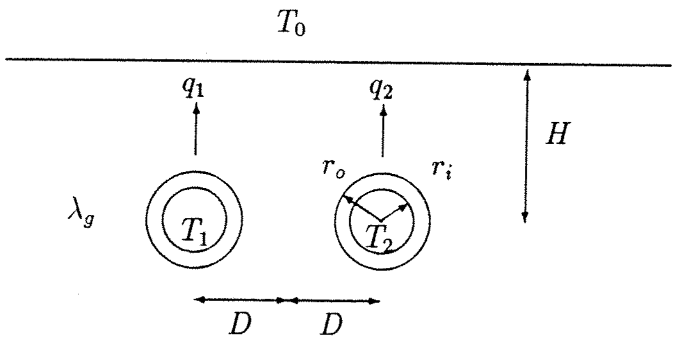

Figure 3.1. Two pipes in the ground.

3.1 Superposition

The original problem can be separated into a symmetrical and anti-symmetrical problem, see Figure 3.2.

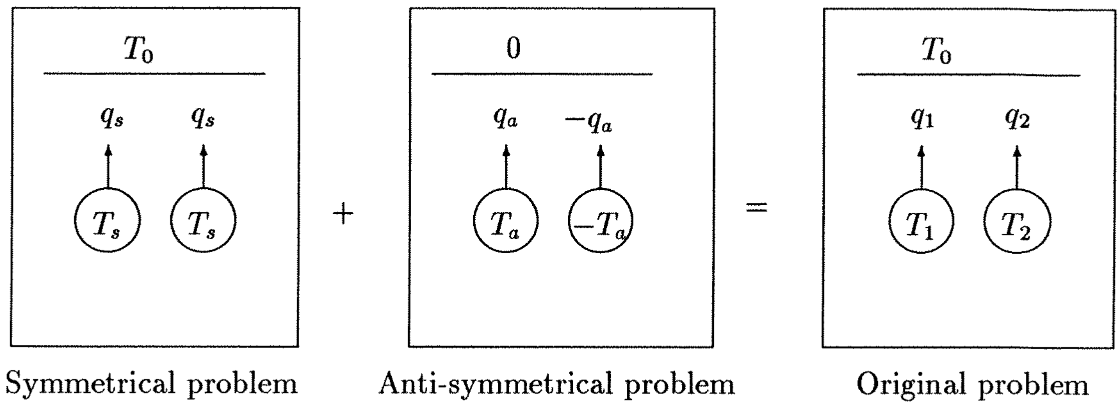

Figure 3.2. Superposition of symmetrical and anti-symmetrical problem.

The temperature in the pipes in the symmetrical problem is T_s. The temperatures in the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{3.1}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{3.2}$$

The subscript s denotes the symmetrical problem of two pipes. The subscript a denotes the anti-symmetrical problem of two pipes. The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_0$ and to the thermal conductivity in the ground λ_g. We may write:

$$q_s = (T_s - T_0) \cdot 2\pi\lambda_g \cdot h_s(H/r_o, D/r_o, \beta) \tag{3.3}$$

Here h_s is the dimensionless heat loss factor for the symmetrical problem. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional to the temperature T_a and to the thermal conductivity in the ground λ_g. We may write:

$$q_a = T_a \cdot 2\pi\lambda_g \cdot h_a(H/r_o, D/r_o, \beta) \tag{3.4}$$

Here h_a is the dimensionless heat loss factor for the anti-symmetrical problem. It should be noted that the temperature T_a connected with q_a in (3.4) is half the temperature difference between the pipes. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{3.5}$$

$$q_2 = q_s - q_a \tag{3.6}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{3.7}$$

Formulae for h_s and h_a are listed below. The heat losses q_1 and q_2 are obtained from (3.5,3.6).

3.2 Approximate formulae

Multipole formulae of zero and first order have been derived for the heat loss factors. One pair of old formulae have been investigated.

3.2.1 Zero-order multipole formulae

$$h_s^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) \tag{3.8}$$

$$h_a^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) \tag{3.9}$$

Here β is the dimensionless thermal resistance parameter from (2.2).

3.2.2 First-order multipole formulae

$$h_s^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 + \frac{r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} + \left(\frac{r_o}{2D}\right)^2} \tag{3.10}$$

$$h_a^{-1} = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 - \frac{3r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} - \left(\frac{r_o}{2D}\right)^2} \tag{3.11}$$

3.2.3 Old formulae

Two old formulae described in [A], based on line sources, are:

$$h_s^{-1} = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta + \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) \tag{3.12}$$

$$h_a^{-1} = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta - \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) \tag{3.13}$$

Formulae (3.12,13) originates from line sources.

3.3 Errors of the formulae

The relative errors in the heat loss, when the old formulae are used, are for district heating pipes typically less than 5 %. The errors, when the zero-order formulae are used, are typically less than 3 %. The errors, when the first-order formulae are used, are typically less than 0.5 %.

4 Two pipes imbedded in a circular insulation

The results in this chapter are presented in [B]. There are two pipes imbedded in a circular insulation. The radius of the pipes is r_i and the radius of the circular insulation is r_c. The distance between the center of the pipes is $2 \cdot D$. The temperatures in the imbedded pipes are T_1 and T_2. The temperature on the circumscribing larger pipe is T_c. The thermal conductivity in the insulation is λ_i. The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes inside the large pipe.

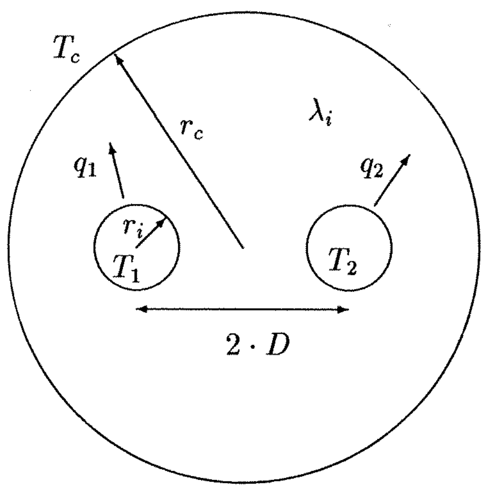

Figure 4.1. Two pipes inside a large pipe.

4.1 Superposition

The original problem is separated into a symmetrical and anti-symmetrical problem, see section 3.1. The temperatures T_s and T_a are defined in (3.1,2). The heat losses q_s and q_a become:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_c, D/r_c) \tag{4.1}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_c, D/r_c) \tag{4.2}$$

Note that h_s and h_a only depends on r_i/r_c and D/r_c and that the thermal conductivity used here is λ_i. As before, the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{4.3}$$

$$q_2 = q_s - q_a \tag{4.4}$$

4.2 Approximate formulae

Multipole formulae of zero and first order have been derived for the heat loss factors. Two old formulae have been investigated.

4.2.1 Zero-order multipole formulae

$$h_s^{-1} = \ln\left(\frac{r_c^2}{2Dr_i}\right) - \ln\left(\frac{r_c^4}{r_c^4 - D^4}\right) \tag{4.5}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_c^2 + D^2}{r_c^2 - D^2}\right) \tag{4.6}$$

4.2.2 First-order multipole formulae

$$h_s^{-1} = \ln\left(\frac{r_c^2}{2Dr_i}\right) - \ln\left(\frac{r_c^4}{r_c^4 - D^4}\right) - \frac{\left(\frac{r_i}{2D} + \frac{2r_i D^3}{r_c^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 - \left(\frac{2r_i r_c^2 D}{r_c^4 - D^4}\right)^2} \tag{4.7}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_c^2 + D^2}{r_c^2 - D^2}\right) - \frac{\left(\frac{r_i}{2D} - \frac{2r_i r_c^2 D}{r_c^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - 2r_i^2 r_c^2 \cdot \frac{r_c^4 + D^4}{(r_c^4 - D^4)^2}} \tag{4.8}$$

4.2.3 Area approximation formula

One of the old formulae investigated in [B] is called the area approximation. The formula calculates the heat loss factor in the symmetrical problem.

$$r_e = \sqrt{\frac{2 \cdot r_c^2}{\pi} \arccos\left(\frac{D}{r_c}\right) - \frac{2 \cdot D}{\pi}\sqrt{r_c^2 - D^2}} \tag{4.9}$$

$$d_e = \frac{\sqrt{r_c^2 - D^2} + r_c}{2} - r_i \tag{4.10}$$

$$R_1 = 2\ln\left(\frac{r_e}{r_i}\right) \tag{4.11}$$

$$R_2 = \frac{\pi d_e}{D} \tag{4.12}$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 + 1/R_2 \tag{4.13}$$

4.2.4 Two-model approximation formula

Another old formula investigated in [B] is called the two-model approximation. The formula calculates the heat loss factor in the symmetrical problem.

$$R_1 = \operatorname{arccosh}\left(\frac{r_i/r_c + r_c/r_i - (r_c/r_i)(D/r_c)^2}{2}\right) \tag{4.14}$$

$$R_2 = 4 \cdot \operatorname{arccosh}\left(2\left(\frac{D}{r_i}\right)^2 - 1\right) \tag{4.15}$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 - 1/R_2 \tag{4.16}$$

4.3 Errors of the formulae

The relative errors in the heat loss, when the zero-order formulae are used, are for district heating pipes typically less than 20% for q_s and less than 10% for q_a. The relative errors in the heat loss, when the first-order formulae are used, are typically less than 0.1% for q_s and less than 5% for q_a. The relative error in the heat loss, when the area approximation formula is used, is typically less than 10% for q_s. The relative error in the heat loss, when the two-model approximation formula is used, is typically less than 5% for q_s.

5 Two pipes in the ground imbedded in a circular insulation

The results in this chapter are presented in [C]. There are two pipes in the ground imbedded in a circular insulation. The radius of the pipes is r_i and the radius of the circular insulation is r_c. The center of the circular insulation lies at the depth H. The distance between the center of the pipes is $2 \cdot D$. The temperatures in the imbedded pipes are T_1 and T_2. The temperature at the ground surface is T_0. The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes inside the large pipe.

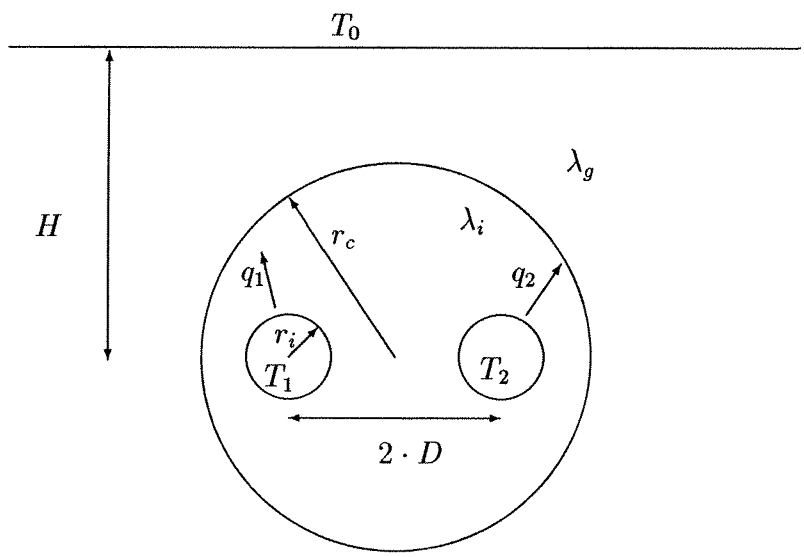

Figure 5.1. Two pipes in the ground imbedded in a circular insulation.

The dimensionless parameter σ will be used in the following:

$$\sigma = \frac{\lambda_i - \lambda_g}{\lambda_i + \lambda_g} \tag{5.1}$$

5.1 Superposition

The original problem is separated into a symmetrical and anti-symmetrical problem, see section 3.1. The temperatures T_s and T_a are defined in (3.1,2). The heat losses q_s and q_a become:

$$q_s = (T_s - T_0) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_c, D/r_c, H/r_c, \lambda_i/\lambda_g) \tag{5.2}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_c, D/r_c, H/r_c, \lambda_i/\lambda_g) \tag{5.3}$$

As before, the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{5.4}$$

$$q_2 = q_s - q_a \tag{5.5}$$

5.2 Approximate formulae

Multipole formulae of zero and first order have been derived for the heat loss factors. Two old formulae have been investigated.

5.2.1 Zero-order multipole formulae

$$h_s^{-1} = \frac{2\lambda_i}{\lambda_g} \ln\left(\frac{2H}{r_c}\right) + \ln\left(\frac{r_c^2}{2Dr_i}\right) + \sigma \cdot \ln\left(\frac{r_c^4}{r_c^4 - D^4}\right) \tag{5.6}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_c^2 + D^2}{r_c^2 - D^2}\right) \tag{5.7}$$

Here σ is defined in (5.1).

5.2.2 First-order multipole formulae

$$h_s^{-1} = \frac{2\lambda_i}{\lambda_g} \ln\left(\frac{2H}{r_c}\right) + \ln\left(\frac{r_c^2}{2Dr_i}\right) + \sigma \cdot \ln\left(\frac{r_c^4}{r_c^4 - D^4}\right) - \frac{\left(\frac{r_i}{2D} - \frac{\sigma 2 r_i D^3}{r_c^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2r_i r_c^2 D}{r_c^4 - D^4}\right)^2} \tag{5.8}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_c^2 + D^2}{r_c^2 - D^2}\right) - \frac{\left(\frac{r_i}{2D} - \gamma\frac{Dr_i}{4H^2} + \frac{2\sigma r_i r_c^2 D}{r_c^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - \gamma\frac{r_i}{2H} + 2\sigma r_i^2 r_c^2 \cdot \frac{r_c^4 + D^4}{(r_c^4 - D^4)^2}} - \gamma\left(\frac{D}{2H}\right)^2 \tag{5.9}$$

$$\gamma = \frac{2(1-\sigma^2)}{1 - \sigma\left(\frac{r_c}{2H}\right)^2} \tag{5.10}$$

5.2.3 Area approximation formula

One of the old formulae investigated in [C] is called the area approximation. The formula calculates the heat loss factor in the symmetrical problem.

$$r_e = \sqrt{\frac{2 \cdot r_c^2}{\pi} \arccos\left(\frac{D}{r_c}\right) - \frac{2 \cdot D}{\pi}\sqrt{r_c^2 - D^2}} \tag{5.11}$$

$$d_e = \frac{\sqrt{r_c^2 - D^2} + r_c}{2} - r_i \tag{5.12}$$

$$R_1 = 2\ln\left(\frac{r_e}{r_i}\right) \tag{5.13}$$

$$R_2 = \frac{\pi d_e}{D} \tag{5.14}$$

$$R_3 = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{H}{r_c} + \sqrt{\frac{H}{r_c}^2 - 1}\right) \tag{5.15}$$

$$h_s^{-1} = \frac{1}{1/R_1 + 1/R_2} + R_3 \tag{5.16}$$

5.2.4 Two-model approximation formula

Another old formula investigated in [C] is called the two-model approximation. The formula calculates the heat loss factor in the symmetrical problem.

$$R_1 = \operatorname{arccosh}\left(\frac{r_i/r_c + r_c/r_i - (r_c/r_i)(D/r_c)^2}{2}\right) \tag{5.17}$$

$$R_2 = 4 \cdot \operatorname{arccosh}\left(2\left(\frac{D}{r_i}\right)^2 - 1\right) \tag{5.18}$$

$$R_3 = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{H}{r_c} + \sqrt{\frac{H}{r_c}^2 - 1}\right) \tag{5.19}$$

$$h_s^{-1} = \frac{1}{1/R_1 - 1/R_2} + R_3 \tag{5.20}$$

5.3 Errors of the formulae

The relative errors in the heat loss, when the zero-order formulae are used, are for district heating pipes typically less than 10% for q_s and less than 20% for q_a. The relative error in the heat loss, when the first-order formulae are used, are typically less than 1% for q_s and less than 5% for q_a. The relative error in the heat loss, when the two-model approximation or the area approximation formula is used, is typically less than 5% for q_s.

5.4 Position of the pipes

There is a general opinion that, for heating district pipes it is better to put the warmer pipe underneath the cooler pipe. This is supposed to reduce the total heat loss from the pipes. It is true that the heat loss is reduced when the pipes are positioned vertically, but this reduction is so small that it is negligible. Calculations show that for district heating pipes the total heat loss is reduced with $< 0.2\%$.

References

A. Wallentén P, Heat loss from one or two insulated pipes in the ground, Notes on Heat Transfer 1-1990, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1990.

B. Wallentén P, Heat loss from two pipes imbedded in a circular insulation, Notes on Heat Transfer 2-1991, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.

C. Wallentén P, Heat loss from two pipes in the ground imbedded in a circular insulation, Notes on Heat Transfer 3-1991, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.

1. Claesson J and Bennet J, Multipole method to compute the conductive heat flows to and between pipes in a cylinder, Notes on Heat Transfer 2-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

2. Claesson J et al, Multipole method to compute the conductive heat flows to and between pipes in a composite cylinder, Notes on Heat Transfer 3-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

3. Hellström G, Ground heat storage, Thermal analyses of duct storage systems (thesis), Depts. of Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.

PART A:

Notes on Heat Transfer 1-1990

HEAT LOSS FROM ONE OR TWO INSULATED PIPES IN THE GROUND

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1 Introduction

This report deals with the calculation of the heat loss from one (q_1) or two (q_1, q_2) insulated pipes in the ground, see Figures 1.1-2.

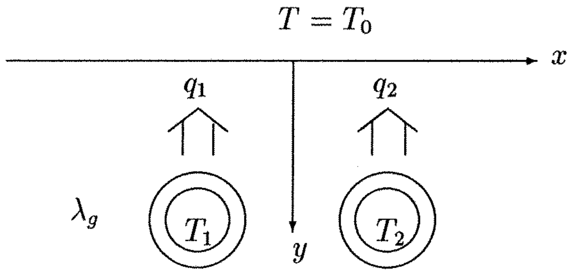

Figure 1.1. Two pipes in the ground.

The pipes with their thermal insulation are indicated by two concentric circles. The temperature inside the pipes are T_1 and T_2. The temperature is T_0 at the ground surface. The thermal conductivity of the ground is λ_g.

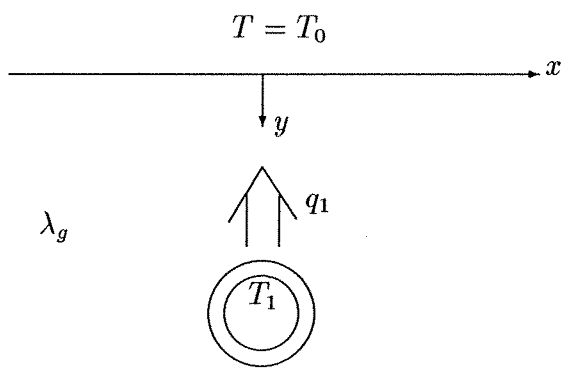

Figure 1.2. One pipe in the ground.

The temperature in the ground $T(x, y)$ and the heat losses q_n (W/m) from the pipes are unknown. The problem is to solve the steady-state heat conduction equation for the temperature and, in particular, to determine the heat flows q_1 and q_2 for given boundary temperatures T_0, T_1 and T_2. The problem is two-dimensional. The steady-state heat conduction equation for the temperature $T(x, y)$ is to be satisfied:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.1}$$

1.1 Traditional methods

Most methods to calculate the heat loss from one or two pipes to a surface with constant temperature are based on the use of line sources and the assumption that the pipe depth and the distance between the pipes are much larger than the diameter of the pipe. If this is true, these methods are accurate enough.

The temperature field from one line source at (x_n, y_n) with the strength q_n (W/m) is well known:

$$T(x, y) = \frac{q_n}{2\pi\lambda_g} \ln\left(\frac{r_c}{\sqrt{(x - x_n)^2 + (y - y_n)^2}}\right)$$

The temperature is zero at the distance r_c from the line source. If another line source is placed at $(x_n, -y_n)$ with the strength $-q_n$, the temperature will be zero at the line $y = 0$:

$$T(x, y) = \frac{q_n}{2\pi\lambda_g} \ln\left(\sqrt{\frac{(x - x_n)^2 + (y + y_n)^2}{(x - x_n)^2 + (y - y_n)^2}}\right) \tag{1.2}$$

A line source with the strength $-q_n$ is called a line sink. The line $y = 0$ is chosen as the ground surface. Each pipe will thus be represented as the sum of one line source and a mirror line sink.

There are several slightly different methods to calculate the heat loss but they are all based on a the line source model: [10], [20] and [26].

In particular it is possible to displace the line source from the center of the pipe to satisfy the boundary condition at the pipe to a better approximation. This we will call the line source displacement method. The displacement method is possible to use for one or two pipes in the ground. The only way to test the error of the various proposed formulae has been to use a finite element or finite difference method, or to make an electrical analogy experiment. These methods have a small but noticeable error. With the multipole method described in the next section the formulae can be tested with arbitrary accuracy.

1.2 The multipole method

A new method, the multipole method, to calculate the heat flow to and between pipes is presented in [1]. The method is implemented as programs for computers of PC-type in [2] and [3]. The program of [2] deals with the heat flow problem when one or more pipes are positioned inside a large pipe with a known constant temperature. The program of

[3] deals with the problem of one or more pipes inside a larger pipe, which in its turn lies in the ground with a another thermal conductivity. A brief summary of the multipole method is given here.

The thermal problem is solved with the use of line sources and what is called multipoles. The problem is solved in the complex plane ($z = x + iy$). The temperature from a line source with the strength q_n at position $z_n = x_n + iy_n$ may be written as the real part of the complex-valued logarithm:

$$T(x,y) = \mathrm{Re}\left[\frac{q_n}{2\pi\lambda_g}\ln\left(\frac{r_c}{z - z_n}\right)\right] \tag{1.3}$$

The radius r_c is introduced for dimensional reasons. The complex-valued derivative of order j of (1.3) with respect to z gives $(z - z_n)^{-j}$, which represents the multipole of order j.

The method uses multipoles at the pipe centers z_n. The temperature from the multipole of order j at pipe n is :

$$T(x,y) = \mathrm{Re}\left[P_{nj}\left(\frac{r_{pn}}{z - z_n}\right)^j\right] \qquad \begin{aligned} n &= 1,2..N \\ j &= 1,2... \end{aligned} \tag{1.4}$$

Here N is the number of pipes. The pipe radius r_{pn} of pipe n is introduced for dimensional reasons. The complex numbers P_{nj} give the strength of the multipoles:

$$P_{nj} = c_{nj} + i \cdot s_{nj} \tag{1.5}$$

The temperature (1.4) satisfies Laplace equation (1.1), since it is the real part of a regular (analytic) function. The temperature (1.4) becomes in polar coordinates ρ_n, ψ_n from z_n:

$$z = z_n + \rho_n \cdot e^{i\psi_n}$$

$$T = \mathrm{Re}\left[P_{nj}\left(\frac{r_{pn}}{\rho_n \cdot e^{i\psi_n}}\right)^j\right] = \left(\frac{r_{pn}}{\rho_n}\right)^j \cdot \{c_{nj}\cdot\cos(j\psi_n) + s_{nj}\cdot\sin(j\psi_n)\} \tag{1.6}$$

The multipole of order j can represent any variation $\cos(j\psi_n)$ and $\sin(j\psi_n)$ around the pipe at $\rho_n = r_{pn}$.

The final temperature is a sum of the temperature fields from all the pipes with multipoles up to order J. The strength of the multipoles P_{nj} and the strength of the line sources q_n are unknown. The boundary conditions of each pipe will give rise to an equation system, from which P_{nj} and q_n are solved. In the limit when $J \to \infty$ the exact solution is found. The error of the calculation can thus be chosen arbitrarily small.

With the multipole method it is possible to derive systematic approximations of increasing accuracy. The traditional method described in section 1.1 is similar to the zero order approximation. This report deals with approximations of the first and, in one case, the second order.

1.3 Survey of literature

Analytical expressions for the thermal or electrical resistance between a cylinder and a parallel plane was given by Foster and Lodge [4] (1875) and Forchheimer [5] (1888). Krischer [6] [7] (1936) was one of the first who used these analytical expressions to calculate the heat loss from a district heating pipe in the ground. Carslaw and Jaeger [8] (1946) gave analytical expressions for the solution of two pipes without insulation in an infinite region. They also studied the time-dependent behaviour of the temperature in the ground. The most used formulae today are the ones they calculated, except for an additive insulation resistance. Jakob [9] (1949) used the same formulae. For one pipe in the ground Louden [10] (1957) used the formulae of [8] with the resistance of the insulation as an additive term. For two pipes he suggested experimentally determined correction factors.

Vidal [11] (1961) analyzed the general problem of heat conduction between any number of surfaces with different constant temperatures. He gave a proof that the relationship between the heat flow from the surfaces and the surface temperatures was given by a symmetric equation system. He also calculated the resistance between a pipe and a surrounding square with constant temperature.

Brauer [12] (1963) used the expressions of Vidal for one pipe with quadratic insulation and determined experimentally expressions for more pipes in a surrounding quadratic insulation. Kutateladze [13] (1963) used the formulae of [10] and compiled a collection of approximate expressions for the heat loss from one up to three pipes in the ground.

Elgeti [14] (1967) proposed a semi-analytic method to calculate the error made when a surface resistance from ground to air was replaced by a equivalent layer of soil. The analysis concerned one non-insulated pipe in the ground. Schwaigerer [15] (1967) used the expressions of [13]. Bosselman [16] (1968) measured the temperature around an insulated pipe in the ground. Franz and Grigull [17] (1969) did experiments to find the minimum of the heat loss from the large pipe, when one small and one large pipe were in the ground. Claesson [18] (1970) calculated the temperature field around an insulated pipe with the use of line source displacement.

Homonnay and Hoffman [19] (1971) studied the dynamic behaviour of the temperature in the pipe, in the direction of the pipe. Jenowski [20] (1973) proposed the use of a small non-insulated pipe with the same thermal resistance to the ground as an insulated pipe, see section 3.6. Merker [21] (1977) calculated analytic expressions for the error when a surface resistance from ground to air was replaced by a equivalent layer of soil, the same problem as Elgeti [14] tried to solve. Merker could however give the exact expressions. The calculations concerned one or more pipes in the ground. Claesson and Dunard [27] proved the same formula but with a shorter argument. Brakelmann [22] (1980) calculated the dependence of the heat loss on the saturation of the ground. Kvisgaard and Hadvig [23] (1980) compiled a collection of formulae for different types of insulated pipes. The formulae were basically those of [13]. They also made finite element calculations to test the approximations.

Zeitler [24] (1980) used the formulae of [13] and tried to make the formulae more exact with correction factors determined experimentally. The corrections concerned different types of insulations of two pipes in the ground. Lunardini [25] (1981) studied the ice formation around a pipe with the use of [10]. Werner [26] (1982) calculated the heat loss from one or two pipes with the use of line source displacement. The proposed formula for the heat loss from two pipes was rather complicated. The displacement was not affected by the insulation thickness, as the formulae in section 3.5 are. Claesson and Dunard [27] (1983) used the formulae of [13] for insulated and non-insulated pipes in the ground. They also calculated the heat loss from a pipe in a ground with two different layers of soil, i.e. different heat conductivity. The dynamic behaviour of the temperature in the ground was studied. They calculated analytical solutions for the effect of ground surface resistance. Formulae for the effect of a ground water stream under a pipe were also calculated. Homonnay et al [28] (1985) proposed a numerical method to calculate the heat loss from two insulated pipes in a rectangular air culvert to the earth with surface resistance from pipe to air and from ground surface to air. The solution was obtained with the use of Schwarz-Christoffel's transform and a not well described complex temperature field. Schneider [29] (1985) proposed formulae and corrections for the heat loss from one pipe with insulation and a ground surface with thermal resistance, based on computer calculations with a finite difference method. Weinspach [30] (1987) used the formulae of [13]. Bøhm [31] (1988) used the formulae of [13] and studied the dynamics of a system of more than one pipe. Hansen [32] (1988) used the finite element method to test the method of Schneider [29].

2 Heat loss from two pipes in the ground

There are two pipes of the same type and at the same depth in the ground. For each pipe there is an insulation with the thermal conductivity λ_i between the radius r_i and r_o, see Figure 2.1. The steady-state heat conduction heat equation (1.1) is to be solved.

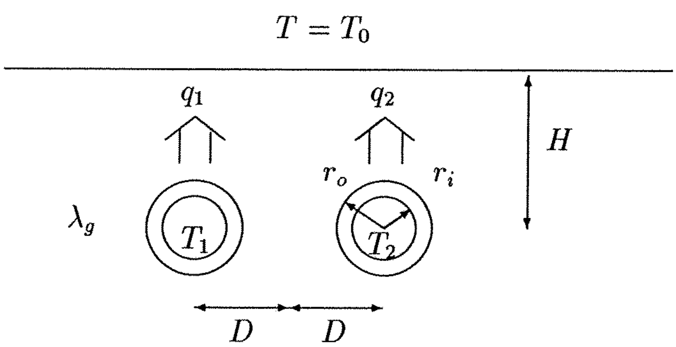

Figure 2.1. Two pipes in the ground.

H = Depth from the ground surface to the center of the pipes (m)
D = Half the distance between the center of the pipes (m)
r_o = Outer radius of the pipe (m)
r_i = Inner radius of the pipe (m)
λ_g = Thermal conductivity of the ground (W/mK)
λ_i = Thermal conductivity of the insulation (W/mK)
T_0 = Temperature on the ground surface (°C)
T_1 = Temperature in pipe 1 (°C)
T_2 = Temperature in pipe 2 (°C)
q_1 = Heat loss from pipe 1 per meter (W/m)
q_2 = Heat loss from pipe 2 per meter (W/m)

2.1 The solved problem

The thermal insulation of the pipes has the finite width $r_o - r_i$. It is customary to replace the insulation annulus by its thermal resistance as a surface resistance at the outer radius r_o. We will use this approximation. Its validity is studied in section 3.2 for the case of a single pipe in the ground. The thermal resistance over the thermal insulation of the pipes is :

$$R_{isol} = \frac{1}{2\pi\lambda_i} \ln\left(\frac{r_o}{r_i}\right) \qquad \text{(mK/W)} \tag{2.1}$$

The corresponding resistance per unit area is $R_{isol} \cdot 2\pi r_o$ (m^{2}K/W). The boundary condition for pipe 1 at its radius $\rho_1 = r_o$ is according to [1]:

$$T_1 = T - r_o\beta\frac{\partial T}{\partial \rho_1} \qquad \rho_1 = r_o, \quad 0 \le \psi_1 \le 2\pi \tag{2.2}$$

$$\beta = 2\pi\lambda_g \cdot R_{isol} = \frac{\lambda_g}{\lambda_i} \ln\left(\frac{r_o}{r_i}\right)$$

Here, ψ_1 denotes the angle around the pipe periphery in accordance with (1.6). The dimensionless thermal resistance parameter β will be used in the following. The new problem is described in Figure 2.2.

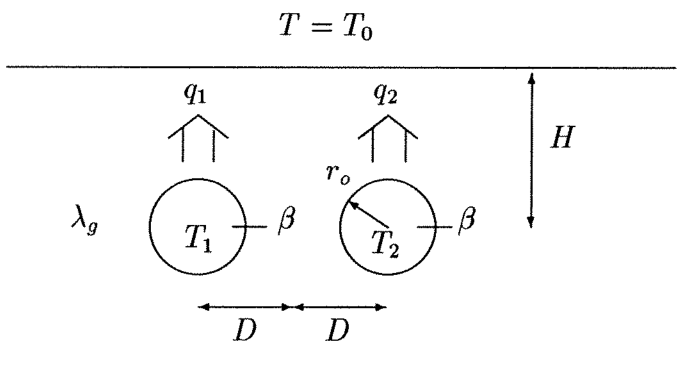

Figure 2.2. Two pipes in the ground with the thermal insulation as a surface resistance β.

2.2 Superposition

For two pipes in the ground one can construct two basic problems, a symmetrical problem and an anti-symmetrical problem, see Figure 2.3. With the use of the superposition principle, every problem concerning different temperatures can be constructed from the solutions of these two problems.

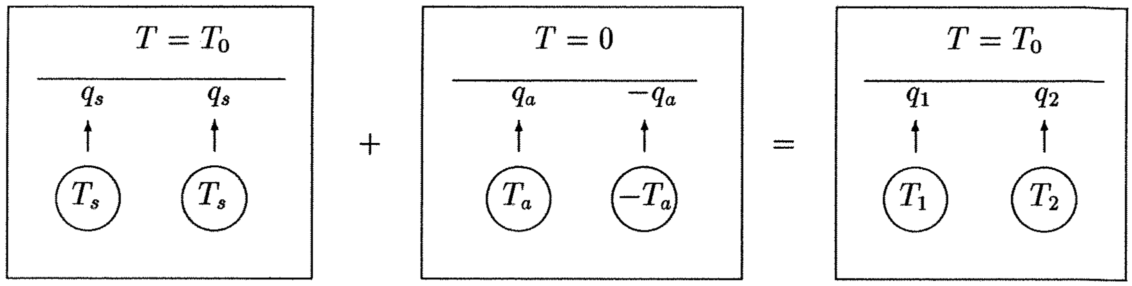

Figure 2.3. Superposition of symmetrical and anti-symmetrical problem.

The original problem is the sum of one symmetrical and one anti-symmetrical problem. The temperatures of the pipes in the symmetrical problem are T_s. The temperatures of the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{2.3}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{2.4}$$

The subscript s denotes the symmetrical problem of two pipes. The subscript a denotes the anti-symmetrical problem of two pipes. The temperatures of the original problem are from (2.3-4):

$$T_1 = T_s + T_a \tag{2.5}$$

$$T_2 = T_s - T_a \tag{2.6}$$

The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_0$. We may write:

$$q_s = \frac{T_s - T_0}{R_s} \tag{2.7}$$

Here R_s (mK/W) is the thermal resistance between one of the pipes and the ground. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional the temperature T_a. We may write:

$$q_a = \frac{T_a}{R_a} \tag{2.8}$$

Here R_a (mK/W) is the thermal resistance associated with the anti-symmetrical problem. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{2.9}$$

$$q_2 = q_s - q_a \tag{2.10}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{2.11}$$

The symmetrical and anti-symmetrical problem are solved separately in this section. Formulae for R_s and R_a are obtained. The heat losses q_1 and q_2 are then obtained from (2.3-11).

2.3 Zero-order approximation

The zero-order multipole approximation uses the line sources and sinks without any multipoles. The zero-order approximations gives the following expressions for the thermal resistances for the symmetrical and anti-symmetrical problem :

$$2\pi\lambda_g R_s = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{2.12}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{2.13}$$

These formulae are not derived in this report, but the derivation is identical to the derivation of the first-order multipole formulae (2.16,17) with the multipoles omitted. Here β is the thermal resistance parameter of the insulation introduced in (2.2). The first and second term on the right side in (2.12,13) is identical to the zero-order multipole approximation (3.5) for one pipe in the ground. The third term represents the thermal influence between the pipes.

2.4 Traditional method

The traditional way of calculating the two resistances are [9], [13] and [23] :

$$2\pi\lambda_g R_s = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta + \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{2.14}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta - \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{2.15}$$

Here, the third term on the right side in (2.14,15) represents the influence between the pipes.

The first term is an exact expression for the thermal resistance between one pipe in the ground without insulation and the ground surface. It is certainly not an exact expression for the thermal resistance between two pipes without insulation and the ground surface. The first term of (2.14,15) originates from a displacement of the line source in the vertical direction, while the first term of (2.12,13) originates from the problem with the line sources at the center of the pipes. Formulae (2.14,15) will become identical to (2.12,13) with the approximation :

$$\frac{H^2 - r_o^2}{r_o^2} \approx \frac{H^2}{r_o^2} \tag{2.16}$$

The displacement of the line source in (2.14,15) originates from the different problem of one pipe in the ground. Therefore, as the comparisons below will confirm, formulae (2.12,13) are better than formulae (2.14,15), when the pipes are insulated.

Werner [26] derived, based on a displacement of the line sources, a formula for the heat loss from two identical non-insulated pipes in the ground. The displacement was done in both the x- and y-directions. The magnitudes of the displacements were found by considering the temperatures at four points on the pipe circle. Werner had to approximate the complicated equations to be able to calculate the displacements. For insulated pipes he just added the insulation to the total thermal resistance. The formula is, in spite of these simplifications, long and complicated. Werner's formula is therefore not studied in this report.

2.5 First-order approximation

With the use of multipoles of the first order, the new formulae (2.17,18) for the thermal resistances from two pipes in the ground are calculated. The derivation of (2.18) is made in detail in section B.6. The derivation of (2.17) is very similar so it is not included in this report.

$$2\pi\lambda_g R_s = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 + \frac{r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} + \left(\frac{r_o}{2D}\right)^2} \tag{2.17}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 - \frac{3r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} - \left(\frac{r_o}{2D}\right)^2} \tag{2.18}$$

The first three terms to the right are the zero-order multipole formula (2.12,13). The fourth term is the first-order multipole compensation. These formulae are derived with the use of a slight approximation for the mirror multipole above the ground surface. The error of this approximation decreases, when the depth or H/D increases. The fourth term in both (2.17) and (2.18) will approach zero when β approaches the value 1.

2.6 Errors of different methods

The multipole program of [2] can calculate the solution to the problem in Figure 2.2 with arbitrary accuracy. In Tables 2.1-7 the order of the highest multipole is 10. Then, the relative errors in the calculations of the heat losses are less than 0.001%, i.e. a relative error of 10^{-5}. Tables 2.1-7 show the error made, when the heat losses q_s and q_a are calculated with formulae (2.12,14,17) and (2.13,15,18), respectively. The error is expressed in per cent. If it is less than 0.001 % the error is set to 0.0 % Tables 2.1-2 show the error made when the heat loss q_s of the symmetrical problem is calculated with the different formulae. In Table 2.1 the ratio r_o/r_i is 1.5 and in Table 2.2 this ratio is 2.0.

$r_o/H = 1.0$		0.7	0.5	0.3	0.1	
(2.14)	12.8	6.60	4.9	3.6	2.73	$r_o/D = 1.0$
(2.12)	6.86	5.43	4.41	3.48	2.72	
(2.17)	1.68	1.80	1.64	1.40	1.13	
	8.86	3.40	2.16	1.31	0.78	0.7
	3.08	2.23	1.65	1.15	0.78	
	0.210	0.087	0.13	0.11	0.076	
	7.99	2.66	1.53	0.80	0.37	0.5
	2.22	1.48	1.03	0.64	0.37	
	0.27	0.032	0.019	0.026	0.016	
	7.49	2.20	1.14	0.50	0.15	0.3
	1.72	1.02	0.63	0.34	0.15	
	0.24	0.050	0.0093	0.0023	0.0021	

Table 2.1. The relative error in per cent for the formulae (2.12,14,17) to calculate q_s. ($r_o/r_i = 1.5$, $\lambda_i/\lambda_g = 1/30$; $\beta = 12.16$)

$r_o/H = 1.0$		0.7	0.5	0.3	0.1	
(2.14)	7.99	4.42	3.38	2.59	2.04	$r_o/D = 1.0$
(2.12)	4.56	3.70	3.07	2.49	2.03	
(2.17)	1.16	1.28	1.20	1.04	0.87	
	5.34	2.17	1.41	0.88	0.57	0.7
	1.97	1.46	1.10	0.78	0.56	
	0.17	0.045	0.083	0.075	0.054	
	4.77	1.67	0.99	0.54	0.28	0.5
	1.41	0.96	0.68	0.44	0.27	
	0.20	0.030	0.0097	0.017	0.011	
	4.45	1.37	0.73	0.33	0.10	0.3
	1.09	0.66	0.41	0.23	0.10	
	0.17	0.038	0.0079	0.0012	0.0014	

Table 2.2. The relative error in per cent for the formulae (2.12,14,17) to calculate q_s. ($r_o/r_i = 2.0$, $\lambda_i/\lambda_g = 1/30$; $\beta = 20.79$)

The error decreases when the insulation increases. Note that in Tables 2.1-2 the error decreases faster for increasing D than for increasing H. The reason for this is that the symmetrical problem of two pipes in an infinite surrounding is a more difficult problem than the anti-symmetrical one, see section 3.6 and 4.3. The error when calculating q_a, which is an anti-symmetrical problem for four pipes, is thus in general less than the error of q_s.

The error of the zero-order multipole formula (2.12) is about half the error of the traditional formula (2.14). The error of the first-order multipole formula (2.17) is about one tenth of the error of the traditional formula. A typical problem for district heating mains may be: $r_o/H = 0.5$, $r_o/D = 0.7$ and $r_o/r_i = 1.5$. From Table 2.1 we see that the error of the traditional formula (2.14) is 2.2 % ,the error of the zero-order multipole formula (2.12) is 1.6 % and the error of the first-order multipole formula (2.17) is 0.13 %.

Tables 2.3-4 show the error made when the heat loss q_a of the anti-symmetrical problem is calculated with the different formulae. In Table 2.3 the ratio r_o/r_i is 1.5 and in Table 2.4 this ratio is 2.0.

The error made when calculating q_a is in general less than the error of q_s.

