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Optimization Based Robust Design of Uncertain SISO Systems

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Abstract. A robust design method for uncertain single-input-single-output systems is presented. Both structured and unstructured uncertainties are considered. Optimal performance is described as maximum achievable bandwidth. A controller for robust optimal performance is determined through a convex optimization problem where the constraints come from frequency domain performance criteria. The theoretical framework is developed. Uniqueness of the solution is shown. The design method is applied on two problems.

Keywords: Robust Control, Optimization, Performance Bounds, Frequency Domain, Convex Programming

1. Introduction

Structured and unstructured uncertainty are convenient ways to describe model mismatches. Structured uncertainty is best for low frequency variations and unstructured uncertainty describes best high frequency uncertainty.

Performance criteria may be given as bounds on time- and/or frequency responses. It is important that all systems described within the uncertainty range of the nominal model fulfills the performance criteria. This is referred as *robust performance* (Morari and Doyle 1986).

This paper describes a design method that guarantees robust performance in frequency domain for systems with a combination of both types of uncertainty. An attempt to control such systems is made by Wei and Yedavalli (1989). They propose an algorithm for finding a stabilizing controller. Nothing is however mentioned about the performance. Boyd et al. (1988) describes a design method using optimization. That method does not consider structured uncertainty. In this paper it is attempted to outline these ideas to obtain a design procedure that gives specified performance for uncertain plants.

2. System Description

The process has three inputs and one output. Using the control signal u, the output y should be controlled to follow a desired reference, while effects from disturbances are minimized. Two disturbances l and d act on the input and the output of the system. It is given by

$$y = G_T(s)(u + l) + d \tag{1}$$

The process transfer function G_T is not completely known. It can be separated into two parts.

$$G_T(s) = G(s)G_u(s) \tag{2}$$

The first part $G(s)$ is a rational transfer function

$$G(s,p) = \frac{B(s,p)}{A(s,p)} = \frac{B_o(s)B_p(s,p)}{A_o(s)A_p(s,p)} \tag{3}$$

where B_o, A_o, B_p and A_p are polynomials of finite degrees in the Laplace operator s. Polynomials B_o and A_o are accurately known. Polynomials B_p and A_p are known to be of the form

$$\begin{aligned} B_p(s,p) &= b_0 s^{d_b} + b_1 s^{d_b-1} + \cdots + b_{d_b} \\ A_p(s,p) &= s^{d_a} + a_1 s^{d_a-1} + \cdots + a_{d_a} \end{aligned} \tag{4}$$

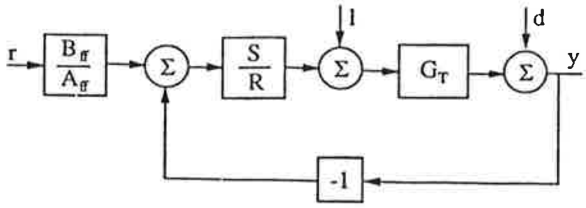

Figure 1. The Closed Loop System

Their coefficients are described by the vector

$$p = (a_1 \quad \cdots \quad a_{d_a} \quad b_0 \quad \cdots \quad b_{d_b})^T \in \mathbb{R}^{d_p} \quad (5)$$

DEFINITION 1
Let p be as in (5). The set

$$\mathcal{P} = \{p : p_i \in (p_i^-, p_i^+) \quad i = 1, \ldots, d_p\}$$

defines the transfer function family $G(s,p)$ with structured uncertainty. □

DEFINITION 2
Let the set $\mathcal{P}_c$ contain the $2^{d_{pc}}$ corners of the set $\mathcal{P}$. □

The second part of the process transfer function is stable and nonparametric.

$$G_u(i\omega) = (1 + \Delta(i\omega)W_2(\omega)) \qquad (6)$$

The quantity $\Delta(i\omega)W_2(\omega)$ describes multiplicative unstructured uncertainty, where $W_2(\omega)$ is the maximum magnitude of the uncertainty and $|\Delta(i\omega)| \leq 1$.

3. Controller and Closed Loop

A two-degree of freedom controller is assumed.

$$u = \frac{S(s)}{R(s)}(y_c - y) = \frac{S(s)}{R(s)}\left(\frac{B_{ff}(s)}{A_{ff}(s)}r - y\right) \quad (7)$$

The feedback compensator $S(s)/R(s)$ is determined to handle disturbances l or d. The feedforward compensator is chosen to give the desired transfer function from r to y. Define the transfer functions

$$\begin{aligned} \mathcal{T}(s,p) &= \frac{BS}{AR+BS} \\ \mathcal{S}_o(s,p) &= \frac{AR}{AR+BS} \\ \mathcal{S}_i(s,p) &= \frac{BR}{AR+BS} \end{aligned} \qquad (8)$$

Neglecting unstructured uncertainties the process output is given by

$$\begin{aligned} y(s) = \mathcal{T}(s,p)&\frac{B_{ff}(s)}{A_{ff}(s)}r(s) \\ &+ \mathcal{S}_o(s,p)d(s) + \mathcal{S}_i(s,p)l(s) \end{aligned} \qquad (9)$$

Disturbances assumed to be the output from a filter $1/A_r(s)$, whose input is an impulse, may be eliminated in steady state provided that A_r is a factor of R. This yields $R = A_rR'$ and the closed loop characteristic polynomial is

$$A_c(s,p) = AA_rR' + BS \qquad (10)$$

Integral action corresponds to $A_r(s) = s$.

Controller Parametrization

If no structured uncertainty is present and if the casual controller S_0/R_0 stabilizes the system B/A, then all stabilizing controllers could be parameterized as

$$\frac{S}{R} = \frac{S_0 + QA_rA}{A_r(R_0' - QB)} \qquad (11)$$

where $Q = N/D$ with N a polynomial and D a stable polynomial. Since (11) contains polynomials instead of proper stable rational transfer functions as in Vidyasagar (1985) Q must fulfill

$$\deg D \geq \deg A - \deg R_0' + \deg N \qquad (12)$$

to yield a proper controller.
If on the other hand A and B may take different values due to structured uncertainty, a slight modification of (11) is required. Select a nominal process model $p_0 \in \mathcal{P}$ yielding $B_0 = B(s,p_0)$ and $A_0 = A(s,p_0)$. Use these as B and A in the controller (11). The characteristic polynomial will now be perturbed so that it is necessary to guarantee stability by another criteria. Such criteria are implied by the robust performance criteria given later.

4. Design of Feedback

The amount of uncertainty in a model of a system limits the achievable performance. Different approaches are required depending on the nature of the disturbances.

Input Disturbance Rejection

Consider the case of minimizing the effects from the input disturbance l. Assume that no structured uncertainty is present. This assumption will be relaxed later. The system response for an input disturbance is

$$y = \frac{BG_uR}{AR + BG_uS}l \qquad (13)$$

The magnitude of the transfer function from l to y will be limited using the weighting function $W_1(\omega)$.

$$W_1 \left| \frac{BRG_u}{AR + BSG_u} \right| \leq 1 \quad \forall\omega, \forall|\Delta| \leq 1 \tag{14}$$

Using (6) and (8) this can be written as

$$W_1 \frac{|S_i(1+\Delta W_2)|}{|1+\Delta W_2 T|} \leq 1 \quad \forall\omega, \forall|\Delta| \leq 1 \tag{15}$$

LEMMA 1
A sufficient condition for (15) is

$$W_1(1+W_2)|S_i| + W_2|T| \leq 1 \quad \forall\omega$$

Proof: It follows that $W_2|T| \leq 1$. If $W_2|T| < 1$ then

$$1 > \frac{W_1(1+W_2)|S_i|}{1 - W_2|T|} \geq \frac{W_1|1+W_2\Delta||S_i|}{|1+\Delta W_2 T|}$$

If $W_2|T| = 1$ then $|S_i| = 0$ and (15) holds. □

Remark. If the closed loop system based on $G(s)$ is stable, strict inequality in lemma 1 implies robust stability for the closed loop system based on $G(s)G_u(s)$. □

In the prescence of structured uncertainty (15) must hold for all $p \in \mathcal{P}$. A modification is proposed such that it is sufficient to check the elements $p \in \mathcal{P}_c$ only. The controller is of the form (11). Define

$$\begin{aligned} \delta_B(s,p) &= B(s,p) - B_0(s) \\ \delta_A(s,p) &= A(s,p) - A_0(s) \end{aligned} \tag{16}$$