$r_o/H =$	1.0	0.7	0.5	0.3	0.1	
(2.15)	6.73	1.99	1.49	1.39	1.44	r_o/D =1.0
(2.13)	0.82	0.75	0.93	1.21	1.42	
(2.18)	0.12	0.054	0.042	0.060	0.081	
	6.59	1.60	0.91	0.71	0.74	0.7
	0.75	0.38	0.37	0.53	72	
	0.014	0.017	0.013	0.020	0.033	
	6.74	1.58	0.72	0.39	0.37	0.5
	0.93	0.37	0.19	0.22	0.35	
	0.092	0.0046	0.0016	0.0039	0.0097	
	7.00	1.73	0.74	0.24	0.13	0.3
	1.21	0.54	0.22	0.068	0.11	
	0.16	0.022	0.0032	0.0	0.0011	

Table 2.3. The relative error in per cent for the formulae (2.13,15,18) to calculate q_a. ($r_o/r_i = 1.5$, $\lambda_i/\lambda_g = 1/30$; $\beta = 12.16$)

$r_o/H =$	1.0	0.7	0.5	0.3	0.1	
(2.15)	3.91	1.20	0.91	0.87	0.90	r_o/D =1.0
(2.13)	0.51	0.47	0.58	0.76	0.89	
(2.18)	0.068	0.030	0.021	0.032	0.047	
	3.85	0.97	0.56	0.45	0.47	0.7
	0.47	0.24	0.24	0.34	0.46	
	0.015	0.0098	0.0070	0.011	0.020	
	3.95	0.96	0.44	0.25	0.24	0.5
	0.58	0.24	0.12	0.14	0.23	
	0.069	0.0048	0.0	0.0022	0.0060	
	4.13	1.06	0.46	0.15	0.084	0.3
	0.76	0.46	0.14	0.045	0.072	
	0.11	0.018	0.0028	0.0	0.0	

Table 2.4. The relative error in per cent for the formulae (2.13,15,18) to calculate q_a. ($r_o/r_i = 2.0$, $\lambda_i/\lambda_g = 1/30$; $\beta = 20.79$)

From Table 2.3 we see that for the typical problem above ($r_o/H = 0.5$, $r_o/D = 0.7$ and $r_o/r_i = 1.5$) the error of the traditional formula (2.15) is 0.91 % , the error of the zero-order multipole formula (2.13) is 0.37 %. and the error of the first-order multipole formula (2.18) is 0.013 %. Table 2.5 shows the error of (2.12-15,17,18) in the most critical case when the two pipes are in contact with the surface and each other, $H = r_o$ and $D = r_o$. The thermal resistance parameter β is varying.

	q_s			q_a		
	(2.14)	(2.12)	(2.17)	(2.15)	(2.13)	(2.18)
β =25.6	6.61	3.84	0.99	3.17	0.42	0.055
12.8	12.24	6.62	1.63	6.39	0.78	0.11
6.4	21.35	10.04	2.36	12.97	1.36	0.22
3.2	34.03	12.12	2.97	26.83	2.04	0.41
1.6	48.01	9.14	3.16	58.55	2.09	0.68
0.8	57.74	1.69	2.06	151	0.43	0.82
0.4	56.20	19.00	2.64	1183	8.13	0.097
0.2	40.40	38.10	13.05	392	21.60	4.05
0.1	14.99	54.94	28.20	212	38.08	13.04

Table 2.5. The relative error in per cent for the formulae (2.12-15,17,18) to calculate q_s and q_a. ($H = r_o$, $D = r_o$)

The errors of (2.17) and (2.18) in Table 2.5 become zero and change signs somewhere between $\beta = 0.8$ and $\beta = 0.4$. The errors of (2.12,13) become zero and change signs somewhere between $\beta = 1.6$ and $\beta = 0.8$. Table 2.6 shows the error of (2.12-15,17,18) when the two pipes are in contact with each other and the depth is twice the radius, $H = 2 \cdot r_o$ and $D = r_o$. The thermal resistance parameter β is varying.

	q_s			q_a		
	(2.14)	(2.12)	(2.17)	(2.15)	(2.13)	(2.18)
β =25.6	2.87	2.62	1.03	0.75	0.48	0.016
12.8	4.76	4.28	1.61	1.42	0.89	0.039
6.4	6.79	5.92	2.08	2.55	1.54	0.098
3.2	7.59	6.20	2.07	4.15	2.24	0.24
1.6	5.78	3.85	1.47	5.50	2.14	0.51
0.8	1.64	0.71	0.61	4.52	0.72	0.63
0.4	3.05	5.65	0.25	1.59	8.54	0.65
0.2	6.85	9.55	0.97	13.77	21.42	5.74
0.1	9.34	12.08	1.49	29.63	36.79	16.25
0.05	10.77	13.55	1.83	45.64	51.61	30.78

Table 2.6. The relative error in per cent for the formulae (2.12-15,17,18) to calculate q_s and q_a. ($H = 2 \cdot r_o$, $D = r_o$)

The errors of (2.14,17) and (2.15,18) in Table 2.6 become zero and change signs somewhere between $\beta = 0.8$ and $\beta = 0.4$. The errors of (2.12,13) become zero and change signs somewhere between $\beta = 1.6$ and $\beta = 0.8$. Table 2.7 shows the error of (2.12-15,17,18) when $H = 2 \cdot r_o$ and $D = 2 \cdot r_o$. The thermal resistance parameter β is varying.

	q_s			q_a		
	(2.14)	(2.12)	(2.17)	(2.15)	(2.13)	(2.18)
β =25.6	0.82	0.57	0.0074	0.36	0.10	0.0
12.8	1.47	0.99	0.018	0.69	0.18	0.0015
6.4	2.38	1.51	0.042	1.24	0.30	0.0039
3.2	3.25	1.80	0.085	2.05	0.38	0.0091
1.6	3.37	1.22	0.12	2.99	0.28	0.015
0.8	2.19	0.605	0.076	3.73	0.18	0.011
0.4	0.096	3.16	0.10	4.04	0.97	0.015
0.2	2.00	5.51	0.36	4.01	1.81	0.058
0.1	3.54	7.19	0.59	3.85	2.46	0.10
0.05	4.48	8.20	0.75	3.72	2.88	0.13

Table 2.7. The relative error in per cent for the formulae (2.12-15,17,18) to calculate q_s and q_a. ($H = 2 \cdot r_o$, $D = 2 \cdot r_o$)

The error of (2.14) in Table 2.7 becomes zero and changes sign somewhere between $\beta = 0.4$ and $\beta = 0.2$. The errors of (2.17,18) become zero and change signs somewhere between $\beta = 0.8$ and $\beta = 0.4$. The errors of (2.12,13) become zero and change signs somewhere between $\beta = 1.6$ and $\beta = 0.8$.

The relative error of the traditional formulae (2.14,15) for the heat loss (q_s) in the symmetrical problem is typically less that 5%. The error of the zero-order multipole formulae (2.12,13) for the heat loss is typically less than 3%. The error of the first-order multipole formulae for the heat loss is typically less than 0.1%. The error of the formulae for the heat loss (q_a) in the anti-symmetrical problem is typically half of the error in the symmetrical problem.

2.7 Asymptotic behaviour

For a large depth H, the first-order multipole formula (2.17) for the symmetrical problem, will have the following behaviour :

$$2\pi\lambda_g R_s \approx \ln\left(\frac{(2H)^2}{2r_oD}\right) + \beta - \frac{\left(\frac{r_o}{2D}\right)^2}{\frac{1+\beta}{1-\beta} + \left(\frac{r_o}{2D}\right)^2} \qquad (2.19)$$

$$H \to \infty$$

This formula is the same as formula (4.3) for the thermal resistance associated with the problem of two pipes with the same temperature inside a large pipe with the large radius $r_c = 2 \cdot H$. This problem is studied in section 4. Calculations made with the multipole program of [2] show that the heat loss from two pipes in the ground with the same temperature, approaches the heat loss from two pipes inside a large pipe with the radius $2 \cdot H$, when the depth increases. For a large depth H, the first-order multipole formula (2.18), for the anti-symmetrical problem, will have the following behaviour :

$$2\pi\lambda_g R_a \approx \ln\left(\frac{2D}{r_o}\right) + \beta - \frac{\left(\frac{r_o}{2D}\right)^2}{\frac{1+\beta}{1-\beta} - \left(\frac{r_o}{2D}\right)^2} \tag{2.20}$$

$$H \to \infty$$

This formula is the same as formula (3.8) for the thermal resistance for a pipe in the ground at the depth D. This problem is studied in section 3.

Calculations made with the multipole program of [2] show that the heat loss from one pipe in the anti-symmetrical problem of two pipes in the ground, approaches the heat loss from one pipe in the ground with the depth D , when the depth H increases.

2.8 Examples

Five examples are chosen from different manufactures of Swedish district heating pipes (Ecopipe, Ecoflex, [26]). Table 2.9 shows the errors of the traditional formulae and the multipole formulae for these five particular cases. The figures in Table 2.9 are calculated for two pipes in the ground with the following data :

$$T_1 = 90 \text{ °C}$$
$$T_2 = 55 \text{ °C}$$
$$T_0 = 8 \text{ °C}$$
$$\lambda_i = 0.04 \text{ W/mK}$$
$$\lambda_g = 1.5 \text{ W/mK}$$

The exact values of q_s are calculated with the multipole model program of [2]. The thermal insulation is replaced by a surface resistance, see Figure 2.2. The order of the multipoles has been 10. This means that the error for q_s is less than 0.001%.

$r_o(m)$	r_i(m)	H(m)	D(m)	q_s(W/m)	error (2.14)(%)	(2.12) (%)	(2.17)(%)
0.394	0.3048	0.994	0.50	49.48	2.37	2.00	0.27
0.344	0.2540	0.944	0.46	43.06	1.80	1.55	0.18
0.100	0.0450	0.600	0.20	18.06	0.25	0.23	0.01
0.080	0.0200	0.580	0.18	10.87	0.12	0.05	0.00
0.400	0.2500	0.900	0.50	30.17	1.77	1.50	0.20

Table 2.9. The error in per cent for q_s.

When the pipes are insulated is the error of the zero-order multipole formula always less than the error of the traditional formula. The error of the traditional method (2.14) is not so large but, since the multipole formula is only slightly more complicated, one can just as well use (2.17) to be on the safe side.

3 Heat loss from one pipe in the ground

There is one insulated pipe in the ground. There is an insulation between the radius r_i and r_o. The temperatures in the pipe and on the surface are known and constant. The steady state heat conduction equation (1.1) is to be solved. The problem is described in Figure 3.1. We have here, contrary to the previous case, included a heat transfer coefficient α_a (W/m^2K) between the air and the ground surface.

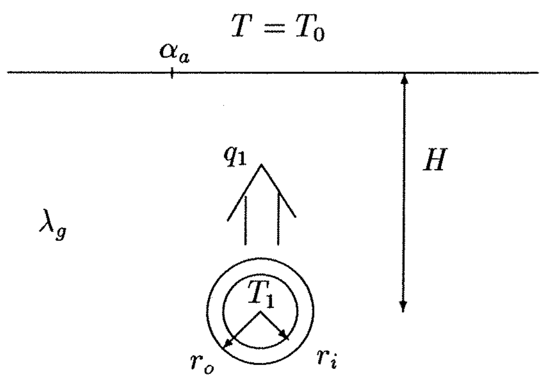

Figure 3.1. One pipe in the ground.

H = Depth between the ground surface and the center of the pipe (m)
r_o = Outer radius of the pipe (m)
r_i = Inner radius of the pipe (m)
λ_g = Thermal conductivity of the ground (W/mK)
λ_i = Thermal conductivity of the insulation (W/mK)
$1/\alpha_a$ = Surface resistance from ground to air (m^2K/W)
T_0 = Temperature of the surface (°C)
T_1 = Temperature in pipe 1 (°C)
q_1 = Heat loss from pipe 1 per meter (W/m)

The heat loss q_1 is proportional to the temperature difference $T_1 - T_0$. We may write:

$$q_1 = \frac{T_1 - T_0}{R_a} \tag{3.1}$$

Here is R_a (mk/W) the thermal resistance from the pipe to the ground surface. The subscript a denotes that the problem is anti-symmetrical with respect to the ground surface.

There are two often used approximations concerning a pipe in the ground. The first one is the replacement of a surface resistance from the ground to the air with an equivalent layer of soil. The second one is the replacement of the thermal insulation of the pipe with an equivalent surface resistance. The multipole program of [2] can solve the simpler problem of a surface resistance replacing the thermal insulation. The multipole program of [3] can however solve the real problem of a thermal insulation with a finite width. The program of [3] is used to calculate the error in this section 3.

3.1 Approximation of the surface resistance

The surface resistance from ground to air ($1/\alpha_a$) is approximated with an equivalent thin layer of soil (d) with the same thermal resistance to the air. We have:

$$d = \frac{\lambda_g}{\alpha_a} \tag{3.2}$$

$$H_{eff} = H + d$$

This effective depth (H_{eff}) is then used instead of H in all the formulae if the surface resistance should be accounted for.

In Appendix A it is proved that the error due to this approximation is small if the surface resistance is not too large. The error is typically 0.01 % or less.

3.2 Approximation of the insulation

The thermal insulation of the pipe is often replaced by a surface resistance. The problem is described in the complex plane in Figure 3.2.

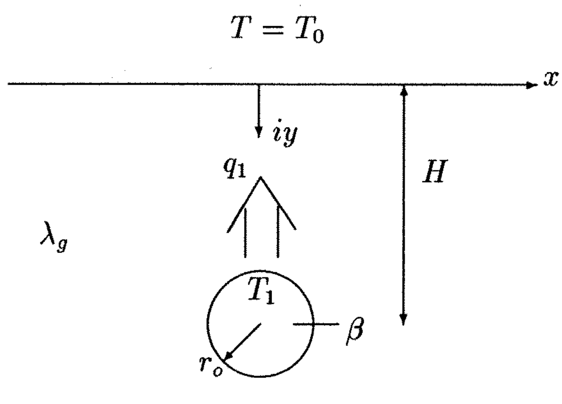

Figure 3.2. One pipe in the ground without insulation thickness.

The thermal surface resistance is described by the dimensionless thermal resistance parameter β from (2.2):

$$\beta = \frac{\lambda_g}{\lambda_i} \ln\left(\frac{r_o}{r_i}\right)$$

The boundary condition of the pipe becomes with the same polar coordinates as in (1.6):

$$z = x + iy \tag{3.3}$$

$$z - iH = \rho e^{i\psi}$$

$$T - \beta r_o \frac{\partial T}{\partial \rho} = T_1 \qquad \rho = r_o \qquad 0 \le \psi \le 2\pi \tag{3.4}$$

The error for the heat loss q_1, when the thermal insulation is replaced by a surface resistance, is shown in Table 3.1-2. This error is calculated with the multipole program of [3]. The order of the highest multipole has been 20, in which case the relative error in the heat loss is less than 10^{-7}.

$r_o/r_i = 1.5$		2.0	4.0	8.0
$\lambda_i/\lambda_g = 1.0$	0.49	0.99	1.80	2.04
0.5	0.34	0.56	0.81	0.85
0.1	0.050	0.059	0.061	0.056
0.05	0.014	0.017	0.017	0.013
0.03	0.0063	0.0066	0.0062	0.0056
0.01	0.00086	0.00077	0.00073	0.00067

Table 3.1. The relative error in per cent when the insulation approximation (3.4) is used. ($r_o/H = 0.8$).

The error of the insulation approximation arises from the fact that the flow in the tangential direction in the insulation is neglected. The approximation will therefore give a smaller heat loss than the real problem. If the ratio λ_i/λ_g is greater than ~ 0.1 the error will increase for increasing thickness of the insulation in Table 3.1. If the ratio is less than ~ 0.1 the error will decrease for increasing thickness of the insulation. The magnitude of the error is obviously influenced by counteracting effects.

r_o/H	error (%)
0.99	0.022
0.90	0.020
0.80	0.017
0.60	0.010
0.40	0.0053
0.20	0.0016
0.10	0.00068

Table 3.2. The relative error in per cent when the insulation approximation (3.4) is used. ($r_o/r_i = 4$, $\lambda_g/\lambda_i = 20$; $\beta = 27.72$).

The error of the insulation approximation is for most applications small. For a pipe in the ground with $\lambda_i/\lambda_g = 0.04/1.5 \approx 0.03$ and $r_o/H < 0.8$, the error is about 0.006 %.

3.3 Zero-order approximation

With the use of a line source and a mirror sink without multipoles, the thermal resistance becomes:

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta. \tag{3.5}$$

This formula is not derived in this report, but the derivation is identical to the derivation of the first-order multipole formulae (3.8) with the multipoles omitted.

Instead of the thermal resistance β one can use a small pipe ($r_o{}^*$) with an equivalent thermal resistance to earth :

$$r_o{}^* = r_o\left(\frac{r_i}{r_o}\right)^{\frac{\lambda_g}{\lambda_i}} = r_o e^{-\beta} \tag{3.6}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o{}^*}\right)$$

This is just another way of writing (3.5). The advantage of this is that it is now possible to use formulae for non-insulated pipes. This is done in [20] for more than one pipe in the ground.

3.4 Traditional method

The traditional method to calculate the thermal resistance is [9],[10] and [13]:

$$2\pi\lambda_g R_a = \ln\left(\frac{H}{r_o} + \sqrt{\left(\frac{H}{r_o}\right)^2 - 1}\right) + \beta \tag{3.7}$$

This is the exact solution if $\beta = 0$. It is a displacement of the line source from the center of the pipe. This displacement is independent of β, but when β increases the displacement should decrease as in (3.9). Therefore (3.5) is better when the pipe is insulated and (3.7) is better when the pipe is non-insulated.

3.5 First-order approximation

The new formula (3.8), for the thermal resistance between a pipe and the ground, is calculated with the use of the first-order multipoles :

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta + \frac{1}{1 - \left(\frac{2H}{r_o}\right)^2 \frac{1+\beta}{1-\beta}} \tag{3.8}$$

The derivation of (3.8) is made in Appendix B.3. The first two terms on the right side give the zero-order approximation (3.5). The third term is the first-order compensation.

With a displacement of the line source it is possible to calculate the temperature field for an insulated pipe up to the variation of the first order. With the use of this displacement δ of the line source the thermal resistance becomes :

$$2\pi\lambda_g R_a = \ln\left(\frac{2H+\delta}{r_o}\right) + \beta \tag{3.9}$$

$$\delta = H\left(\sqrt{1 - \frac{1-\beta}{1+\beta}\left(\frac{r_o}{H}\right)^2} - 1\right)$$

This formula is derived in Appendix B.4. Note that the displacement δ depends on the insulation. Formula (3.9) will become identical to the traditional formula (3.7) when β is set to zero. Formula (3.8) and (3.9) are both obtained by making the first-order variation become zero on the pipe. In formula (3.8) this is done by adjusting the strength of the first-order multipole, see (1.6). In formula (3.9) this is done with a displacement of the line source from the center of the pipe. Therefore, as the comparison below will confirm, formula (3.8) and (3.9) will have the same behaviour.

It is also possible to express formula (3.8) as a displacement of the line source:

$$2\pi\lambda_g R_a = \ln\left(\frac{2H+\delta^*}{r_o}\right) + \beta \tag{3.10}$$

$$\delta^* = 2H\left(e^{\left(1 - \frac{1+\beta}{1-\beta}\left(\frac{2H}{r_o}\right)^2\right)^{-1}} - 1\right)$$

In Table 3.3 the ratio between δ and δ^* is listed for different values of β.

$H/r_o = 1.0$		2.0	3.0
$\beta = 0$	1.764	1.039	1.015
0.5	1.056	1.011	1.005
2	0.967	0.990	0.996
4	0.951	0.983	0.992
8	0.943	0.979	0.990
16	0.939	0.977	0.989
32	0.937	0.975	0.988
64	0.936	0.975	0.988
128	0.936	0.974	0.987
10^6	0.935	0.974	0.987

Table 3.3. δ/δ^* for different values of β and H/r_o.

The ratio between R_a^{mult} of the multipole method and R_a^{disp} of the displacement method is :

$$\frac{R_a^{mult}}{R_a^{disp}} = \frac{\ln(\frac{2H}{r_o}) + \beta + \ln\left(1 + \frac{\delta^*}{2H}\right)}{\ln(\frac{2H}{r_o}) + \beta + \ln\left(1 + \frac{\delta}{2H}\right)} \tag{3.11}$$

In Table 3.4 the expression $100 \cdot (R_a^{mult}/R_a^{disp} - 1)$ is listed for different β and H/r_o. From Table 3.4 it is clear that the multipole method of the first order and the displacement of the line source are very similar. When the insulation is zero the displacement method will however give the exact solution.

$H/r_o = 1.0$		2.0	3.0
$\beta = 0$	∞	0.203	0.0250
0.5	0.486	0.0129	0.00197
2	0.0873	0.00579	0.00108
4	0.126	0.0110	0.0.00222
8	0.0977	0.0101	0.0.00215
16	0.0601	0.00683	0.00151
32	0.0333	0.00397	0.0
64	0.0175	0.00214	0.0
128	0.00897	0.00111	0.0

Table 3.4. $100 \cdot (R_a^{mult}/R_a^{disp} - 1)$ for different values of β and H/r_o.

3.6 Second-order approximation

With the use of multipoles up to the second order the thermal resistance becomes:

$$2\pi\lambda_g R_a = \ln(\frac{2H}{r_o}) + \beta + \tag{3.12}$$

$$\left[1 + \frac{1}{2}\frac{(1+\beta)}{(1-\beta)}\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2 - \frac{3(1-2\beta)}{2(1+2\beta)}(\frac{r_o}{2H})^4\right] \cdot$$

$$\left[1 - \left((\frac{2H}{r_o})^2 - 3\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2\right)\frac{(1+\beta)}{(1-\beta)} - \frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^4\right]^{-1}$$

This formula is derived in Appendix B.5. The calculations are longer than for the derivation of the first-order approximation (3.8), but not essentially more complex.

3.7 Errors of different methods

We have introduced four different formulae to calculate the heat loss from one pipe in the ground:

(3.7)	= traditional method
(3.5)	= zero-order multipole method
(3.8)	= first-order multipole method
(3.12)	= second-order multipole method

The errors of the formulae above, compared to the exact solution calculated with the multipole program of [3], are listed in Table 3.5. The problem is described in Figure 3.1, the surface resistance from ground to air is zero. If the error is less than 0.001% the error is set to zero.

	$r_o/H = 0.9$	0.7	0.5	0.3	0.1	
(3.7)	7.80	3.77	1.65	0.538	0.0500	r_o/r_i =1.2
(3.5)	2.10	1.26	0.622	0.210	0.0203	($\beta = 5.47$)
(3.8)	0.140	0.0644	0.0179	0.00184	0.0	
(3.12)	0.00151	0.0	0.00178	0.0	0.0	
	3.85	1.93	0.895	0.300	0.0304	1.5
	1.19	0.741	0.379	0.134	0.0139	(12.16)
	0.0603	0.0287	0.00768	0.0	0.0	
	0.00498	0.00423	0.00291	0.00116	0.0	
	2.32	1.19	0.560	0.191	0.0199	2.0
	0.751	0.473	0.245	0.0880	0.00938	(20.79)
	0.0310	0.0147	0.00329	0.0	0.0	
	0.00760	0.00538	0.00331	0.00129	0.0	
	1.18	0.614	0.293	0.101	0.0108	4.0
	0.393	0.250	0.131	0.0475	0.00518	(41.59)
	0.0113	0.00460	0.0	0.0	0.0	
	0.00809	0.00570	0.00334	0.00131	0.0	
	0.794	0.413	0.198	0.0686	0.00737	8.0
	0.265	0.169	0.0888	0.0324	0.00353	(83.18)
	0.00552	0.00170	0.0	0.0	0.0	
	0.00747	0.00523	0.00308	0.00115	0.0	

Table 3.5. The relative error in per cent for formulae (3.7,5,8,12).($\lambda_g/\lambda_i = 30$)

The errors will increase if the thermal conductivity of the insulation increases, as in Table 3.1.

The error of formula (3.8) will always be lower than 0.140 % if the ratio between the radius and the depth (r_o/H) is less than 0.9, if the ratio between the outer and inner radius is greater then 1.2 and if the ratio between the thermal conductivity of the ground and the insulation (λ_g/λ_i) is greater then 30. The error is for this case at least 50 times less than for the traditional formula (3.7).

The error of the traditional method to calculate the heat loss from one pipe in the ground is for district heating mains typically less than 1%. The error of the first-order multipole formula is typically less than 0.05% and the error of the second-order multipole method is typically less than 0.01%

4 Heat loss from two pipes to a large pipe

In section 2 it was stated that the heat loss from two pipes in the ground with the same temperature approaches the heat loss from two pipes inside a large pipe with the radius $r_c = 2 \cdot H$, when the depth increases. In this section 4 the heat loss from two pipes inside a large pipe to the large pipe is studied. The outer boundary condition is in this case that the temperature is constant on the large pipe instead of on the ground surface. The problem is thus that there are two insulated pipes, with the insulation parameter β, inside a large pipe, see section 2.1. The temperature is T_s in both pipes. It is T_c on the large outer pipe.

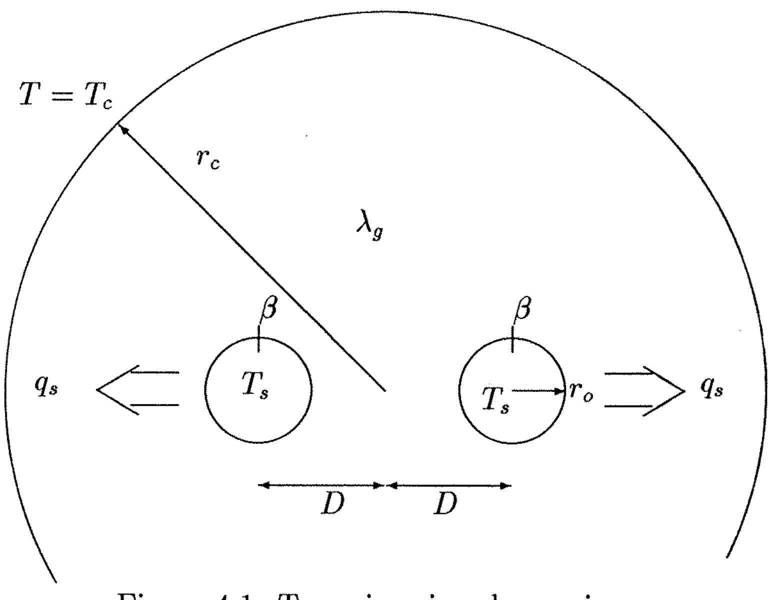

Figure 4.1. Two pipes in a large pipe.

The heat loss q_s is proportional to the temperature difference $T_s - T_c$. We may write:

$$q_s = \frac{T_s - T_c}{R_s} \tag{4.1}$$

Here is R_s (mK/W) the thermal resistance from one of the pipes to the large pipe.

4.1 Zero-order approximation

The zero-order approximation uses only the line source and a mirror sink. The zero-order approximation of the thermal resistance becomes :

$$2\pi\lambda_g R_s = \ln\left(\frac{{r_c}^2}{2Dr_o}\right) + \beta \tag{4.2}$$

4.2 First-order approximation

The thermal resistance becomes with use of multipoles of the first order:

$$2\pi\lambda_g R_s = \ln\left(\frac{{r_c}^2}{2Dr_o}\right) + \beta - \frac{1}{1+\left(\frac{2D}{r_o}\right)^2\frac{1+\beta}{1-\beta}} \tag{4.3}$$

The derivation of (4.3) is not included in this report. But the derivation is very similar to the derivation of (3.8), which is made in Appendix B.3. The temperatures and the multipoles are here symmetrical and not anti-symmetrical as in the problem of one pipe in the ground, (3.8). With a displacement of the line source it is possible to calculate the temperature on the insulated pipe up to the variation of the first order. With the use of this displacement δ of the line source the thermal resistance becomes :

$$2\pi\lambda_g R_s = \ln\left(\frac{{r_c}^2}{(2D+\delta)r_o}\right) + \beta \tag{4.4}$$

$$\delta = D\left(\sqrt{1+\frac{1-\beta}{1+\beta}\left(\frac{r_o}{D}\right)^2} - 1\right)$$

The derivation of (4.4) is not included in this report. But the derivation is very similar to the derivation of (3.9), which is made in Appendix B.4 Formula (4.3) and (4.4) are both obtained by making the first-order variation become zero on the pipe. In formula (4.3) this is done by adjusting the strength of the first-order multipole, see (1.6). In formula (4.4) this is done with a displacement of the line source from the center of the pipe. Therefore, as the comparison below will confirm, formula (4.3) and (4.4) will have the same behaviour.

It is possible to express formula (4.4) as a displacement of the line source :

$$2\pi\lambda_g R_s^* = \ln\left(\frac{{r_c}^2}{(2D+\delta^*)r_o}\right) + \beta \tag{4.5}$$

$$\delta^* = 2D\left(e^{\left(1+\frac{1+\beta}{1-\beta}\left(\frac{2D}{r_o}\right)^2\right)^{-1}} - 1\right)$$

In Table 4.1 the ration between δ and δ^* is listed for different values of β.

$D/r_o = 1.0$		2.0	3.0	4.0
$\beta = 0$	0.935	0.974	0.987	0.993
0.5	0.967	0.990	0.996	0.997
2	1.056	1.011	1.005	1.003
4	1.136	1.021	1.009	1.005
8	1.232	1.029	1.012	1.006
16	1.333	1.033	1.013	1.007
32	1.426	1.036	1.014	
64	1.506	1.037	1.015	
128	1.572	1.038	1.015	
10^4	1.739	1.039	1.015	
10^6	1.761	1.039	1.015	

Table 4.1. δ/δ^* for different values of β and D/r_o.

The ratio between R_s^{mult} of the multipole method and R_s^{disp} of the displacement method is:

$$\frac{R_s^{mult}}{R_s^{disp}} = \frac{\ln\left(\frac{r_c^2}{2Dr_o}\right) + \beta - \ln\left(1 + \frac{\delta^*}{2D}\right)}{\ln\left(\frac{r_c^2}{2Dr_o}\right) + \beta - \ln\left(1 + \frac{\delta}{2D}\right)} \tag{4.6}$$

In Table 4.2 the expression $100 \cdot (1 - R_s^{mult}/R_s^{disp})$ is listed for $r_c/r_o = 100$.

$D/r_o = 1.0$		2.0	3.0
$\beta =$	0.14	0.019	0.0046
0.5	0.027	0.0024	0.0052
2	0.050	0.0024	0.0047
4	0.21	0.0071	0.0013
8	0.39	0.0095	0.0017
16	0.46	0.0084	0.0014
32	0.41	0.0058	0.0
64	0.29	0.0035	0.0
128	0.18	0.0019	0.0

Table 4.2. $100 \cdot (1 - R_s^{mult}/R_s^{disp})$ for different values of β and D/r_o. ($r_c/r_o = 100$)

It is clear that the multipole method of the first order and the displacement of the line source give similar thermal resistance except when the pipes are in contact with each other.

4.3 Errors of different methods

The errors of formulae (4.2) and (4.3) are listed in Table 4.3. The radius of the large pipe is $r_c = 1000 \cdot r_o$. The exact solution is calculated with the multipole program of [2], with the use of multipoles up to order 10.

$r_o/D = 1.0$		0.9	0.7	0.5	0.1	
(4.2) (4.3)	1.60 0.0775	1.36 0.0660	0.891 0.0367	0.486 0.0126	0.0231 0.0	r_o/r_i =1.0 (β =0)
	0.122 0.122	0.0669 0.0669	0.0211 0.0211	0.00520 0.00517	0.0 0.0	1.0339 (1.0)
	1.83 0.762	1.06 0.239	0.515 0.0495	0.237 0.00936	0.00924 0.0	1.5 (12.16)
	1.533 0.661	0.864 0.197	0.413 0.0394	0.189 0.00742	0.00723 0.0	2.0 (20.79)
	1.0341 0.460	0.570 0.131	0.269 0.0254	0.122 0.00475	0.00480 0.0	4.0 (41.59)

Table 4.3. The relative error in per cent for formula (4.2,3).($\lambda_g/\lambda_i = 30$)

5 Summary

With the use of the multipole method, approximate formulae with successively increasing accuracy can be calculated for the heat flow to and between pipes. In this report new and better formulae have been derived for the heat loss from two identical pipes in the ground and the heat loss from one pipe in the ground.

The new and the traditional formulae for these heat losses have been studied with the multipole programs [2] and [3]. The errors of the formulae have been calculated for different values of the used parameters.

5.1 Heat loss from two pipes in the ground

There are two insulated pipes in the ground. The radius and insulation are identical for the two pipes. The insulation lies between the radii r_i and r_o. The temperatures in the pipes and at the ground surface are given and constant. The steady-state heat losses from the pipes are to be calculated. The problem is described in Figure 5.1.

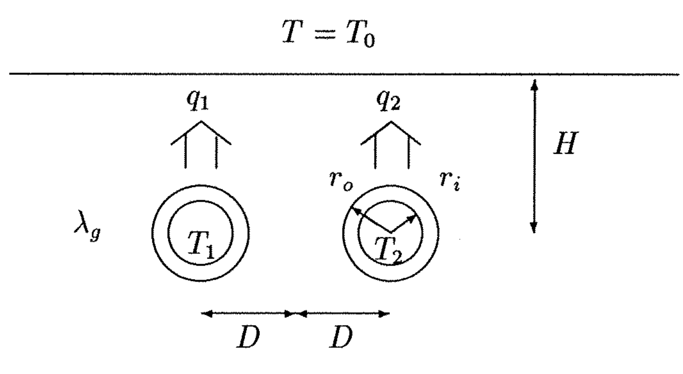

Figure 5.1. Two pipes in the ground.

H	= Depth from the ground surface to the center of the pipes (m)
D	= Half the distance between the center of the pipes (m)
r_o	= Outer radius of the pipe (m)
r_i	= Inner radius of the pipe (m)
λ_g	= Thermal conductivity of the ground (W/mK)

λ_i	=	Thermal conductivity of the insulation (W/mK)
T_0	=	Temperature on the ground surface (°C)
T_1	=	Temperature in pipe 1 (°C)
T_2	=	Temperature in pipe 2 (°C)
q_1	=	Heat loss from pipe 1 per meter (W/m)
q_2	=	Heat loss from pipe 2 per meter (W/m)

The problem is separated in a symmetrical and an anti-symmetrical part, see Figure 2.3. The symmetrical problem has the temperature T_s in both pipes and T_0 at the ground surface:

$$T_s = \frac{T_1 + T_2}{2} \tag{5.1}$$

The heat loss q_s from one of the pipes in the symmetrical problem is proportional to the temperature difference $T_s - T_0$. We may write :

$$q_s = \frac{T_s - T_0}{R_s} \tag{5.2}$$

Here R_s (mK/W) is the thermal resistance associated with the symmetrical problem.

The anti-symmetrical problem has the temperature T_a and $-T_a$ in the two pipes and zero at the ground surface:

$$T_a = \frac{T_1 - T_2}{2} \tag{5.3}$$

The heat loss q_a from the pipe with $T = T_a$ is proportional to T_a. We may write :

$$q_a = \frac{T_a}{R_a} \tag{5.4}$$

Here R_a (mK/W) is the thermal resistance associated with the anti-symmetrical problem. The original problem is the sum of the symmetrical and anti-symmetrical problems. We have in accordance with Figure 2.3 :

$$q_1 = q_s + q_a \tag{5.5}$$

$$q_2 = q_s - q_a \tag{5.6}$$

The zero-order multipole formulae for the thermal resistances for the symmetrical and anti-symmetrical problems are:

$$2\pi\lambda_g R_s = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{5.7}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1 + \left(\frac{H}{D}\right)^2}\right) \tag{5.8}$$

Here β is the dimensionless thermal resistance parameter from (2.2) :

$$\beta = \frac{\lambda_g}{\lambda_i} \ln\left(\frac{r_o}{r_i}\right) \tag{5.9}$$

The first-order multipole formulae for the thermal resistances for the symmetrical and anti-symmetrical problem are :

$$2\pi\lambda_g R_s = \ln\left(\frac{2H}{r_o}\right) + \beta + \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 + \frac{r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} + \left(\frac{r_o}{2D}\right)^2} \tag{5.10}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta - \ln\left(\sqrt{1+\left(\frac{H}{D}\right)^2}\right) - \frac{\left(\frac{r_o}{2D}\right)^2 + \left(\frac{r_o}{2H}\right)^2 - \frac{3r_o^2}{4(D^2+H^2)}}{\frac{1+\beta}{1-\beta} - \left(\frac{r_o}{2D}\right)^2} \tag{5.11}$$

The relative errors in the heat loss, when the traditional formulae are used, are typically less than 5 %. The errors, when the zero-order formulae are used, are typically less than 3 %. The errors, when the first-order formulae are used, are typically less than 0.5 %.

5.2 Heat loss from one pipe in the ground

There is one insulated pipe in the ground. There is an insulation between the radii r_i and r_o. The temperatures in the pipe and on the surface are given and constant. The steady-state heat loss from the pipe is to be calculated. The problem is described in Figure 5.2.

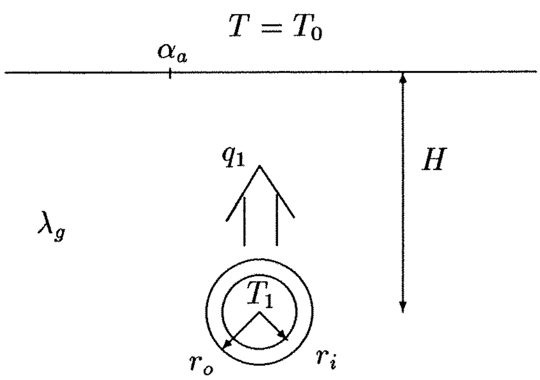

Figure 5.2. One pipe in the ground.

H	= Depth between the ground surface and the center of the pipe (m)
r_o	= Outer radius of the pipe (m)
r_i	= Inner radius of the pipe (m)
λ_g	= Thermal conductivity of the ground (W/mK)
λ_i	= Thermal conductivity of the insulation (W/mK)
$1/\alpha_a$	= Surface resistance from ground to air (m^2K/W)

T_0 = Temperature of the surface ($°C$)
T_1 = Temperature in pipe 1 (°C)
q_1 = Heat loss from pipe 1 per meter (W/m)

The heat loss q_1 is proportional to the temperature difference $T_1 - T_0$. We may write :

$$q_1 = \frac{T_1 - T_0}{R_a} \tag{5.12}$$

Here is R_a (mK/W) the thermal resistance from the pipe to the ground surface. The zero-order multipole formula for the thermal resistance is :

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta \tag{5.13}$$

The first-order multipole formula for the thermal resistance is :

$$2\pi\lambda_g R_a = \ln\left(\frac{2H}{r_o}\right) + \beta + \frac{1}{1 - (\frac{2H}{r_o})^2 \frac{1+\beta}{1-\beta}} \tag{5.14}$$

The second-order multipole formula for the thermal resistance is :

$$2\pi\lambda_g R_a = \ln(\frac{2H}{r_o}) + \beta + \tag{5.15}$$

$$\left[1 + \frac{1}{2}\frac{(1+\beta)}{(1-\beta)}\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2 - \frac{3(1-2\beta)}{2(1+2\beta)}(\frac{r_o}{2H})^4\right] \cdot$$

$$\left[1 - \left((\frac{2H}{r_o})^2 - 3\frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^2\right)\frac{(1+\beta)}{(1-\beta)} - \frac{(1-2\beta)}{(1+2\beta)}(\frac{r_o}{2H})^4\right]^{-1}$$

The relative error in the heat loss, when the traditional formula is used, is typically less than 1 %. The error, when the zero-order formula is used, is typically less than 0.5 %. The error, when the first-order formula is used, is typically less than 0.05 %. The error, when the second-order formula is used, is typically less than 0.01 %.

5.3 Approximation of the insulation

The thermal insulation in Figure 5.2 is often replaced by a surface resistance described by the dimensionless thermal resistance parameter β, see (5.9). The error of this approximation is typically 0.006 % for the ratio $\lambda_i/\lambda_g \approx 0.03$.

5.4 Approximation of the surface resistance

A surface resistance $1/\alpha_a$ is introduced between the ground surface and the air. This resistance may be approximated with an equivalent layer of soil :

$$d = \frac{\lambda_g}{\alpha_a} \tag{5.16}$$

$$H_{eff} = H + d \tag{5.17}$$

The depth H_{eff} is used instead of H in the formulae. The error in the temperature field, when this approximation is used, is typically less than 0.01%.

References

1. J.Claesson and J.Bennet, Multipole method to compute the conductive heat flows to and between pipes in a cylinder, Notes on Heat Transfer 2-1987, Dept. of Building Technology and Mathematical Physics, University of Lund, Box 118,S-221 00 Lund, Sweden, 1987.

2. J.Arfvidsson, Multipole method to compute the conductive heat flows to and between pipes in a cylinder. PC-version of computer model, Notes on Heat Transfer 2.1-1987, Dept. of Building Technology and Mathematical Physics, University of Lund, Box 118,S-221 00 Lund, Sweden, 1987.

3. Myrehed M, Multipole method to compute the conductive heat flows to and between pipes in a composite region: pipes in a circle in a half plane, Notes on Heat Transfer 1-1989, Dept. of Building Technology and Mathematical Physics, Lunds Institute of Technology, Sweden.

4. * Foster G.C and Lodge O.J, On the flow of electricity in a uniform plane conducting surface, Phil Mag 49(1875), p385-400, p453-471, 50(1875) , p475-489.

5. * Forchheimer ,Über die Erwärmung des Wassers in Leitungen, Zeitschrift des Architekten- ud Ingenieurvereins zu Hannover, 34(1888), p175-192.

6. * Krischer O, Wärmeverluste von Erdleitungen und Auskülungsverluste von Rohrleitungen, Die Wärme, 53(1930), p759-762.

7. * Krischer O, Die Berechnung der Wärmeverluste im Erdreich verlegten Rohrleitungen, Z. Wärme- und Kältetechnik 38(1936) Heft 6 p 7ff.

8. Carslaw and Jaeger, Conduction of heat in solids, Oxford University Press (1946) 1959.

9. Jakob M, Heat Transfer Vol 1, p 322-342, John Wiley New York, 1949.

10. Louden A.G, Heat loss from underground heating mains, IHVE Journal (1957) p 196-203.

11. Vidal J, Etude par analogie électrique des pertes calorifiques dans les canalisations de transport d'enthalpie. L'Universite de Paris, Dissertation 1961

12. Brauer H, Berechnung der Wärmeverluste von im Erdreich verlegten Rohrleitungen, Energie, Jahrg 15 Nr 9,September 1963.

13. Kutateladze S.S, Fundamentals of heat transfer, Edward Arnold Ltd (1963) London.

14. Elgeti K, Der Wärmeverlust einer erdverlegten Rohrleitung im stationären Zustand unter dem Einflußder Wärmeübergangzahl an der Erdoberfläche, Forsch. ing -ws Bd 33 (1967) Nr 4, p101-132.

15. Schwaigerer S, Rohrleitungen, Springer Verlag 1967.

16. Bosselman W, Temperaturfelder um Wärmequellen im Erdreich, Energie ,Jahrg 20,Nr 1,Januari 1968.

17. Franz G and Grigull U, Wärmeverluste von beheizten Rohrleitungen im Erdboden, Wärme- und Stoffübertragung 2(1969),p109-117.

18. Claesson J, The temperature around insulated pipes in the ground, Lund University, Dept. of Building physics, 1970:2, Sweden. (in Swedish)

19. Homonnay G and Hoffman A, Wärmeverluste von Fernheiznetzen bei periodischen Temperaturänderungen, HLH 22(1971):12, p376-381.

20. Jenowski M, Die neue Berechnungsmethode der Wärmeverluste in den Fernheizleitungen, 2:nd Int. Distr. Heat. Conv, Budapest 1973, sess 4.

21. Merker G and Schällig R, Berechnung des Temperaturfeldes un eine im Erdboden verlegte Punkt- und Liniequelle bei Randbedingung III. Art an der Erdoberfläche, Wärme- und Stoffübertragung 10(1977), p145-152.

22. Brakelmann H, Wärmewiderstand von Kabelgräben bei partieller Boden austrocknung-Physicalisches Modell und Rechenverfahren, Elekrizitätswirtschaft Jg 79(1980), Heft 8.

23. B.Kvisgaard and S.Hadvig, Heat loss from Pipelines in District Heating Systems, Laboratory of Heating and Air Conditioning, The Technical University of Denmark, Denmark, 1980.

24. M.Zeitler, Berechnungsverfahren zur Bestimmung des Wärmeverlustes von vershiedenen Verlegesystemen erdverlegter Rohrleitungen, Fernwärme international-FWI,Jg.9(1980),p170-179.

25. Lunardini V.J, Phase change around insulated buried pipes. Quasi-steady method, Journal of Energy Resources Technology, sept 1981,Vol 103,p201-207.

26. Werner S, Heat losses from district heating nets, Rapport A82-109,Ins of Energy Technology, Chalmers Institute of Technology (1982), Göteborg, Sweden. (in Swedish)

27. J.Claesson and A.Dunand, Heat extraction from the ground by horizontal pipes, Swedish Council for Building Research ,D1:1983 ,1983.

28. Homonnay G et al, Modellierung der Wärmeverluste untererdischer Fernleitungen, 22:nd Unichal-Congress (1985), Technische Universität Budapest.

29. Schneider G.E, An investigation into heat loss characteristics of buried pipes, Journal of Heat Transfer, Transactions of ASME, vol 107, August 1985, p 696-699.

30. Weinspach P.M, Vergleich unterschiedlicher Verlegesysteme für Fernwärme- , Universität Dortmund, Abteilung Chemietechnik, Lehrstuhl für Thermische Verfahrenstechnik, Februar 1987.

31. Bøhm B, Energy-economy of Danish district heating systems, Laboratory of Heating and Air conditioning. Technical University of Denmark, 1988.

32. Hansen J.C, Control of finite element results, 1988. (in Danish)

33. Abramowitz and Stegun, Handbook of Mathematical Functions, AMS 55, National Bureau of Standards, 1970.

I have not read the references with an asterisk *. They are described in other references.

Appendix A
Approximation of the surface resistance

We will here calculate the error made when a thin layer of soil is used instead of a surface resistance $1/\alpha_a$ (m^2K/W) between the ground and the air.

A.1 Surface resistance

Consider a non-insulated pipe in the ground. The thermal conductivity between the ground and the air is α_a. The heat loss from the pipe is q_1, see Figure A.1. With this known heat loss we will calculate the temperature on the pipe T_1.

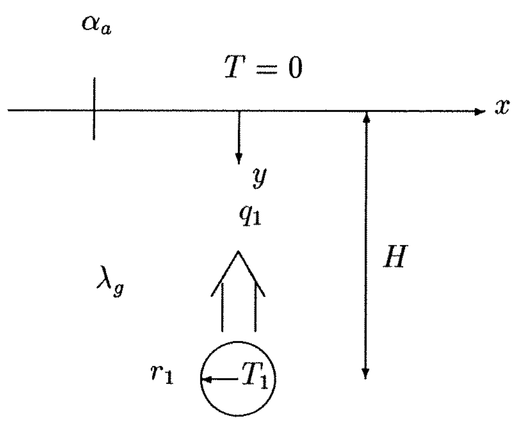

Figure A.1. Pipe in ground with surface resistance.

H	= Depth between the ground surface and the center of the pipe (m)
r_1	= Radius of the pipe (m)
λ_g	= Thermal conductivity of the ground (W/mK)
$1/\alpha_a$	= Surface resistance from ground to air (m^2K/W)
T_1	= Temperature on the pipe (°C)

q_1 = Heat loss from the pipe per meter (W/m)

The steady-state heat conduction equation for the temperature $T(x, y)$ is to be solved :

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{A.1}$$

The boundary condition at the pipe is:

$$T(\rho = r_1) = T_1 \qquad \rho = \sqrt{x^2 + (y - H)^2} \tag{A.2}$$

The boundary condition at the surface is :

$$T - \frac{\lambda_g}{\alpha_a}\frac{\partial T}{\partial y} = 0 \qquad y = 0 \tag{A.3}$$

The temperature at a non-insulated pipe with the radius r_1 with a surface resistance $(1/\alpha_a)$ at the ground is according to [27] :

$$T_1 = \frac{q_1}{2\pi\lambda_g}\left[\ln(\frac{2H}{r_1}) + 2e^{\frac{2H\alpha_a}{\lambda_g}} \cdot E_1(\frac{2H\alpha_a}{\lambda_g})\right] \tag{A.4}$$

With the assumption :

$$r_1 \ll H$$

Where $E_1(x)$ is the exponent integral :

$$E_1(x) = \int_x^\infty \frac{1}{x} e^{-x} dx \tag{A.5}$$

A.2 Thin layer of soil

Consider the same problem as in Figure A.1, but instead of the surface resistance there is a thin layer of soil, see Figure A.2.

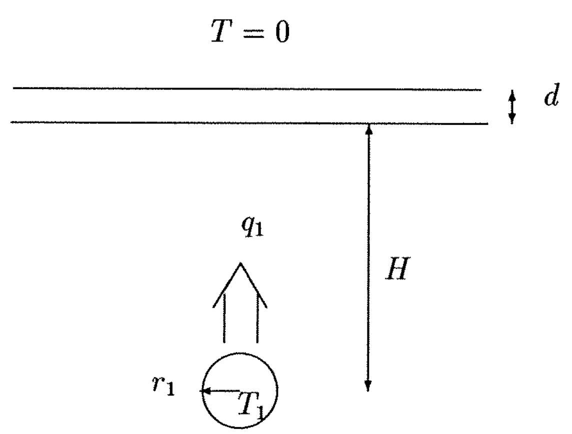

Figure A.2. Pipe in ground with total depth = $H + d$.

The thickness (d) of the layer is chosen so that the thermal resistance is equivalent with the surface resistance $(1/\alpha_a)$:

$$d = \frac{\lambda_g}{\alpha_a} \tag{A.6}$$

The temperature at the pipe becomes :

$$T_1{}^* = \frac{q_1}{2\pi\lambda_g}\ln(\frac{2(H+d)}{r_1}) = \frac{q_1}{2\pi\lambda_g}\left[\ln(\frac{2H}{r_1}) + \ln(1+\frac{\lambda_g}{H\alpha_a})\right] \tag{A.7}$$

With the assumption :

$$r_1 \ll H$$

A.3 Error estimation

The ratio between the temperature with a thin layer of soil and the temperature with a surface resistance is $T_1{}^*/T_1$.

$$K_1 = \ln(\frac{2H}{r_1}) \tag{A.8}$$

The dimensionless length parameter $\hat{d}$ will be used :

$$\hat{d} = \frac{H\alpha_a}{\lambda_g} = \frac{H}{d} \tag{A.9}$$

$$\frac{T_1{}^*}{T_1} = \left[\frac{K_1}{2e^{2\hat{d}}E_1(2\hat{d})} + \frac{\ln(1+\hat{d}^{-1})}{2e^{2\hat{d}}E_1(2\hat{d})}\right]\left[\frac{K_1}{2e^{2\hat{d}}E_1(2\hat{d})} + 1\right]^{-1} \tag{A.10}$$

With a dimensionless parameter η the ratio becomes :

$$\eta = \frac{\ln(1+\hat{d}^{-1})}{2e^{2\hat{d}}E_1(2\hat{d})}$$

$$\frac{T_1{}^*}{T_1} = \left[\frac{R_m}{2e^{2\hat{d}}E_1(2\hat{d})} - \eta\right]\left[\frac{R_m}{2e^{2\hat{d}}E_1(2\hat{d})} - 1\right]^{-1} \tag{A.11}$$

For large $\hat{d}$ the following approximations are true [33]:

$$\ln(1+\hat{d}^{-1}) \approx \frac{1}{1+\hat{d}} \tag{A.12}$$

$$2e^{2\hat{d}}E_1(2\hat{d}) \approx 2e^{2\hat{d}} \cdot \frac{e^{-2\hat{d}}}{2\hat{d}+1} = \frac{2}{2\hat{d}+1} \tag{A.13}$$

With (A.14) the parameter η becomes :

$$\eta = \frac{\ln(1+\hat{d}^{-1})}{2e^{2\hat{d}}E_1(2\hat{d})} \rightarrow \frac{2x+1}{2x+2} \rightarrow 1 \qquad \text{when} \qquad \hat{d} \rightarrow \infty \tag{A.14}$$

This means that:

$$\frac{T_1{}^*}{T_1} \to 1 \qquad \text{when} \qquad \frac{H\alpha_a}{\lambda_g} \to \infty \tag{A.15}$$

Table A.1 and A.2 shows $100 \cdot (T_1 - T_1{}^*)/T_1$ and $100 \cdot (1 - \eta)$ for different values of $\hat{d}$.

$\hat{d}$	5.0	10.0	15.0	20.0	25.0
$100 \cdot (1 - \eta)$	0.4396	0.1333	0.0635	0.0370	0.0242
$100 \cdot (T_1 - T_1{}^*)/T_1$	0.0820	0.0142	0.0047	0.0021	0.0011

$r_1/H = 0.9$

Table A.1. $100 \cdot (T_1 - T_1{}^*)/T_1$ and $100 \cdot (1 - \eta)$

$\hat{d}$	5.0	10.0	15.0	20.0	25.0
$100 \cdot (1 - \eta)$	0.4396	0.1333	0.0635	0.0370	0.0242
$100 \cdot (T_1 - T_1{}^*)/T_1$	0.0513	0.0086	0.0028	0.0013	0.0

$r_1/H = 0.5$

Table A.2. $100 \cdot (T_1 - T_1{}^*)/T_1$ and $100 \cdot (1 - \eta)$

According to [23] a usual value of α_a is 14.6(W/m^{2}K). A typical value of the thermal conductivity of the ground (λ_g) is 1.5(W/mK). This means that $\hat{d}$ is about $H \cdot 9.73$, the error in the temperature on the pipe is then approximately 0.01 %. Tables A.1-2 show that the error decreases fast if the depth (H) increases.

Appendix B
Derivation of the multipole formulae

The derivations of the different multipole formulae are very similar, and therefore only the derivation of (3.8),(3.9),(3.12) and (2.18) will be described here.

B.1 The multipole problem

Consider the problem in Figure B.1. There are N pipes with different radii r_n and surface resistance β_n inside a large pipe with the radius r_c and surface resistance β_c. The surface resistances are described by the dimensionless thermal resistance parameters β_n and β_c, see (2.1-2). The temperature at the large pipe is T_c. The temperatures in the pipes inside the large pipe are T_n.

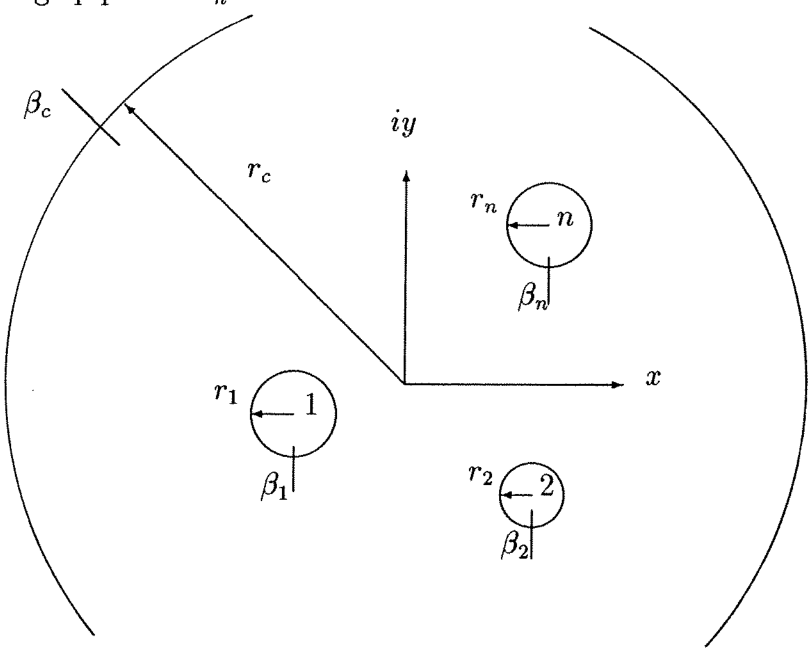

Figure B.1. N pipes in a large pipe in the complex plane.

The stationary heat conduction equation for the temperature $T(x, y)$ is to be solved.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{B.1}$$

The problem is solved in the complex plane ($z = x + iy$). Here the imaginary unit is denoted $i = \sqrt{-1}$. Polar coordinates will be used:

$$z = x + iy = r \cdot e^{i\phi}$$

The center of the pipe n is z_n.

$$z_n = x_n + i \cdot y_n$$

We will also use the local polar coordinates ρ_n,ψ_n from the center of any pipe n.

$$z - z_n = \rho_n e^{i\psi_n} \tag{B.2}$$

The boundary condition at each pipe n is:

$$T - \beta_n r_n \frac{\partial T}{\partial \rho_n} = T_n \qquad \rho_n = r_n, \qquad 0 \leq \psi_n \leq 2\pi \tag{B.3}$$

Inside each pipe n there is a line source with the strength q_n, and multipoles of strength $P_{n,j}$. The temperature field consists of these two parts and a constant temperature level T_0.

$$T(x,y) = T_0 + T_q(x,y) + T_p(x,y) \tag{B.4}$$

$$T_q(x,y) = \Re\left[\sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_g} \ln\left(\frac{r_c}{z - z_n}\right)\right]$$

$$T_p(x,y) = \Re\left[\sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{n,j}\left(\frac{r_n}{z - z_n}\right)^j\right] + \Re\left[\sum_{j=1}^{\infty} P_{c,j}\left(\frac{z}{r_c}\right)^j\right]$$

The last terms are multipoles at infinity with the strengths $P_{c,j}$. For a more detailed discussion of the multipoles see [1].

The temperature field satisfies the heat conduction equation because it is the real part of a sum of analytical functions. The quantities T_0,q_n,$P_{n,j}$ and $P_{c,j}$ are determined by the boundary conditions. The expression (B.4) is inserted in the boundary condition of each pipe.

The line source q_n and the multipoles $P_{n,j}$ of pipe n can represent any solution of the heat conduction equation in the region outside pipe n.

To solve the boundary condition problem of pipe m we must express the line sources of the pipes and the multipoles in the local polar coordinates of pipe m. From [1] we get these expressions. The number of the pipe with line source or multipole is n and the number of the pipe whose boundary condition is to be satisfied is m.

$\boxed{n = m}$

$$\Re\left[\ln\left(\frac{r_c}{z - z_m}\right)\right] = \ln\left(\frac{r_c}{\rho_m}\right) \tag{B.5}$$

$$\Re\left[P_{m,j}\left(\frac{r_m}{z - z_m}\right)^j\right] = \Re\left[P_{m,j}\left(\frac{r_m}{\rho_m}\right)^j e^{-i \cdot j\psi_m}\right] \tag{B.6}$$

$\boxed{n \neq m}$

$$\Re\left[\ln\left(\frac{r_c}{z - z_n}\right)\right] = \Re\left[\ln\left(\frac{r_c}{z_m - z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m}\right] \tag{B.7}$$

$$\Re\left[P_{n,j}\left(\frac{r_n}{z - z_n}\right)^j\right] = \tag{B.8}$$

$$\Re\left[P_{n,j}\left(\frac{r_n}{z_m - z_n}\right)^j \sum_{k=0}^{\infty}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m}\right]$$

$$\Re\left[P_{c,j}\left(\frac{z}{r_c}\right)^j\right] = \Re\left[P_{c,j}\sum_{k=0}^{j}\begin{pmatrix} j \\ k \end{pmatrix}\frac{{\rho_m}^k \cdot {z_m}^{j-k}}{{r_c}^j} \cdot e^{ik\psi_m}\right] \tag{B.9}$$

These expressions are put in the boundary condition (B.3). For the boundary condition of pipe m we get:

$$T_m = T_0 + \frac{q_m}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_m}\right) + \beta_m\right) \tag{B.10}$$

$$+ \sum_{n\neq m}\frac{q_n}{2\pi\lambda_g}\Re\left[\ln\left(\frac{r_c}{z_m - z_n}\right) + \sum_{k=1}^{\infty}(1 - \beta_m k)\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k \cdot e^{ik\psi_m}\right]$$

$$+ \sum_{j=1}^{\infty}\Re\left[(1 + \beta_m j)P_{m,j}e^{-ij\psi_m}\right]$$

$$+ \sum_{n\neq m}\sum_{j=1}^{\infty}\Re\left[P_{n,j}\left(\frac{r_n}{z_m - z_n}\right)^j\left\{1 + \sum_{k=1}^{\infty}(1 - \beta_m k)\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{r_m}{z_n - z_m}\right)^k e^{ik\psi_m}\right\}\right]$$

$$+ \sum_{j=1}^{\infty}\Re\left[P_{c,j}\left(\frac{z_m}{r_c}\right)^j + P_{c,j}\sum_{k=1}^{j}(1 - \beta_m k)\begin{pmatrix} j \\ k \end{pmatrix}\frac{{r_m}^k {z_m}^{j-k}}{{r_c}^j}e^{ik\psi_m}\right]$$

$$0 \leq \psi_m \leq 2\pi$$

The constant part of (B.10), independent of ψ_m is:

$$T_m - T_0 = \frac{q_m}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_m}\right) + \beta_m\right) + \sum_{n\neq m}\Re\left[\frac{q_n}{2\pi\lambda_g}\ln\left(\frac{r_c}{z_m - z_n}\right)\right. \tag{B.11}$$

$$\left. + \sum_{j=1}^{\infty}P_{n,j}\left(\frac{r_n}{z_m - z_n}\right)^j\right] + \Re\left[\sum_{j=1}^{\infty}P_{c,j}\left(\frac{z_m}{r_c}\right)^j\right]$$

$$m = 1, 2 \ldots N$$

In the part of (B.10) depending on ψ_m, the summations are made over exponents with both positive and negative signs. But only the real part of these expressions are used. Therefore it is possible to complex conjugate the terms in order to get the same summation factors:

$$\Re\left[P_{m,j}e^{ik\psi_m}\right] = \Re\left[\overline{P_{m,j}}e^{-ik\psi_m}\right]$$

Equation (B.10), omitting the constant part (B.11), has now the following form:

$$0 = \Re\left[\sum_{k=1}^{\infty} Z_k \cdot e^{-ik\psi_m}\right] = \sum_{k=1}^{\infty}\left\{\Re\left[Z_k\right]\cos(k\psi_m) + \Im\left[Z_k\right]\sin(k\psi_m)\right\} \tag{B.12}$$

$$0 \leq \psi_m \leq 2\pi$$

Here Z_k is a complex number. This equation should be valid for every ψ_m. This means that all the complex numbers Z_k must be zero. We have for any k:

$$0 = (1+\beta_m k)\overline{P_{m,k}} + (1-\beta_m k)\cdot\left\{\sum_{n\neq m}\frac{q_n}{2\pi\lambda_g}\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k\right. \tag{B.13}$$

$$+\sum_{n\neq m}\sum_{j=1}^{\infty} P_{n,j}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{r_n}{z_m - z_n}\right)^j\left(\frac{r_m}{z_n - z_m}\right)^k$$

$$\left.+\sum_{k=k}^{\infty} P_{c,k}\begin{pmatrix} j \\ k \end{pmatrix}\frac{{r_m}^k {z_m}^{j-k}}{{r_c}^k}\right\}$$

$$m = 1, 2\ldots, N \qquad k = 1, 2, \ldots$$

There is also a set of equations for the outer pipe (r_c). The result from the ψ_m independent part is:

$$T_0 = T_c + \sum_{n=1}^{N}\frac{q_n}{2\pi\lambda_g}\beta_c \tag{B.14}$$

The part depending on ψ_m is listed in (B.17).

B.2 Final equation system

The equation systems (B.11,13,14) and the equation system for the outer pipe must all be truncated. We consider multipoles at the pipes and at infinity up to order J. Here is J a positive integer or in the lowest approximation zero, in which case only the line sources are used.

The sine- and cosine-variation around the pipes and the outer circle can be made zero up to order J only.

We get the following equation system:

$$\boxed{m = 1, \ldots, N}$$

$$T_m - T_c = \sum_{n=1}^{N} q_n \cdot R^{\circ}_{mn} + \Re\left[\sum_{n\neq m}\sum_{j=1}^{J} P_{n,j}\left(\frac{r_n}{z_m - z_n}\right)^j + \sum_{j=1}^{J} P_{c,j}\left(\frac{z_m}{r_c}\right)^j\right] \tag{B.15}$$

$$\boxed{m = 1, \ldots, N \qquad k = 1, \ldots, J}$$

$$0 = \overline{P_{mk}} + \frac{1 - \beta_m k}{1 + \beta_m k} \cdot \left\{ \sum_{n \neq m} \frac{q_n}{2\pi\lambda_g} \frac{1}{k} \left(\frac{r_m}{z_n - z_m} \right)^k \right. \tag{B.16}$$

$$\left. + \sum_{n \neq m} \sum_{j=1}^{J} P_{n,j} \begin{pmatrix} k + j - 1 \\ j - 1 \end{pmatrix} \left(\frac{r_n}{z_m - z_n} \right)^j \left(\frac{r_m}{z_n - z_m} \right)^k + \sum_{j=k}^{J} P_{c,k} \begin{pmatrix} j \\ k \end{pmatrix} \frac{{r_m}^k {z_m}^{j-k}}{{r_c}^j} \right\}$$

$$\boxed{k = 1, \ldots J}$$

$$0 = \overline{P_{ck}} + \frac{1 - \beta_c k}{1 + \beta_c k} \cdot \left\{ \sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_g} \frac{1}{k} \left(\frac{z_n}{r_c} \right)^k \right. \tag{B.17}$$

$$\left. + \sum_{n=1}^{N} \sum_{j=1}^{k} P_{n,j} \begin{pmatrix} k - 1 \\ j - 1 \end{pmatrix} \frac{{r_n}^j {z_n}^{k-j}}{{r_c}^k} \right\}$$

The thermal resistances R°_{mn} (K/(W/m)) in (B.15) are given by:

$$R^{\circ}_{mm} = \frac{1}{2\pi\lambda_g} \left(\ln \left(\frac{r_c}{r_m} \right) + \beta_m + \beta_c \right) \tag{B.18}$$

$$R^{\circ}_{mn} = \frac{1}{2\pi\lambda_g} \left(\ln \left(\frac{r_c}{|z_m - z_n|} \right) + \beta_c \right) \tag{B.19}$$

These are the equations that completely determine the coefficients of the multipoles and the line sources.

B.3 Derivation of (3.8)

We will here derive the multipole formula of the first-order for the heat loss from one pipe in the ground. The problem of one pipe in the ground is equivalent to the anti-symmetrical problem of two pipes in a large pipe. This problem is described in Figure B.2. Consider two pipes in a large pipe. The temperatures in the pipes have the same magnitude but opposite signs. The heat losses from the pipes are to be determined.

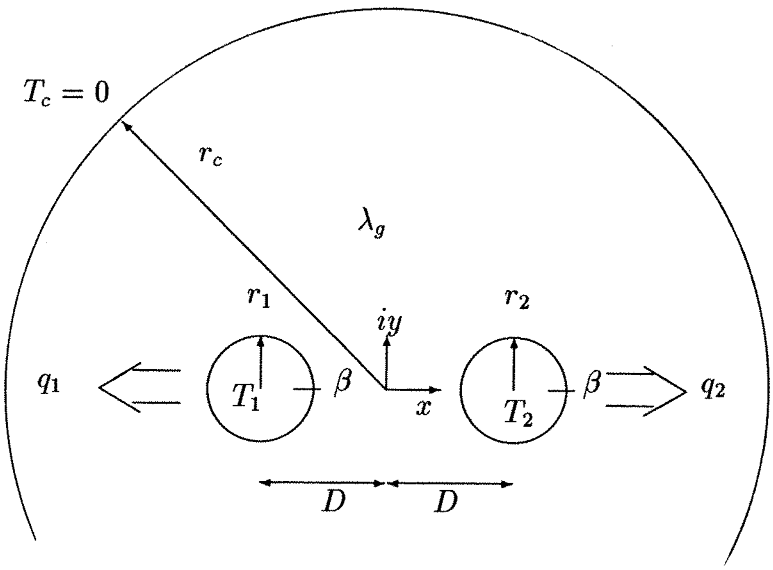

Figure B.2. Two pipes in a large circle.

The positions of the pipes are

$$z_1 = -D \qquad z_2 = D \qquad \text{(m)} \tag{B.20}$$

The pipes have the same radii and are equally insulated.

$$r_1 = r_o \qquad r_2 = r_o \qquad \text{(m)} \tag{B.21}$$

The insulation is described with the dimensionless thermal resistance parameter β, see (2.1-2). The temperatures are:

$$T_1 = T_a \qquad \text{(K)} \tag{B.22}$$

$$T_2 = -T_a$$

$$T_c = 0$$

This means that the heat losses from the pipes are:

$$q_1 = q_a \qquad q_2 = -q_a \qquad \text{(W/m)} \tag{B.23}$$

The temperature field consists of two parts, the line source part (T_q) and the multipole part (T_p).

$$T(x,y) = T_q(x,y) + T_p(x,y) \tag{B.24}$$

$$T_q(x,y) = \Re\left[\frac{q_o}{2\pi\lambda_g}\ln\left(\frac{r_c}{z+D}\right) - \frac{q_a}{2\pi\lambda_g}\ln\left(\frac{r_c}{z-D}\right)\right] \tag{B.25}$$

$$= \Re\left[\frac{q_a}{2\pi\lambda_g}\ln\left(\frac{z-D}{z+D}\right)\right]$$

$$T_p(x,y) = \Re\left[P_1\frac{r_o}{z+D} + P_2\frac{r_o}{z-D}\right] \tag{B.26}$$

Here is P_1 the strength of the first multipole of pipe 1 and P_2 is the strength of the first multipole of pipe 2. There is also a multipole at infinity with the strength $P_{c,j}$. But from (B.17) it is clear that the strength of this multipole approaches zero when the radius r_c increases. A temperature field consisting of these two parts satisfies Laplace equation because it is the real part of a sum of analytical functions.

Due to the symmetry the temperature field must satisfy:

$$T(z) = -T(-\overline{z}) \tag{B.27}$$

$$z = x + iy$$

$$\overline{z} = x - iy$$

This means that the strength of the multipoles must satisfy:

$$P_1 = \overline{P_2} \tag{B.28}$$

The real and imaginary part of P_1 is called N_1 and M_1:

$$P_1 = N_1 + iM_1 \tag{B.29}$$

$$P_2 = N_1 - iM_1 \tag{B.30}$$

The multipole part (B.26) becomes:

$$T_p(x,y) = \Re\left[P_1\frac{r_o}{z+D} + \overline{P_1}\frac{r_o}{z-D}\right] \tag{B.31}$$

At pipe m the boundary condition described in polar coordinates is:

$$z = z_m + \rho_m e^{i\psi_m} \tag{B.32}$$

$$T(z) - r_m\beta\frac{\partial T(z)}{\partial \rho_m} = T_m \qquad \rho_m = r_m \tag{B.33}$$

Here T_m is the temperature at pipe m. The part of (B.33) not depending on ψ_m is, from (B.15):

$$T_m = \Re\left[\frac{q_m}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_m}\right) + \beta\right) + \sum_{n\neq m}\left(\frac{q_n}{2\pi\lambda_g}\ln\left(\frac{r_c}{z_m - z_n}\right) + P_n\frac{r_n}{z_m - z_n}\right)\right] \tag{B.34}$$

This is true for each pipe m. For $m = 1$ and (B.23,30) we get:

$$T_a = \Re\left[\frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_o}\right) + \beta\right) - \frac{q_a}{2\pi\lambda_g}\ln\left(\frac{r_c}{-2D}\right) + \overline{P_1}\frac{r_o}{2D}\right] \tag{B.35}$$

The large radius r_c disappears and (B.35) becomes:

$$T_a = \Re\left[\frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{2D}{r_o}\right) + \beta\right) - \overline{P_1}\frac{r_o}{2D}\right] \tag{B.36}$$

Expressed in real parts equation (B.36) becomes:

$$T_a = \frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{2D}{r_o}\right) - \beta\right) + N_1\frac{r_o}{2D} \tag{B.37}$$

The part of (B.33) depending on ψ_m is, from (B.16):

$$0 = \overline{P_m}(1+\beta) \tag{B.38}$$

$$+(1-\beta)\sum_{n\neq m}\left(\frac{q_n}{2\pi\lambda_g}\frac{r_o}{z_n - z_m} + P_n\frac{{r_o}^2}{(z_m - z_n)(z_n - z_m)}\right)$$

For $m = 1$ and (B.23,30) we get:

$$0 = \overline{P_1}(1+\beta) + (1-\beta)\left(\frac{q_a}{2\pi\lambda_g}\frac{r_o}{2D} - \overline{P_1}\frac{{r_o}^2}{4D^2}\right) \tag{B.39}$$

This produces two real equations:

$$0 = N_1(1+\beta) + (1-\beta)\left(-\frac{q_a}{2\pi\lambda_g}\frac{r_o}{2D} - N_1\frac{{r_o}^2}{4D^2}\right) \tag{B.40}$$

$$0 = -M_1(1+\beta) + (1-\beta)M_1\frac{{r_o}^2}{4D^2} \tag{B.41}$$

Equation (B.41) means that the imaginary part of the strength of the multipoles are zero.

$$M_1 = 0 \tag{B.42}$$

N_1 is solved from (B.40).

$$N_1 = \frac{q_a}{2\pi\lambda_g}\cdot\frac{r_o}{2D}\cdot\left(\frac{1+\beta}{1-\beta} - \frac{{r_o}^2}{4D^2}\right)^{-1} \tag{B.43}$$

When (B.43) is used in (B.37) one gets:

$$T_a = \frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{2D}{r_o}\right) + \beta + \left(1 - \frac{1+\beta}{1-\beta}\left(\frac{2D}{r_o}\right)^2\right)^{-1}\right) \tag{B.44}$$

The derivation of (4.3) for the symmetrical problem of two pipes in a large pipe is very similar to the derivation of (3.8) made above. The temperatures for this symmetrical problem are listed below:

$$T_1 = T_m \tag{B.45}$$

$$T_2 = T_m$$

$$T_c = T_0$$

If the pipes are positioned as in Figure B.2, the temperature field must satisfy the following equation:

$$T(z) = T(-\overline{z}) \tag{B.46}$$

This means that the strength of the multipoles must satisfy:

$$P_1 = -\overline{P_2} \tag{B.47}$$

The strengths of the multipoles at infinity will approach zero when the large radius r_c increases. The radius will however be included in the formula and not disappear as in the equations above.

B.4 Derivation of (3.9)

The problem is the same as in the previous section B.3. There are two identical insulated pipes in a large pipe, see Figure B.2. The temperatures in the pipes have the same magnitude but opposite signs. The heat loss from the pipes are to be determined, see (B.20-23).

The temperature field consists of two line sources. The positions of these line sources are not in the center of the pipes.

$$T(x,y) = \Re\left[\frac{q_a}{2\pi\lambda_g}\ln\left(\frac{r_c}{z-l_1}\right) - \frac{q_a}{2\pi\lambda_g}\ln\left(\frac{r_c}{z-l_2}\right)\right] \tag{B.48}$$

$$= \frac{q_a}{2\pi\lambda_g}\Re\left[\ln\left(\frac{z-l_2}{z-l_1}\right)\right]$$

Here l_1 and l_2 are the positions of the line sources.

$$l_1 = -D - \delta$$

$$l_2 = D + \delta$$

With the formulae from [1] one can express (B.48) in polar coordinates with respect to the center of pipe m.

$$z = z_m + \rho_m e^{i\psi_m}$$

Here z_m is the center of pipe m and l_n is the position of the line source.

$\rho_m < |l_n - z_m|$,

$$\ln\left(\frac{r_c}{z-l_n}\right) = \ln\left(\frac{r_c}{z_m-l_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\rho_m}{l_n-z_m}\right)^k e^{ik\psi_m} \tag{B.49}$$

$\rho_m > |l_n - z_m|$,

$$\ln\left(\frac{r_c}{z-l_n}\right) = \ln\left(\frac{r_c}{\rho_m}e^{i\psi_m}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{l_n-z_m}{\rho_m}\right)^k e^{-ik\psi_m} \tag{B.50}$$

The temperature field expressed in polar coordinates with respect to the center of pipe 1 is:

$$T(z_1+\rho_1 e^{i\psi_1}) = \frac{q_a}{2\pi\lambda_g}\Re\left[\ln\left(\frac{r_c}{\rho_1}\right) - \ln\left(e^{i\psi_1}\right)\right. \tag{B.51}$$

$$\left. + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{l_1-z_1}{\rho_1}\right)^k e^{-ik\psi_1} - \ln\left(\frac{r_c}{z_1-l_2}\right) - \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\rho_1}{l_2-z_1}\right)^k e^{ik\psi_1}\right]$$

This expression for the temperature field is used in the boundary condition of pipe 1:

$$T(\rho_1, \psi_1) - \beta r_o \frac{\partial T(\rho_1, \psi_1)}{\partial \rho_1} = T_a \qquad \rho_1 = r_o \tag{B.52}$$

$$T_a \cdot \frac{2\pi\lambda_g}{q_a} = \Re\left[\ln\left(\frac{r_c}{\rho_1}\right) - \ln\left(e^{i\psi_1}\right) - \ln\left(\frac{r_c}{z_1 - l_2}\right)\right. \tag{B.53}$$

$$+\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{l_1 - z_1}{\rho_1}\right)^k e^{-ik\psi_1} - \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\rho_1}{l_2 - z_1}\right)^k e^{ik\psi_1}$$

$$\left.-\beta r_o\left\{-\frac{1}{\rho_1} + \sum_{k=1}^{\infty}\left(\frac{l_1 - z_1}{\rho_1}\right)^k\left(-\frac{1}{\rho_1}\right)e^{-ik\psi_1} - \sum_{k=1}^{\infty}\left(\frac{\rho_1}{l_2 - z_1}\right)^k\frac{1}{\rho_1}e^{ik\psi_1}\right\}\right]$$

$$\rho_1 = r_o$$

$$l_1 - z_1 = -\delta$$

$$l_2 - z_1 = 2D + \delta$$

$$\Re[\ln(e^{i\psi_1})] = 0$$

It is possible to divide (B.53) in two parts, one part that depends on ψ_1 and one part that is independent of ψ_1. The independent part is:

$$T_a \cdot \frac{2\pi\lambda_g}{q_a} = \ln\left(\frac{2D + \delta}{r_o}\right) + \beta \tag{B.54}$$

The part that depends on ψ_1 is:

$$0 = \Re\left[\sum_{k=1}^{\infty}\left\{\left(\frac{1}{k}(-\delta)^k + \beta(-\delta)^k\right) r_o{}^{-k} e^{-ik\psi_1} + \right.\right. \tag{B.55}$$

$$\left.\left.\left(\beta \cdot \frac{1}{(2D+\delta)^k} - \frac{1}{k}\cdot\frac{1}{(2D+\delta)^k}\right) r_o{}^k e^{ik\psi_1}\right\}\right] \qquad 0 \leq \psi_m \leq 2\pi$$

With the use of:

$$e^{ik\psi_1} = \cos(k\psi_1) + i\sin(k\psi_1),$$

(B.55) becomes:

$$0 = \sum_{k=1}^{\infty}\left\{(1 + \beta k)(-\delta)^k r_o{}^{-k} + (\beta k - 1)(2D + \delta)^{-k} r_o{}^k\right\}\cos(k\psi_1) \tag{B.56}$$

$$0 \leq \psi_m \leq 2\pi$$

This equation is not possible to solve for every k. One can only satisfy this equation for one value of k. If one chooses $k = 1$ one gets the following displacement of the line source:

$$\delta = -D + \sqrt{D^2 - \frac{1-\beta}{1+\beta} r_o{}^2} \tag{B.57}$$

From (B.54) we get an expression for the heat loss q_a from pipe 1:

$$q_a = \frac{T_a}{R_m} \tag{B.58}$$

$$2\pi\lambda_g R_m = \ln\left(\frac{2D+\delta}{r_o}\right) + \beta$$

When the pipe is non-insulated this is the exact solution.

The derivation of (4.4) for the displacement of the line source for the symmetrical problem of two pipes in a large pipe is very similar to the derivation of (3.9) made above. The solution will however not be exact for the non-insulated pipe.

B.5 Derivation of 3.12

We will here derive the multipole formula of the second order for the heat loss from one pipe in the ground. The problem of one pipe in the ground is equivalent to the anti-symmetrical problem of two pipes in a large pipe. This problem is described in Figure B.3.

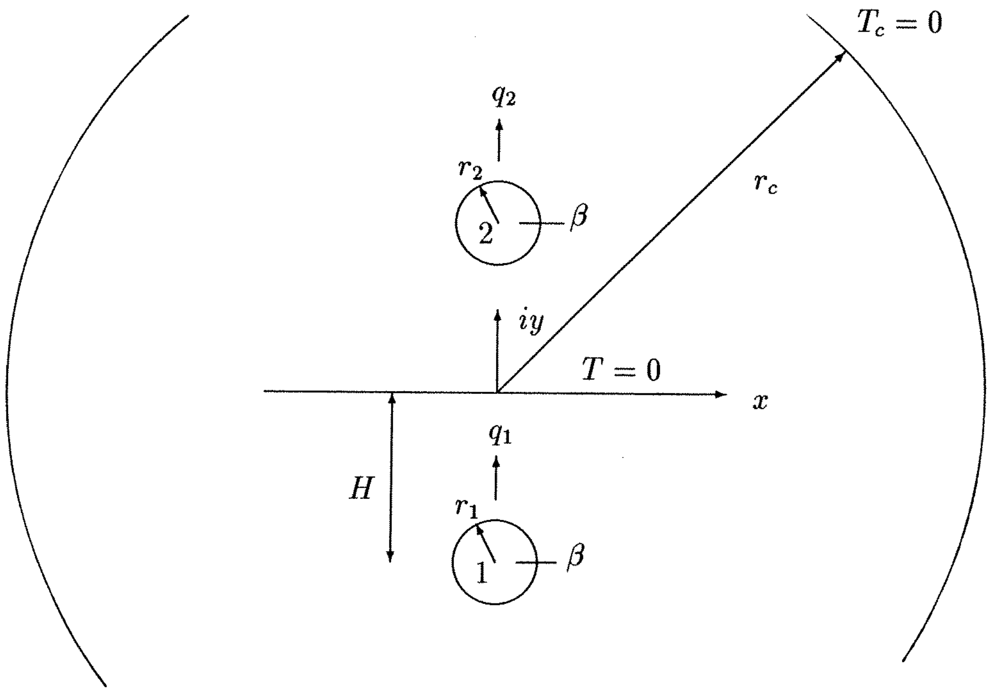

Figure B.3. Two pipes in a large pipe.

$$\begin{array}{llll} T_1 = T_a & q_1 = q_a & z_1 = -iH & r_1 = r_o \\ T_2 = -T_a & q_2 = -q_a & z_2 = iH & r_2 = r_o \end{array} \tag{B.59}$$

The temperature at the large pipe is zero.

$$T_c = 0 \tag{B.60}$$

The temperature field consists of two parts: the line source part and the multipole part.

$$T(x,y) = T_q(x,y) + T_p(x,y); \tag{B.61}$$

$$T_q(x,y) = \frac{q_a}{2\pi\lambda_g}\Re\left[\ln\left(\frac{r_c}{z+iH}\right) - \ln\left(\frac{r_c}{z-iH}\right)\right] \tag{B.62}$$

$$T_p(x,y) = \Re\left[P_{11}\frac{r_o}{z+iH} + P_{12}\left(\frac{r_o}{z+iH}\right)^2 \right. \tag{B.63}$$

$$\left. +P_{21}\frac{r_o}{z-iH} + P_{22}\left(\frac{r_o}{z-iH}\right)^2\right]$$

There is a multipole at infinity also but from (B.17) we see that the strength P_{cj} of this multipole is reduced to zero when the radius r_c increases. The radius r_c disappears from the equations if the sum of the strength of line sources is zero.