Write the characteristic polynomial (10) as

$$A_c(s,p) = A_{c0}(s) + \delta_{Ac}(s,p,Q) \tag{17}$$

where

$$A_{c0}(s) = A_0 A_r R_0' + B_0 S_0 \tag{18}$$

is independent of p and Q and

$$\begin{aligned} \delta_{Ac}(s,p,Q) = {} & \delta_A A_r R_0' + \delta_B S_0 \\ & + Q A_r (A_0 \delta_B - B_0 \delta_A) \end{aligned} \tag{19}$$

is an affine function in p and in Q. The inequality

$$|A_c| = |A_{c0} + \delta_{Ac}| \geq |A_{c0}| \left(1 + \mathrm{Re}\, \frac{\delta_{Ac}}{A_{c0}}\right) \tag{20}$$

used in the condition

$$W_1(1+W_2)|\frac{BR}{A_{c0}}| + W_2|\frac{BS}{A_{c0}}| \leq \left(1 + \mathrm{Re}\, \frac{\delta_{Ac}}{A_{c0}}\right) \tag{21}$$

implies (15) for the system p. Insertion of the controller in (21) yields the constraint

$$g(\omega,p,Q) \leq 1 \qquad \forall\omega \tag{22}$$

with the constraint function

$$\begin{aligned} g = {} & W_1 f_1(\omega,p,Q) + f_2(\omega,p,Q) \\ = {} & W_1(1+W_2) \left| \frac{BA_r B_0}{A_{c0}} \right| \left| \frac{R_0'}{B_0} - Q \right| \\ & + W_2 \left| \frac{BA_r A_0}{A_{c0}} \right| \left| \frac{-S_0}{A_0 A_r} - Q \right| - \mathrm{Re}\, \frac{\delta_{Ac}}{A_{c0}} \end{aligned} \tag{23}$$

Condition (22) implies that the perturbation $\delta_{Ac}(s,p,Q)$ will not destabilize the characteristic polynomial $A_c(s,p)$, since the real part of $\delta_{Ac}(i\omega,p,Q)/A_{c0}(i\omega)$ always is larger than -1. Now one main result can be stated.

THEOREM 1
If (22) is fulfilled $\forall p \in \mathcal{P}_c$ then it is fulfilled $\forall p \in \mathcal{P}$.

Proof: $g(\omega,p,Q)$ is a sum of three functions. The two first are convex in p because a function $f(z) = |P(i\omega,z)|$ where P is a polynomial is convex for a fixed ω. The third is affine in p and thus convex. Therefore $g(\omega,p,Q)$ is convex in p. If the inequality is satisfied for the extreme p's then it is satisfied for convex combinations of the extreme p's by the definition of convex functions. □

Output Disturbance Rejection

The robust performance criterion for output disturbance rejection is derived in a similar way. The only difference is the function f_1 in (23). It is here given by

$$f_1(\omega,p,Q) = \left| \frac{AA_r B_0}{A_{c0}} \right| \left| \frac{R_0'}{B_0} - Q \right| \tag{24}$$

Selection of Q

Let $N(s,x)$ be the numerator of the rational function Q. The constraint function g is then convex in x, i.e. the coefficients of N. However g is not convex in the coefficients of $D(s)$. Therefore the optimization problem is formulated to determine the coefficient vector $x \in \mathbb{R}^n$ for an appropriate choice of $D(s)$.
Select a stable polynomial $D(s)$ so that $1/D(i\omega)$ is a filter with cut-off frequency where the system is supposed to operate. This choice may be revised from the the frequency response $Q(i\omega)$ and the constraints g in a previous design. The choice of $D(s)$ is not crucial provided that deg N is sufficiently high. A similar approach is taken in Boyd et al. (1988). Their discrete $D(q)$ has all zeros at the origin.

Performance Optimization

It is of interest to achieve optimal performance. This it is equivalent to maximize the bandwidth while regarding uncertainty.