The temperature field is anti-symmetrical with respect to the x-axis.

$$T(z) = -T(\overline{z}) \tag{B.64}$$

$$z = x + iy$$

$$\overline{z} = x - iy$$

This means that the strength of the multipoles must satisfy the following equations:

$$\Re[P_{11}] = -\Re[P_{21}] \qquad \Im[P_{11}] = \Im[P_{21}] \tag{B.65}$$

$$\Re[P_{12}] = -\Re[P_{22}] \qquad \Im[P_{12}] = \Im[P_{22}] \tag{B.66}$$

Here $\Re[P_{11}]$ means the real part of P_{11} and $\Im[P_{11}]$ means the imaginary part. From (B.15) and (B.56,60,65,66) we get for $m = 1$:

$$T_a = \frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_o}\right) + \beta - \ln\left(\frac{r_c}{2H}\right)\right) + \tag{B.67}$$

$$\Re\left[-P_{21}\frac{r_o}{2iH} + P_{22}\left(\frac{r_o}{2iH}\right)^2\right]$$

For $(m = 2)$ we get:

$$-T_a = -\frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_o}\right) + \beta - \ln\left(\frac{r_c}{2H}\right)\right) + \tag{B.68}$$

$$\Re\left[P_{11}\frac{r_o}{2iH} + P_{12}\left(\frac{r_o}{2iH}\right)^2\right]$$

We will use the following notations:

$$P_{11} = N_1 + iM_1 \tag{B.69}$$

$$P_{21} = -N_1 + iM_1 \tag{B.70}$$

$$P_{21} = N_2 + iM_2 \tag{B.71}$$

$$P_{22} = -N_2 + iM_2 \tag{B.72}$$

With (B.67-72) we get:

$$T_a = \frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{2H}{r_o}\right) + \beta\right) - M_1 \cdot \frac{r_o}{2H} + N_2 \cdot \left(\frac{r_o}{2H}\right)^2 \tag{B.73}$$

From (B.16) with $m = 1$ and $k = 1$ we get:

$$0 = \overline{P_{11}} + \frac{1-\beta}{1+\beta} \cdot \left(-\frac{q_a}{2\pi\lambda_g}\frac{r_o}{2iH} + P_{21}\left(\frac{r_o}{2H}\right)^2 + P_{22} \cdot 2i\left(\frac{r_o}{2H}\right)^3\right) \tag{B.74}$$

This equation is separated in a real and an imaginary part:

$$0 = N_1 + \frac{1-\beta}{1+\beta} \cdot \left(-N_1\left(\frac{r_o}{2H}\right)^2 - 2M_2\left(\frac{r_o}{2H}\right)^3\right) \tag{B.75}$$

$$0 = -M_1 + \frac{1-\beta}{1+\beta} \cdot \left(\frac{q_a}{2\pi\lambda_g} \cdot \left(\frac{r_o}{2H}\right) + M_1\left(\frac{r_o}{2H}\right)^2 - 2N_2\left(\frac{r_o}{2H}\right)^3\right) \tag{B.76}$$

No new equations will be produced if $m = 2$ is used instead of $m = 1$ in equation (B.16).
With $m = 1$ and $k = 2$ in (B.16) one gets:

$$0 = \overline{P_{12}} + \frac{1-2\beta}{1+2\beta}\left(\frac{q_a}{2\pi\lambda_g} \cdot \frac{1}{2} \cdot \left(\frac{r_o}{2H}\right)^2 - P_{21} \cdot i\left(\frac{r_o}{2H}\right)^3 + 3P_{22}\left(\frac{r_o}{2H}\right)^4\right) \tag{B.77}$$

The real part of (B.77) becomes:

$$0 = N_2 + \frac{1-2\beta}{1+2\beta}\left(\frac{q_a}{2\pi\lambda_g} \cdot \frac{1}{2} \cdot \left(\frac{r_o}{2H}\right)^2 + M_1\left(\frac{r_o}{2H}\right)^3 - 3N_2\left(\frac{r_o}{2H}\right)^4\right) \tag{B.78}$$

The imaginary part of (B.77) becomes:

$$0 = -M_2 + \frac{1-2\beta}{1+2\beta}\left(N_1\left(\frac{r_o}{2H}\right)^3 + 3M_2\left(\frac{r_o}{2H}\right)^4\right) \tag{B.79}$$

Equations (B.75) and (B.79) give:

$$0 = M_2 = N_1 \tag{B.80}$$

We will use two coefficients K_1 and K_2:

$$K_1 = \left(\frac{2H}{r_o}\right)\frac{1+\beta}{1-\beta} - \left(\frac{r_o}{2H}\right) \tag{B.81}$$

$$K_2 = \left(\frac{2H}{r_o}\right)^2\frac{1+2\beta}{1-2\beta} - 3\left(\frac{r_o}{2H}\right)^2 \tag{B.82}$$

Equations (B.76) and (B.78) become with these notations:

$$0 = M_1 \cdot K_1 - \frac{q_a}{2\pi\lambda_g} + 2N_2\left(\frac{r_o}{2H}\right)^2 \tag{B.83}$$

$$0 = N_2 \cdot K_2 + \frac{q_a}{2\pi\lambda_g} \cdot \frac{1}{2} + M_1\left(\frac{r_o}{2H}\right) \tag{B.84}$$

From (B.83) and (B.84) the real part of the strength of the second order multipole N_2 is calculated:

$$N_2 = \frac{q_a}{2\pi\lambda_g} \cdot \frac{\left(\frac{r_o}{2H}\right) + \frac{K_1}{2}}{2\left(\frac{r_o}{2H}\right)^3 - K_1 K_2} = \frac{q_a}{2\pi\lambda_g} K_3 \tag{B.85}$$

Here we have introduced the coefficient K_3. The imaginary part of the strength of the first order multipole M_1 becomes:

$$M_1 = \left(\frac{2H}{r_o}\right)\left(-\frac{q_a}{2\pi\lambda_g} \cdot \frac{1}{2} - N_2 \cdot K_2\right) = -\frac{q_a}{2\pi\lambda_g}\left(\frac{2H}{r_o}\right)\left(\frac{1}{2} + K_2 K_3\right) \tag{B.86}$$

Equation (B.73) becomes:

$$T_a = \frac{q_a}{2\pi\lambda_g}\left[\ln\left(\frac{2H}{r_o}\right) + \beta + \frac{1}{2} + K_2 \cdot K_3 + \left(\frac{r_o}{2H}\right)^2 K_3\right] \tag{B.87}$$

After some simplifications equation (B.87) becomes:

$$T_a = \frac{q_a}{2\pi\lambda_g} \cdot \frac{1 - \frac{3}{2}\left(\frac{r_o}{2H}\right)^4\left(\frac{1-2\beta}{1+2\beta}\right) + \frac{1}{2}\left(\frac{r_o}{2H}\right)^2 \frac{(1+\beta)(1-2\beta)}{(1-\beta)(1+2\beta)}}{1 - \left(\frac{r_o}{2H}\right)^4\left(\frac{1-2\beta}{1+2\beta}\right) + \left(3\left(\frac{r_o}{2H}\right)^2\left(\frac{1-2\beta}{1+2\beta}\right) - \left(\frac{2H}{r_o}\right)^2\right)\left(\frac{1+\beta}{1-\beta}\right)} \tag{B.88}$$

B.6 Derivation of (2.18)

We will here derive the first-order multipole formula for the heat loss from two pipes in the ground. The temperatures in the pipes have the same magnitude but opposite signs. This problem is equivalent to the anti-symmetrical problem of four pipes in a large pipe. This new problem is anti-symmetrical with respect to both the x-axis and the y-axis. Every pipe has the radius r_1 and a surface resistance β to the ground. The temperature on the x-axis is zero ($T = 0$ on $y = 0$). The problem is described in Fig B.4.

Because the sum of the strength of the line sources is zero, the large pipe will not affect the formulae if the radius is large enough ($r_c \gg r_n$).

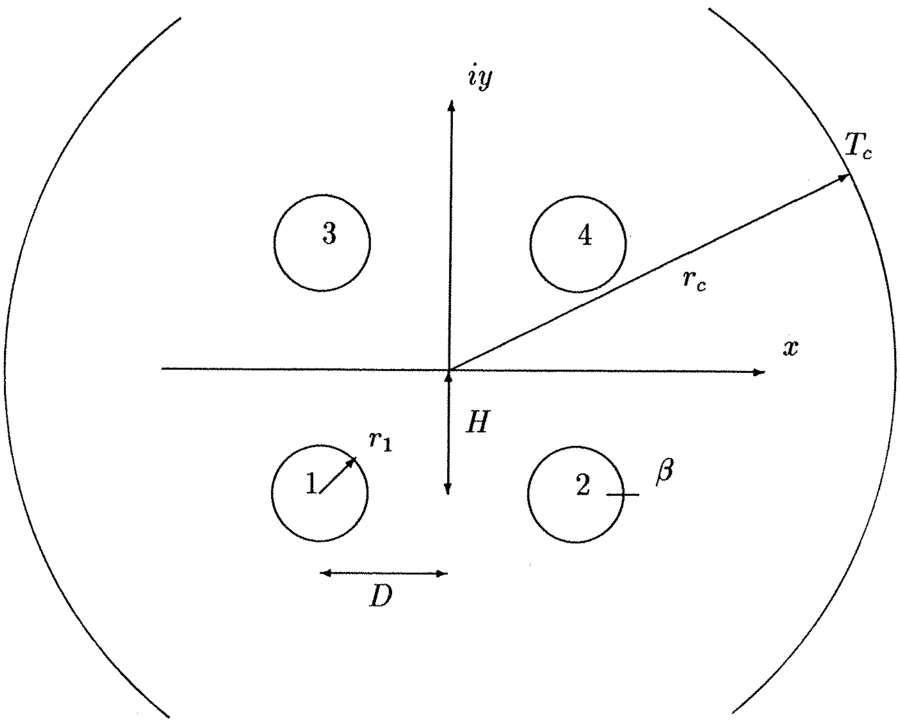

Figure B.4. Four pipes in an infinite surrounding.

$$z_1 = -D - iH \qquad z_2 = D - iH \tag{B.89}$$

$$z_3 = -D + iH \qquad z_4 = D + iH$$

The temperatures in the pipes are given:

$$T_1 = T_a \qquad T_2 = -T_a \tag{B.90}$$

$$T_3 = -T_a \qquad T_4 = T_a$$

$$T_c = 0$$

The temperature field consists of two parts: the line source part and the multipole part.

$$T(z) = T_q(z) + T_p(z) \tag{B.91}$$

$$T_q(z) = \Re\left[\sum_{n=1}^{4} q_n \ln\left(\frac{r_c}{z - z_n}\right)\right] \tag{B.92}$$

$$T_p(z) = \Re\left[\sum_{n=1}^{4} P_n \left(\frac{r_1}{z - z_n}\right)\right] \tag{B.93}$$

There is a line source ($q_1 \ldots q_4$) and a multipole ($P_1 \ldots P_4$) of the first order in the center of each pipe. There is a multipole at infinity also but from (B.17) we see that the strength P_{cj} of this multipole is reduced to zero when the radius r_c increases. The radius r_c disappears from the equations if the sum of the strength of line sources is zero.

Due to the symmetry the line sources must satisfy the following equations:

$$q_1 = q_a \tag{B.94}$$

$$q_2 = -q_a \tag{B.95}$$

$$q_3 = -q_a \tag{B.96}$$

$$q_4 = q_a \tag{B.97}$$

The problem is anti-symmetrical with respect to both the x-axis and the y-axis:

$$T(z) = -T(\overline{z}) \tag{B.98}$$

$$T(z) = -T(-\overline{z}) \tag{B.99}$$

To satisfy (B.98,99) the strength of the multipoles must satisfy the following equations:

$$P_2 = \overline{P_1} \tag{B.100}$$

$$P_3 = -\overline{P_1} \tag{B.101}$$

$$P_4 = -P_1 \tag{B.102}$$

With the use of (B.5-9) the line sources and multipoles are expressed in polar coordinates with respect to the center of pipe m. These expressions are put in the boundary condition for pipe m.

$$z = z_m + \rho_m e^{i\psi_m}$$

$$T(\rho_m, \psi_m) - \beta r_1 \frac{\partial T(\rho_m, \psi_m)}{\partial \rho_m} = T_m \qquad \rho_m = r_m = r_1 \tag{B.103}$$

From (B.15) we get the part of (B.103) independent of ψ_m:

$$T_m = \Re\left[\frac{q_m}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_1}\right) + \beta\right) + \sum_{n \neq m}\left\{\frac{q_n}{2\pi\lambda_g}\ln\left(\frac{r_c}{z_m - z_n}\right) + P_n\frac{r_1}{z_m - z_n}\right\}\right] \tag{B.104}$$

From (B.16) we get the part of (B.103) dependent on ψ_m:

$$0 = \overline{P_m}(1+\beta) + (1-\beta)\sum_{n \neq m}\left\{\frac{q_n}{2\pi\lambda_g}\frac{r_c}{(z_n - z_m)} - P_n\frac{{r_1}^2}{(z_m - z_n)^2}\right\} \tag{B.105}$$

For $m = 1$ and (B.100-102) equation (B.104) becomes:

$$T_a = \Re\left[\frac{q_a}{2\pi\lambda_g}\left(\ln\left(\frac{r_c}{r_1}\right) + \beta - \ln\left(\frac{r_c}{-2D}\right) - \ln\left(\frac{r_c}{2iH}\right) + \ln\left(\frac{r_c}{-2(D+iH)}\right)\right) + \overline{P_1}\left(-\frac{r_1}{2D} + \frac{r_1}{2iH}\right) + P_1\frac{r_1}{2(D+iH)}\right] \tag{B.106}$$

Equation (B.105) becomes for $m = 1$:

$$0 = \overline{P_1}\left(\frac{1+\beta}{1-\beta} - \left(\frac{r_1}{2D}\right)^2 - \left(\frac{r_1}{2H}\right)^2\right) \tag{B.107}$$

$$+P_1\left(\frac{r_1}{2(D+iH)}\right)^2 + \frac{q_a}{2\pi\lambda_g}\left(-\frac{r_1}{2D} - \frac{r_1}{2iH} + \frac{r_1}{2(D+iH)}\right)$$

To reduce (B.107) the two terms with H^2 in the denominator are excluded. This means that the mirror multipoles are neglected in the calculation of the angular dependent equation. The mirror multipoles are not neglected for the angular independent equation (B.106). This approximation makes the strength of the first order multipole $\overline{P_1}$, easy to solve.

$$\overline{P_1} = \frac{q_a}{2\pi\lambda_g}\left(\frac{r_1}{2D} + \frac{r_1}{2iH} - \frac{r_1}{2(D+iH)}\right)\left(\frac{1+\beta}{1-\beta} - \left(\frac{r_1}{2D}\right)^2\right)^{-1} \tag{B.108}$$

The expression for $\overline{P_1}$ is put in (B.106).

$$T_a \cdot \frac{2\pi\lambda_g}{q_a} = \Re\left[\ln\left(\frac{2DH}{r_1\sqrt{D^2+H^2}}\right) + \beta\right. \tag{B.109}$$

$$+\left\{\left(\frac{r_1}{2iH} - \frac{r_1}{2D}\right)\left(\frac{r_1}{2D} + \frac{r_1}{2iH} - \frac{r_1}{2(D+iH)}\right)\right.$$

$$\left.\left.+\frac{r_1}{2(D+iH)}\left(\frac{r_1}{2D} - \frac{r_1}{2iH} - \frac{r_1}{2(D-iH)}\right)\right\} \cdot \left(\frac{1+\beta}{1-\beta} - \left(\frac{r_1}{2D}\right)^2\right)^{-1}\right]$$

After some simplifications the expression for the heat loss q_a becomes:

$$q_a = \frac{T_a}{R_a} \tag{B.110}$$

$$2\pi\lambda_g R_a = \ln\left(\frac{2DH}{r_1\sqrt{D^2+H^2}}\right) + \beta$$

$$-\left[\left(\frac{r_1}{2D}\right)^2 + \left(\frac{r_1}{2H}\right)^2 - \frac{3}{4}\cdot\frac{{r_1}^2}{D^2+H^2}\right]\left[\frac{1+\beta}{1-\beta} - \left(\frac{r_1}{2D}\right)^2\right]^{-1}$$

The derivation of (2.17) is similar to the derivation of (2.18) made above. The temperatures are different and the strength of the multipoles must satisfy a different set of equations.

$$T_1 = T_m \qquad T_2 = T_m \tag{B.111}$$

$$T_3 = -T_m \qquad T_4 = -T_m \tag{B.112}$$

If the pipes are positioned as in Figure B.4, the temperature field must satisfy the following equation:

$$T(z) = -T(\overline{z}) \tag{B.113}$$

The strength of the multipoles must satisfy the following equations:

$$P_2 = -\overline{P_1} \tag{B.114}$$

$$P_3 = -\overline{P_1} \tag{B.115}$$

$$P_4 = P_1 \tag{B.116}$$

PART B:

Notes on Heat Transfer 2-1991

HEAT LOSS FROM TWO PIPES IMBEDDED IN A CIRCULAR INSULATION

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May 1991
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1 Introduction

This report deals with the problem of determining the steady-state heat losses from two pipes imbedded in a circular insulation. A summary of the results is presented in chapter 8.

In Claesson [1] a new method, the *multipole method*, is presented which can solve this problem with arbitrary accuracy. With the use of the multipole method, new formulae with improved accuracy have been derived for the heat losses. The errors of the new formulae and two old formulae have been determined with the multipole method.

The formulae are mainly derived for district heating pipes. They can be used on any problem with the same boundary conditions, but the listed errors of the formulae are valid for dimensions usual for district heating pipes in the ground.

1.1 Two pipes imbedded in a circular insulation

There are two pipes with the radius r_i imbedded in a circular insulation, see Figure 1.1. The temperatures in the imbedded pipes are T_1 and T_2. The temperature on the circumscribing larger pipe is T_0. The thermal conductivity in the insulation is λ_i. The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes inside the large pipe. The pipes are assumed to be long. It is therefore sufficient to study a vertical cross-section of the pipes. The temperature $T(x, y)$ satisfies the steady-state heat conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.1}$$

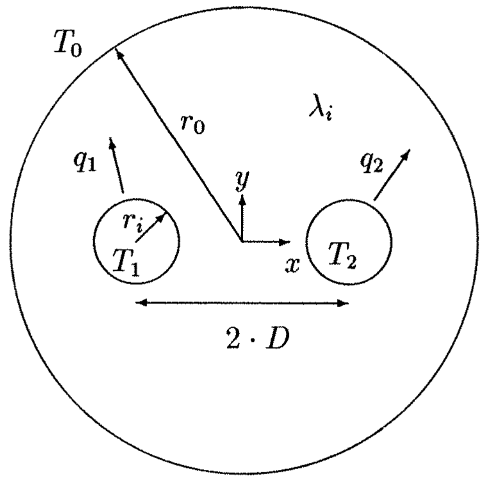


Figure 1.1. Two pipes imbedded in a circular insulation.

D = Half the distance between the center of the pipes (m)
r_0 = Radius of the circumscribing large pipe (m)
r_i = Radius of the imbedded pipes (m)
q_1 = Heat flow from pipe 1 per meter (W/m)
q_2 = Heat flow from pipe 2 per meter (W/m)
T_0 = Temperature on the larger pipe (°C)
T_1 = Temperature in pipe 1 (°C)
T_2 = Temperature in pipe 2 (°C)
λ_i = Thermal conductivity of the insulation (W/mK)

1.2 Solution method

A new method, the *multipole method*, to calculate the heat flow to and between pipes inside a larger pipe with constant temperature is presented in [1]. The method is implemented as programs for computers of PC-type.

A brief summary of the *multipole method* is given here. A more detailed description is made in chapter 5.

There are N pipes inside a larger pipe. The temperature on the larger pipe is T_0. The temperature in pipe number n inside the larger pipe is T_n. The steady-state heat conduction equation is to be solved. The problem is solved in the complex plane ($z = x + iy$). Here is i used to describe the imaginary unit ($i^2 = -1$). The temperature field in the large pipe consists of a line source part, a multipole part and the temperature on the large pipe T_0.

$$T(x, y) = T_0 + T_q(x, y) + T_p(x, y) \tag{1.2}$$

There is a line source with the strength q_n in the center of each pipe. In order to satisfy the boundary condition at the large pipe there is, for each pipe, a mirror line source with the strength $-q_n$ at $r_0^2/\bar{z}_n$. The function $T_q(z)$ will always be zero at $x^2+y^2=r_0^2$.

$$T_q(x,y) = \mathrm{Re}\left[\sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_i} \cdot W_{n0}(z)\right] \tag{1.3}$$

$$W_{n0}(z) = \ln\left(\frac{r_0}{z-z_n}\right) - \ln\left(\frac{r_0^2}{r_0^2 - \bar{z}z_n}\right) \tag{1.4}$$

The complex-valued derivative of order j of W_{n0} with respect to z_n is called a multipole of order j. We will use the function W_{nj}. The complex strength of each multipole is P_{nj}.

$$T_p(x,y) = \mathrm{Re}\left[\sum_{n=1}^{N}\sum_{j=1}^{J} P_{nj} \cdot r_n^j \cdot W_{nj}(z)\right] \tag{1.5}$$

$$P_{nj} = c_{nj} + i \cdot s_{nj} \tag{1.6}$$

$$W_{nj}(z) = \frac{1}{(j-1)!} \cdot \frac{\partial^j}{\partial z_n^j}(W_{n0}) = \frac{1}{(z-z_n)^j} - \left(\frac{\bar{z}}{r_0^2 - \bar{z}z_n}\right)^j \tag{1.7}$$

The real part of (1.4) and (1.7) satisfy the heat conduction equation (1.1). The multipole of order j can represent any variation $\cos(j\psi_n)$ and $\sin(j\psi_n)$ around pipe n. The final temperature is a sum of the temperature fields from all the pipes with multipoles up to order J. The strength of the multipoles P_{nj} and the strength of the line sources q_n are unknown. The boundary conditions from all the pipes will produce an equation system, from which P_{nj} and q_n are solved. In the limit when $J \to \infty$ the exact solution is found. Thus the error of the calculation can be chosen arbitrarily small.

With the multipole method it is possible to derive systematic approximations of increasing accuracy. This report deals with approximations of the zero, first and second order.

1.3 Previous reports

There exist several reports describing different types of *multipole methods*. The method used in this report is based on Claesson [1], in which the problem is to determine the heat flows between pipes in a composite cylinder, i.e. two concentric cylinders with different thermal conductivities. The report of Wallentén [6] is also based on Claesson [1] and presents explicit formulae of the zero and first order for the heat loss from two pipes in the ground imbedded in a circular insulation.

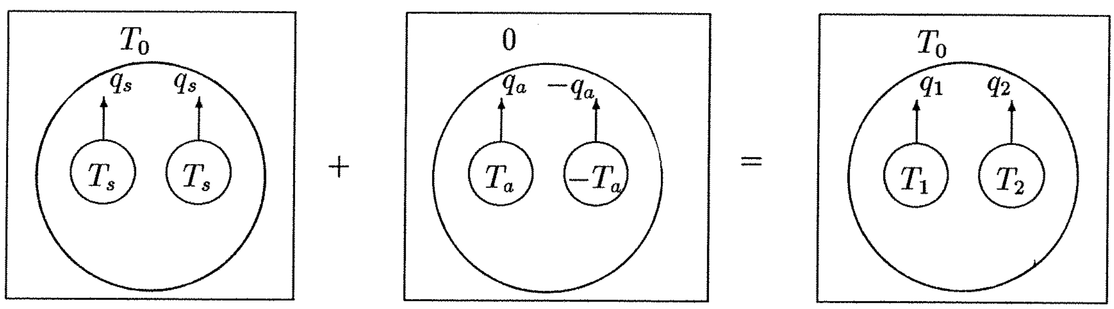
Claesson [2] presents a *multipole method* without any mirror line sinks. That method is not suitable for the problem in this report because the boundary condition at the larger pipe will not be sufficiently satisfied for small J. The method described in [2] is implemented in [3] and [4]. The program of [3] deals with the heat flow problem when one or more pipes are positioned inside a large pipe. The program of [4] deals with the problem of one or more pipes inside a larger pipe, which in its turn lies in the ground with another thermal conductivity. The report of Wallentén [5] is based on Claesson [2] and presents explicit formulae of the zero and first order for the heat loss from one or two pipes in the ground.

2 Mathematical formulation

The problem described in Figure 1.1 can, with the use of the superposition principle described in section 2.1, be separated into two problems. These two problems are easier to solve than the original problem. The solution is expressed in the new temperatures T_s, T_a, T_0 and the dimensionless heat loss factors h_s and h_a.

2.1 Superposition principle

For the problem described in Figure 1.1 one can construct two basic problems, a symmetrical problem and an anti-symmetrical problem, see Figure 2.1. With the use of the superposition principle, every problem concerning different temperatures can be constructed from the solutions of these two problems.



Symmetrical problem Anti-symmetrical problem Original problem

Figure 2.1. The superposition principle.

The temperature in the pipes in the symmetrical problem is T_s. The temperatures in the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{2.1}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{2.2}$$

The subscript s denotes the symmetrical problem of two pipes. The subscript a denotes the anti-symmetrical problem of two pipes. The temperatures in the original problem are from (2.1-2):

$$T_1 = T_s + T_a \tag{2.3}$$

$$T_2 = T_s - T_a \tag{2.4}$$

The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_0$. We may write:

$$q_s = \frac{T_s - T_0}{R_s} \tag{2.5}$$

Here R_s (mK/W) is the thermal resistance between one of the pipes and the larger pipe. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional to the temperature T_a. We may write:

$$q_a = \frac{T_a}{R_a} \tag{2.6}$$

Here R_a (mK/W) is the thermal resistance associated with the anti-symmetrical problem. It should be noted that the temperature T_a connected with R_a in (2.6) is half the temperature difference between the pipes. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{2.7}$$

$$q_2 = q_s - q_a \tag{2.8}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{2.9}$$

The symmetrical and anti-symmetrical problems are solved separately. Formulae for R_s and R_a are obtained. The heat losses q_1 and q_2 are then obtained from (2.7-8).

2.2 Dimensional analysis

The thermal resistances R_a and R_s are both inversely proportional to the thermal conductivity λ_i. The heat losses q_s and q_a are therefore proportional to λ_i. The heat losses q_s and q_a are also proportional to $(T_s - T_0)$ and T_a respectively.

It is convenient to introduce the dimensionless heat loss factors h_s and h_a. This is done to separate the dependence on the temperatures and thermal conductivity from the dependence on the geometry:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s \tag{2.10}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a \tag{2.11}$$

The factor $2\pi\lambda_i$ is introduced to make the expressions for h_s and h_a simpler.

The geometry is described by three lengths: r_i, r_0 and D. The number of parameters necessary to describe the geometry is reduced from three to two by scaling with the radius of the outer pipe r_0. The heat loss factors h_s and h_a only depend on the parameters r_i/r_0 and D/r_0:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0) \tag{2.12}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0) \tag{2.13}$$

The geometry of the problem gives the following inequalities:

$$0 < r_i/r_0 < D/r_0 \tag{2.14}$$

$$r_i/r_0 + D/r_0 < 1 \tag{2.15}$$

The thermal resistances R_s and R_a can be expressed in h_s and h_a.

$$R_s = \frac{1}{2\pi\lambda_i \cdot h_s} \tag{2.16}$$

$$R_a = \frac{1}{2\pi\lambda_i \cdot h_a} \tag{2.17}$$

3 Symmetrical problem

For the symmetrical problem the temperature in the pipes is $T_s = (T_1 + T_2)/2$ and the temperature on the large pipe is T_0. The problem is described in Figures 1.1 and 2.1. The heat loss from the pipes is q_s for both pipes.

3.1 Exact solution

The exact solution to the problem was obtained with the use of the program described in [1]. The order of the highest used multipole has been at least 8. This means that the error in the heat losses is approximately less than 0.001 %. Figure 3.1 and Table 3.1 show the computed heat loss factor $h_s(r_i/r_0, D/r_0)$. From equation (2.12) we get the heat loss q_s:

$$q_s = (T_s - T_0) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0) \tag{3.1}$$

The heat loss q_s will decrease for decreasing D/r_0, while the anti-symmetrical heat loss q_a discussed in chapter 4 has a minimum for $D/r_0 \approx 0.5$, see Figure 4.1. The heat loss q_s is strongly dependent on the ratio r_i/r_0. For small pipes the heat loss is only weakly dependent on the positions of the pipes (D/r_0).

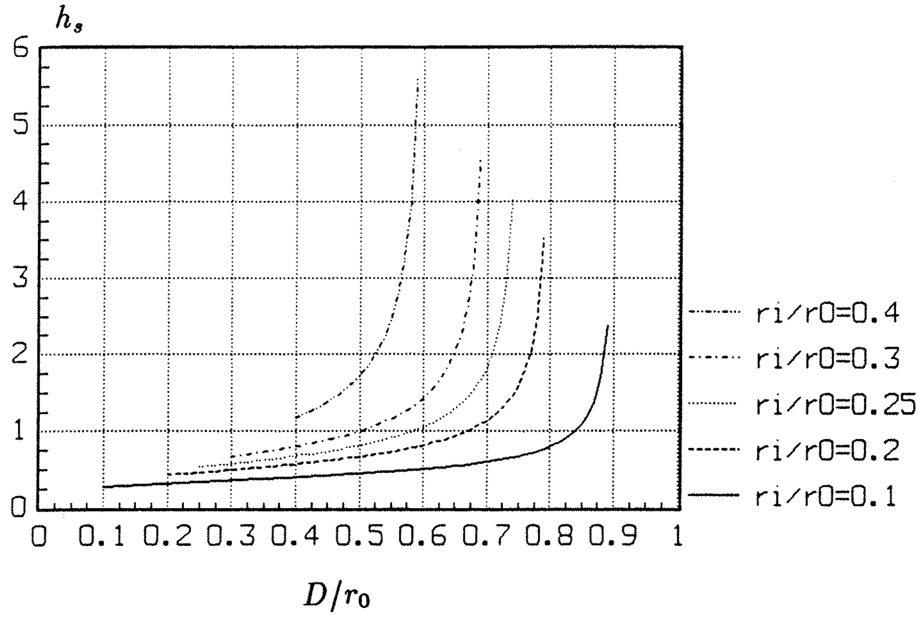


Figure 3.1. Heat loss factor h_s for different values of D/r_0 and r_i/r_0.

	h_s								
	$r_i/r_0 = 0.05$	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$D/r_0 = 0.10$	0.2200	0.2701							
0.15	0.2397	0.2940	0.3461						
0.20	0.2568	0.3168	0.3730	0.4326					
0.25	0.2721	0.3388	0.4003	0.4652	0.5380				
0.30	0.2864	0.3602	0.4282	0.4998	0.5804	0.6749			
0.35	0.3002	0.3815	0.4570	0.5368	0.6273	0.7351	0.8685		
0.40	0.3137	0.4031	0.4873	0.5773	0.6806	0.8060	0.9663	1.1829	
0.45	0.3273	0.4257	0.5201	0.6227	0.7430	0.8934	1.0942	1.3857	1.8699
0.50	0.3416	0.4501	0.5567	0.6755	0.8193	1.0073	1.2765	1.7199	2.6986
0.55	0.3571	0.4773	0.5993	0.7400	0.9186	1.1692	1.5766	2.4681	
0.60	0.3745	0.5093	0.6515	0.8240	1.0598	1.4358	2.2485		
0.65	0.3950	0.5487	0.7199	0.9441	1.2928	2.0336			
0.70	0.4205	0.6008	0.8183	1.1434	1.8174				
0.75	0.4544	0.6764	0.9829	1.5946					
0.80	0.5039	0.8038	1.3582						
0.85	0.5879	1.0970							
0.90	0.7833								

Table 3.1. Heat loss factor h_s for different D/r_0 and r_i/r_0.

3.2 Approximate formulae

With the use of the multipole method described in [1] approximate formulae have been derived for the heat losses from the pipes in the symmetrical problem. Formulae of order J employs the solution of a equation system of order J. The formula will therefore be very complicated for $J > 2$. The general solution is described in chapter 7.

3.2.1 Zero-order approximation

The zero-order multipole approximation uses the line sources and sinks without any multipoles. The zero-order approximation gives the following expressions for the heat loss factor h_s (or thermal resistance R_s) for the symmetrical problem:

$$h_s^{-1} = 2\pi\lambda_i R_s = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{3.2}$$

3.2.2 First-order approximation

With the use of multipoles of the first order, the following new formula is obtained:

$$h_s^{-1} = 2\pi\lambda_i R_s = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) - \frac{\left(\frac{r_i}{2D} + \frac{2r_i D^3}{r_0^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 - \left(\frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2} \tag{3.3}$$

Formula (3.3) is derived in section 6.2.

3.2.3 Second-order approximation

For higher order multipole formulae a general method is described in section 7.1. The second-order formula is shown here. The coefficients c_{11}, c_{21}, c_{22}, d_1, d_2 and A are introduced to make the formula more comprehensible.

$$c_{11} = 1 + \left(\frac{r_i}{2D}\right)^2 - \left(\frac{r_i D}{r_0^2 + D^2}\right)^2 - \left(\frac{r_i D}{r_0^2 - D^2}\right)^2 + \frac{r_i^2}{r_0^2 + D^2} - \frac{r_i^2}{r_0^2 - D^2}$$

$$c_{21} = \left(\frac{r_i}{2D}\right)^3 - \left(\frac{r_i D}{r_0^2 + D^2}\right)^3 + \left(\frac{r_i D}{r_0^2 - D^2}\right)^3 + \frac{r_i^3 D}{(r_0^2 + D^2)^2} + \frac{r_i^3 D}{(r_0^2 - D^2)^2}$$

$$c_{22} = 1 + 3\left(\frac{r_i}{2D}\right)^4 - 3\left(\frac{r_i D}{r_0^2 + D^2}\right)^4 - 3\left(\frac{r_i D}{r_0^2 - D^2}\right)^4$$

$$+4\frac{r_i^4 D^2}{(r_0^2 + D^2)^3} - 4\frac{r_i^4 D^2}{(r_0^2 - D^2)^3} - \left(\frac{r_i^2}{r_0^2 + D^2}\right)^2 - \left(\frac{r_i^2}{r_0^2 - D^2}\right)^2$$

$$d_1 = -\frac{r_i}{2D} + \frac{r_i D}{r_0^2 + D^2} - \frac{r_i D}{r_0^2 - D^2}$$

$$d_2 = \frac{1}{2}\left(-\left(\frac{r_i}{2D}\right)^2 + \left(\frac{r_i D}{r_0^2 + D^2}\right)^2 + \left(\frac{r_i D}{r_0^2 - D^2}\right)^2\right)$$

$$A = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right)$$

$$h_s^{-1} = 2\pi\lambda_i R_s = A + \frac{4d_1 d_2 c_{21} - d_1^2 c_{22} - 2d_2^2 c_{11}}{c_{11}c_{22} - 2c_{21}^2} \tag{3.4}$$

3.2.4 Area approximation

One old formula described in [7] we will here the area approximation. This formula is based on the assumption that the resistance of the insulation can be separated into two resistances coupled in parallel. The resistance R_1 originates from the resistance of a circular insulation with radius r_e. The resistance R_2 originates from the resistance of a rectangular insulation with height d_e and width D.

$$r_e = \sqrt{\frac{2 \cdot r_0^2}{\pi}\arccos\left(\frac{D}{r_0}\right) - \frac{2 \cdot D}{\pi}\sqrt{r_0^2 - D^2}}$$

$$d_e = \frac{\sqrt{r_0^2 - D^2} + r_0}{2} - r_i$$

$$R_1 = 2 \cdot \ln\left(\frac{r_e}{r_i}\right)$$

$$R_2 = \frac{\pi d_e}{D}$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 + 1/R_2 \tag{3.5}$$

3.2.5 Two-model approximation

Another old formula described in [8] we will here call the two-model approximation. This formula is based on the assumption that the problem can be separated into two problems which each has an analytical solution. The resistance R_1 originates from the resistance between a pipe with the radius r_i inside a pipe with the radius r_0. The resistance R_2 originates from the resistance between two pipes with the radius r_i which centers are $2D$ apart.

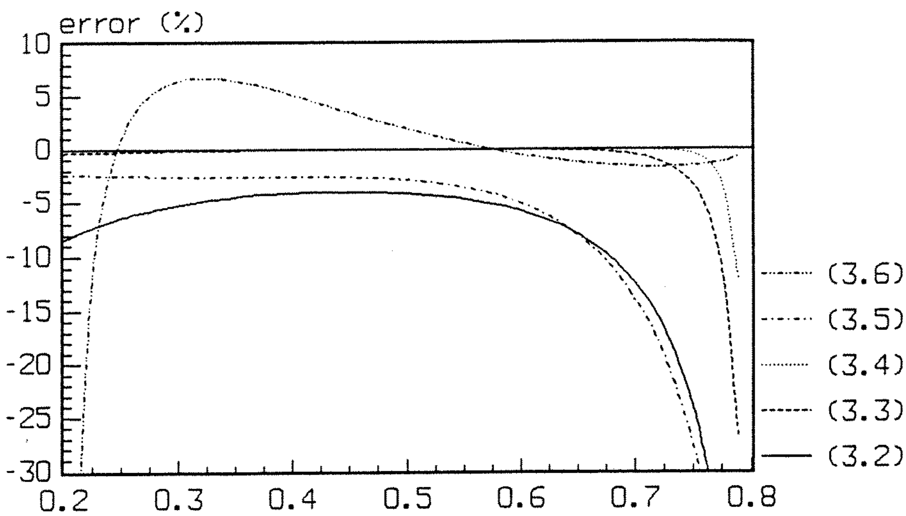
$$R_1 = \operatorname{arccosh}\left(\frac{r_i/r_0 + r_0/r_i - (r_0/r_i)(D/r_0)^2}{2}\right)$$

$$R_2 = 4 \cdot \operatorname{arccosh}\left(2\left(\frac{D}{r_i}\right)^2 - 1\right)$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 - 1/R_2 \tag{3.6}$$

3.3 Errors of the formulae

The errors of the different formulae have been studied with the use of the multipole program [1]. Figures 3.2-7 and Table 3.2 show the error made when the heat loss q_s is calculated with formulae (3.2-6). The error is expressed in per cent. A positive relative error means that the formula gives too large a heat loss.



D/r_0

Figure 3.2. The relative error (%) of the different formulae to calculate q_s for different values of D/r_0 ($r_i/r_0 = 0.2$).

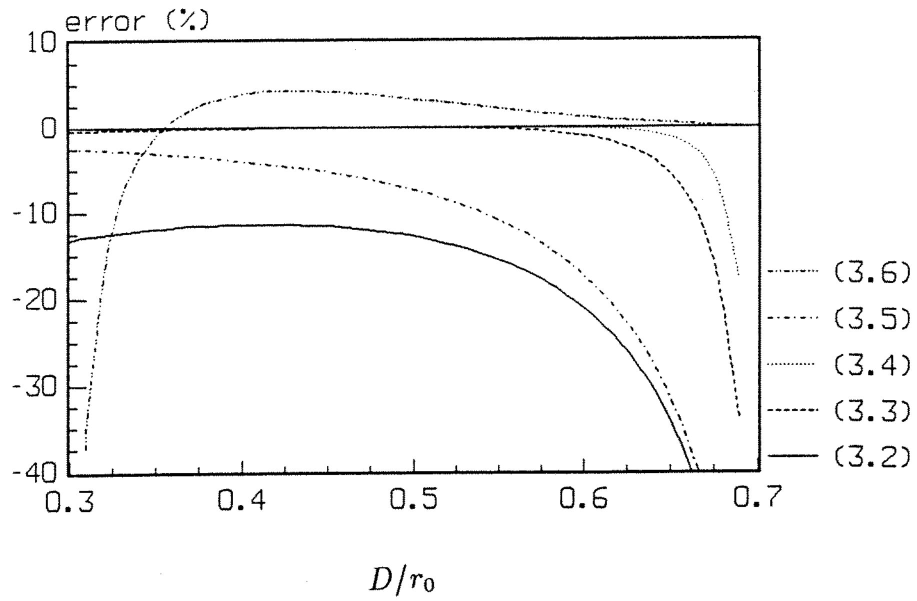


Figure 3.3. The relative error (%) of the different formulae to calculate q_s for different values of D/r_0 ($r_i/r_0 = 0.3$).

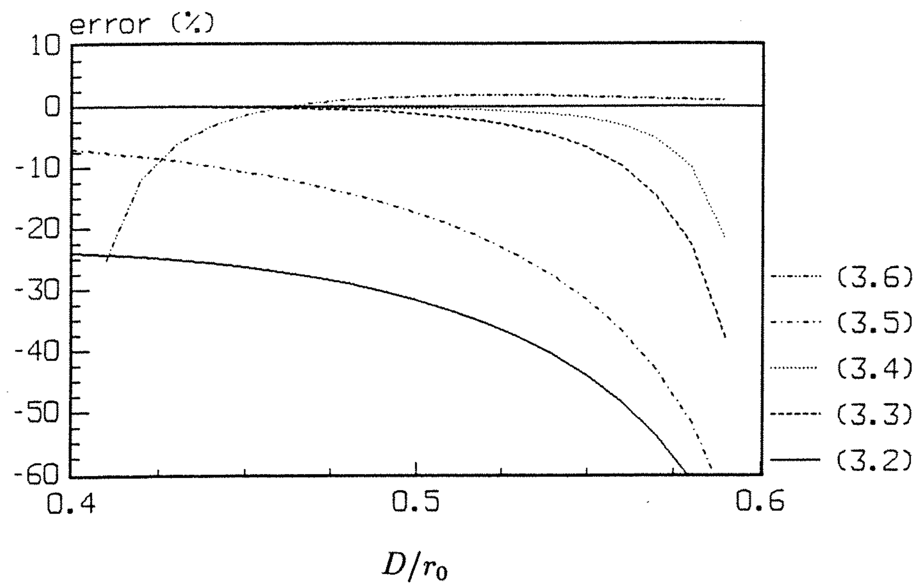


Figure 3.4. The relative error (%) of the different formulae to calculate q_s for different values of D/r_0 ($r_i/r_0 = 0.4$).

From Figures 3.2-4 one can see that all the formulae except the two-model approximation (3.6) "collapse" when the pipes lie close to the large pipe $(D + r_i)/r_0 \approx 1$. The zero-order (3.2) and area approximation formula (3.5) have a similar behavior for large D/r_0. The errors of the first (3.3) and second-order (3.4) formulae are very small except when the radius r_i is large.

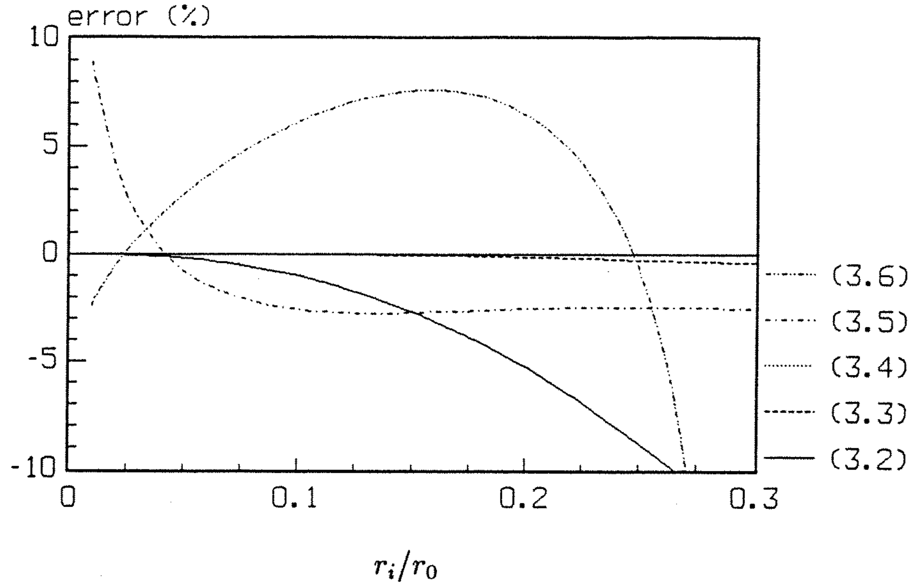


Figure 3.5. The relative error (%) of the different formulae to calculate q_s for different values of r_i/r_0 ($D/r_0 = 0.3$).

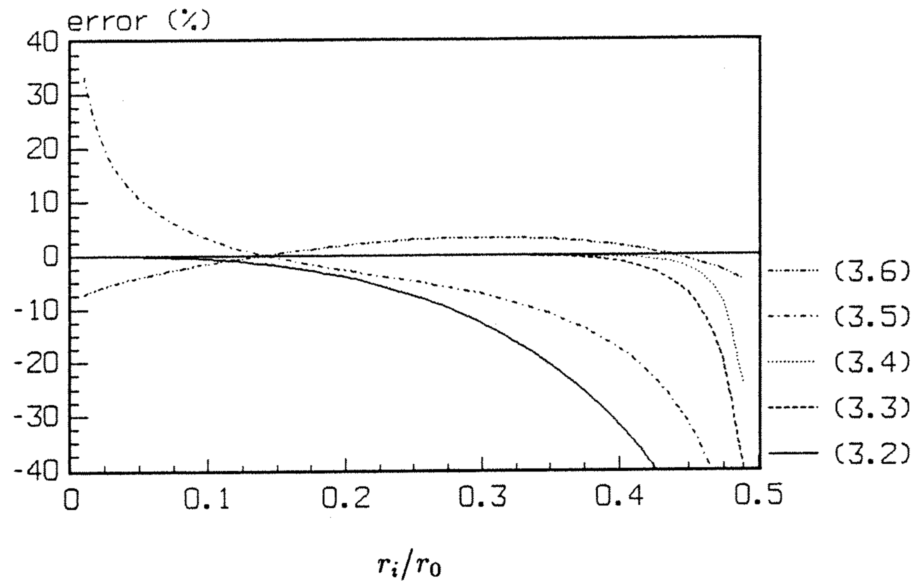


Figure 3.6. The relative error (%) of the different formulae to calculate q_s for different values of r_i/r_0 ($D/r_0 = 0.5$).

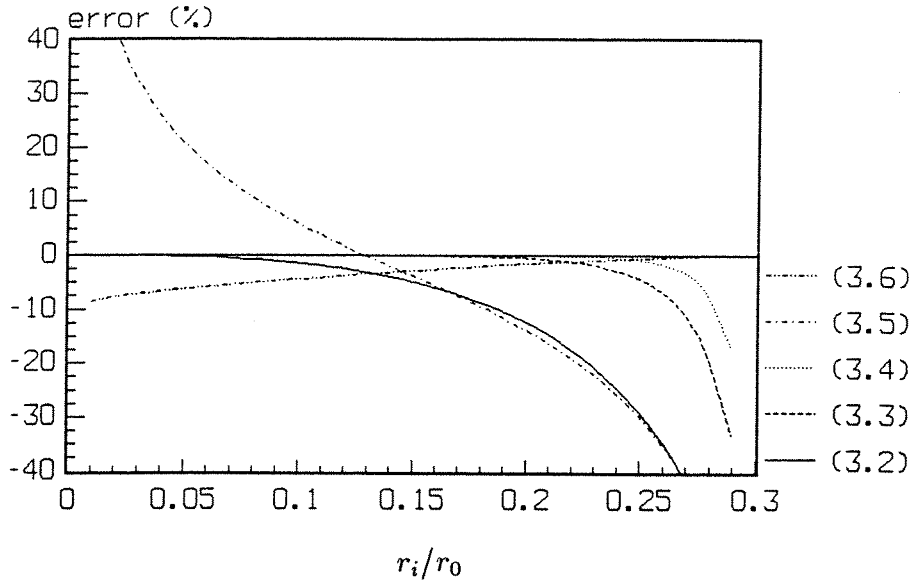


Figure 3.7. The relative error (%) of the different formulae to calculate q_s for different values of r_i/r_0 ($D/r_0 = 0.7$).

In Figure 3.5 the error of formula (3.4) is so small that the line disappears. It is notable that the formulae mostly underestimate the heat loss. From Figures 3.5-7 one can see that the errors of formulae (3.2-4) all approach zero when r_i/r_0 decreases. Table 3.2 shows the error of the formulae for some typical values of D/r_0 and r_i/r_0.

Relative error (%)			
	$r_i/r_0 = 0.25$	0.35	formula
$D/r_0 = 0.4$	-7.2	-17	(3.2)
	-0.084	-0.13	(3.3)
	-0.012	-0.048	(3.4)
	-3.2	-5.2	(3.5)
	5.3	-1.4	(3.6)
0.5	-7.7	-21	(3.2)
	-0.0090	-0.25	(3.3)
	-0.0090	-0.11	(3.4)
	-5.0	-11	(3.5)
	2.8	2.9	(3.6)

Table 3.2. The relative error (%) of the different formulae to calculate q_s for different values of r_i/r_0 and D/r_0.

The first-order formula (3.3) seems to be the best choice for practical use. It is a simple formula with only a small error.

4 Anti-symmetrical problem

For the anti-symmetrical problem the temperatures in the pipes are $T_a = (T_1 - T_2)/2$ and $-T_a$, see Figures 1.1 and 2.1. The temperature for the large pipe is $T_0 = 0$. The heat losses from the pipes are q_a and $-q_a$.

4.1 Exact solution

The exact solution to the problem was obtained with the use of the program described in [1]. The order of the highest used multipole has been at least 8. This means that error in the heat losses is approximately less than 0.001 %. Figure 4.1 and Table 4.1 show the computed heat loss factor $h_a(r_i/r_0, D/r_0)$. From equation (2.13) we get the heat loss q_a:

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0) \tag{4.1}$$

The heat loss q_a has a minimum for $D/r_0 \approx 0.5$. The heat loss is strongly dependent on r_i/r_0. The minimum gets thinner when r_i/r_0 increases.

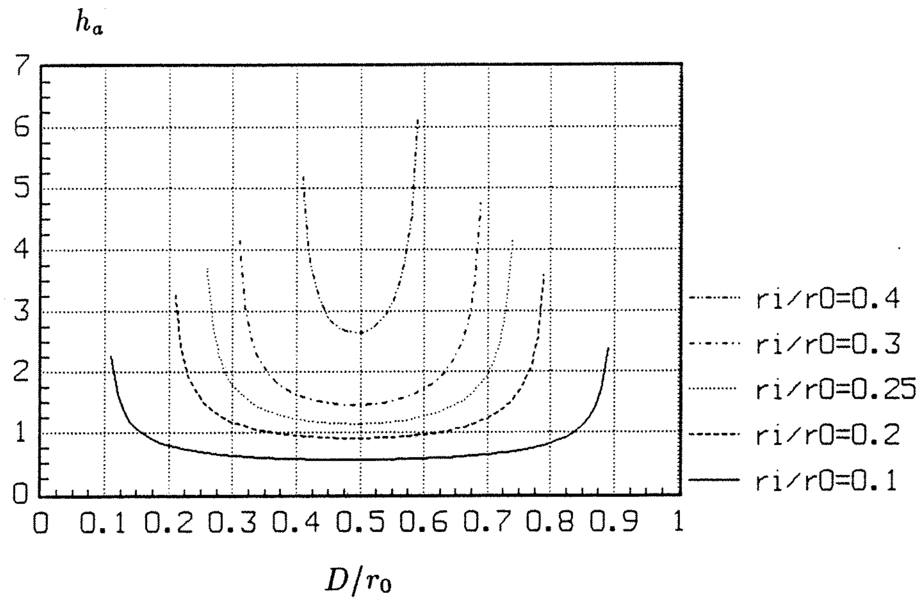


Figure 4.1. Heat loss factor h_a for different values of D/r_0 and r_i/r_0.

	h_a								
	$r_i/r_0 = 0.05$	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$D/r_0 = 0.10$	0.7681								
0.15	0.5805	1.0674							
0.20	0.5030	0.7964	1.3186						
0.25	0.4605	0.6852	0.9858	1.5545					
0.30	0.4345	0.6254	0.8512	1.1733	1.7946				
0.35	0.4181	0.5903	0.7812	1.0232	1.3770	2.0608			
0.40	0.4083	0.5699	0.7430	0.9503	1.2217	1.6232	2.3931		
0.45	0.4033	0.5599	0.7247	0.9171	1.1580	1.4827	1.9720	2.9052	
0.50	0.4024	0.5582	0.7216	0.9115	1.1471	1.4591	1.9090	2.6531	4.3040
0.55	0.4055	0.5642	0.7325	0.9305	1.1815	1.5262	2.0614	3.1398	
0.60	0.4126	0.5787	0.7588	0.9782	1.2723	1.7224	2.6315		
0.65	0.4246	0.6037	0.8059	1.0686	1.4650	2.2654			
0.70	0.4431	0.6438	0.8867	1.2434	1.9567				
0.75	0.4712	0.7094	1.0365	1.6742					
0.80	0.5159	0.8285	1.3996						
0.85	0.5961	1.1149							
0.90	0.7884								

Table 4.1. Heat loss factor h_a for different D/r_0 and r_i/r_0.

4.2 Approximate formulae

With the use of the multipole method described in [1] approximate formulae have been derived for the heat loss from the pipes in the anti-symmetrical problem. The derived formulae are of zero and first order. The general solution is described in 7.2.

4.2.1 Zero-order approximation

The zero-order multipole approximation uses the line sources and sinks without any multipoles. The zero-order approximation gives the following expression for the heat loss factor h_a (or thermal resistance R_a) for the anti-symmetrical problem:

$$h_a^{-1} = 2\pi\lambda_i R_a = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{4.2}$$

4.2.2 First-order approximation

With the use of multipoles of the first order, the following new formula is obtained:

$$h_a^{-1} = 2\pi\lambda_i R_a = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) - \frac{\left(\frac{r_i}{2D} - \frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - 2r_i^2 r_0^2 \cdot \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} \tag{4.3}$$

Formula (4.3) is derived in section 6.3.

4.2.3 Second-order approximation

For higher order multipole formulae, a general method is described in section 7.2. The second-order formula is shown here. The coefficients c_{11}, c_{21}, c_{22}, d_1, d_2 and A are introduced to make the formula more comprehensible.

$$c_{11} = 1 - \left(\frac{r_i}{2D}\right)^2 + \left(\frac{r_i D}{r_0^2 + D^2}\right)^2 - \left(\frac{r_i D}{r_0^2 - D^2}\right)^2 - \frac{r_i^2}{r_0^2 + D^2} - \frac{r_i^2}{r_0^2 - D^2}$$

$$c_{21} = -\left(\frac{r_i}{2D}\right)^3 + \left(\frac{r_i D}{r_0^2 + D^2}\right)^3 + \left(\frac{r_i D}{r_0^2 - D^2}\right)^3 - \frac{r_i^3 D}{(r_0^2 + D^2)^2} + \frac{r_i^3 D}{(r_0^2 - D^2)^2}$$

$$c_{22} = 1 - 3\left(\frac{r_i}{2D}\right)^4 + 3\left(\frac{r_i D}{r_0^2 + D^2}\right)^4 - 3\left(\frac{r_i D}{r_0^2 - D^2}\right)^4$$

$$-4\frac{r_i^4 D^2}{(r_0^2 + D^2)^3} - 4\frac{r_i^4 D^2}{(r_0^2 - D^2)^3} + \left(\frac{r_i^2}{r_0^2 + D^2}\right)^2 - \left(\frac{r_i^2}{r_0^2 - D^2}\right)^2$$

$$d_1 = -\frac{r_i}{2D} + \frac{r_i D}{r_0^2 + D^2} + \frac{r_i D}{r_0^2 - D^2}$$

$$d_2 = \frac{1}{2}\left(-\left(\frac{r_i}{2D}\right)^2 + \left(\frac{r_i D}{r_0^2 + D^2}\right)^2 - \left(\frac{r_i D}{r_0^2 - D^2}\right)^2\right)$$

$$A = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right)$$

$$h_a^{-1} = 2\pi\lambda_i R_a = A + \frac{4d_1 d_2 c_{21} - d_1^2 c_{22} - 2d_2^2 c_{11}}{c_{11}c_{22} - 2c_{21}^2} \tag{4.4}$$

4.3 Errors of the formulae

The errors of the different formulae have been studied with the use of the multipole program [1]. Figures 4.2-7 and Table 4.3 show the error made when the heat loss q_a is calculated with formulae (4.2-4). The error is expressed in per cent. A positive relative error means that the formula gives too large a heat loss.

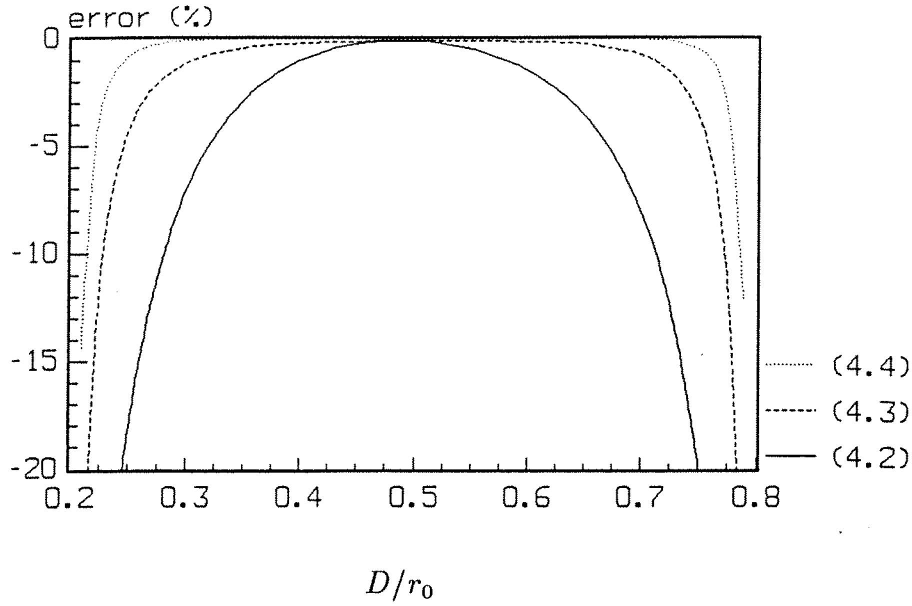


Figure 4.2. The relative error (%) of the different formulae to calculate q_a for different values of D/r_0 ($r_i/r_0 = 0.2$).

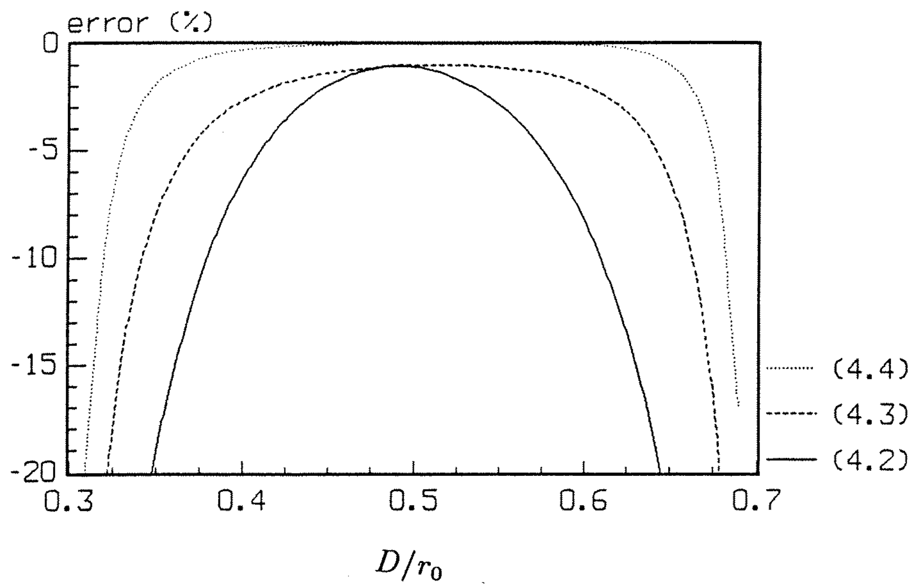


Figure 4.3. The relative error (%) of the different formulae to calculate q_a for different values of D/r_0 ($r_i/r_0 = 0.3$).

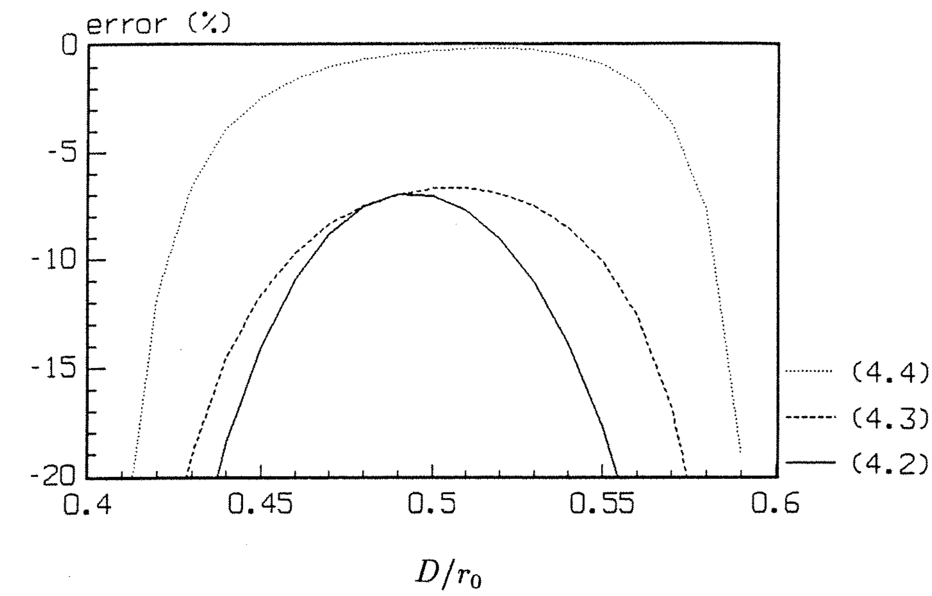


Figure 4.4. The relative (%) error of the different formulae to calculate q_a for different values of D/r_0 ($r_i/r_0 = 0.4$).

The absolute error of all the formulae has a minimum when $D/r_0 \approx 0.5$. Close to this minimum is the zero order formula (4.2) equal to the first-order formula (4.3).

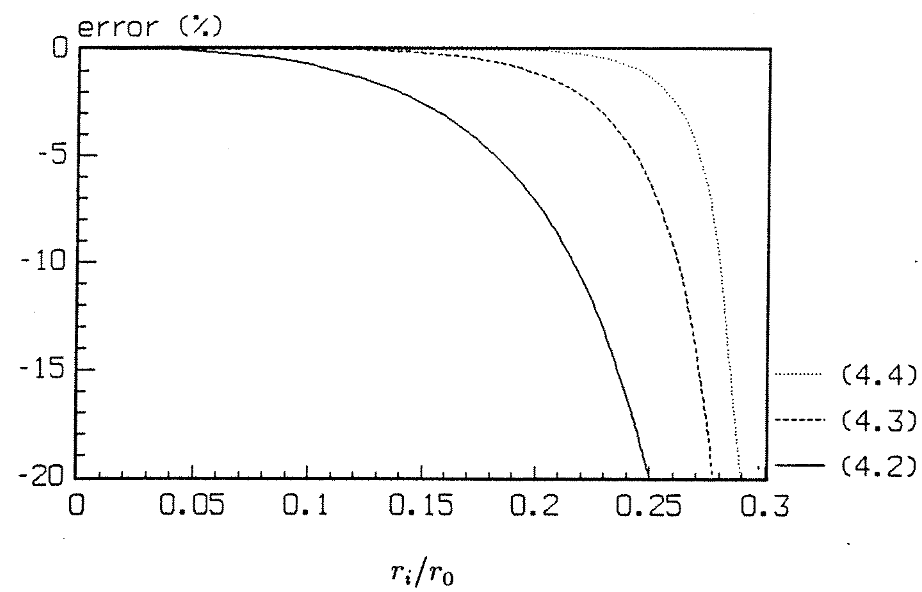


Figure 4.5. The relative error (%) of the different formulae to calculate q_a for different values of r_i/r_0 ($D/r_0 = 0.3$).

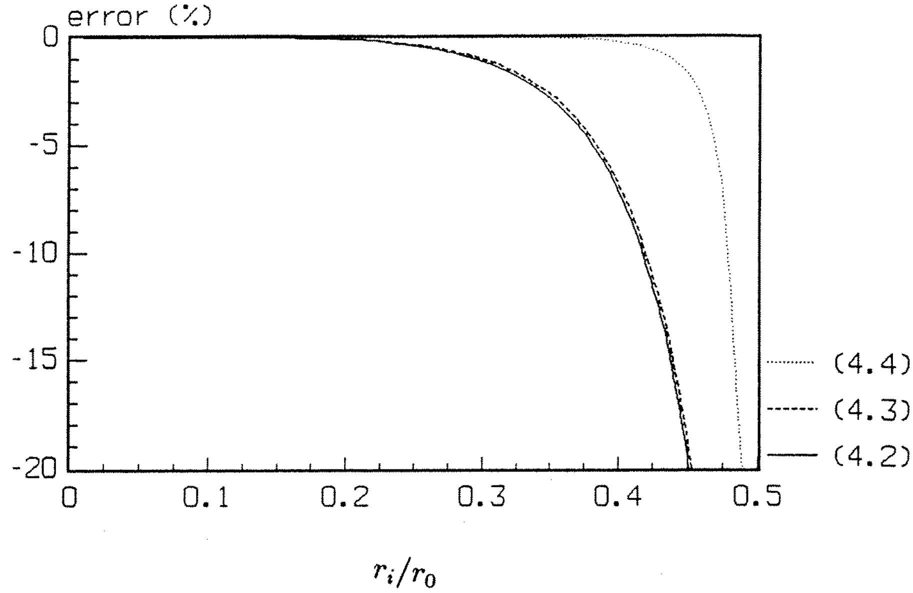


Figure 4.6. The relative error (%) of the different formulae to calculate q_a for different values of r_i/r_0 ($D/r_0 = 0.5$).

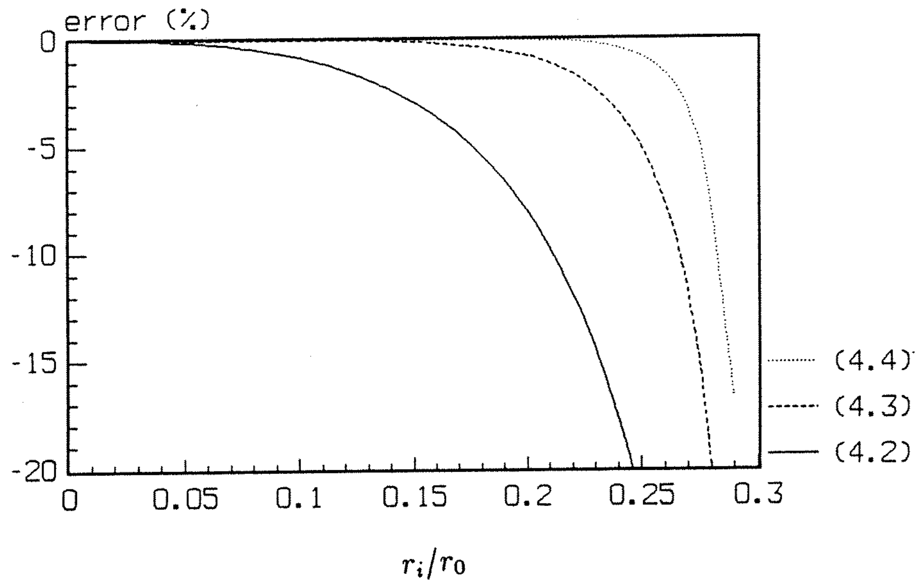


Figure 4.7. The relative error (%) of the different formulae to calculate q_a for different values of r_i/r_0 ($D/r_0 = 0.7$).

The error of all the formulae increases when r_i/r_0 increases. All the formulae underestimate the heat loss. Table 4.2 shows the error of the formulae for some typical values of D/r_0 and r_i/r_0.

Relative error (%)			
	r_i/r_0 = 0.25	0.35	formula
D/r_0= 0.4	-2.6	-17	(4.2)
	-0.83	-9.6	(4.3)
	-0.055	-2.3	(4.4)
0.5	-0.42	-2.8	(4.2)
	-0.38	-2.6	(4.3)
	-0.0041	-0.060	(4.4)

Table 4.2. The relative error (%) of the different formulae to calculate q_a for different values of r_i/r_0 and D/r_0.

The first-order formula (4.3) seems to be the best choice for practical use.

5 Multipole method

The general multipole method can, with some adjustments, be used to solve many different problems. The only restriction is that the boundaries are circular. In the following the multipole method used in this report is described. This method is based on [1] that deals with the problem of pipes in a composite cylinder. The method used here is acquired from [1] by letting the thermal conductivity in the outer circle approach infinity.

5.1 Thermal problem

Consider the problem in Figure 5.1. There are N pipes with different radii r_n inside a large pipe with the radius r_0. The temperature at the large pipe is T_0. The temperatures in the pipes inside the large pipe are T_n. The thermal conductivity in the large pipe is λ_i.

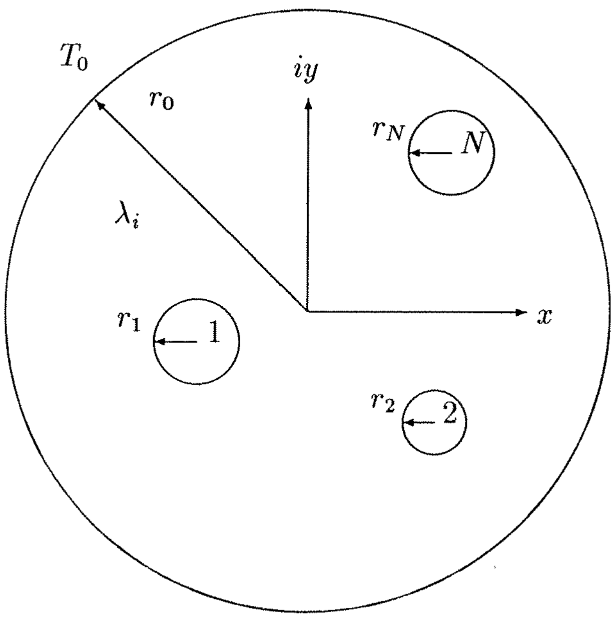


Figure 5.1. N pipes in a pipe in the complex plane.

The stationary heat conduction equation for $T(x, y)$ is to be solved:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{5.1}$$

Equation (5.1) is also called the Laplace equation.

The problem is solved in the complex plane ($z = x + i \cdot y$). Here i is used to describe the imaginary unit ($i^2 = -1$). The complex conjugate of z is denoted $\overline{z}$.

The center of the pipe n is z_n.

$$z_n = x_n + i \cdot y_n$$

Local polar coordinates ρ_n,ψ_n from the center of pipe n is used:

$$z - z_n = \rho_n e^{i\psi_n} \tag{5.2}$$

The boundary condition at each pipe n is:

$$T = T_n \qquad \rho_n = r_n, \qquad 0 \leq \psi_n \leq 2\pi \tag{5.3}$$

The boundary condition at the outer pipe is:

$$T = T_0 \qquad \rho = r_0, \qquad 0 \leq \psi_n \leq 2\pi \tag{5.4}$$

5.2 The multipole method

According to the multipole method the temperature field in the large pipe consists of a line source part, a multipole part and the temperature on the large pipe T_0:

$$T(x,y) = T_0 + T_q(x,y) + T_p(x,y) \tag{5.5}$$

There is a line source with the strength q_n in the center of each pipe. In order to satisfy the boundary condition at the large pipe there is, for each pipe, a mirror line source with the strength $-q_n$ at $r_0^2/\bar{z}_n$. The function $T_q(z)$ will always be zero at $x^2 + y^2 = r_0^2$.

$$T_q(x,y) = \Re\left[\sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_i} \cdot W_{n0}(z)\right] \tag{5.6}$$

$$W_{n0}(z) = \ln\left(\frac{r_0}{z - z_n}\right) - \ln\left(\frac{r_0^2}{r_0^2 - \overline{z}z_n}\right) \tag{5.7}$$

Both the real and imaginary parts of W_{n0} each satisfy the Laplace equation (5.1) and the boundary condition (5.4).

The complex-valued derivative of order j of W_{n0} with respect to z_n is called a multipole of order j. We will use notation W_{nj}. The complex strength of each multipole is P_{nj}.

$$T_p(x,y) = \Re\left[\sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj} \cdot r_n^j \cdot W_{nj}(z)\right] \tag{5.8}$$

$$W_{nj}(z) = \frac{1}{(j-1)!} \cdot \frac{\partial^j}{\partial z_n^j}(W_{n0}) = \frac{1}{(z - z_n)^j} - \left(\frac{\overline{z}}{r_0^2 - \overline{z}z_n}\right)^j \tag{5.9}$$

Both the real and imaginary parts of W_{nj} each satisfy the Laplace equation (5.1) and the boundary condition (5.4). The quantities q_n and P_{nj} are determined by the boundary conditions. The expression (5.5) is inserted in the boundary condition (5.3) of each pipe.

The line source q_n and the multipoles P_{nj} of pipe n can represent any solution of the heat conduction equation in the region outside pipe n. The boundary condition at the large pipe is satisfied for every choice of q_n and P_{nj}.

5.3 Boundary condition at pipe m

To solve the boundary condition problem of pipe m we need expressions for the line sources of the pipes and the multipoles in the local polar coordinates of pipe m. From [1] we get these expressions. In the following, n is the number of the pipe with line source or multipole and m is the number of the pipe which boundary condition is to be satisfied. We will use polar coordinates from pipe m:

$$z = z_m + \rho_m e^{i \cdot \psi_m} \tag{5.10}$$

$\boxed{n = m}$

$$\ln\left(\frac{r_0}{z - z_m}\right) = \ln\left(\frac{r_0}{\rho_m}\right) - i\psi_m \tag{5.11}$$

$$\left(\frac{r_m}{z - z_m}\right)^j = \left(\frac{r_m}{\rho_m}\right)^j e^{-i \cdot j \psi_m} \tag{5.12}$$

$\boxed{n \neq m}$

$$\ln\left(\frac{r_0}{z - z_n}\right) = \ln\left(\frac{r_0}{z_m - z_n}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m} \tag{5.13}$$

$$\left(\frac{r_n}{z - z_n}\right)^j = \left(\frac{r_n}{z_m - z_n}\right)^j \sum_{k=0}^{\infty} \begin{pmatrix} k + j - 1 \\ j - 1 \end{pmatrix} \left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m} \tag{5.14}$$

$\boxed{\text{every } n \text{ and } m}$

$$\left(\frac{r_n \overline{z}}{r_0^2 - \overline{z} z_n}\right)^j = \tag{5.15}$$

$$\sum_{k=0}^{\infty} \sum_{j'=0}^{min(j,k)} \begin{pmatrix} j \\ j' \end{pmatrix} \begin{pmatrix} k + j - j' - 1 \\ j - 1 \end{pmatrix} \frac{r_n^j \cdot \overline{z}_m^{j-j'} \cdot z_n^{k-j'}}{(r_0^2 - z_n \overline{z}_m)^{j+k-j'}} \cdot \rho_m^k e^{-ik\psi_m}$$

$$\ln\left(\frac{r_0^2}{r_0^2 - \overline{z} z_n}\right)^j = \tag{5.16}$$

$$\ln\left(\frac{r_0^2}{r_0^2 - \overline{z_m} z_n}\right)^j + \sum_{k=1}^{\infty}\left(\frac{z_n \rho_m}{r_0^2 - z_n \overline{z}_m}\right)^k \cdot e^{-ik\psi_m}$$

With (5.5-5.16) the boundary condition (5.3) of pipe m becomes:

$$\boxed{m = 1,2\ldots N \qquad 0 \leq \psi_m < 2\pi}$$

$$T_m = T_0 + \frac{q_m}{2\pi\lambda_i}\ln\left(\frac{r_0}{r_m}\right) \tag{5.17}$$

$$+\Re\left[\sum_{n\neq m}\frac{q_n}{2\pi\lambda_i}\left\{\ln\left(\frac{r_0}{z_m - z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k \cdot e^{ik\psi_m}\right\}\right.$$

$$-\sum_{n=1}^{N}\frac{q_n}{2\pi\lambda_i}\left\{\ln\left(\frac{r_0^2}{r_o^2 - \overline{z}_m z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_n r_m}{r_o^2 - \overline{z}_m z_n}\right)^k \cdot e^{-ik\psi_m}\right\}$$

$$+\sum_{j=1}^{\infty} P_{mj} e^{-ij\psi_m}$$

$$+\sum_{n\neq m}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} P_{nj}\left(\frac{r_n}{z_m - z_n}\right)^j \binom{j+k-1}{j-1}\left(\frac{r_m}{z_n - z_m}\right)^k e^{ik\psi_m}$$

$$\left.-\sum_{n=1}^{N}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j'=0}^{min(j,k)} P_{nj}\binom{j}{j'}\binom{j+k-j'-1}{j-1}\frac{r_n^j \cdot \overline{z}_m^{j-j'} \cdot z_n^{k-j'}}{(r_0^2 - z_n\overline{z}_m)^{j+k-j'}} \cdot \rho_m^k e^{-ik\psi_m}\right]$$

The summation index on the fourth line (concerning P_{mj}) is changed into k. The dependence on ψ_m lies in the exponents $e^{i\cdot k\psi_m}$ and $e^{-i\cdot k\psi_m}$. The latter terms may be changed into the positive exponent $e^{i\cdot k\psi_m}$ by taking the complex conjugate.

Equation (5.17) can now be separated into a part independent of ψ_m, and a part depending on ψ_m. The part independent of ψ_m is:

$$\boxed{m = 1,2\ldots N}$$

$$T_m - T_0 = \sum_{n=1}^{N} q_n \cdot R^{\circ}_{mn} \tag{5.18}$$

$$+\Re\left[\sum_{n\neq m}\sum_{j=1}^{\infty} P_{nj}\left(\frac{r_n}{z_m - z_n}\right)^j - \sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj}\left(\frac{r_n\overline{z}_m}{r_0^2 - z_n\overline{z}_m}\right)^j\right]$$

The thermal resistances R°_{mn} (K/(W/m)) in (5.18) are given by:

$$R^{\circ}_{mm} = \frac{1}{2\pi\lambda_i}\left(\ln\left(\frac{r_0}{r_m}\right) - \ln\left(\frac{r_0^2}{r_0^2 - |z_m|^2}\right)\right) \tag{5.19}$$

$$R^{\circ}_{mn} = \frac{1}{2\pi\lambda_i}\left(\ln\left(\frac{r_0}{|z_m - z_n|}\right) - \ln\left(\frac{r_0^2}{|r_0^2 - \overline{z}_n z_m|}\right)\right) \tag{5.20}$$

The part depending on ψ_m ($e^{i\psi_m}$) is:

$$\boxed{m = 1,2\ldots N \qquad k = 1,2\ldots}$$

$$0 = \overline{P}_{mk} + \sum_{n\neq m} \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k - \sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m\overline{z}_n}{r_0^2 - \overline{z}_n z_m}\right)^k \tag{5.21}$$

$$+\sum_{n\neq m}\sum_{j=1}^{\infty} P_{nj}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{r_n}{z_m - z_n}\right)^j\left(\frac{r_m}{z_n - z_m}\right)^k$$

$$-\sum_{n=1}^{N}\sum_{j=1}^{\infty}\sum_{j'=0}^{min(j,k)} \overline{P}_{nj}\begin{pmatrix} j \\ j' \end{pmatrix}\begin{pmatrix} k+j-j'-1 \\ j-1 \end{pmatrix}\cdot\frac{r_n^j\cdot r_m^k\cdot z_m^{j-j'}\cdot \overline{z}_n^{k-j'}}{(r_0^2 - \overline{z}_n z_m)^{j+k-j'}}$$

5.4 Final equation system

The equation system (5.18,5.21) must be truncated. We consider multipoles at the pipes up to order J. Here J is a positive integer or in the lowest approximation zero, in which case only the line sources and mirror line sinks are used. The sine- and cosine-variation around the pipes can be made zero up to order J only. We get the following equation system:

$$\boxed{m = 1,2\ldots N}$$

$$T_m - T_0 = \sum_{n=1}^{N} q_n \cdot R^{\circ}_{mn} \tag{5.22}$$

$$+\Re\left[\sum_{n\neq m}\sum_{j=1}^{J} P_{nj}\left(\frac{r_n}{z_m - z_n}\right)^j - \sum_{n=1}^{N}\sum_{j=1}^{J} P_{nj}\left(\frac{r_n\overline{z}_m}{r_0^2 - z_n\overline{z}_m}\right)^j\right]$$

$$\boxed{m = 1,\ldots,N \qquad k = 1,\ldots,J}$$

$$0 = \overline{P}_{mk} + \sum_{n\neq m} \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k - \sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m\overline{z}_n}{r_0^2 - \overline{z}_n z_m}\right)^k \tag{5.23}$$

$$+\sum_{n\neq m}\sum_{j=1}^{J} P_{nj}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{r_n}{z_m - z_n}\right)^j\left(\frac{r_m}{z_n - z_m}\right)^k$$

$$-\sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{j'=0}^{min(j,k)} \overline{P}_{nj}\begin{pmatrix} j \\ j' \end{pmatrix}\begin{pmatrix} k+j-j'-1 \\ j-1 \end{pmatrix}\cdot\frac{r_n^j\cdot r_m^k\cdot z_m^{j-j'}\cdot \overline{z}_n^{k-j'}}{(r_0^2 - \overline{z}_n z_m)^{j+k-j'}}$$

The thermal resistances R°_{mn} (K/(W/m)) in (5.22) are given by (5.19,5.20). These are the equations that completely determine the strength of the multipoles and the line sources.

6 Derivation of the first-order formulae

We will here derive the first-order formulae from equation system (5.22,23). Due to the symmetry of the problem the multipole strengths have a simple relationship that can be calculated. This is done in section 6.1. In section 6.2 formula for the symmetrical problem (3.3) is derived and in section 6.3 formula for the anti-symmetrical problem (4.3) is derived.

6.1 Symmetry analysis

Figure 6.1 describes the problem. The parameters are defined in section 1.1.

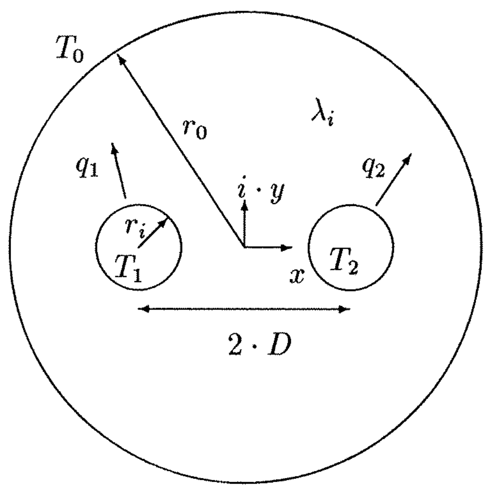


Figure 6.1. Two pipes inside a large pipe.

The position of the pipes are:

$$z_1 = -D \qquad\qquad z_2 = D \tag{6.1}$$

The temperature field is in accordance with (5.5) divided into three parts:

$$T(x, y) = T_0 + T_q(z) + T_p(z) \tag{6.2}$$

Here $T_q(z)$ is the temperature field from the line sources, and $T_p(z)$ is the temperature field from the multipoles. The temperature on the large pipe is T_0.

$$T_q(z) = \Re\left[\sum_{n=1}^{2} \frac{q_n}{2\pi\lambda_i} \cdot W_{n0}(z)\right] \tag{6.3}$$

$$W_{n0}(z) = \ln\left(\frac{r_0}{z - z_n}\right) - \ln\left(\frac{r_0^2}{r_0^2 - \overline{z}z_n}\right) \tag{6.4}$$

We consider multipoles up to order J.

$$T_p(z) = \Re\left[\sum_{n=1}^{2}\sum_{j=1}^{J} P_{nj} \cdot r_n^j \cdot W_{nj}(z)\right] \tag{6.5}$$

$$W_{nj}(z) = \frac{1}{(z - z_n)^j} - \left(\frac{\overline{z}}{r_0^2 - \overline{z}z_n}\right)^j \tag{6.6}$$

From (6.1) and (6.6) we get:

$$W_{2j}(z) = (-1)^j W_{1j}(-z) \tag{6.7}$$

$$\overline{W}_{nj}(z) = W_{nj}(\overline{z}) \tag{6.8}$$

6.1.1 Symmetrical problem

For the symmetrical problem we have:

$$T_1 = T_2 = T_s \tag{6.9}$$

$$q_1 = q_2 = q_s \tag{6.10}$$

The temperature field must be symmetric with respect to the real and imaginary axis:

$$T(z) = T(-\overline{z}) \tag{6.11}$$

$$T(z) = T(\overline{z}) \tag{6.12}$$

The multipole part $T_p(z)$ of equation (6.11) becomes with (6.7):

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(z) + P_{2j}(-1)^j W_{1j}(-z)\right\} r_i^j\right] = \tag{6.13}$$

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(-\overline{z}) + P_{2j}(-1)^j W_{1j}(\overline{z})\right\} r_i^j\right]$$

This must be true for every J and hence for every j. When the right side of equation (6.13) is complex conjugated and equation (6.8) is used, one gets:

$$\Re\left[W_{1j}(z)\left(P_{1j} - \overline{P}_{2j} \cdot (-1)^j\right) + W_{1j}(-z)\left(-\overline{P}_{1j} + P_{2j} \cdot (-1)^j\right)\right] = 0 \tag{6.14}$$

If equation (6.14) is to be satisfied for every j and z the following must be true:

$$P_{1j} = (-1)^j \overline{P}_{2j} \tag{6.15}$$

The multipole part of equation (6.12) becomes with (6.7):

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(z)+P_{2j}(-1)^{j}W_{1j}(-z)\right\}r_i^j\right] = \tag{6.16}$$

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(\overline{z})+P_{2j}(-1)^{j}W_{1j}(-\overline{z})\right\}r_i^j\right]$$

This must be true for every J and hence for all j. When the right side of equation (6.16) is complex conjugated and equation (6.8) is used, one gets:

$$\Re\left[W_{1j}(z)\left(P_{1j}-\overline{P}_{1j}\right)+W_{1j}(-z)\cdot(-1)^{j}\left(P_{2j}-\overline{P}_{2j}\right)\right]=0 \tag{6.17}$$

From (6.17) we see that the following is true for the symmetrical problem:

$$\Im\left[P_{1j}\right]=\Im\left[P_{2j}\right]=0 \tag{6.18}$$

From (6.15) and (6.18) we get:

$$\Re\left[P_{1j}\right]=(-1)^{j}\Re\left[P_{2j}\right] \tag{6.19}$$

6.1.2 Anti-symmetrical problem

For the anti-symmetrical problem equations we have:

$$T_1=-T_2=-T_a \tag{6.20}$$

$$q_1=-q_2=-q_a \tag{6.21}$$

The temperature field must be symmetric with respect to the real axis thus equation (6.18) must be true:

$$T(z)=T(\overline{z})\Rightarrow\Im\left[P_{1j}\right]=\Im\left[P_{2j}\right]=0 \tag{6.22}$$

The temperature field must be anti-symmetric with respect to the imaginary axis:

$$T(z)=-T(-\overline{z}) \tag{6.23}$$

The multipole part of equation (6.23) becomes with (6.7):

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(z)+P_{2j}(-1)^{j}W_{1j}(-z)\right\}r_i^j\right] = \tag{6.24}$$

$$-\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(-\overline{z})+P_{2j}(-1)^{j}W_{1j}(-z)\right\}r_i^j\right]$$

This must be true for every J and hence for all j. When the right side of equation (6.24) is complex conjugated and equation (6.8) is used, one gets:

$$\Re\left[W_{1j}(z)\left(P_{1j}+\overline{P}_{2j}\cdot(-1)^{j}\right)+W_{1j}(-z)\left(P_{2j}\cdot(-1)^{j}+\overline{P}_{1j}\right)\right]=0 \tag{6.25}$$

From (6.22) and (6.25) we see that the following is true for the anti-symmetrical problem:

$$\Re\left[P_{1j}\right] = (-1)^{j+1}\Re\left[P_{2j}\right] \tag{6.26}$$

$$\Im\left[P_{1j}\right] = \Im\left[P_{2j}\right] = 0 \tag{6.27}$$

6.2 Derivation of formula (3.3)

We will here derive the first-order multipole formula for the symmetrical problem. The problem is described in Figure 6.1 with equations (6.9-10). For the multipoles equations (6.18-19) with $J = j = 1$ are true. We will use the following notations:

$$P_{11} = M_1 \tag{6.28}$$

$$P_{21} = -M_1 \tag{6.29}$$

Here M_1 is a non-complex constant. When equations (6.28-29) are used in (5.19,5.20,5.22) with $m = 1$ one gets:

$$T_s - T_0 = \frac{q_s}{2\pi\lambda_i}\left(\ln\left(\frac{r_0}{r_i}\right) - \ln\left(\frac{r_0^2}{r_0^2 - D^2}\right)\right) + \ln\left(\frac{r_0}{2D}\right) - \ln\left(\frac{r_0^2}{r_0^2 + D^2}\right) \tag{6.30}$$

$$+M_1\left\{\frac{r_i}{2D} + \frac{r_iD}{r_0^2 - D^2} - \frac{r_iD}{r_0^2 + D^2}\right\}$$

Equation (6.30) reduces to the following equation:

$$T_s - T_0 = \frac{q_s}{2\pi\lambda_i}\left(\ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right)\right) \tag{6.31}$$

$$+M_1\left(\frac{r_i}{2D} + \frac{2r_iD^3}{r_0^4 - D^4}\right)$$

When equations (6.28-29) are used in (5.19,5.20,5.23) with $m = 1, k = 1$ one gets:

$$0 = M_1 + \frac{q_s}{2\pi\lambda_i}\left\{\frac{r_i}{2D} + \frac{r_iD}{r_0^2 - D^2} - \frac{r_iD}{r_0^2 + D^2}\right\} \tag{6.32}$$

$$+M_1\left\{\left(\frac{r_i}{2D}\right)^2 - \frac{r_i^2D^2}{(r_0^2 - D^2)^2} - \frac{r_i^2}{r_0^2 - D^2} - \frac{r_i^2D^2}{(r_0^2 + D^2)^2} + \frac{r_i^2}{r_0^2 + D^2}\right\}$$

After some simplifications equation (6.32) reduces to

$$M_1 = -\frac{q_s}{2\pi\lambda_i}\left(\frac{r_i}{2D} + \frac{2r_iD^3}{r_0^4 - D^4}\right)\cdot\left(1 + \left(\frac{r_i}{2D}\right)^2 - \left(\frac{2r_ir_0D}{r_0^4 - D^4}\right)^2\right)^{-1} \tag{6.33}$$

This expression for M_1 is used in equation (6.31).

$$\frac{2\pi\lambda_i(T_s - T_0)}{q_s} = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{6.34}$$

$$-\left(\frac{r_i}{2D}+\frac{2r_iD^3}{r_0^4-D^4}\right)^2\cdot\left(1+\left(\frac{r_i}{2D}\right)^2-\left(\frac{2r_ir_0D}{r_0^4-D^4}\right)^2\right)^{-1}$$

Equation (6.34) is the same as (3.3).