DEFINITION 3
The bandwidth of the closed loop system is the frequency ω_b specifying the weighting function

$$W_1(\omega, \omega_b) = \max\left((\frac{\omega_b}{\omega})^\sigma, \frac{1}{M_p} \right)$$

M_p is the maximum peak of the frequency function from disturbance to output and σ is its low frequency slope. □

This W_1 requires that BA_r, in the input disturbance case, or AA_r, in the output disturbance case, has σ zeros at the origin.
With this W_1, it is convenient to split the constraint g into two functions

$$\begin{aligned} g_1 &= \left(\frac{\omega_b}{\omega}\right)^\sigma f_1(\omega,p,x) + f_2(\omega,p,x) \\ g_2 &= \frac{1}{M_p} f_1(\omega,p,x) + f_2(\omega,p,x) \end{aligned} \tag{25}$$

Optimal bandwidth is found through the convex optimization problem.

$$\begin{array}{lll} \text{maximize} & \omega_b^\sigma & \\ \text{subject to} & g_1(\omega_b^\sigma, \omega, p, x) & \leq 1 \\ & g_2(\omega,p,x) & \leq 1 \\ & & \forall\omega, \forall p \in \mathcal{P}_c \end{array} \tag{26}$$

5. Optimization Issues

Constraint functions define a set in the parameter space. The properties of this set of x depend on the constraint functions.

DEFINITION 4
Let $g_i(y,x) = yf_{1i}(x) + f_{2i}(x)$ with $f_{ji}(x)$ convex and non-negative. Define for fixed $y > 0$ the convex set

$$\Omega(y) = \{x : g_i(y,x) \leq 1, \forall i\}$$

□

Note that this formulation also covers g_2 in (25) by letting $f_{1i} = 0$. A lemma relates the sets $\Omega(y)$ for different values of y.

LEMMA 2
If $y_2 > y_1$ then $\Omega(y_2) \subset \Omega(y_1)$ for the set $\Omega(\cdot)$ of definition 4. □

Now it is possible to state the other main result.

THEOREM 2
If a feasible solution exists for the optimization problem

$$\begin{array}{ll} \text{maximize} & y \\ \text{subject to} & x \in \Omega(y) \\ & y > 0 \end{array} \tag{27}$$

with $\Omega(\cdot)$ of definition 4 then the set $X^* \in \mathbb{R}^n$ corresponding to the maximum $y = y^*$ is convex and any relative maximum is a global maximum.

Proof: Given an initial $y_0 > 0$, assume that $\Omega(y_0) \neq \emptyset$, i.e. a feasible solution exists. Increasing y generates ordered subsets of $\Omega(y_0)$ according to lemma 2. If $\exists\hat{y} : \Omega(\hat{y}) = \emptyset$ the optimal value $y^* = \inf\hat{y}$. Optimum is then achieved for the convex set $X^* = \Omega(y^*)$. If $\Omega(\hat{y}) \neq \emptyset, \forall\hat{y} > y_0$ then y^* is unbounded and the extremum is achieved on the convex set $X^* = \{x : f_{2i}(x) \leq 1, \forall i\}$. □

Implementation

The optimization problem stated in section 4 is of the type in theorem 2 when $y = \omega_b^\sigma$. It is a semi-infinite problem as discussed in Polak et al. (1984) since the constraint functions are functions of ω as well. To simplify, it is reformulated to a finite optimization problem using the following assumption.

ASSUMPTION
The functions $f_1(\cdot)$, and $f_2(\cdot)$ are assumed to be sufficiently smooth so that if the constraints are satisfied at a finite number of frequencies $\omega_j, j = \{1,..m\}$, they are satisfied for all ω. □

In practice the chosen frequencies ω_j should reflect the assumed operating range of the system. Each element $p \in \mathcal{P}_c$ produces $2m$ constraint functions. The optimization problem thus has $m2^{d_{pc}+1}$ convex constraint functions. It is

$$\begin{array}{lll} \text{maximize} & \omega_b^\sigma & \\ \text{subject to} & g_1(\omega_b^\sigma, \omega_j, p_k, x) & \leq 1 \\ & g_2(\omega_j, p_k, x) & \leq 1 \\ & j = 1..m & \\ & k = 1..2^{d_{pc}} & \end{array} \tag{28}$$

Initial Feasible Solution

An initial feasible solution could be found from a slightly modified problem.

$$\begin{aligned} \text{maximize} \quad & \gamma \\ \text{subject to} \quad & 0 \le \gamma \le 1 \\ & \gamma g_1(\omega_{b0}^{\sigma}, \omega_j, p_k, x) \le 1 \\ & \gamma g_2(\omega_j, p_k, x) \qquad \le 1 \\ & \qquad j = 1..m \\ & \qquad k = 1..2^{d_{pc}} \end{aligned} \tag{29}$$

Given a desired bandwidth ω_{b0} a feasible solution is achieved if optimal $\gamma^* = 1$. If $\gamma^* < 1$ no feasible solution is found. Investigation of $Q(i\omega)$ and $g.(\cdot, \omega_j, \cdot, \cdot)$ will suggest a new $D(s)$ for which the problem may have a feasible solution. This iteration over $D(s)$ also is useful in the search for optimal bandwidth.
This optimization problem may have any performance function W_1.