6.3 Derivation of formula (4.3)

For the anti-symmetrical problem equations (6.20-21) are true. For the multipoles equations (6.26-27) with $J=j=1$ are true. The temperature on the larger pipe is zero:

$$T_0=0 \tag{6.35}$$

We will use the following notations:

$$P_{11}=M_1 \tag{6.36}$$

$$P_{21}=M_1 \tag{6.37}$$

Here M_1 is a non-complex constant. When equations (6.36-37) are used in (5.19,5.20,5.22) with $m=1$ one gets:

$$T_a=\frac{q_a}{2\pi\lambda_i}\left(-\ln\left(\frac{r_0}{2D}\right)+\ln\left(\frac{r_0^2}{r_0^2+D^2}\right)+\ln\left(\frac{r_0}{r_i}\right)-\ln\left(\frac{r_0^2}{r_0^2-D^2}\right)\right) \tag{6.38}$$

$$+M_1\left\{\frac{r_i}{2D}-\frac{r_iD}{r_0^2-D^2}-\frac{r_iD}{r_0^2+D^2}\right\}$$

Equation (6.38) reduces to the following equation:

$$T_a=\frac{q_a}{2\pi\lambda_i}\left(\ln\left(\frac{2D}{r_i}\right)-\ln\left(\frac{r_0^2+D^2}{r_0^2-D^2}\right)\right) \tag{6.39}$$

$$+M_1\left(\frac{r_i}{2D}-\frac{2r_iDr_0^2}{r_0^4-D^4}\right)$$

When equations (6.36-37) are used in (5.19,5.20,5.23) with $m=1, k=1$ one gets:

$$0=M_1+\frac{q_a}{2\pi\lambda_i}\left\{\frac{r_i}{2D}-\frac{r_iD}{r_0^2-D^2}-\frac{r_iD}{r_0^2+D^2}\right\} \tag{6.40}$$

$$+M_1\left\{-\left(\frac{r_i}{2D}\right)^2-\frac{r_i^2D^2}{(r_0^2-D^2)^2}-\frac{r_i^2}{r_0^2-D^2}+\frac{r_i^2D^2}{(r_0^2+D^2)^2}-\frac{r_i^2}{r_0^2+D^2}\right\}$$

After some simplifications equation (6.40) reduces to

$$M_1=-\frac{q_a}{2\pi\lambda_i}\left(\frac{r_i}{2D}-\frac{2r_iDr_0^2}{r_0^4-D^4}\right)\cdot\left(1-\left(\frac{r_i}{2D}\right)^2-2r_i^2r_0^2\frac{r_0^4+D^4}{(r_0^4-D^4)^2}\right)^{-1} \tag{6.41}$$

The expression for M_1 is used in equation (6.39).

$$\frac{2\pi\lambda_i T_a}{q_a} = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{6.42}$$

$$-\left(\frac{r_i}{2D} - \frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2 \cdot \left(1 - \left(\frac{r_i}{2D}\right)^2 - 2r_i^2 r_0^2 \frac{r_0^4 + D^4}{\left(r_0^4 - D^4\right)^2}\right)^{-1}$$

Equation (6.42) is the same as (4.3).

7 Derivation of the general formulae

In this chapter general expressions for the J order multipole formulae for the heat loss factors h_s and h_a are shown. In section 7.1 the formula for the heat loss factor in the symmetrical problem h_s is shown and in section 7.2 the formula for the heat loss factor for the anti-symmetrical problem h_a is shown. The formulae are expressed with the use of matrices.

7.1 The symmetrical problem

The problem is described in Figure 6.1. For the symmetrical problem equations we have:

$$T_1 = T_2 = T_s \tag{7.1}$$

$$q_1 = q_2 = q_s \tag{7.2}$$

From section 6.1 we see that the strength of the multipoles satisfies the following equations:

$$\mathrm{Re}\,[P_{1j}] = (-1)^j \mathrm{Re}\,[P_{2j}] \tag{7.3}$$

$$\mathrm{Im}\,[P_{1j}] = \mathrm{Im}\,[P_{2j}] = 0 \tag{7.4}$$

We will use the following notations:

$$P_{1j} = M_j \tag{7.5}$$

$$P_{2j} = (-1)^j M_j \tag{7.6}$$

Here M_j is a non-complex constant. When equations (7.1-6) are used in (5.19,5.20,5.22) with $m = 1$ one gets:

$$T_s - T_0 = \frac{q_s}{2\pi\lambda_i}\left(\ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right)\right) \tag{7.7}$$

$$+\sum_{j=1}^{J} M_j \left\{\left(\frac{r_i}{2D}\right)^j - (-1)^j\left(\frac{r_iD}{r_0^2 - D^2}\right)^j - \left(\frac{r_iD}{r_0^2 + D^2}\right)^j\right\}$$

When equations (7.1-6) are used in (5.19,5.20,5.23) with $m = 1$ one gets:

$$\boxed{k = 1, \dots, J}$$

$$0 = M_k + \frac{q_s}{2\pi\lambda_i} \cdot \frac{1}{k} \left\{ \left(\frac{r_i}{2D}\right)^k - (-1)^k \left(\frac{r_i D}{r_0^2 - D^2}\right)^k - \left(\frac{r_i D}{r_0^2 + D^2}\right)^k \right\} \tag{7.8}$$

$$+ \sum_{j=1}^{J} M_j \left\{ \binom{j+k-1}{j-1} \left(\frac{r_i}{2D}\right)^{k+j} - \sum_{j'=0}^{min(j,k)} r_i^{j+k} D^{j+k-2j'} \right.$$

$$\left. \cdot \left(\frac{(-1)^{j+k-2j'}}{(r_0^2 - D^2)^{j+k-j'}} + \frac{(-1)^{2j-j'}}{(r_0^2 + D^2)^{j+k-j'}} \right) \binom{j}{j'} \binom{j+k-j'-1}{j-1} \right\}$$

7.1.1 General solution to equation system

If the equation system (7.7-8) is expressed in matrix notation one gets:

$$T_s - T_0 = \frac{q_s}{2\pi\lambda_i} \cdot A + \mathbf{B} \cdot \mathbf{M} \tag{7.9}$$

$$\mathbf{C} \cdot \mathbf{M} = \frac{q_s}{2\pi\lambda_i} \mathbf{D} \tag{7.10}$$

$$A = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{7.11}$$

$$\mathbf{M} = \begin{pmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \\ M_J \end{pmatrix} \qquad \mathbf{B} = (b_1 \quad b_2 \quad \cdot \quad \cdot \quad b_J) \tag{7.12}$$

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1J} \\ c_{21} & c_{22} & \cdots & c_{2J} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{J1} & c_{J2} & \cdots & c_{JJ} \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ d_J \end{pmatrix} \tag{7.13}$$

The solution can then be expressed in these matrices:

$$q_s = (T_s - T_0) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0) \tag{7.14}$$

$$h_s^{-1} = A + \mathbf{B} \cdot \mathbf{C}^{-1} \cdot \mathbf{D} \tag{7.15}$$

7.1.2 The elements in the matrices

The elements in the matrices are:

$$b_k = \left(\frac{r_i}{2D}\right)^k - (-1)^k \left(\frac{r_i D}{r_0^2 - D^2}\right)^k - \left(\frac{r_i D}{r_0^2 + D^2}\right)^k \tag{7.16}$$

$$c_{kk} = 1 + \begin{pmatrix} 2k-1 \\ k-1 \end{pmatrix} \left(\frac{r_i}{2D}\right)^{2k} \tag{7.17}$$

$$- \sum_{j'=0}^{k} r_i^{2k} D^{2k-2j'} \cdot \left(\frac{(-1)^{2k-2j'}}{(r_0^2 - D^2)^{2k-j'}} + \frac{(-1)^{2k-j'}}{(r_0^2 + D^2)^{2k-j'}} \right) \begin{pmatrix} k \\ j' \end{pmatrix} \begin{pmatrix} 2k-j'-1 \\ k-1 \end{pmatrix}$$

$$c_{kj}\,[j \neq k] = \begin{pmatrix} j+k-1 \\ j-1 \end{pmatrix} \left(\frac{r_i}{2D}\right)^{k+j} - \sum_{j'=0}^{min(j,k)} r_i^{j+k} D^{j+k-2j'} \tag{7.18}$$

$$\cdot \left(\frac{(-1)^{j+k-2j'}}{(r_0^2 - D^2)^{j+k-j'}} + \frac{(-1)^{2j-j'}}{(r_0^2 + D^2)^{j+k-j'}} \right) \begin{pmatrix} j \\ j' \end{pmatrix} \begin{pmatrix} j+k-j'-1 \\ j-1 \end{pmatrix}$$

$$d_k = -\frac{1}{k} \left\{ \left(\frac{r_i}{2D}\right)^k - (-1)^k \left(\frac{r_i D}{r_0^2 - D^2}\right)^k - \left(\frac{r_i D}{r_0^2 + D^2}\right)^k \right\} \tag{7.19}$$

7.2 The anti-symmetrical problem

The anti-symmetrical problem is described in Figure 6.1 together with the following equations

$$T_1 = -T_2 = -T_a \tag{7.20}$$

$$q_1 = -q_2 = -q_a \tag{7.21}$$

The temperature on the larger pipe is zero:

$$T_0 = 0 \tag{7.22}$$

From section 6.1 we see that the strength of the multipoles satisfies the following equations:

$$\mathrm{Re}\,[P_{1j}] = (-1)^{j+1} \mathrm{Re}\,[P_{2j}] \tag{7.23}$$

$$\mathrm{Im}\,[P_{1j}] = \mathrm{Im}\,[P_{2j}] = 0 \tag{7.24}$$

We will use the following notations:

$$P_{1j} = M_j \tag{7.25}$$

$$P_{2j} = (-1)^{j+1} M_j \tag{7.26}$$

Here M_j is a non-complex constant. When equations (7.20-7.26) are used in (5.19,5.20,5.22) with $m = 1$ one gets:

$$T_a = \frac{q_a}{2\pi\lambda_i} \left(\ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{(r_0^2 - D^2)}\right) \right) \tag{7.27}$$

$$- \sum_{j=1}^{J} M_j \left\{ -\left(\frac{r_i}{2D}\right)^j - (-1)^j \left(\frac{r_i D}{r_0^2 - D^2}\right)^j + \left(\frac{r_i D}{r_0^2 + D^2}\right)^j \right\}$$

When equations (7.20-7.26) are used in (5.19,5.20,5.23) with $m = 1$ one gets:

$$\boxed{k = 1, \ldots, J}$$

$$0 = M_k + \frac{q_a}{2\pi\lambda_i} \cdot \frac{1}{k} \left\{ \left(\frac{r_i}{2D}\right)^k + (-1)^k \left(\frac{r_i D}{r_0^2 - D^2}\right)^k - \left(\frac{r_i D}{r_0^2 + D^2}\right)^k \right\} \tag{7.28}$$

$$+ \sum_{j=1}^{J} M_j \left\{ - \binom{j+k-1}{j-1} \left(\frac{r_i}{2D}\right)^{k+j} - \sum_{j'=0}^{min(j,k)} r_i^{j+k} D^{j+k-2j'} \right.$$

$$\left. \cdot \left(\frac{(-1)^{j+k-2j'}}{(r_0^2 - D^2)^{j+k-j'}} - \frac{(-1)^{2j-j'}}{(r_0^2 + D^2)^{j+k-j'}} \right) \binom{j}{j'} \binom{j+k-j'-1}{j-1} \right\}$$

7.2.1 General solution to equation system

If the equation system (7.27,7.28) is expressed in matrix notation one gets:

$$T_a = \frac{q_a}{2\pi\lambda_i} \cdot A + \mathbf{B} \cdot \mathbf{M} \tag{7.29}$$

$$\mathbf{C} \cdot \mathbf{M} = \frac{q_s}{2\pi\lambda_i} \mathbf{D} \tag{7.30}$$

$$A = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{7.31}$$

Here **M**,**B** and **C** are defined in (7.12,7.13). The solution can then be expressed in these matrices:

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0) \tag{7.32}$$

$$h_a^{-1} = A + \mathbf{B} \cdot \mathbf{C}^{-1} \cdot \mathbf{D} \tag{7.33}$$

7.2.2 The elements in the matrices

The elements in the matrices are:

$$b_k = \left(\frac{r_i}{2D}\right)^k + (-1)^k \left(\frac{r_i D}{r_0^2 - D^2}\right)^k - \left(\frac{r_i D}{r_0^2 + D^2}\right)^k \tag{7.34}$$

$$c_{kk} = 1 - \binom{2k-1}{k-1} \left(\frac{r_i}{2D}\right)^{2k} \tag{7.35}$$

$$- \sum_{j'=0}^{k} r_i^{2k} D^{2k-2j'} \cdot \left(\frac{(-1)^{2k-2j'}}{(r_0^2 - D^2)^{2k-j'}} - \frac{(-1)^{2k-j'}}{(r_0^2 + D^2)^{2k-j'}} \right) \binom{k}{j'} \binom{2k-j'-1}{k-1}$$

$$c_{kj}\,[j \neq k] = - \binom{j+k-1}{j-1} \left(\frac{r_i}{2D}\right)^{k+j} - \sum_{j'=0}^{min(j,k)} r_i^{j+k} D^{j+k-2j'} \tag{7.36}$$

$$\cdot\left(\frac{(-1)^{j+k-2j'}}{(r_0^2-D^2)^{j+k-j'}}-\frac{(-1)^{2j-j'}}{(r_0^2+D^2)^{j+k-j'}}\right)\binom{j}{j'}\binom{j+k-j'-1}{j-1}$$

$$d_k=-\frac{1}{k}\left\{-\left(\frac{r_i}{2D}\right)^k-(-1)^k\left(\frac{r_iD}{r_0^2-D^2}\right)^k+\left(\frac{r_iD}{r_0^2+D^2}\right)^k\right\} \tag{7.37}$$

8 Summary

For a reader who is only interested in formulae for practical use it is enough to read this chapter.

8.1 Two pipes imbedded in a circular insulation

There are two pipes imbedded in a circular insulation. The temperatures in the imbedded pipes are T_1 and T_2. The temperature on the circumscribing larger pipe is T_0. The thermal conductivity in the insulation is λ_i. The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes inside the large pipe. The temperature $T(x, y)$ in a vertical cross-section of the pipes satisfies the steady-state heat conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{8.1}$$

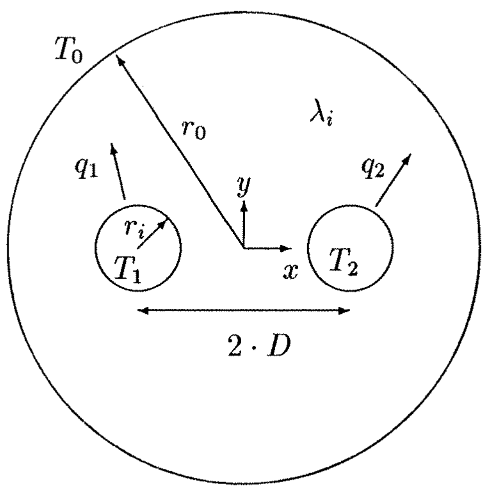


Figure 8.1. Two pipes inside a large pipe.

D = Half the distance between the center of the pipes (m)
r_0 = Radius of the circumscribing large pipe (m)
r_i = Radius of the imbedded pipes (m)

q_1 = Heat flow from pipe 1 per meter (W/m)
q_2 = Heat flow from pipe 2 per meter (W/m)
T_0 = Temperature on the larger pipe (°C)
T_1 = Temperature in pipe 1 (°C)
T_2 = Temperature in pipe 2 (°C)
λ_i = Thermal conductivity of the insulation (W/mK)

8.2 Mathematical formulation

The original problem can be separated into a symmetrical and anti-symmetrical problem. The temperature in the pipes in the symmetrical problem is T_s. The temperatures in the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{8.2}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{8.3}$$

The subscript s denotes the symmetrical problem of two pipes. The subscript a denotes the anti-symmetrical problem of two pipes. The temperatures of the original problem are from (8.2-3):

$$T_1 = T_s + T_a \tag{8.4}$$

$$T_2 = T_s - T_a \tag{8.5}$$

The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_0$ and the thermal conductivity λ_i. We may write:

$$q_s = (T_s - T_0) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0) \tag{8.6}$$

Here h_s is the dimensionless heat loss factor for the symmetrical problem. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional to the temperature T_a and the thermal conductivity λ_i. We may write:

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0) \tag{8.7}$$

Here h_a is the dimensionless heat loss factor for the anti-symmetrical problem. It should be noted that the temperature T_a connected with q_a in (8.7) is half the temperature difference between the pipes. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{8.8}$$

$$q_2 = q_s - q_a \tag{8.9}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{8.10}$$

The symmetrical and anti-symmetrical problems are solved separately. Formulae for h_s and h_a are obtained. The heat losses q_1 and q_2 are then obtained from (8.8-9).

8.3 Approximate formulae

Approximate formulae of the zero, first and second order have been derived. The zero and first order formulae are shown below together with two old formulae.

8.3.1 Zero-order approximation

The zero-order multipole approximation uses the line sources and sinks without any multipoles. The zero-order approximation gives the following expressions for the thermal resistances:

$$h_s^{-1} = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{8.11}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{8.12}$$

The relative errors in the heat loss, when the zero-order formulae are used, are typically less than 20% for q_s and less than 10% for q_a.

8.3.2 First-order approximation

With the use of multipoles of the first order, the following new formulae are obtained:

$$h_s^{-1} = \ln\left(\frac{r_0^2}{2Dr_i}\right) - \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) - \frac{\left(\frac{r_i}{2D} + \frac{2r_iD^3}{r_0^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 - \left(\frac{2r_ir_0^2D}{r_0^4 - D^4}\right)^2} \tag{8.13}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) - \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) - \frac{\left(\frac{r_i}{2D} - \frac{2r_ir_0^2D}{r_0^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - 2r_i^2r_0^2 \cdot \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} \tag{8.14}$$

The relative errors in the heat loss, when first zero-order formulae are used, are typically less than 0.1% for q_s and less than 5% for q_a.

8.3.3 Area approximation

An old formulae from [7] described in this report is here called the area approximation. The formula calculates the symmetrical heat loss q_s.

$$r_e = \sqrt{\frac{2 \cdot r_0^2}{\pi} \arccos\left(\frac{D}{r_0}\right) - \frac{2 \cdot D}{\pi}\sqrt{r_0^2 - D^2}}$$

$$d_e = \frac{\sqrt{r_0^2 - D^2} + r_0}{2} - r_i$$

$$R_1 = 2 \cdot \ln\left(\frac{r_e}{r_i}\right)$$

$$R_2 = \frac{\pi d_e}{D}$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 + 1/R_2 \tag{8.15}$$

The relative error in the heat loss, when the area approximation formula is used, is typically less than 10% for q_s.

8.3.4 Two-model approximation

Another old formula described in [8] is here called the two-model approximation. The formula calculates the symmetrical heat loss q_s.

$$R_1 = \operatorname{arccosh}\left(\frac{r_i/r_0 + r_0/r_i - (r_0/r_i)(D/r_0)^2}{2}\right)$$

$$R_2 = 4 \cdot \operatorname{arccosh}\left(2\left(\frac{D}{r_i}\right)^2 - 1\right)$$

$$h_s = \frac{1}{2\pi\lambda_i R_s} = 1/R_1 - 1/R_2 \tag{8.16}$$

The relative error in the heat loss, when the two-model approximation formula is used, is typically less than 5% for q_s.

References

1. Claesson J, Bennet J and Hellström G, Multipole method to compute the conductive heat flows to and between pipes in a composite cylinder, Notes on Heat Transfer 3-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

2. Claesson J and Bennet J, Multipole method to compute the conductive heat flows to and between pipes in a cylinder, Notes on Heat Transfer 2-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

3. Arfvidsson J, Multipole method to compute the conductive heat flows to and between pipes in a cylinder. PC-version of computer model, Notes on Heat Transfer 2.1-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118,S-221 00 Lund, Sweden, 1987.

4. Myrehed M, Multipole method to compute the conductive heat flows to and between pipes in a composite region: Pipes in a circle in a half plane, Notes on Heat Transfer 1-1989, Depts. of Building Technology and Mathematical Physics, Lunds Institute of Technology, Sweden.

5. Wallentén P, Heat loss from one or two insulated pipes in the ground, Notes on Heat Transfer 1-1990, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1990.

6. Wallentén P, Heat loss from two pipes in the ground imbedded in a circular insulation, Notes on Heat Transfer 3-1991, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.

7. Carlsson S and Eneborg I, Heat losses from underground heating mains, Byggforskningens handlingar nr 42, 1963, Stockholm, Sweden. (in Swedish)

8. Jarfelt U, Heat losses from district heating mains, thesis, Department of Building Technology, Chalmers University of Technology, 1986, Göteborg, Sweden. (in Swedish)

PART C:

Notes on Heat Transfer 3-1991

HEAT LOSS FROM TWO PIPES IN THE GROUND IMBEDDED IN A CIRCULAR INSULATION

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1 Introduction

This report deals with the problem of determining the steady-state heat losses from two pipes in the ground imbedded in a circular insulation. A summary of the results is presented in chapter 8.

In Claesson [2] a new method, the *multipole method*, is presented that can solve steady-state heat transfer problems with circular boundaries. With the use of the multipole method, new formulae with improved accuracy have been derived for the heat losses from two pipes in the ground imbedded in a circular insulation.

The errors of the new formulae and two already existing formulae have been determined with the multipole method implemented on a computer of PC-type. The already existing formulae are in this report called old formulae.

The formulae are mainly derived for district heating pipes. They can be used on any problem with the same boundary conditions, but the listed errors of the formulae are valid for dimensions usual for district heating pipes in the ground.

1.1 Two pipes in the ground imbedded in a circular insulation

There are two pipes with the radius r_i in the ground imbedded in a circular insulation, see Figure 1.1. The temperatures in the imbedded pipes are T_1 and T_2. The temperature at the ground surface is T_c. The thermal conductivity in the insulation is λ_i. The thermal conductivity in the ground is λ_g.

The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes. The pipes are assumed to be long. It is therefore enough to study a vertical cross-section of the ground. The temperature $T(x, y)$ satisfies the steady-state heat conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1.1}$$

The dimensionless parameter σ will be used in the following:

$$\sigma = \frac{\lambda_i - \lambda_g}{\lambda_i + \lambda_g} \tag{1.2}$$

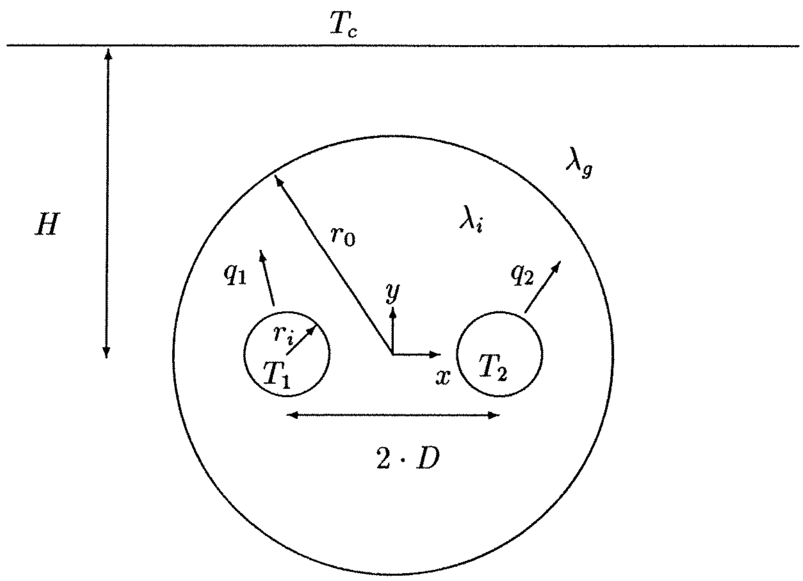

Figure 1.1. Two pipes in the ground imbedded in a circular insulation.

D = Half the distance between the center of the pipes (m)
H = Distance from the center of the large pipe to the ground surface (m)
r_0 = Radius of the circumscribing large pipe (m)
r_i = Radius of the imbedded pipes (m)
q_1 = Heat flow from pipe 1 per meter (W/m)
q_2 = Heat flow from pipe 2 per meter (W/m)
T_c = Temperature at the ground surface (°C)
T_1 = Temperature in pipe 1 (°C)
T_2 = Temperature in pipe 2 (°C)
λ_i = Thermal conductivity of the insulation (W/mK)
λ_g = Thermal conductivity of the ground (W/mK)

1.2 Solution method

In Claesson [2] a new method, the *multipole method*, is presented that can solve steady-state heat transfer problems with circular boundaries. The multipole method is a semi-analytical method. Myrehed [4] has written a PC program, based on [2], which solves the heat flow problem of pipes in a cylinder in the ground with arbitrary accuracy. In this report the heat losses from the pipes are calculated with [4].

In Claesson [1] a method is described that solves the problem of finding the heat flow to and between pipes in a composite cylinder. The formulae in this report are derived with the use of [1] and not [4]. The reason for this is that the zero and first order formulae derived from [1] are simpler and have a smaller error than the formulae derived from [4].

The *multipole method* can solve two-dimensional steady-state heat transfer problems with circular boundaries. The solution is found with the use of the complex plane. Complex line sources in the center of each pipe are used. Only the real part of these line sources contributes to the temperature field. The complex-valued derivative of order j of the line source with respect to the position of the pipe is called a *multipole* of order j. The temperature field is a sum of the line sources and multipoles up to order J at each pipe. In the limit when $J \to \infty$ the exact solution is found. Thus the error of the calculation can be chosen arbitrarily small.

With the multipole method it is possible to derive systematic approximations of increasing accuracy. This report deals with approximations of the zero and first order.

A detailed description of the *multipole method* is made in chapter 6.

1.3 Previous reports

There exist several reports describing different types of *multipole methods*. The method used in this report is given in Claesson [1], in which the problem is to determine the heat flows between pipes in a composite cylinder, i.e. two concentric cylinders with different thermal conductivity. The report of Wallentén [6] is based on Claesson [1] and presents explicit formulae of the zero and first order for the heat flow from two pipes to a larger surrounding pipe. Hellström [9] presents similar formulae based on [1] to be used in ground heat storage problems.

Claesson [2] presents a multipole method without any mirror line sinks. The method described in [2] is implemented in [3] and [4]. The program of [3] deals with the heat flow problem when one or more pipes are positioned inside a large pipe. The program of [4] deals with the problem of one or more pipes inside a larger pipe, which in its turn lies in the ground with another thermal conductivity. The report of Wallentén [5] is based on Claesson [2] and presents explicit formulae of the zero and first order for the heat loss from one or two pipes in the ground.

2 Mathematical formulation

The problem described in Figure 1.1 can, with the use of the superposition principle described in section 2.1, be separated into two problems. These two problems are easier to solve than the original problem. The solution is expressed in the new temperatures T_s, T_a, T_c and the dimensionless heat loss factors h_s and h_a.

2.1 Superposition

For the problem described in Figure 1.1 one can construct two basic problems, a symmetrical problem and an anti-symmetrical problem, see Figure 2.1. With the use of the superposition principle, every problem concerning different temperatures can be constructed from the solutions of these two problems.

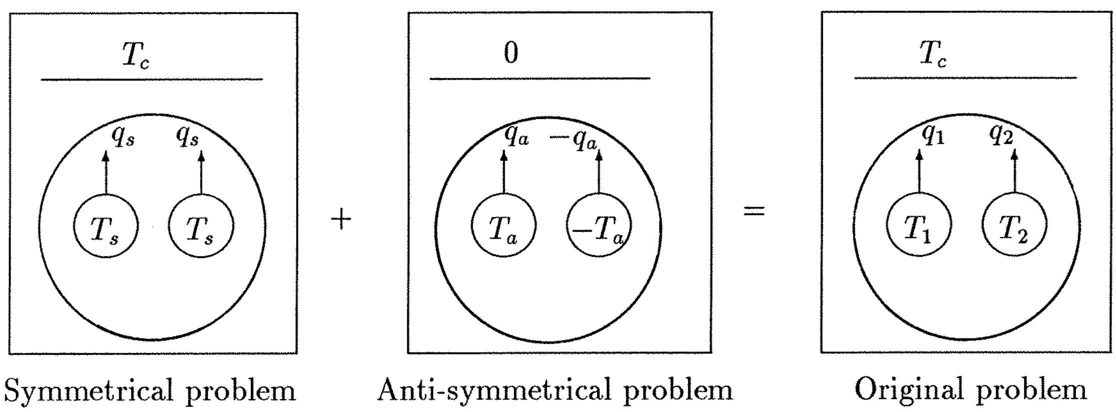

Figure 2.1. Superposition of symmetrical and anti-symmetrical problem.

The temperature in the pipes in the symmetrical problem is T_s. The temperatures in the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{2.1}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{2.2}$$

The subscript s denotes the symmetrical problem. The subscript a denotes the anti-symmetrical problem. The temperatures in the original problem are from (2.1-2):

$$T_1 = T_s + T_a \tag{2.3}$$

$$T_2 = T_s - T_a \tag{2.4}$$

The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_c$. We may write:

$$q_s = \frac{T_s - T_c}{R_s} \tag{2.5}$$

Here R_s (mK/W) is the thermal resistance between one of the pipes and the ground surface. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional to the temperature T_a. We may write:

$$q_a = \frac{T_a}{R_a} \tag{2.6}$$

Here R_a (mK/W) is the thermal resistance associated with the anti-symmetrical problem. It should be noted that the temperature T_a connected with R_a in (2.6) is half the temperature difference between the pipes. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{2.7}$$

$$q_2 = q_s - q_a \tag{2.8}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{2.9}$$

The symmetrical and anti-symmetrical problems are solved separately. Formulae for R_s and R_a are obtained. The heat losses q_1 and q_2 are then obtained from (2.7-8).

2.2 Dimensional analysis

The heat losses q_s and q_a are proportional to $(T_s - T_c)$ and T_a respectively. It is convenient to introduce the dimensionless heat loss factors h_s and h_a. This is done to separate the dependence on the temperatures from the dependence on the geometry and thermal conductivity of the problem.

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s \tag{2.10}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a \tag{2.11}$$

The factor $2\pi\lambda_i$ is introduced to make the expressions for h_s and h_a simpler.

The geometry is described by four lengths: r_i, r_0, D and H. The number of parameters necessary to describe the geometry is reduced from four to three by scaling with the radius of the outer pipe r_0. The heat loss factors h_s and h_a only depend on the parameters r_i/r_0, D/r_0, H/r_0 plus the ratio λ_i/λ_g:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{2.12}$$

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{2.13}$$

The geometry of the problem gives the following inequalities:

$$0 < r_i/r_0 < D/r_0 \tag{2.14}$$

$$r_i/r_0 + D/r_0 < 1 \tag{2.15}$$

The thermal resistances R_s and R_a can be expressed in h_s and h_a.

$$R_s = \frac{1}{2\pi\lambda_i \cdot h_s} \tag{2.16}$$

$$R_a = \frac{1}{2\pi\lambda_i \cdot h_a} \tag{2.17}$$

3 Symmetrical problem

The problem is described in Figures 1.1 and 2.1. For the symmetrical problem the temperature in the pipes is $T_s = (T_1 + T_2)/2$ and the temperature at the ground surface is T_c. The heat loss from the pipes is q_s for both pipes.

3.1 Exact solution

The exact solution to the problem was obtained with the use of the program described in [4]. The order of the highest used multipole has been at least 10. This means that the error in the heat losses is approximately less than 0.01 %. From equation (2.12) we get the heat loss q_s:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{3.1}$$

Figure 3.1 shows the computed heat loss factor $h_s(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g)$ for $H/r_0 = 2$. The thermal conductivity of the ground is 2 W/mK and of the thermal conductivity of the insulation is 0.04 W/mK. The ratio λ_i/λ_g is therefore 0.02 and σ is -0.96078, see equation (1.2).

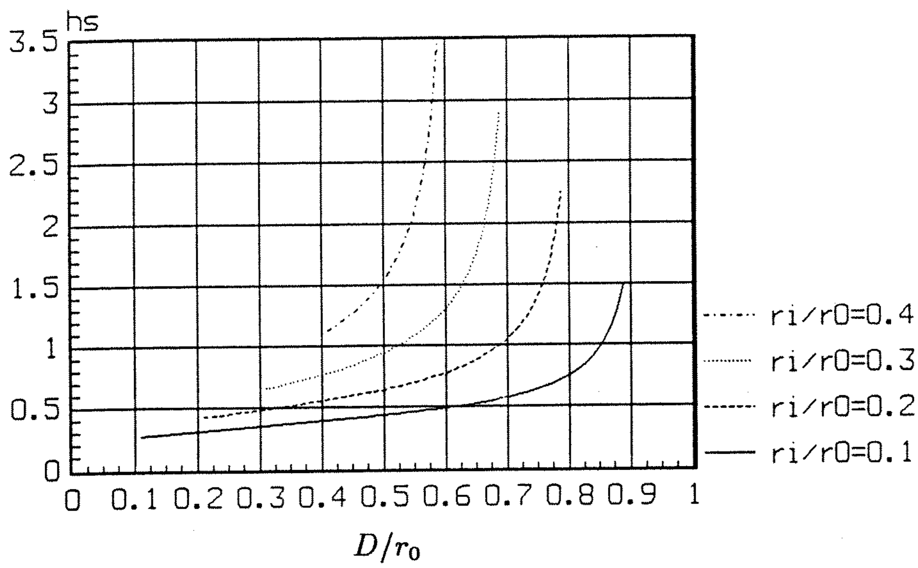

Figure 3.1. Heat loss factor h_s for different values of D/r_0 and r_i/r_0 ($H/r_0 = 2, \lambda_i/\lambda_g = 0.02$).

The heat loss q_s will increase for increasing D/r_0. The heat loss q_s is strongly dependent on the ratios r_i/r_0 and D/r_0. For small pipes the heat loss is only weakly dependent on the positions of the pipes (D/r_0). The heat loss is only weakly dependent on the ratio H/r_0.

The heat loss factor h_s is shown in Figures 3.2-5 for different values of the parameters.

3.2 Approximate formulae

With the use of the multipole method described in [1] approximate formulae of the zero and first order have been derived for the heat losses from the pipes. A formula of order J employs the solution of a J order equation system. The formulae will therefore be very complicated for $J > 1$. Formulae (3.2) and (3.3) are derived in chapter 7.

3.2.1 Zero-order approximation

The zero-order multipole approximation uses the line sources and sinks without any multipoles. The zero-order approximation gives the following expression for the heat loss factor h_s (or thermal resistance R_s) for the symmetrical problem:

$$h_s^{-1} = 2\pi\lambda_i R_s = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{2H}{r_0}\right) + \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma\cdot\ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{3.2}$$

Here is $\sigma = (\lambda_i - \lambda_g)/(\lambda_i + \lambda_g)$.

3.2.2 First-order approximation

With the use of multipoles of the first order, the following new formula is obtained:

$$h_s^{-1} = 2\pi\lambda_i R_s = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{2H}{r_0}\right) + \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma\cdot\ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{3.3}$$
$$-\frac{\left(\frac{r_i}{2D} - \frac{\sigma 2 r_i D^3}{r_0^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2}$$

The last term is the correction to the zero-order formula. Note that the depth H is not used in this term.

3.2.3 Area approximation

There exists an old formula that we here will call the area approximation formula [7]. The formula is based on the assumption that the resistance of the insulation can be separated into two resistances coupled in parallel. The resistance of the ground is added in series to the other two resistances.

$$r_e = \sqrt{\frac{2r_0^2}{\pi}\arccos\left(\frac{D}{r_0}\right) - \frac{2D}{\pi}\sqrt{r_0^2 - D^2}} \tag{3.4}$$

$$d_e = \frac{\sqrt{r_0^2 - D^2} + r_0}{2} - r_i \tag{3.5}$$

$$R_1 = 2\ln\left(\frac{r_e}{r_i}\right) \tag{3.6}$$

$$R_2 = \frac{\pi d_e}{D} \tag{3.7}$$

$$R_3 = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{H}{r_0} + \sqrt{\frac{H}{r_0}^2 - 1}\right) \tag{3.8}$$

$$h_s^{-1} = 2\pi\lambda_i R_s = \frac{1}{1/R_1 + 1/R_2} + R_3 \tag{3.9}$$

The resistance R_1 originates from the resistance of a circular insulation with inner radius r_i and outer radius r_e. The resistance R_2 originates from the resistance of a rectangular insulation with height d_e and width D. The circular and rectangular insulations are defined so that the total insulated area is the same as in the real problem. The resistance R_3 is the resistance between one pipe in the ground, with the radius r_0, and the ground surface.

3.2.4 Two-model approximation

Another old formula investigated here we will call the two-model approximation. It is a formula from [8] but with a additional resistance for the ground R_3. This formula is based on the assumption that the problem can be separated into two problems which each has an analytical solution.

The resistance R_1 originates from the resistance between a pipe with the radius r_i and a circumscribing pipe with radius r_0. The resistance R_2 originates from the resistance between two pipes with the radius r_i whose centers are $2D$ apart. The resistance R_3 is the same as in (3.8).

$$R_1 = \operatorname{arccosh}\left(\frac{r_i/r_0 + r_0/r_i - (r_0/r_i)(D/r_0)^2}{2}\right) \tag{3.10}$$

$$R_2 = 4 \cdot \operatorname{arccosh}\left(2\left(\frac{D}{r_i}\right)^2 - 1\right) \tag{3.11}$$

$$R_3 = \frac{2\lambda_i}{\lambda_g}\ln\left(\frac{H}{r_0} + \sqrt{\frac{H}{r_0}^2 - 1}\right) \tag{3.12}$$

$$h_s^{-1} = 2\pi\lambda_i R_s = \frac{1}{1/R_1 - 1/R_2} + R_3 \tag{3.13}$$

3.3 Errors of the formulae

The errors of the different formulae have been studied with the use of the multipole program [4]. Figures 3.2-5 show the error made when the heat loss q_s is calculated with formulae (3.2,3,9,13). The error is expressed in per cent. A positive relative error means that the formula gives a too large heat loss. In the figures the heat loss factor h_s is also shown. The right ordinate shows the heat loss factor and the left one the error.