6. Design of Feedforward

The feedback design above determines $\mathcal{T}(s,p)$. Command signal following may be improved by a feedforward filter $B_{ff}(s)/A_{ff}(s)$. It should have low-pass characteristics so that the command signal not will excite high frequency uncertainty. The process transfer function

$$\mathcal{T}_T(s,p,\Delta) = \left| \frac{\frac{BS}{AR}(1+\Delta W_2)}{1+\frac{BS}{AR}(1+\Delta W_2)} \right| \tag{30}$$

belongs to a certain interval for each frequency. Calculate supremum $\mathcal{T}_T^+(\omega)$ and infimum $\mathcal{T}_T^-(\omega)$ of $\mathcal{T}_T(i\omega, p, \Delta)$ for $|\Delta| \le 1$ and $p \in \mathcal{P}_c$. Infimum is zero if $W_2(\omega) \ge 1$. Robust stability implies that the denominator of (30) is nonzero.
Now a feedforward filter may be designed giving

$$\mathcal{T}_T^+(\omega)\frac{B_{ff}(i\omega)}{A_{ff}(i\omega)} \quad \text{and} \quad \mathcal{T}_T^-(\omega)\frac{B_{ff}(i\omega)}{A_{ff}(i\omega)} \tag{31}$$

both desired shape.

7. Examples

The examples are taken from Masten and Cohen (1989). Feedback design is made for input disturbance rejection. The reference signal $r = 1$ for time $t < 8$ and $r = -1$ for larger t. An input disturbance signal $l = -1$ for $t < 6$, $l = 2$ for $6 \le t < 15$ and $l = 1$ for larger t. An output disturbance $d = 1$ affects the plant from $t = 20$. It, however, is not designed for.

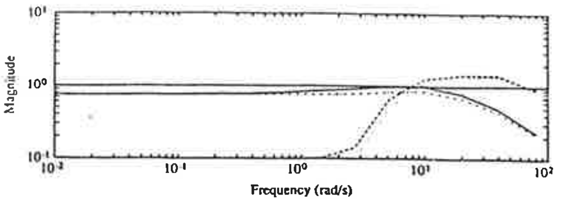

Figure 2. Constraints for Stabilizing Controller

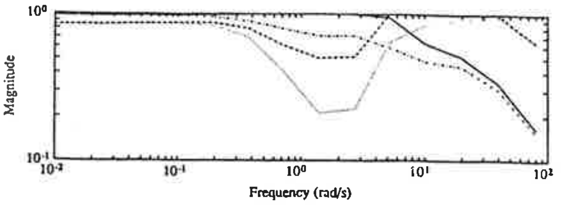

Figure 3. Constraints for Optimal Controller

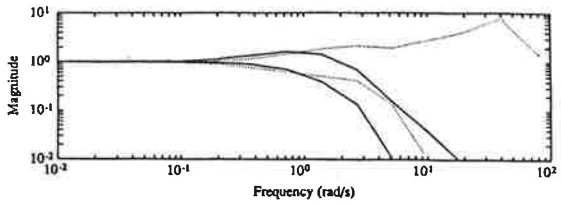

Figure 4. Feedforward Filter Design

Unmodeled dynamics is present in both examples. It is assumed to be $G_u(s) = e^{-sL}/(1+sT)$ with $L \in (0, 0.05)$ and $T \in (0, 0.10)$. This gives

$$W_2(\omega) = \left| \frac{e^{-i0.05\omega}}{(1+i0.1\omega)} - 1 \right| \tag{32}$$