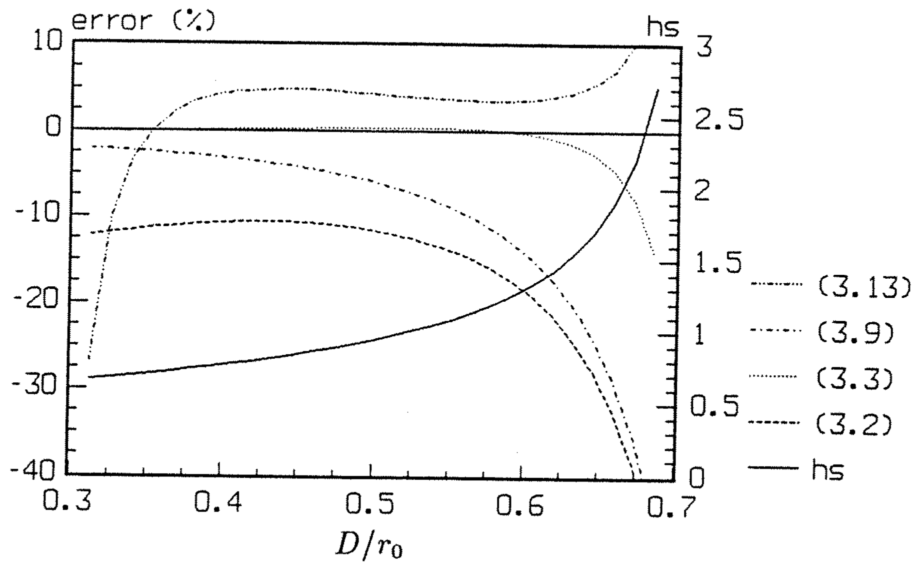

Figure 3.2. The relative error (%) of the formulae to calculate q_s and the heat loss factor h_s for different values of D/r_0 ($H/r_0 = 2$, $r_i/r_0 = 0.3$, $\lambda_i/\lambda_g = 0.02$).

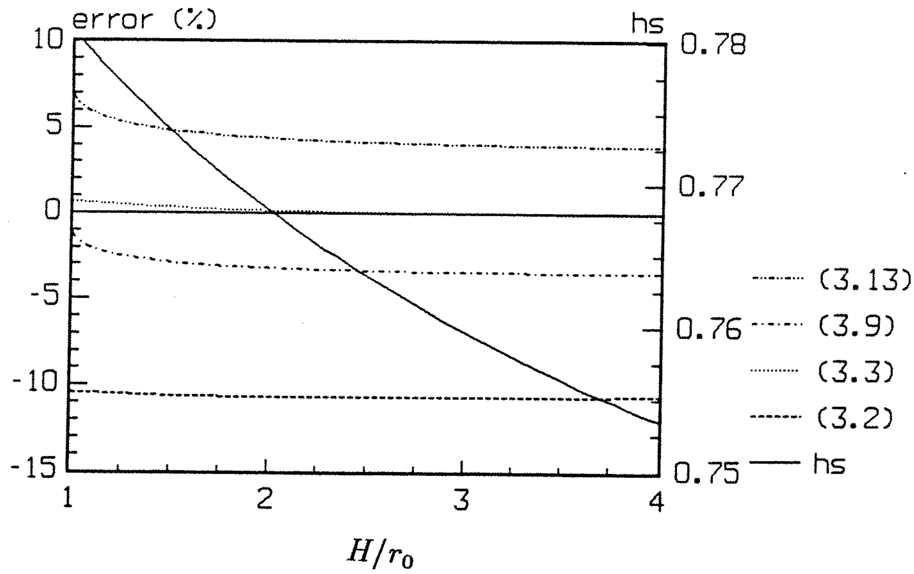

Figure 3.3. The relative error (%) of the formulae to calculate q_s and the heat loss factor h_s for different values of H/r_0 ($D/r_0 = 0.4$, $r_i/r_0 = 0.3$, $\lambda_i/\lambda_g = 0.02$).

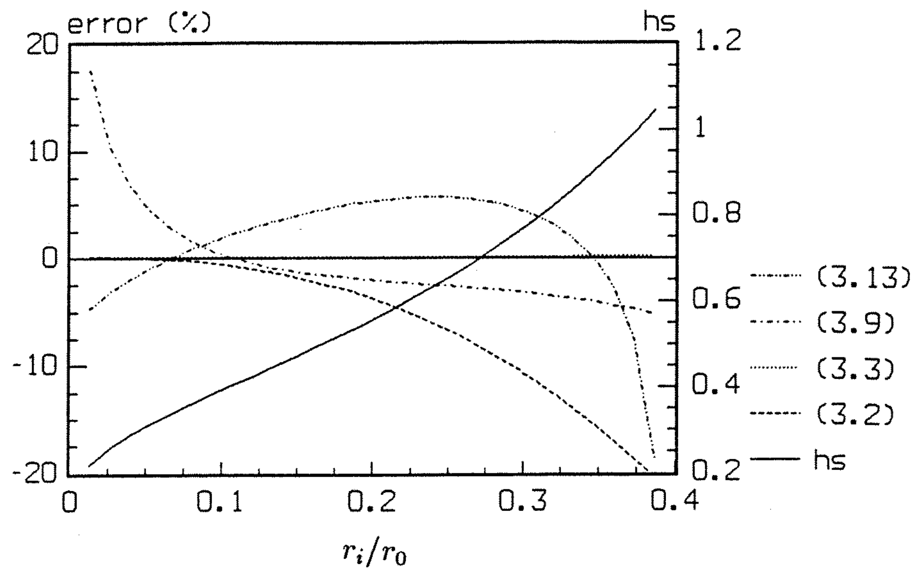

Figure 3.4. The relative error (%) of the formulae to calculate q_s and the heat loss factor h_s for different values of r_i/r_0 ($H/r_0 = 2$, $D/r_0 = 0.4$, $\lambda_i/\lambda_g = 0.02$).

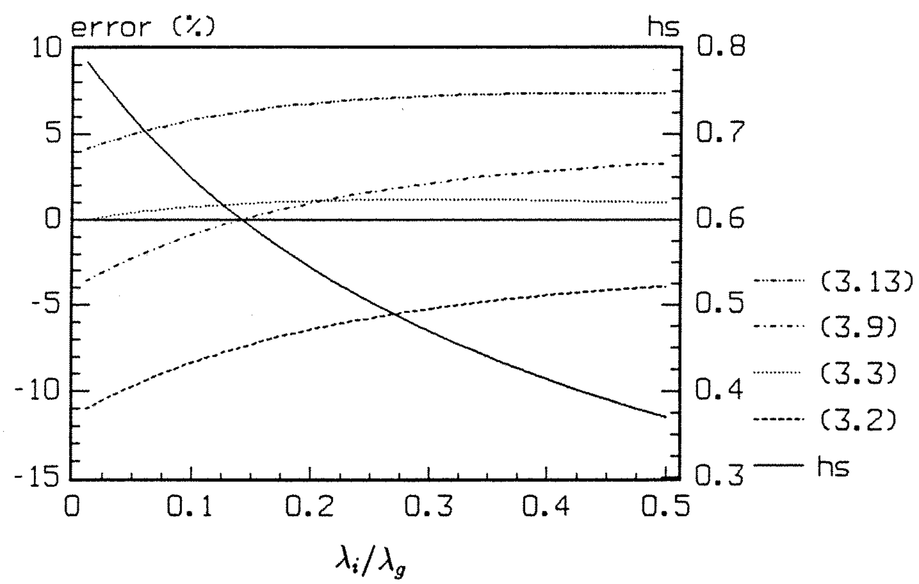

Figure 3.5. The relative error (%) of the formulae to calculate q_s and the heat loss factor h_s for different values of λ_i/λ_g ($H/r_0 = 2$, $r_i/r_0 = 0.3$, $D/r_0 = 0.4$).

Figure 3.3 shows that the errors and the heat loss factor are obviously only weakly dependent on H/r_0. From Figures 3.2-5 one can see that all the formulae "collapse" when the pipes lie close to the large pipe $(D + r_i)/r_0 \approx 1$. Formula (3.13) also collapses when the pipes are too close to each other ($r_i/D \approx 1$). The error of the first-order formula (3.3) is very small compared to the other formulae. The first-order formula (3.3) seems to be the best choice for practical use. It is a simple formula with only a small error.

4 Anti-symmetrical problem

The problem is described in Figures 1.1 and 2.1. For the anti-symmetrical problem the temperatures in the pipes are $T_a = (T_1 - T_2)/2$ and $-T_a$. The temperature at the ground surface is $T_c = 0$. The heat losses from the pipes are q_a and $-q_a$.

4.1 Exact solution

The exact solution to the problem was obtained with the use of the program described in [4]. The order of the highest used multipole has been at least 10. This means that the error in the heat losses is approximately less than 0.01 %. From equation (2.13) we get the heat loss q_a:

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{4.1}$$

Figure 4.1 shows the computed heat loss factor $h_a(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g)$ for $H/r_0 = 2$. In Figure 4.1 the thermal conductivity of the ground is 2 W/mK and of the thermal conductivity of the insulation is 0.04 W/mK. The ratio λ_i/λ_g is therefore 0.02 and σ is -0.96078, see equation (1.2) The heat loss q_a has a minimum for $D/r_0 \approx 0.5$. The heat loss is only weakly dependent on the ratio H/r_0.

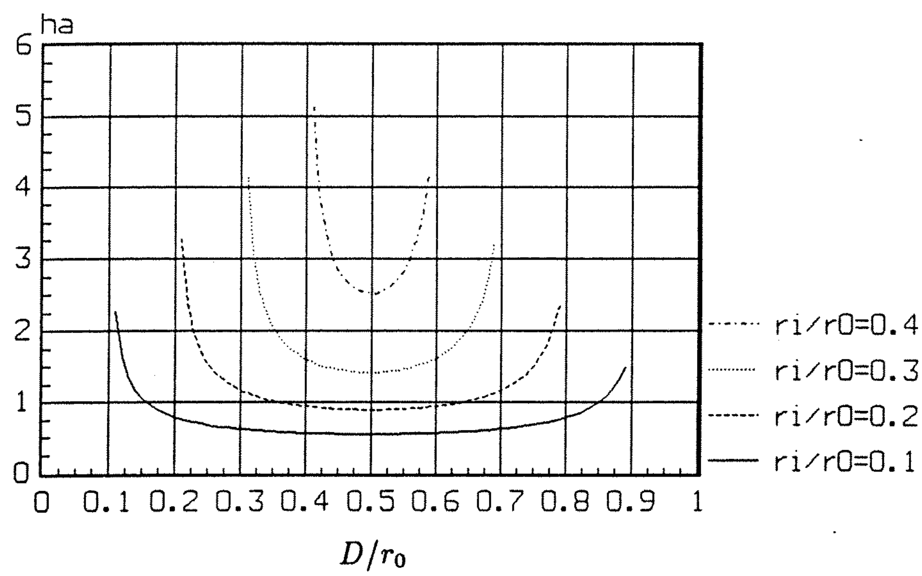

Figure 4.1. Heat loss factor h_a for different values of D/r_0 and r_i/r_0 $(H/r_0 = 2, \lambda_i/\lambda_g = 0.02)$.

The heat loss factor h_a is shown in Figures 4.2-5 for different values of the parameters.

4.2 Approximate formulae

With the use of the multipole method described in [1] approximate formulae of the zero and first order have been derived for the heat loss from the pipes. Formula (4.2) and (4.3) are derived in chapter 7.

4.2.1 Zero-order approximation

The zero-order multipole approximation uses line sources without any multipoles. The zero-order approximation gives the following expression for the heat loss factor h_a (or thermal resistance R_a) for the anti-symmetrical problem:

$$h_a^{-1} = 2\pi\lambda_i R_a = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{4.2}$$

4.2.2 First-order approximation

With the use of multipoles of the first order, the following new formula is obtained:

$$h_a^{-1} = 2\pi\lambda_i R_a = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{4.3}$$

$$-\frac{\left(\frac{r_i}{2D} - \gamma\frac{Dr_i}{4H^2} + \frac{2\sigma r_i r_0^2 D}{r_0^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - \gamma\frac{r_i}{2H} + 2\sigma r_i^2 r_0^2 \cdot \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} - \gamma\left(\frac{D}{2H}\right)^2$$

$$\gamma = \frac{2(1-\sigma^2)}{1 - \sigma\left(\frac{r_0}{2H}\right)^2} \tag{4.4}$$

4.3 Errors of the formulae

The errors of the different formulae have been studied with the use of the multipole program [4]. Figures 4.2-5 show the error made when the heat loss q_s is calculated with formulae (3.2,3). The error is expressed in per cent. A positive relative error means that the formula gives a too large heat loss. In the figures the heat loss factor h_a is also shown. The right ordinate shows the heat loss factor and the left one the error.

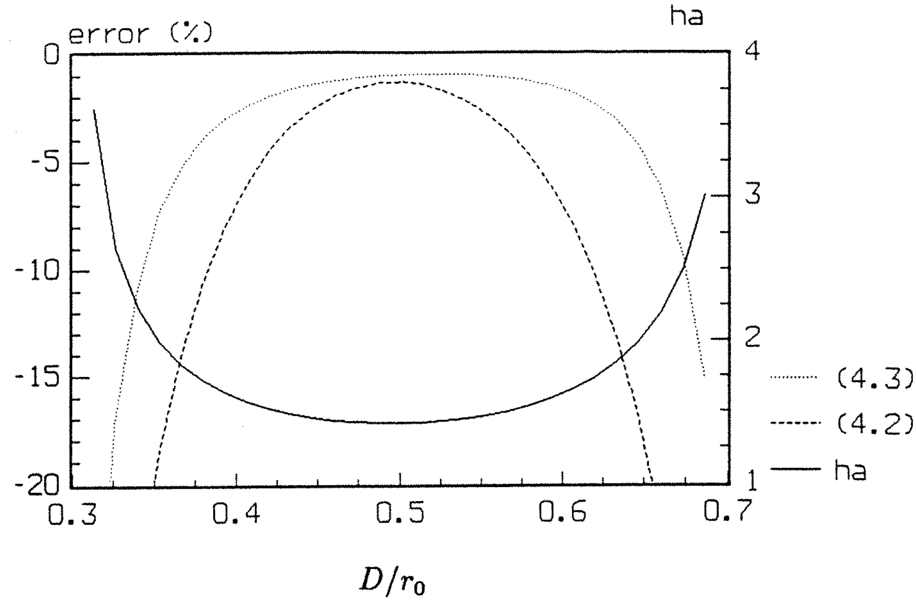

Figure 4.2. The relative error (%) of the formulae to calculate q_a and the heat loss factor h_a for different values of D/r_0 ($H/r_0 = 2$, $r_i/r_0 = 0.3$, $\lambda_i/\lambda_g = 0.02$).

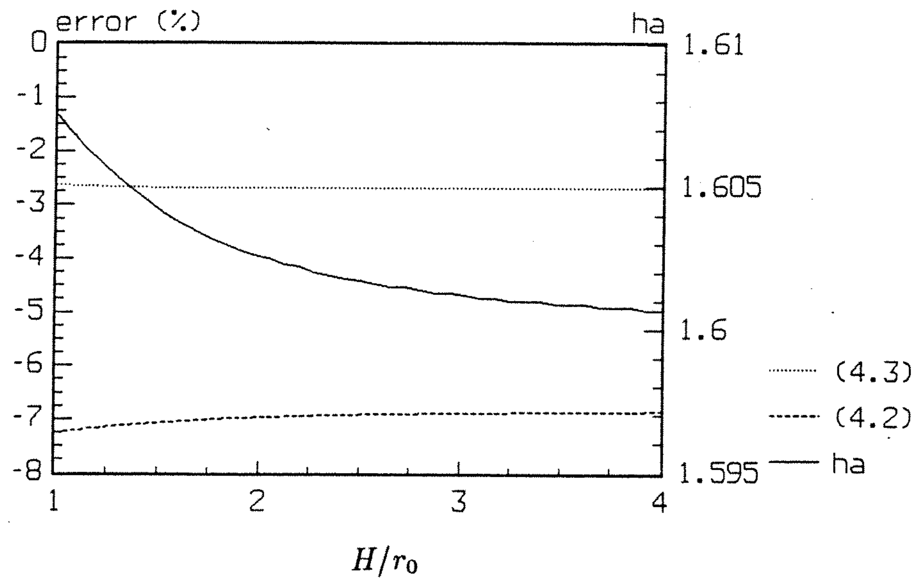

Figure 4.3. The relative error (%) of the formulae to calculate q_a and the heat loss factor h_a for different values of H/r_0 ($D/r_0 = 0.4$, $r_i/r_0 = 0.3$, $\lambda_i/\lambda_g = 0.02$).

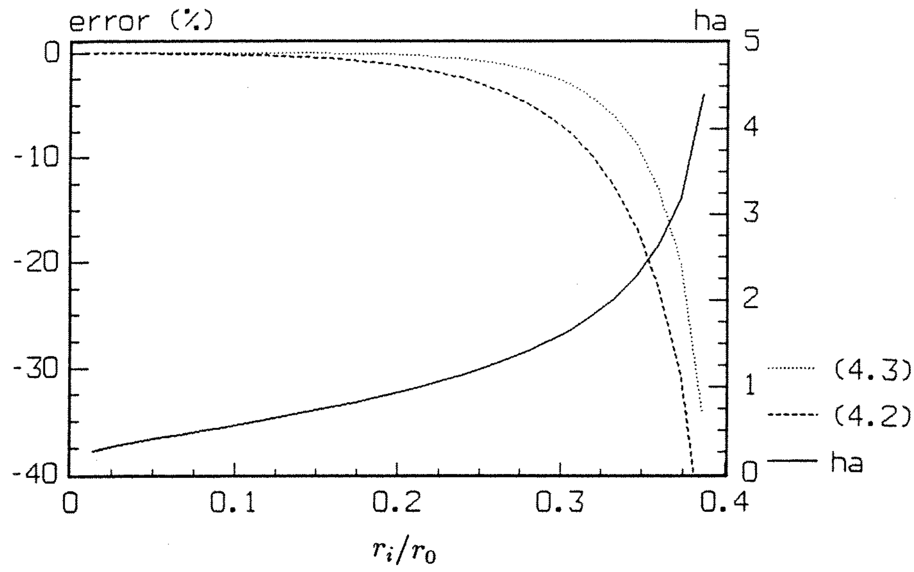

Figure 4.4. The relative error (%) of the formulae to calculate q_a and the heat loss factor h_a for different values of r_i/r_0 ($H/r_0 = 2$, $D/r_0 = 0.4$, $\lambda_i/\lambda_g = 0.02$).

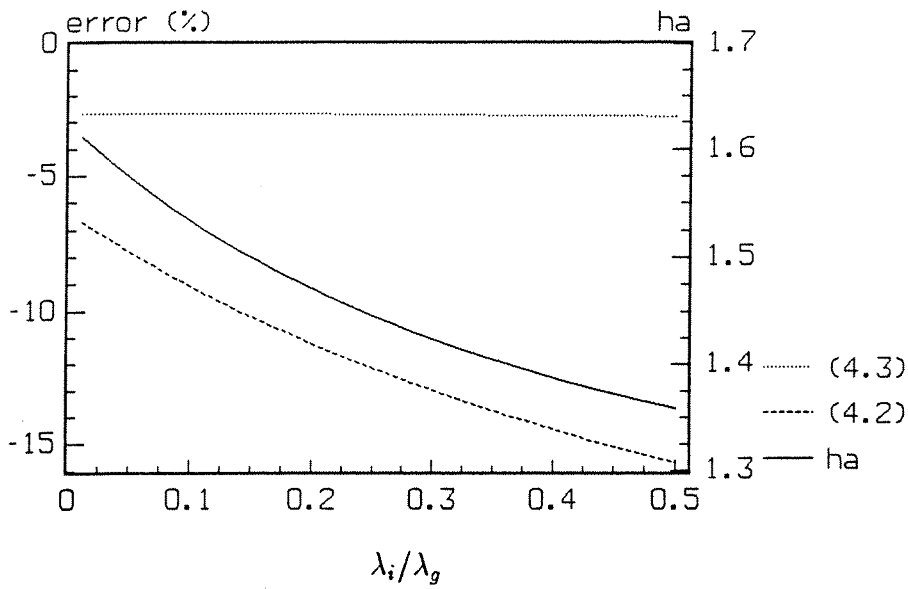

Figure 4.5. The relative error (%) of the formulae to calculate q_a and the heat loss factor h_a for different values of λ_i/λ_g ($H/r_0 = 2$, $r_i/r_0 = 0.3$, $D/r_0 = 0.4$).

From Figures 4.2-5 one can see that the formulae "collapse" when the pipes lie close to the large pipe $(D + r_i)/r_0 \approx 1$ and when the pipes lie too close to each other ($D/r_i \approx 1$). The formulae always underestimate the heat loss. The first-order formula (4.3) seems to be the best choice for practical use. It is a relatively simple formula with only a small error.

5 Position of the pipes

There is a general opinion that, for district heating pipes it is better to position the pipes vertically than horizontally, with the warmer pipe underneath the colder pipe. This is supposed to reduce the total heat loss from the pipes. Figure 5.1 shows the two cases ($T_1 > T_2$).

With the use of Myrehed [4] this problem has been investigated. The result is that the total heat loss is reduced when the pipes are positioned vertically, but for district heating pipes this reduction is so small that it is negligible. The total heat loss in the calculated examples is reduced with $< 0.2\%$.

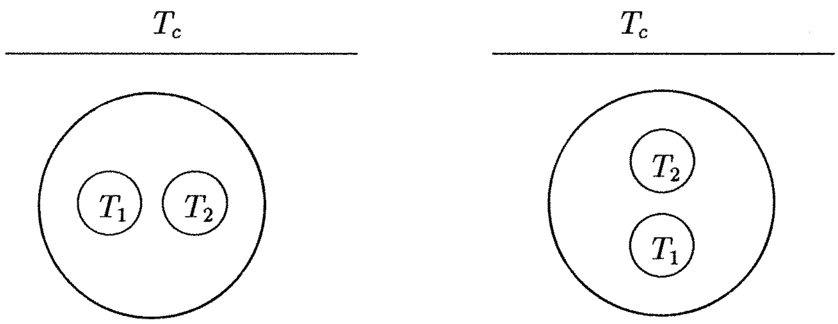

Figure 5.1. Vertical and horizontal positioning of the two pipes inside the larger pipe.

5.1 Calculations

The difference between the heat losses in the two cases has been studied. In Figures 5.2-5 the importance of the ratios H/r_0, D/r_0, r_i/r_0 and $lambda_i/\lambda_g$ are shown. The non-free parameters in each figure are listed below.

$T_1 = 70$ °C
$T_2 = 40$ °C
$T_c = 8$ °C
$\lambda_i = 0.04$ W/mK
$\lambda_g = 2$ W/mK
$\Rightarrow \sigma = -0.96078$

$D = 0.4$ m
$H = 2$ m
$r_0 = 1$ m
$r_i = 0.2$ m

The following definitions are used:

q_1^h = heat loss from pipe 1 in the horizontal problem (W/m)
q_2^h = heat loss from pipe 2 in the horizontal problem (W/m)
q_1^v = heat loss from pipe 1 in the vertical problem (W/m)
q_2^v = heat loss from pipe 2 in the vertical problem (W/m)

$$q_{tot}^h = q_1^h + q_2^h \qquad q_{tot}^v = q_1^v + q_2^v \tag{5.1}$$

Note that pipe 1 is always the warmer pipe. Figure 5.2 shows how the ratios q_{tot}^v/q_{tot}^h, q_1^v/q_1^h and q_2^v/q_2^h are dependent on H/r_0. The heat loss from the warmer pipe is reduced with < 1% when the pipes are positioned vertically and the heat loss from the colder pipe is increased < 2.5%. The total heat loss is therefore reduced with only < 0.2% when the pipes are positioned vertically.

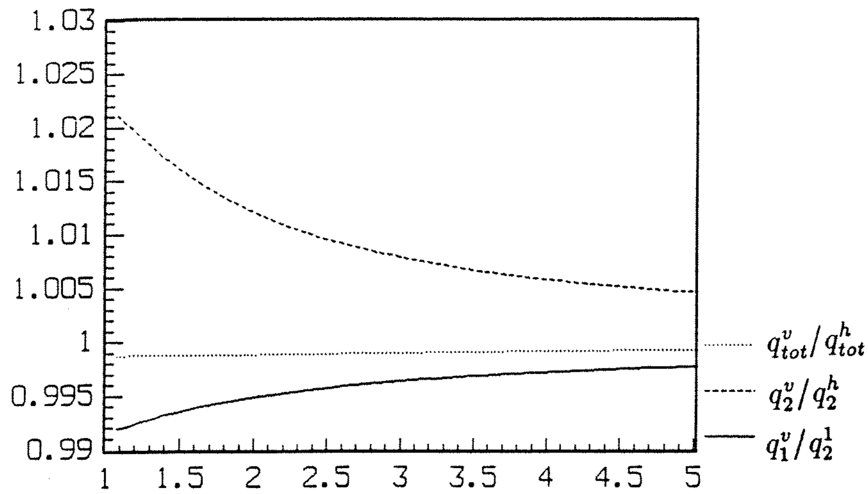

Figure 5.2. The ratios q_{tot}^v/q_{tot}^h, q_1^v/q_1^h and q_2^v/q_2^h as dependent on H/r_0.

Figure 5.3-4 show how the ratio q_{tot}^v/q_{tot}^h, is dependent on D/r_0 and r_i/r_0. The difference between the heat loss from vertically positioned pipes and horizontally positioned pipes is < 0.2% except when the pipes are very close to the outer pipe.

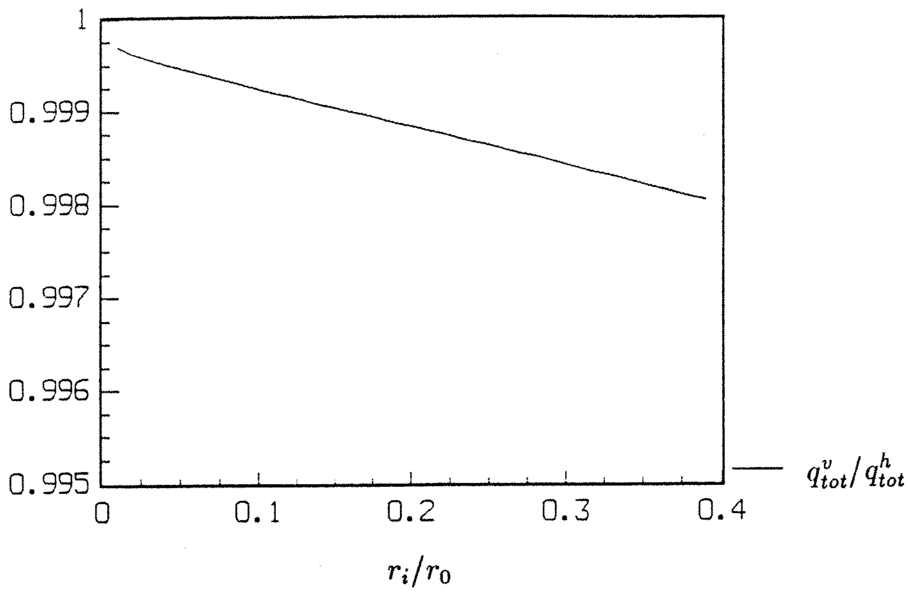

Figure 5.3. The ratio q^v_{tot}/q^h_{tot} as dependent on r_i/r_0.

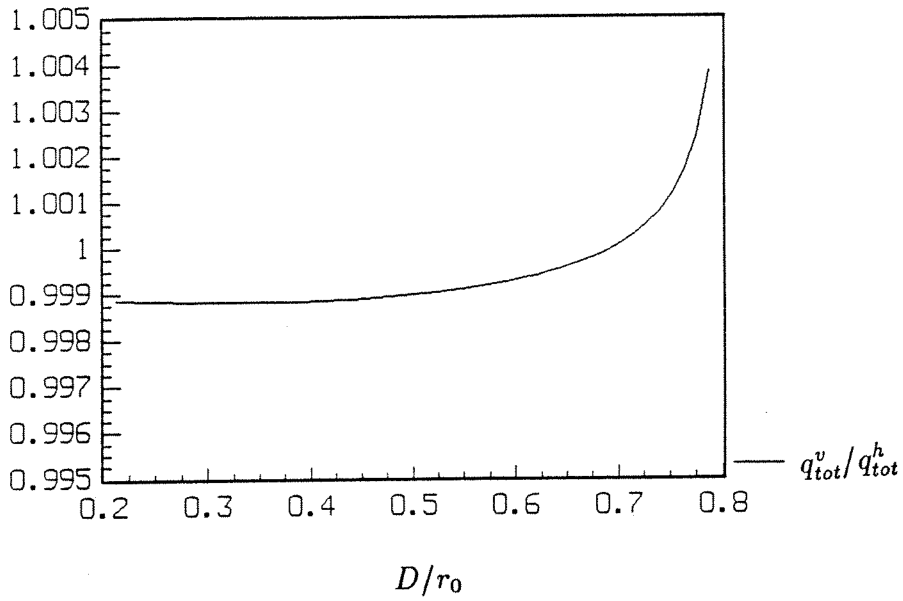

Figure 5.4. The ratio q^v_{tot}/q^h_{tot} as dependent on D/r_0.

Figure 5.5 shows how the ratio q^v_{tot}/q^h_{tot}, is dependent on λ_i/λ_g. Even when $\lambda_i/\lambda_g \approx 0.5$ the difference between the heat loss from vertically positioned pipes and horizontally positioned pipes is $< 2\%$.

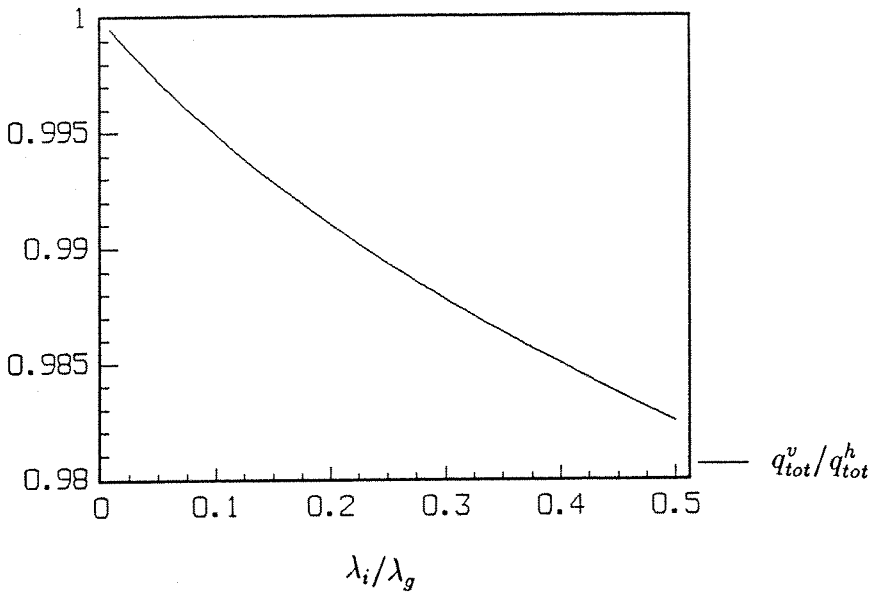

Figure 5.5. The ratio q^v_{tot}/q^h_{tot} as dependent on λ_i/λ_g.

6 Multipole method

The general multipole method can, with some adjustments, be used to solve many different problems. The only restriction is that the boundaries are circular. There exists a number of different multipole methods applicable to different problems. The multipole method used to derive the formulae in this report is described in this chapter.

6.1 Outer circle instead of ground surface

For the problem of two pipes imbedded in a circular insulation in the ground there exists a specific multipole method described in [4]. In this report the heat losses from the pipes are calculated with [4].

Claesson [1] presents a variation of the multipole method that solves the problem of pipes in a composite cylinder, i.e. two concentric cylinders with different thermal conductivity. The border of the large cylinder is the outer boundary.

The formulae in this report are derived from [1]. The formulae are thus derived to satisfy a different problem than the one we are interested in. The reason for this is that the zero and first order formulae derived from [1] are simpler and have a smaller error than the formulae derived from [4]. For higher orders, the formulae derived from [4] will have a smaller error than those derived from [1], but these formulae are much more complicated. The method described in this chapter therefore solves the problem of pipes in a composite cylinder.

Formulae for the problem of pipes in the ground are derived from the problem of pipes in a composite cylinder by simply substituting the radius of the large cylinder r_c with twice the depth $2 \cdot H$.

6.2 Thermal problem

There are N pipes, which lie in a circular region with radius r_0. The circle is surrounded by an annular region of another material. The outer circle has the radius r_c. The problem is described in the complex plane in Figure 6.1.

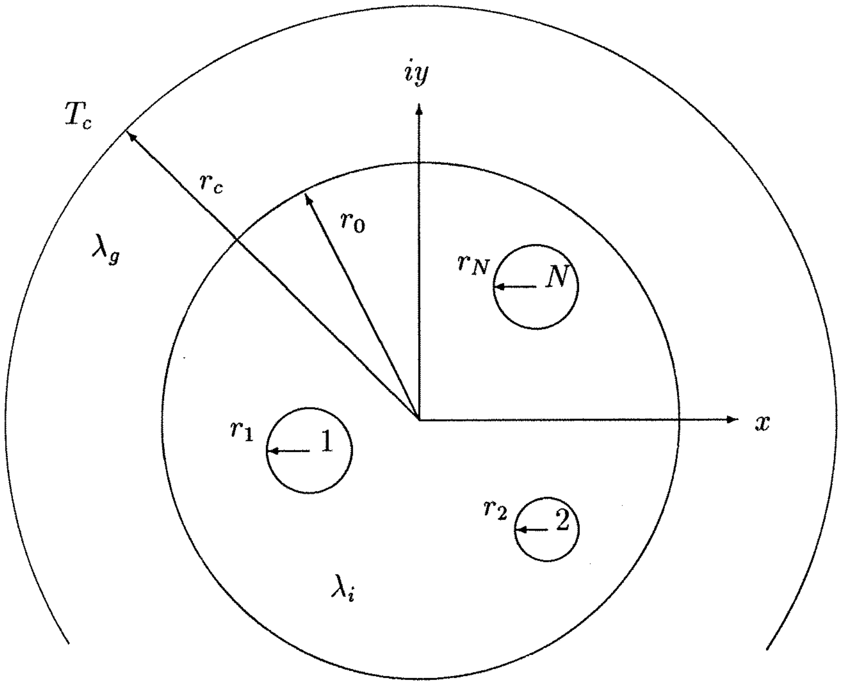

Figure 6.1. N pipes in a composite cylinder in the complex plane.

The temperature at the outer circle is T_c. The temperatures in the pipes inside the inner circle are T_n. The annular region $r_0 < r < r_c$ is homogeneous with the thermal conductivity λ_g. The inner circular region has the thermal conductivity λ_i. The steady state temperature $T(x,y)$ satisfies the heat conduction equation (Laplace equation) in the annular region and in the inner circle.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{6.1}$$

$$r = \sqrt{x^2 + y^2} \tag{6.2}$$

The problem is solved in the complex plane ($z = x + i \cdot y$). The imaginary unit is denoted i, ($i^2 = -1$). The complex conjugate of z is denoted $\overline{z}$. The center of the pipe n is z_n.

$$z_n = x_n + i \cdot y_n \tag{6.3}$$

Local polar coordinates ρ_n,ψ_n from the center of pipe n will be used:

$$z - z_n = \rho_n e^{i\psi_n} \tag{6.4}$$

The boundary condition at each pipe n is:

$$T = T_n \qquad \rho_n = r_n, \qquad 0 < \psi_n \leq 2\pi \tag{6.5}$$

The temperature and radial heat flux are continuous at the inner boundary $r = r_0$:

$$T \Big|_{r_0 - 0} = T \Big|_{r_0 + 0} \tag{6.6}$$

$$\lambda_i \frac{\partial T}{\partial r} \Big|_{r_0 - 0} = \lambda_g \frac{\partial T}{\partial r} \Big|_{r_0 + 0} \tag{6.7}$$

The boundary condition at the outer circle is :

$$T = T_c \qquad r = r_c \qquad 0 \le \psi_n \le 2\pi \tag{6.8}$$

6.3 The multipole method

The temperature field consists of a line source part $T_q(x,y)$, a multipole part at the pipes $T_p(x,y)$, a multipole part at the outer circle $T_c(x,y)$ and a constant temperature level T_0.

$$T(x,y) = T_0 + T_q(x,y) + T_p(x,y) + T_c(x,y) \tag{6.9}$$

From [1] we get the expressions for the line source term $T_q(x,y)$:

$$T_q(x,y) = \Re\left[\sum_{n=1}^{N} \frac{q_n}{2\pi\lambda_i} \cdot W_{n0}(z)\right] \tag{6.10}$$

$$r \le r_0 \qquad W_{n0}(z) = \ln\left(\frac{r_0}{z - z_n}\right) + \sigma \ln\left(\frac{r_0^2}{r_0^2 - \overline{z}z_n}\right) \tag{6.11}$$

$$r \ge r_0 \qquad W_{n0}(z) = (1+\sigma)\ln\left(\frac{r_0}{z - z_n}\right) + \frac{\lambda_i}{\lambda_g}\sigma \ln\left(\frac{r_0}{z}\right) \tag{6.12}$$

Where the dimensionless parameter σ is used $\sigma = (\lambda_i - \lambda_g)/(\lambda_i + \lambda_g)$. The complex function W_{n0} is introduced to make the equations more homogeneous. The first part of equation (6.11) represents a line source with the strength q_n at (x_n, y_n) in a material with the thermal conductivity λ_i. The second term is due to the fact that the thermal conductivity is λ_g for $r > r_0$. This term represents a line source with the strength $\sigma \cdot q_n$ situated at the mirror point $(x_n r_0^2/z_n\overline{z_n}, y_n r_0^2/z_n\overline{z_n})$. The mirror point lies on the same radius as (x_n, y_n). The temperature field (6.12) in the outer region $r \ge r_0$ consists of a line source with the strength $(1-\sigma)q_n$ at (x_n, y_n) and another one with the strength $\sigma \cdot q_n$ at $(0,0)$.

Both the real and imaginary parts of W_{n0} each satisfy the Laplace equation (6.1) and the boundary condition (6.7). The real part of W_{n0} also satisfies the boundary condition (6.6). The function $T_q(x,y)$ defined in (6.10) therefore always satisfies the boundary conditions (6.6-6.8) and the Laplace equation (6.1).

The complex-valued derivative of order j of W_{n0} with respect to z_n is called a multipole of order j. We will use the function W_{nj}. The complex strength of each multipole is P_{nj}.

$$T_p(x,y) = \Re\left[\sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj} \cdot r_n^j \cdot W_{nj}(z)\right] \tag{6.13}$$

$$W_{nj}(z) = \frac{1}{(j-1)!} \cdot \frac{\partial^j}{\partial z_n^j}(W_{n0}) \tag{6.14}$$

$$r \le r_0 \qquad W_{nj}(z) = \frac{1}{(z - z_n)^j} + \sigma \cdot \left(\frac{\overline{z}}{r_0^2 - \overline{z}z_n}\right)^j \tag{6.15}$$

$$r \ge r_0 \qquad W_{nj}(z) = (1+\sigma)\frac{1}{(z - z_n)^j} \tag{6.16}$$

Both the real and imaginary parts of W_{nj} each satisfy the Laplace equation (6.1) and the boundary conditions at the inner boundary, (6.6-6.7).

The line source and multipoles at z_n can be used to represent an arbitrary solution outside pipe n. We need a corresponding representation for the outer boundary circle $r = r_c$. The expression is, from [1]:

$$T_c(x,y) = \Re\left[\sum_{j=1}^{\infty} P_{cj} \cdot r_c^{-j} \cdot W_{cj}(z)\right] \tag{6.17}$$

$$r \le r_0 \qquad W_{cj} = (1-\sigma) \cdot z^j \tag{6.18}$$

$$r \ge r_0 \qquad W_{cj} = z^j - \sigma \cdot \left(r_0^2/\overline{z}\right)^j \tag{6.19}$$

We call these multipoles at infinity. They are needed to satisfy the boundary condition at the outer circle (6.8). Both the real and imaginary parts of W_{cj} each satisfy the Laplace equation (6.1) and the boundary conditions at the inner boundary, (6.6-6.7).

6.3.1 General expression for the temperature

The general expression for the temperature is described below. Instead of q_n the following notation will be used:

$$P_n = \frac{q_n}{2\pi\lambda_i} \tag{6.20}$$

$r \le r_0$

$$\begin{aligned} T = T_0 + \Re\Bigg[&\sum_{n=1}^{N} P_n \left\{ \ln\left(\frac{r_0}{z-z_n}\right) + \sigma \ln\left(\frac{r_0^2}{r_0^2 - \overline{z} z_n}\right)\right\} \\ &+ \sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj} \left\{ \left(\frac{r_n}{z-z_n}\right)^j + \sigma \cdot \left(\frac{r_n \overline{z}}{r_0^2 - \overline{z} z_n}\right)^j \right\} \\ &+ \sum_{j=1}^{\infty} P_{cj}(1-\sigma)\left(\frac{z}{r_c}\right)^j \Bigg] \end{aligned} \tag{6.21}$$

$r \ge r_0$

$$\begin{aligned} T = T_0 + \Re\Bigg[&\sum_{n=1}^{N} P_n \left\{ (1+\sigma) \cdot \ln\left(\frac{r_0}{z-z_n}\right) + \frac{\lambda_i}{\lambda_g}\sigma \ln\left(\frac{r_0}{z}\right)\right\} \\ &+ \sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj} \cdot (1+\sigma)\left(\frac{r_n}{z-z_n}\right)^j + \sum_{j=1}^{\infty} P_{cj} \left\{ \left(\frac{z}{r_c}\right)^j - \sigma\left(\frac{r_0^2}{r_c \overline{z}}\right)^j \right\} \Bigg] \end{aligned} \tag{6.22}$$

The temperature field (6.21,6.22) satisfies the heat conduction equation (6.1). The quantities T_0, P_n, P_{nj} and P_{cj} are determined by the boundary conditions. The expressions (6.21) and (6.22) are inserted in the boundary conditions (6.5) and (6.8 respectively.

6.3.2 Boundary condition at the outer circle

In this section the boundary condition (6.8) at the outer circle is examined. The temperature must be expressed in polar coordinates: $T = T(r, \psi)$.

$$z = r \cdot e^{i\psi} \tag{6.23}$$

From [1] we get the following expressions:

$$\ln\left(\frac{r_0}{z - z_n}\right) = \ln\left(\frac{r_0}{z}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{z_n}{z}\right)^k \qquad |z_n| < |z| \tag{6.24}$$

$$\left(\frac{r_n}{z - z_n}\right)^j = \left(\frac{r_n}{z}\right)^j \cdot \sum_{k'=0}^{\infty} \left(\begin{array}{c} j + k' - 1 \\ j - 1 \end{array}\right) \left(\frac{z_n}{z}\right)^{k'} \qquad |z_n| < |z| \tag{6.25}$$

The multipoles (6.25) are summed over j in (6.21,6.22). We need to rearrange the double sum in the following way:

$$\sum_{j=1}^{\infty} P_{nj}\left(\frac{r_n}{z - z_n}\right)^j = \sum_{j=1}^{\infty}\sum_{k'=0}^{\infty} P_{nj}\left(\frac{r_n}{z}\right)^j \left(\begin{array}{c} j + k' - 1 \\ j - 1 \end{array}\right) \left(\frac{z_n}{z}\right)^{k'} \tag{6.26}$$

$$= [j + k' = k] = \sum_{k=1}^{\infty}\sum_{j=1}^{k} P_{nj} \left(\begin{array}{c} k - 1 \\ j - 1 \end{array}\right) \frac{r_n^j z_n^{k-j}}{z^k}$$

With the use of (6.24), (6.25), (6.26) and $z = re^{i\psi}$ equation (6.22) becomes:

$r \geq r_0$

$$\begin{aligned} T(r, \psi) = T_0 &+ \sum_{n=1}^{N} P_n \frac{\lambda_i}{\lambda_g} \cdot \ln\left(\frac{r_0}{r}\right) + \Re\left[\sum_{n=1}^{N}\sum_{k=1}^{\infty} \frac{q_n}{2\pi\lambda_i}(1 + \sigma)\frac{1}{k}\left(\frac{z_n}{r}\right)^k e^{-i\cdot k\psi}\right. \\ &+ \sum_{n=1}^{N}\sum_{k=1}^{\infty}\sum_{j=1}^{k} P_{nj} \cdot (1 + \sigma) \left(\begin{array}{c} k - 1 \\ j - 1 \end{array}\right) \frac{r_n^j z_n^{k-j}}{r^k} e^{-i\cdot k\psi} \\ &+ \left.\sum_{k=1}^{\infty} P_{cj}\left\{\left(\frac{r}{r_c}\right)^k - \sigma\left(\frac{r_0^2}{r_c r}\right)^k\right\} e^{i\cdot k\psi}\right] \end{aligned} \tag{6.27}$$

The summation index is in the last line changed from j to k. The temperature depends on $e^{i\cdot k\psi}$ and $e^{-i\cdot k\psi}$. The former term may be changed into the negative exponent $e^{-i\cdot k\psi}$ by taking the complex conjugate.

$$\Re\left[P_{ck} \cdot e^{i\cdot k\psi}\right] = \Re\left[\overline{P}_{ck} \cdot e^{-i\cdot k\psi}\right] \tag{6.28}$$

The boundary condition (6.8) can now, with the use of (6.27) and (6.28), be separated into a part independent of ψ, and a part depending on ψ. The part independent of ψ is:

$$T_c = T_0 + \sum_{n=1}^{N} P_n \frac{\lambda_i}{\lambda_g} \cdot \ln\left(\frac{r_0}{r_c}\right) \tag{6.29}$$

The part depending on ψ is:

$$\boxed{k = 1, 2, \ldots}$$

$$0 = \overline{P}_{ck} \cdot \left\{1 - \sigma\left(\frac{r_0}{r_c}\right)^{2k}\right\} \tag{6.30}$$

$$+(1+\sigma) \cdot \left\{\sum_{n=1}^{N} P_n \frac{1}{k}\left(\frac{z_n}{r_c}\right)^k + \sum_{n=1}^{N}\sum_{j=1}^{k} P_{nj} \begin{pmatrix} k-1 \\ j-1 \end{pmatrix} \frac{r_n^j z_n^{k-j}}{r_c^k}\right\}$$

6.3.3 Boundary condition at pipe m

To solve the boundary condition problem of pipe m we need expressions for the line sources of the pipes and the multipoles in local polar coordinates of pipe m. From [1] we get these expressions. In the following, n is the number of the pipe with the line source or multipole and m is the number of the pipe whose boundary condition is to be satisfied. We will use polar coordinates from pipe m:

$$z = z_m + \rho_m e^{i \cdot \psi_m} \tag{6.31}$$

$$\boxed{n = m}$$

$$\ln\left(\frac{r_0}{z - z_m}\right) = \ln\left(\frac{r_0}{\rho_m}\right) - i\psi_m \tag{6.32}$$

$$\left(\frac{r_m}{z - z_m}\right)^j = \left(\frac{r_m}{\rho_m}\right)^j e^{-i \cdot j \psi_m} \tag{6.33}$$

$$\boxed{n \neq m}$$

$$\ln\left(\frac{r_0}{z - z_n}\right) = \ln\left(\frac{r_0}{z_m - z_n}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m} \tag{6.34}$$

$$\left(\frac{r_n}{z - z_n}\right)^j = \left(\frac{r_n}{z_m - z_n}\right)^j \sum_{k=0}^{\infty} \begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix} \left(\frac{\rho_m}{z_n - z_m}\right)^k e^{ik\psi_m} \tag{6.35}$$

$$\boxed{\text{every } n \text{ and } m}$$

$$\left(\frac{r_n \overline{z}}{r_0^2 - \overline{z} z_n}\right)^j = \tag{6.36}$$

$$\sum_{k=0}^{\infty} \sum_{j'=0}^{min(j,k)} \begin{pmatrix} j \\ j' \end{pmatrix} \begin{pmatrix} k+j-j'-1 \\ j-1 \end{pmatrix} \frac{r_n^j \cdot \overline{z}_m^{j-j'} \cdot z_n^{k-j'}}{(r_0^2 - z_n \overline{z}_m)^{j+k-j'}} \cdot \rho_m^k e^{-ik\psi_m}$$

$$\ln\left(\frac{r_0^2}{r_0^2-\overline{z}z_n}\right)^j = \tag{6.37}$$

$$\ln\left(\frac{r_0^2}{r_0^2-\overline{z_m}z_n}\right)^j + \sum_{k=1}^{\infty}\left(\frac{z_n\rho_m}{r_0^2-z_n\overline{z}_m}\right)^k \cdot e^{-ik\psi_m}$$

With (6.31) the multipoles at the outer circle becomes:

$$\sum_{j=1}^{\infty} P_{cj}(1-\sigma)\left(\frac{z}{r_c}\right)^j = \tag{6.38}$$

$$\sum_{k=0}^{\infty}\sum_{j=max(1,k)}^{\infty} P_{cj}(1-\sigma)\begin{pmatrix} j \\ k \end{pmatrix}\frac{z_m^{j-k}\rho^k}{r_c^j}\cdot e^{ik\psi_m}$$

With (6.32-6.38) the boundary condition (6.5) of pipe m become:

$$\boxed{m = 1, 2, \ldots N}$$

$$\begin{aligned}
T_m &= T_0 + \frac{q_m}{2\pi\lambda_i}\ln\left(\frac{r_0}{r_m}\right) \\
&+\Re\left[\sum_{n\neq m} P_n\left\{\ln\left(\frac{r_0}{z_m-z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{r_m}{z_n-z_m}\right)^k \cdot e^{ik\psi_m}\right\}\right. \\
&+\sum_{n=1}^{N} P_n\sigma\left\{\ln\left(\frac{r_0^2}{r_o^2-\overline{z}_m z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_n r_m}{r_o^2-\overline{z}_m z_n}\right)^k \cdot e^{-ik\psi_m}\right\} \\
&+\sum_{j=1}^{\infty} P_{mj}e^{-ij\psi_m} \\
&+\sum_{n\neq m}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} P_{nj}\left(\frac{r_n}{z_m-z_n}\right)^j\begin{pmatrix} j+k-1 \\ j-1 \end{pmatrix}\left(\frac{r_m}{z_n-z_m}\right)^k e^{ik\psi_m} \\
&+\sum_{n=1}^{N}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j'=0}^{min(j,k)} P_{nj}\sigma\begin{pmatrix} j \\ j' \end{pmatrix}\begin{pmatrix} j+k-j'-1 \\ j-1 \end{pmatrix}\frac{r_n^j\cdot\overline{z}_m^{j-j'}\cdot z_n^{k-j'}}{(r_0^2-z_n\overline{z}_m)^{j+k-j'}}\cdot\rho_m^k e^{-ik\psi_m} \\
&\left.+\sum_{k=0}^{\infty}\sum_{j=max(1,k)}^{\infty} P_{cj}(1-\sigma)\begin{pmatrix} j \\ k \end{pmatrix}\frac{z_m^{j-k}\rho^k}{r_c^j}\cdot e^{ik\psi_m}\right]
\end{aligned} \tag{6.39}$$

$$0 \le \psi_m < 2\pi$$

The summation index on the fourth line (concerning P_{mj}) is changed into k. The dependence on ψ_m lies in the exponents $e^{i\cdot k\psi_m}$ and $e^{-i\cdot k\psi_m}$. The latter term may be changed into the positive exponent $e^{i\cdot k\psi_m}$ by taking the complex conjugate. This is possible because only the real part of the expression is used in (6.39). Equation (6.39) can now be separated into a part independent of ψ_m, and a part depending on ψ_m. The part independent of ψ_m is:

$$\boxed{m = 1, 2, \dots N}$$

$$\begin{aligned} T_m - T_0 = {} & \sum_{n=1}^{N} q_n \cdot R^{\circ}_{mn} \\ & + \Re\left[\sum_{n \neq m} \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_n}{z_m - z_n} \right)^j - \sum_{n=1}^{N} \sum_{j=1}^{\infty} P_{nj} \sigma \left(\frac{r_n \overline{z}_m}{r_0^2 - z_n \overline{z}_m} \right)^j \right. \\ & \left. + \sum_{j=1}^{infty} P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right] \end{aligned} \tag{6.40}$$

The thermal resistances R°_{mn} (K/(W/m)) in (6.40) are given by:

$$R^{\circ}_{mm} = \frac{1}{2\pi\lambda_i} \left\{ \ln\left(\frac{r_0}{r_m}\right) + \sigma \ln\left(\frac{r_0^2}{r_0^2 - |z_m|^2}\right) \right\} + \frac{1}{2\pi\lambda_g} \cdot \ln\left(\frac{r_c}{r_0}\right) \tag{6.41}$$

$$R^{\circ}_{mn} = \frac{1}{2\pi\lambda_i} \left\{ \ln\left(\frac{r_0}{|z_m - z_n|}\right) + \sigma \ln\left(\frac{r_0^2}{|r_0^2 - \overline{z}_n z_m|}\right) \right\} + \frac{1}{2\pi\lambda_g} \cdot \ln\left(\frac{r_c}{r_0}\right) \tag{6.42}$$

The part depending on ψ_m is:

$$\boxed{m = 1, 2, \dots N \qquad k = 1, 2 \dots}$$

$$\begin{aligned} 0 = {} & \overline{P}_{mk} + \sum_{n \neq m} \frac{q_n}{2\pi\lambda_i} \frac{1}{k} \left(\frac{r_m}{z_n - z_m} \right)^k + \sum_{n=1}^{N} \sigma \cdot \frac{q_n}{2\pi\lambda_i} \frac{1}{k} \left(\frac{r_m \overline{z}_n}{r_0^2 - \overline{z}_n z_m} \right)^k \\ & + \sum_{n \neq m} \sum_{j=1}^{\infty} P_{nj} \binom{k + j - 1}{j - 1} \left(\frac{r_n}{z_m - z_n} \right)^j \left(\frac{r_m}{z_n - z_m} \right)^k \\ & \sum_{n=1}^{N} \sum_{j=1}^{\infty} \sum_{j'=0}^{min(j,k)} \overline{P}_{nj} \cdot \sigma \cdot \binom{j}{j'} \binom{k + j - j' - 1}{j - 1} \cdot \frac{r_n^j \cdot r_m^k \cdot z_m^{j-j'} \cdot \overline{z}_n^{k-j'}}{(r_0^2 - \overline{z}_n z_m)^{j+k-j'}} \\ & + \sum_{j=k}^{\infty} P_{cj} (1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} r_m^k}{r_c^j} \end{aligned} \tag{6.43}$$

6.4 Final equation system

The equation system (6.29,6.30,6.39,6.43) must be truncated. We consider multipoles at the pipes up to order J. Here J is a positive integer or in the lowest approximation zero, in which case only the line sources are used. The sine- and cosine-variation around the pipes can be made zero up to order J only. We get the following equation system:

$$\boxed{m = 1, \ldots, N}$$

$$
\begin{aligned}
T_m - T_c = &\sum_{n=1}^{N} q_n \cdot R^{\circ}_{mn} \\
&+\Re\left[\sum_{n\neq m}\sum_{j=1}^{J} P_{nj}\left(\frac{r_n}{z_m - z_n}\right)^j + \sum_{n=1}^{N}\sum_{j=1}^{J} P_{nj}\sigma\left(\frac{r_n \overline{z}_m}{r_0^2 - z_n\overline{z}_m}\right)^j\right. \\
&\left.+\sum_{j=1}^{J} P_{cj}(1-\sigma)\left(\frac{z_m}{r_c}\right)^j\right]
\end{aligned}
\tag{6.44}
$$

$$\boxed{m = 1, \ldots, N \qquad k = 1, \ldots, J}$$

$$
\begin{aligned}
0 = &\overline{P}_{mk} + \sum_{n\neq m} \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m}{z_n - z_m}\right)^k + \sum_{n=1}^{N}\sigma \cdot \frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{r_m\overline{z}_n}{r_0^2 - \overline{z}_n z_m}\right)^k \\
&+\sum_{n\neq m}\sum_{j=1}^{J} P_{nj}\begin{pmatrix} k+j-1 \\ j-1 \end{pmatrix}\left(\frac{r_n}{z_m - z_n}\right)^j\left(\frac{r_m}{z_n - z_m}\right)^k \\
&+\sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{j'=0}^{min(j,k)} \overline{P}_{nj}\sigma\begin{pmatrix} j \\ j' \end{pmatrix}\begin{pmatrix} k+j-j'-1 \\ j-1 \end{pmatrix} \cdot \frac{r_n^j \cdot r_m^k \cdot z_m^{j-j'} \cdot \overline{z}_n^{k-j'}}{(r_0^2 - \overline{z}_n z_m)^{j+k-j'}} \\
&+\sum_{j=1}^{J} P_{cj}(1-\sigma)\begin{pmatrix} j \\ k \end{pmatrix}\frac{z_m^{j-k} \cdot r_m^k}{r_c}
\end{aligned}
\tag{6.45}
$$

$$\boxed{k = 1, 2, \ldots}$$

$$
\begin{aligned}
0 = &\overline{P}_{ck} \cdot \left\{1 - \sigma\left(\frac{r_0}{r_c}\right)^{2k}\right\} \\
&+(1+\sigma)\cdot\left\{\sum_{n=1}^{N}\frac{q_n}{2\pi\lambda_i}\frac{1}{k}\left(\frac{z_n}{r_c}\right)^k + \sum_{n=1}^{N}\sum_{j=1}^{k} P_{nj}\begin{pmatrix} k-1 \\ j-1 \end{pmatrix}\frac{r_n^j z_n^{k-j}}{r_c^k}\right\}
\end{aligned}
\tag{6.46}
$$

The thermal resistances R°_{mn} (K/(W/m)) in (6.45) are given by (6.41,6.42). These are the equations that completely determine the strength of the multipoles and the line sources.

7 Derivation of the first-order formulae

We will here derive the first-order formulae from equation system (6.44-46). Due to the symmetry of the problem it is possible to reduce one unknown multipole strength. This is done in section 7.1. In section 7.2 the formula for the symmetrical problem (3.3) is derived and in section 7.3 the formula for the anti-symmetrical problem (4.3) is derived.

7.1 Symmetry analysis

Figure 7.1 describes the problem. The parameters are defined in section 1.1.

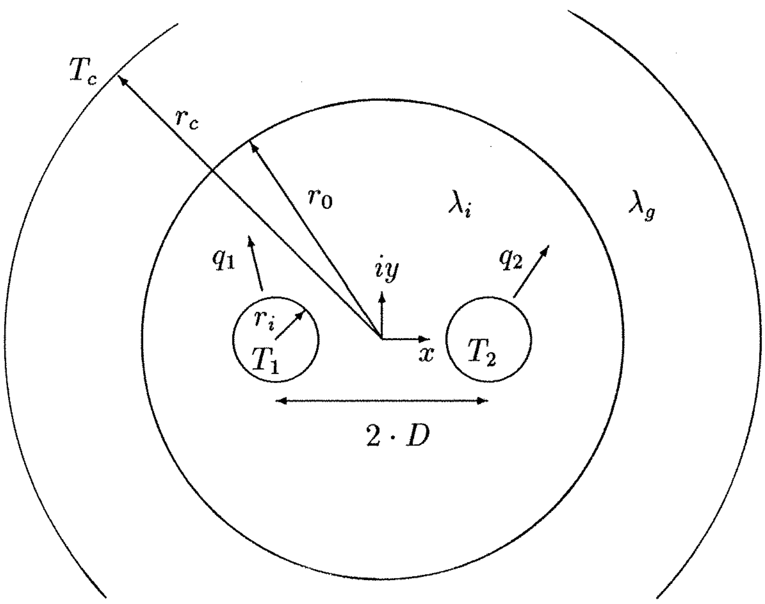

Figure 7.1. Two pipes inside a composite pipe.

The center of pipe 1 are at z_1 and the center of pipe 2 are z_2. The center of the two circumscribing circles are at $z = 0$.

$$z_1 = -D \qquad\qquad z_2 = D \tag{7.1}$$

The temperature field is in accordance with (6.9) divided into four parts:

$$T(x, y) = T_0 + T_q(z) + T_p(z) + T_c(z) \tag{7.2}$$

Here $T_q(z)$ is the temperature field from the line sources, $T_p(z)$ is the temperature field from the multipoles at the pipes and $T_c(z)$ is the temperature field from the multipoles at infinity. The field is superimposed on a constant temperature level T_0.

The line source part is from (6.10):

$$T_q(z) = \Re\left[\sum_{n=1}^{2} P_n \cdot W_{n0}(z)\right] \tag{7.3}$$

$$W_{n0}(z)\Big|_{r<r_0} = \ln\left(\frac{r_0}{z-z_n}\right) + \sigma \ln\left(\frac{r_0^2}{r_0^2 - \overline{z}z_n}\right) \tag{7.4}$$

$$W_{n0}(z)\Big|_{r>r_0} = (1+\sigma)\ln\left(\frac{r_0}{z-z_n}\right) + \sigma\frac{\lambda_i}{\lambda_g}\ln\left(\frac{r_0}{z}\right) \tag{7.5}$$

The multipole part is from (6.13), with multipoles up to order J:

$$T_p(z) = \Re\left[\sum_{n=1}^{2}\sum_{j=1}^{J} P_{nj} \cdot r_n^j \cdot W_{nj}(z)\right] \tag{7.6}$$

$$W_{nj}(z)\Big|_{r<r_0} = \frac{1}{(z-z_n)^j} + \sigma\left(\frac{\overline{z}}{r_0^2 - \overline{z}z_n}\right)^j \tag{7.7}$$

$$W_{nj}(z)\Big|_{r>r_0} = (1+\sigma)\frac{1}{(z-z_n)^j} \tag{7.8}$$

From (7.7,7.8) we get:

$$W_{2j}(z) = (-1)^j W_{1j}(-z) \tag{7.9}$$

$$\overline{W}_{nj}(z) = W_{nj}(\overline{z}) \tag{7.10}$$

The temperature field from the multipoles at infinity is $T_c(z)$:

$$T_c(z) = \Re\left[\sum_{j=1}^{J} P_{cj} \cdot r_c^{-j} \cdot W_{cj}(z)\right] \tag{7.11}$$

$$W_{cj}(z)\Big|_{r<r_0} = (1-\sigma)z^j \tag{7.12}$$

$$W_{cj}(z)\Big|_{r>r_0} = z^j - \sigma\left(\frac{r_0^2}{\overline{z}}\right)^j \tag{7.13}$$

From (7.12,7.13) we get:

$$W_{cj}(z) = (-1)^j W_{cj}(-z) \tag{7.14}$$

$$\overline{W}_{cj}(z) = W_{cj}(\overline{z}) \tag{7.15}$$

The constant temperature level T_0 is, from (6.31)

$$T_0 = T_c - \sum_{n=1}^{2} \frac{q_n}{2\pi\lambda_g} \cdot \ln\left(\frac{r_0}{r_c}\right) \tag{7.16}$$

7.1.1 Symmetrical problem

For the symmetrical problem we have:

$$T_1 = T_2 = T_s \tag{7.17}$$

$$q_1 = q_2 = q_s \tag{7.18}$$

The temperature field must be symmetric with respect to the real and imaginary axis:

$$T(z) = T(-\overline{z}) \tag{7.19}$$

$$T(z) = T(\overline{z}) \tag{7.20}$$

The multipole part $T_p(z)$ of equation (7.19) becomes with (7.9):

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(z) + P_{2j}(-1)^j W_{1j}(-z)\right\} r_i^j\right] = \tag{7.21}$$

$$\Re\left[\sum_{j=1}^{J}\left\{P_{1j}W_{1j}(-\overline{z}) + P_{2j}(-1)^j W_{1j}(\overline{z})\right\} r_i^j\right]$$

This must be true for every J and hence for every j. When the right side of equation (7.21) is complex conjugated and equation (7.10) is used, one gets:

$$\Re\left[W_{1j}(z)\left(P_{1j} - \overline{P}_{2j}\cdot(-1)^j\right) + W_{1j}(-z)\left(-\overline{P}_{1j} + P_{2j}\cdot(-1)^j\right)\right] = 0 \tag{7.22}$$

If equation (7.22) is to be satisfied for every j and z the following must be true:

$$P_{1j} = (-1)^j \overline{P}_{2j} \tag{7.23}$$

The multipole part of equation (7.20) gives, with a procedure similar to the above described:

$$\Re\left[W_{1j}(z)\left(P_{1j} - \overline{P}_{1j}\right) + W_{1j}(-z)\cdot(-1)^j\left(P_{2j} - \overline{P}_{2j}\right)\right] = 0 \tag{7.24}$$

If equation (7.24) is to be satisfied for every j and z the following must be true:

$$\Im\left[P_{1j}\right] = \Im\left[P_{2j}\right] = 0 \tag{7.25}$$

From (7.23) and (7.25) we get:

$$P_{1j} = M_j \qquad P_{2j} = (-1)^j M_j \tag{7.26}$$

Here M_j is a non-complex constant.

For the multipoles at infinity, equation (7.19) gives with (7.14, 7.15):

$$P_{cj} = (-1)^j \overline{P_{cj}} \tag{7.27}$$

Equation (7.20) gives with (7.14, 7.15):

$$P_{cj} = \overline{P_{cj}} \tag{7.28}$$

The strength of the multipoles at infinity will always be non-complex. It will only be nonzero when the order j is even.

$$P_{cj} = \frac{1}{2}(1 + (-1)^j)M_{cj} \tag{7.29}$$

Here M_{cj} is a non-complex constant.

7.1.2 Anti-symmetrical problem

For the anti-symmetrical problem we have:

$$T_1 = -T_2 = -T_a \tag{7.30}$$

$$q_1 = -q_2 = -q_a \tag{7.31}$$

The temperature field must be symmetric with respect to the real axis and anti-symmetric with respect to the imaginary axis:

$$T(z) = T(\overline{z}) \tag{7.32}$$

$$T(z) = -T(-\overline{z}) \tag{7.33}$$

Equations (7.32) and (7.20) are identical. This means that the imaginary part of the strength of the multipole is zero.

$$\Im\left[P_{1j}\right] = \Im\left[P_{2j}\right] = 0 \tag{7.34}$$

The multipole part of equation (7.33) gives with (7.9,7.10):

$$\Re\left[W_{1j}(z)\left(P_{1j} + \overline{P}_{2j} \cdot (-1)^j\right) + W_{1j}(-z)\left(P_{2j} \cdot (-1)^j + \overline{P}_{1j}\right)\right] = 0 \tag{7.35}$$

If equation (7.35) is to be satisfied for every j and z the following must be true:

$$P_{2j} = (-1)^{j+1}P_{1j} \tag{7.36}$$

From (7.34) and (7.36) we see that the following is true for the anti-symmetrical problem:

$$P_{1j} = M_j \qquad P_{2j} = (-1)^{j+1}M_j \tag{7.37}$$

Here M_j is a non-complex constant.

For the multipoles at infinity, equation (7.32) gives with (7.14, 7.15):

$$P_{cj} = (-1)^{j+1}\overline{P_{cj}} \tag{7.38}$$

Equation (7.33) gives with (7.14, 7.15):

$$P_{cj} = \overline{P_{cj}} \tag{7.39}$$

The strength of the multipoles at infinity will always be non-complex. It will only be nonzero when the order j is odd.

$$P_{cj} = \frac{1}{2}(1 + (-1)^{j+1})M_{cj} \tag{7.40}$$

Here M_{cj} is a non-complex constant.

7.2 Derivation of formula (3.3)

We will here derive the first-order multipole formula for the symmetrical problem. The problem is described in Figure 7.1 with equations (7.17-7.20). For the multipoles equation (7.26) and (7.29) are true. We will use the following definitions:

$$P_{11} = M_1 \tag{7.41}$$

$$P_{21} = -M_1 \tag{7.42}$$

$$P_{c1} = 0 \tag{7.43}$$

Here M_1 is a non-complex constant. When equations (7.41-7.43) are used in (6.44) one gets, for $m = 1$:

$$T_s - T_c = \frac{q_s}{2\pi\lambda_i}\left(\ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma\ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) + 2\frac{\lambda_i}{\lambda_g}\ln\left(\frac{r_c}{r_0}\right)\right) \tag{7.44}$$

$$+M_1\left(\frac{r_i}{2D} + -\frac{\sigma 2r_iD^3}{r_0^4 - D^4}\right)$$

When equations (7.41-7.43) are put in (6.45) one gets:

$$0 = M_1 + \frac{q_s}{2\pi\lambda_i}\left\{\frac{r_i}{2D} - \sigma\frac{r_iD}{r_0^2 - D^2} + \sigma\frac{r_iD}{r_0^2 + D^2}\right\} \tag{7.45}$$

$$+M_1\left\{\left(\frac{r_i}{2D}\right)^2 + \sigma\frac{r_i^2D^2}{(r_0^2 - D^2)^2} + \sigma\frac{r_i^2D^2}{(r_0^2 + D^2)^2} + \sigma\frac{r_i^2}{r_0^2 - D^2} - \sigma\frac{r_i^2}{r_0^2 + D^2}\right\}$$

After some simplifications equation (7.45) is reduced to:

$$M_1 = -\frac{q_s}{2\pi\lambda_i}\left(\frac{r_i}{2D} - \frac{\sigma 2r_iD^3}{r_0^4 - D^4}\right)\cdot\left(1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2r_ir_0D}{r_0^4 - D^4}\right)^2\right)^{-1} \tag{7.46}$$

This expression for M_1 is used in equation (7.44).

$$\frac{2\pi\lambda_i(T_s - T_0)}{q_s} = \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma\ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) + 2\frac{\lambda_i}{\lambda_g}\ln\left(\frac{r_c}{r_0}\right) \tag{7.47}$$

$$-\left(\frac{r_i}{2D} - \frac{\sigma 2r_iD^3}{r_0^4 - D^4}\right)^2\cdot\left(1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2r_ir_0D}{r_0^4 - D^4}\right)^2\right)^{-1}$$

Formula (7.44) is valid for two pipes in a composite cylinder. The formula for two pipes in a cylinder in the ground is acquired by replacing r_c with $2H$:

$$\frac{2\pi\lambda_i(T_s - T_0)}{q_s} = \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma\ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) + 2\frac{\lambda_i}{\lambda_g}\ln\left(\frac{2H}{r_0}\right) \tag{7.48}$$

$$-\left(\frac{r_i}{2D} - \frac{\sigma 2r_iD^3}{r_0^4 - D^4}\right)^2\cdot\left(1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2r_ir_0D}{r_0^4 - D^4}\right)^2\right)^{-1}$$

The zero order formula is acquired by setting M_1 to zero in (7.44).

7.3 Derivation of formula (4.3)

The anti-symmetrical problem is described in Figure 7.1 with equations (7.30-7.31). For the multipoles equations (7.37) and 7.40) are true. The temperature on the outer pipe is zero:

$$T_c = 0 \tag{7.49}$$

We will use the following definitions:

$$P_{11} = M_1 \tag{7.50}$$

$$P_{21} = M_1 \tag{7.51}$$

$$P_{c1} = M_c \tag{7.52}$$

Here M_1 and M_c are non-complex constants. When equations (7.49-7.52) are used in (6.44) one gets after some simplifications, for $m = 1$:

$$T_a = \frac{q_a}{2\pi\lambda_i}\left(\ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right)\right) \tag{7.53}$$

$$+M_1\left\{\frac{r_i}{2D} + \sigma\frac{2r_0^2 r_i D}{r_0^4 - D^4}\right\} + M_c(1-\sigma)\frac{D}{r_c}$$

When equations (7.49-7.52) are used in (6.45) one gets after some simplifications, for $m = 1$ and $k = 1$:

$$0 = \frac{q_a}{2\pi\lambda_i}\left\{\frac{r_i}{2D} + \sigma\frac{2r_0^2 r_i D}{r_0^4 - D^4}\right\} \tag{7.54}$$

$$+M_1\left\{1 - \left(\frac{r_i}{2D}\right)^2 + \sigma 2r_0^2 r_i^2 \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}\right\}$$

$$+M_c(1-\sigma)\frac{r_i}{r_c}$$

Equation (6.46) becomes with (7.49-7.52), for $k = 1$:

$$0 = M_c(1 - \sigma(r_0/r_c)^2) + (1+\sigma)\left(\frac{q_a}{2\pi\lambda_i}\frac{2D}{r_c} + M_1\frac{2r_i}{r_c}\right) \tag{7.55}$$

With the use of the dimensionless parameter γ equation (7.55) becomes as follows:

$$\gamma = \frac{2(1+\sigma)}{1 - \sigma(r_0/r_c)^2} \tag{7.56}$$

$$M_c = \gamma\left(\frac{q_a}{2\pi\lambda}\frac{D}{r_c} + M_1\frac{r_i}{r_c}\right) \tag{7.57}$$

Equation (7.54) becomes with (7.57):

$$M_1 = -\frac{\frac{q_a}{2\pi\lambda_i}\left(-\gamma\frac{D}{r_c} + \frac{r_i}{2D} + \sigma\frac{2r_0^2 r_i D}{r_0^4 - D^4}\right)}{1 - \gamma\frac{r_i}{r_c} - \left(\frac{r_i}{2D}\right)^2 + \sigma 2r_i^2 r_0^2 \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} \tag{7.58}$$

When (7.58) is used in (7.53) one gets, after some calculations:

$$\frac{2\pi\lambda_i T_a}{q_a} = \ln\left(\frac{2D}{r_i}\right) + \sigma\ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{7.59}$$

$$-\frac{\left(\frac{r_i}{2D} - \gamma\frac{Dr_i}{r_c^2} + \sigma\frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - \gamma\frac{r_i}{r_c} + \sigma 2r_i^2 r_0^2 \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} - \gamma\left(\frac{D}{r_c}\right)^2$$

Formula (7.59) is valid for two pipes in a composite cylinder. The formula for two pipes in a cylinder in the ground is acquired by replacing r_c with $2H$.

$$\frac{2\pi\lambda_i T_a}{q_a} = \ln\left(\frac{2D}{r_i}\right) + \sigma\ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{7.60}$$

$$-\frac{\left(\frac{r_i}{2D} - \gamma\frac{Dr_i}{(2H)^2} + \sigma\frac{2r_i r_0^2 D}{r_0^4 - D^4}\right)^2}{1 - \left(\frac{r_i}{2D}\right)^2 - \gamma\frac{r_i}{2H} + \sigma 2r_i^2 r_0^2 \frac{r_0^4 + D^4}{(r_0^4 - D^4)^2}} - \gamma\left(\frac{D}{2H}\right)^2$$

The zero order formula is acquired by setting M_1 and M_c to zero in (7.53).

8 Summary

For a reader who is only interested in formulae for practical use it is enough to read this chapter.

8.1 Two pipes in the ground imbedded in a circular insulation

There are two pipes in the ground imbedded in a circular insulation. The temperatures in the imbedded pipes are T_1 and T_2. The temperature at the ground surface is T_c. The thermal conductivity in the insulation is λ_i. The thermal conductivity in the ground is λ_g. The problem is to determine the steady-state heat losses (q_1, q_2) per unit length from the two pipes inside the large pipe. The temperature $T(x, y)$ in a vertical cross-section of the ground satisfies the steady-state heat conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{8.1}$$

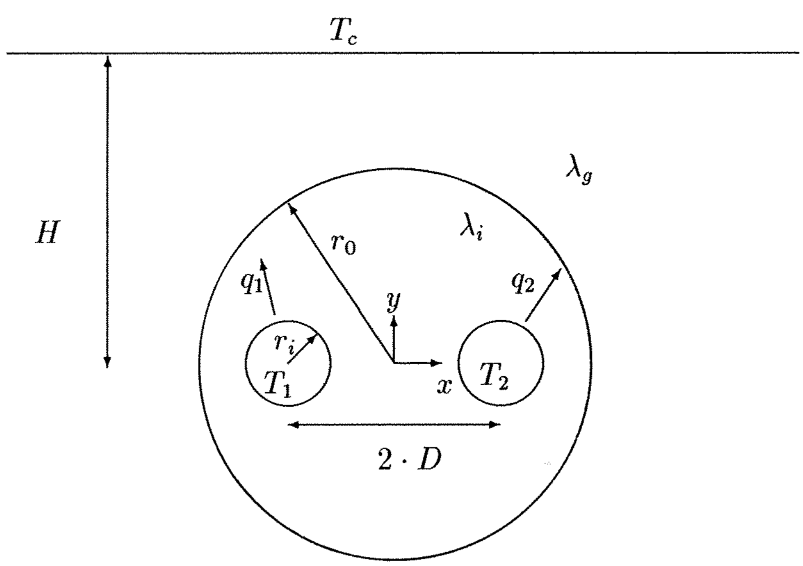

Figure 8.1. Two pipes in the ground imbedded in a circular insulation.

D = Half the distance between the center of the pipes (m)
H = Distance from the center of the large pipe to the ground surface (m)
r_0 = Radius of the circumscribing large pipe (m)
r_i = Radius of the imbedded pipes (m)
q_1 = Heat loss from pipe 1 per meter (W/m)
q_2 = Heat loss from pipe 2 per meter (W/m)
T_c = Temperature at the ground surface (°C)
T_1 = Temperature in pipe 1 (°C)
T_2 = Temperature in pipe 2 (°C)
λ_i = Thermal conductivity of the insulation (W/mK)
λ_g = Thermal conductivity of the ground (W/mK)

The dimensionless parameter σ will be used in the following:

$$\sigma = \frac{\lambda_i - \lambda_g}{\lambda_i + \lambda_g} \tag{8.2}$$

8.2 Mathematical formulation

The original problem can be separated into a symmetrical and anti-symmetrical problem. The temperature in the pipes in the symmetrical problem is T_s. The temperatures in the pipes in the anti-symmetrical problem are T_a and $-T_a$. These temperatures are defined as follows:

$$T_s = \frac{T_1 + T_2}{2} \tag{8.3}$$

$$T_a = \frac{T_1 - T_2}{2} \tag{8.4}$$

The subscript s denotes the symmetrical problem of two pipes. The subscript a denotes the anti-symmetrical problem of two pipes. The temperatures of the original problem are from (8.3-4):

$$T_1 = T_s + T_a \tag{8.5}$$

$$T_2 = T_s - T_a \tag{8.6}$$

The heat loss q_s (W/m) from one pipe in the symmetrical problem is proportional to the temperature difference $T_s - T_c$. We may write:

$$q_s = (T_s - T_c) \cdot 2\pi\lambda_i \cdot h_s(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{8.7}$$

Here h_s is the dimensionless heat loss factor for the symmetrical problem. The heat loss q_a (W/m) from one of the pipes in the anti-symmetrical problem is proportional to the temperature T_a. We may write:

$$q_a = T_a \cdot 2\pi\lambda_i \cdot h_a(r_i/r_0, D/r_0, H/r_0, \lambda_i/\lambda_g) \tag{8.8}$$

Here h_a is the dimensionless heat loss factor for the anti-symmetrical problem. It should be noted that the temperature T_a connected with q_a in (8.8) is half the temperature difference between the pipes. By superposition the heat losses q_1 and q_2 become:

$$q_1 = q_s + q_a \tag{8.9}$$

$$q_2 = q_s - q_a \tag{8.10}$$

The total heat loss $(q_1 + q_2)$ depends on the symmetrical part only:

$$q_1 + q_2 = 2 \cdot q_s \tag{8.11}$$

The symmetrical and anti-symmetrical problems are solved separately. Formulae for h_s and h_a are obtained. The heat losses q_1 and q_2 are then obtained from (8.9-10).

8.3 Approximate formulae

Approximate formulae of the zero and first order have been derived with the multipole method. Two old formulae have been investigated. The typical relative errors mentioned below are valid for district heating pipes in the ground.

8.3.1 Zero-order approximation

The zero-order multipole approximation uses the line sources without any multipoles. The zero-order approximation gives the following expressions for the heat loss factors:

$$h_s^{-1} = \frac{2\lambda_i}{\lambda_g} \ln\left(\frac{2H}{r_0}\right) + \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma \cdot \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) \tag{8.12}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{8.13}$$

The relative errors in the heat loss, when the zero-order formulae are used, are typically less than 10% for q_s and less than 20% for q_a.

8.3.2 First-order approximation

With the use of multipoles of the first order, the following new formulae are obtained for the heat loss factors:

$$h_s^{-1} = \frac{2\lambda_i}{\lambda_g} \ln\left(\frac{2H}{r_0}\right) + \ln\left(\frac{r_0^2}{2Dr_i}\right) + \sigma \cdot \ln\left(\frac{r_0^4}{r_0^4 - D^4}\right) - \frac{\left(\frac{r_i}{2D} - \frac{\sigma 2 r_i D^3}{r_0^4 - D^4}\right)^2}{1 + \left(\frac{r_i}{2D}\right)^2 + \sigma\left(\frac{2 r_i r_0^2 D}{r_0^4 - D^4}\right)^2} \tag{8.14}$$

$$h_a^{-1} = \ln\left(\frac{2D}{r_i}\right) + \sigma \ln\left(\frac{r_0^2 + D^2}{r_0^2 - D^2}\right) \tag{8.15}$$

$$-\frac{\left(\frac{r_i}{2D}-\gamma\frac{Dr_i}{4H^2}+\frac{2\sigma r_i r_0^2 D}{r_0^4-D^4}\right)^2}{1-\left(\frac{r_i}{2D}\right)^2-\gamma\frac{r_i}{2H}+2\sigma r_i^2 r_0^2\cdot\frac{r_0^4+D^4}{(r_0^4-D^4)^2}}-\gamma\left(\frac{D}{2H}\right)^2$$

$$\gamma=\frac{2(1-\sigma^2)}{1-\sigma\left(\frac{r_0}{2H}\right)^2} \tag{8.16}$$

The relative error in the heat loss, when the first zero-order formulae are used, are typically less than 1% for q_s and less than 5% for q_a.

8.3.3 Area approximation

One old formula described in this report is here called the area approximation [7]. The formula calculates the heat loss in the symmetrical problem q_s.

$$r_e=\sqrt{\frac{2\cdot r_0^2}{\pi}\arccos\left(\frac{D}{r_0}\right)-\frac{2\cdot D}{\pi}\sqrt{r_0^2-D^2}} \tag{8.17}$$

$$d_e=\frac{\sqrt{r_0^2-D^2}+r_0}{2}-r_i \tag{8.18}$$

$$R_1=2\ln\left(\frac{r_e}{r_i}\right) \tag{8.19}$$

$$R_2=\frac{\pi d_e}{D} \tag{8.20}$$

$$R_3=\frac{2\lambda_i}{\lambda_g}\ln\left(\frac{H}{r_0}+\sqrt{\frac{H}{r_0}^2-1}\right) \tag{8.21}$$

$$h_s^{-1}=\frac{1}{1/R_1+1/R_2}+R_3 \tag{8.22}$$

The relative error in the heat loss, when the area approximation formula is used, is typically less than 5% for q_s.

8.3.4 Two-model approximation

Another old formula described in this report is here called the two-model approximation [8]. The formula calculates the heat loss in the symmetrical problem q_s.

$$R_1=\text{arccosh}\left(\frac{r_i/r_0+r_0/r_i-(r_0/r_i)(D/r_0)^2}{2}\right) \tag{8.23}$$

$$R_2=4\cdot\text{arccosh}\left(2\left(\frac{D}{r_i}\right)^2-1\right) \tag{8.24}$$

$$R_3 = \frac{2\lambda_i}{\lambda_g} \ln\left(\frac{H}{r_0} + \sqrt{\frac{H}{r_0}^2 - 1}\right) \tag{8.25}$$

$$h_s^{-1} = \frac{1}{1/R_1 - 1/R_2} + R_3 \tag{8.26}$$

The relative error in the heat loss, when the two-model approximation formula is used, is typically less than 5% for q_s.

8.4 Position of the pipes

There is a general opinion that, for heating district pipes it is better to put the the warmer pipe underneath the colder pipe. This is supposed to reduce the total heat loss from the pipes. It is true that the heat loss is reduced when the pipes are positioned vertically, but this reduction is so small that it is negligible. Calculations show that the total heat loss is reduced with $< 0.2\%$.

References

1. Claesson J, Bennet J and Hellström G, Multipole method to compute the conductive heat flows to and between pipes in a composite cylinder, Notes on Heat Transfer 3-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

2. Claesson J and Bennet J, Multipole method to compute the conductive heat flows to and between pipes in a cylinder, Notes on Heat Transfer 2-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

3. Arfvidsson J, Multipole method to compute the conductive heat flows to and between pipes in a cylinder. PC-version of computer model, Notes on Heat Transfer 2.1-1987, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118,S-221 00 Lund, Sweden, 1987.

4. Myrehed M, Multipole method to compute the conductive heat flows to and between pipes in a composite region: Pipes in a circle in a half plane, Notes on Heat Transfer 1-1989, Depts. of Building Technology and Mathematical Physics, Lunds Institute of Technology, Sweden.

5. Wallentén P, Heat loss from two pipes imbedded in a circular insulation, Notes on Heat Transfer 2-1991, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.

6. Wallentén P, Heat loss from one or two insulated pipes in the ground, Notes on Heat Transfer 1-1990, Depts. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1990.

7. Carlsson S and Eneborg I, Heat losses from underground heating mains, Byggforskningens handlingar nr 42, 1963, Stockholm, Sweden. (in Swedish)

8. Jarfelt U, Heat losses from district heating mains, thesis, Department of Building Technology, Chalmers University of Technology, 1986, Göteborg, Sweden. (in Swedish)

9. Hellström G, Ground heat storage, Thermal analyses of duct storage systems (thesis), Depts. of Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1991.