First Order System

First the system $G(s,p) = b/(s+a)$ with $b \in (0.5, 3.0)$ and $a \in (-1.0, 3.0)$ is considered. The set $\mathcal{P}_c$ has four elements. Characteristics for these will be shown in the figures.
A stabilizing controller is $S_0/R_0 = (17s+14)/s$. It yields constraint functions $g(i\omega, p, 0)$ in (23) for $p \in \mathcal{P}_c$ with the performance function W_1 in definition 3 where $M_p = 2.0$, $\sigma = 1$ and $\omega_b = 2.0$. These functions are shown in figure 2. The robust performance criterion is not fulfilled and higher bandwidth is also desired. The polynomial $D(s)$ with poles $\{-1,-1,-3,-3,-5,-5\}$ is selected for the optimization problem (28). The optimal solution is found for bandwidth $\omega_b = 3.6$. Figure 3 shows $\max(g_1(i\omega,p,x), g_2(i\omega,p,x))$ in (25) for the optimal controller.
Another denominator $D(s) = (s+10)^6$ gave approximately the same bandwidth, confirming that the choice of D is not crucial.
A feedforward filter is designed. Figure 4 shows $\mathcal{T}_T^+$ and $\mathcal{T}_T^-$ (dotted lines) and when they are cascaded with B_{ff}/A_{ff} (solid lines).

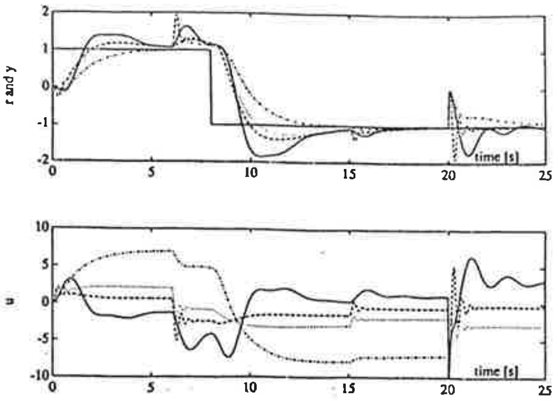

Figure 5. Simulation of First Order System

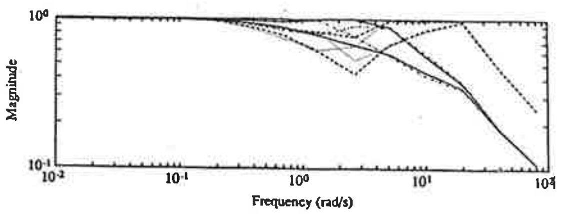

Figure 6. Constraints for Optimal Controller

The process is simulated in figure 5. The input disturbance is well eliminated for all structured and unstructured perturbations.

Second Order System

The parametric model is here described by $G(s,p) = b/(s^2+a_1 s+a_2)$ where $b \in (1.13, 2.37)$, $a_1 \in (0.4, 2.4)$ and $a_2 \in (-0.5, 2.5)$. The set $\mathcal{P}_c$ has eight elements.

The same $D(s)$, σ and M_p is used as for the first order example. The optimal bandwidth is $\omega_b = 5.7$. The corresponding constraint functions $\max(g_1(i\omega, p, x),\ g_2(i\omega, p, x))$ are found in figure 6. The simulation of the system in figure 7 shows that the effect from the input disturbance is small. Nice reference signal following is achieved by the use of a feedforward filter. Interesting is the poor output disturbance rejection, but that was not regarded in the design.

8. Conclusions

A design method guaranteeing robust performance for systems with both structured and unstructured uncertainty is described. The feedback controller design is formulated as a convex optimization problem.

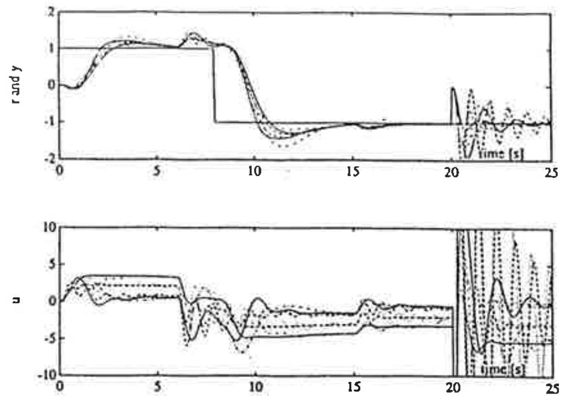

Figure 7. Simulation of Second Order System

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