Test of Common Factors of Identified Models. Application to the Generalized Least Squares Method

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Abstract

Tests of common factors of two polynomials are considered. When the coefficients of the polynomials are given only with some certain accuracy, the problem is not trivial. A systematic way to perform such tests taking the accuracies of the coefficients into account, is proposed. Different ways to implement the proposed algorithm are discussed. It is further proven that applied to models obtained with least squares identification, the resulting algorithm is equivalent to Clarke's generalized least squares method. The proposed method can in fact be considered as an alternative to get the generalized least squares estimate.

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I. INTRODUCTION

In the analysis of models obtained by process identification it is often desirable to perform a test of common factors. A typical example is models of multivariable systems, where the number of common factors may have a drastic influence on the final choice of the order as well as the structure of the model. Another typical example is the so called repeated least squares method, see Åström-Eykhoff(1971). This method means simply the ordinary least squares method but with a high order model in order to include also the noise in a proper way in the model. In the papers by Unbehauen-Göhring (1973) and van den Boom-van den Enden (1973) tests of common factors are used to determine suitable orders for models of single input single output systems.

Although interest has been shown concerning tests of common factors in identified models, there is not presented any systematic way to perform such a test. A usual method to do the test is to compute the zeros of the two polynomials and compare them "visually" or by a quite arbitrary "criterion." In this report the purpose is to give and discuss a systematic way to perform tests of this nature.

The problem that will be considered can be formulated as follows:
Assume that the two polynomials

$$\hat{A}(z) = \hat{a}_0 z^{n_a} + \hat{a}_1 z^{n_a-1} + \ldots \hat{a}_{n_a}$$

$$\hat{B}(z) = \hat{b}_0 z^{n_b} + \hat{b}_1 z^{n_b-1} + \ldots \hat{b}_{n_b}$$

are given. It is possible to allow that some of the coefficients are known. A usual case is $a_0 = 1$. It is assumed that the remaining, unknown elements have been estimated and that the covariance matrix of these estimates is known or estimated as well.

The problem is now to decide if the polynomials $\hat{A}(z)$ and $\hat{B}(z)$ have common factors or not taking the uncertainties into account. Especially, the test can be done for k common factors, and then the test can be repeated with k = 1,2.. etc. as long as common factors are found.

The report is organized as follows. In the next chapter a way of testing common factors is given. It leads to an optimization problem with a complicated constraint. However, it is shown in chapter III that it is easy to remove the constraint completely through a parameterization. Also different approximate solutions of the optimization problem are discussed. In chapter IV the algorithm is discussed for application to the repeated least squares algorithm. It is shown that it leads to the general least squares method proposed by Clarke (1967). The properties of this method were analyzed in Söderström (1972) and some of them are discussed in chapter IV as well. Finally, chapter V contains some numerical examples of the proposed algorithm.

II. A SYSTEMATIC WAY OF TESTING COMMON FACTORS.

In this chapter a basic idea how to test and compute common factors will be presented. It is assumed that the coefficients of the polynomials are known with a given accuracy. An algorithm for performing the test is proposed and it is further discussed in the next chapter, which also includes some approximate versions of the algorithm.

The problem is to test if the two polynomials $\hat{A}(z)$ and $\hat{B}(z)$ have k common factors. This problem will first be considered in a geometric way and the algorithm will be formulated geometrically as well. Then the same algorithm is presented in an analytical form. Finally, the chapter will contain some statistical interpretations of the algorithm.

Let the vector $\hat{x}$ consist of the estimates of all the unknown coefficients of the polynomials. Assume that $\hat{x}$ is n-dimensional. An arbitrary point x in R^n corresponds also to two polynomials. Introduce now the set S_k, which is a subset of R^n, defined by S_k = {x, such that the corresponding polynomials have at least k common factors}. Thus the problem is to examine if $\hat{x} \in S_k$. If the statistical nature of the problem is not considered, there are several well-known ways to perform the test and the problem is rather trivial. One method is to compute all the zeros of the two polynomials and compare them one by one. Another way to perform the examination is to use the Euclidean algorithm. It will be shown in the next chapter how these two methods can be used also in a more sophisticated way including the consideration of the statistical nature of the problem.

Taking the accuracies into account, it is necessary to allow not only the point $\hat{x}$ but also points close to $\hat{x}$ when the set S_k is examined. It is convenient in similar applications to consider points in an ellipsoid with centre in $\hat{x}$. In order to define this type of set in a proper way, assume that $\hat{x}$ is an unbiased estimate of an unknown vector m, and that $\hat{x}$ has the known covariance matrix P. Then the ellipsoid given by

$$C = \{x;\ (x-\hat{x})^T P^{-1} (x-\hat{x}) \leq \gamma_n\}$$

is called the ellipsoid of concentration, see Cramér (1946). A typical ellipsoid of concentration for n = 2 is shown in figure 2.1.

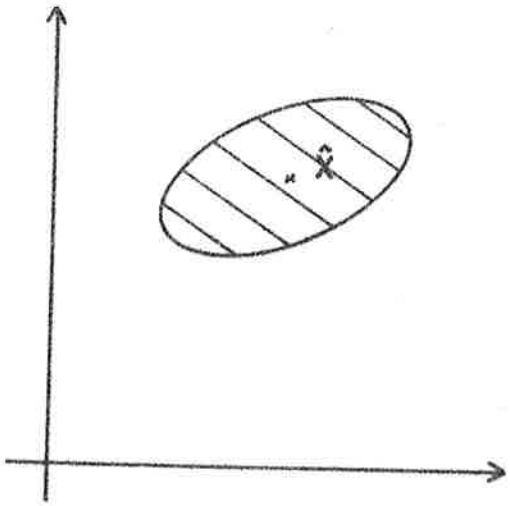

Figure 2.1 An ellipsoid of concentration

There are several ways to determine a "suitable" value of γ_n. Of course, the intended use of C influences the choice. Some natural choices are the following. Introduce an artificial stochastic variable ξ with mean $\hat{x}$ and covariance matrix P.

1. Following Cramér(1946) γ_n can be defined through the condition that ξ is uniformly distributed in C. It is shown in Cramér (1946) that this gives $\gamma_n = n+2$.
2. The number γ_n can be chosen as $E\ (\xi-\hat{x})^T P^{-1} (\xi-\hat{x}) = E\ \mathrm{tr}\ P^{-1} (\xi-\hat{x}) (\xi-\hat{x})^T = \mathrm{tr}\ P^{-1} P = n$

3. Assume that ξ is gaussian distributed. Then γ_n can be chosen so that the probability of $\xi \in C$ is an arbitrary number $1-\beta$ $(0 < \beta < 1)$. Since $(\xi-\hat{x})^T P^{-1} (\xi-\hat{x})$ is $\chi^2(n)$ distributed, this definition leads to $\gamma_n = \chi^2_\beta(n)$. Of course, also other distributions are possible to use for a definition of γ_n.

The concept of ellipsoids of concentration will now be used for proposing an algorithm for a test of common factors. The algorithm can be formulated as:

> Examine if there is any x in C which also belongs to S_k. If there are such points, then the estimate $\hat{x}$ can be modified to $\hat{m}$, which is the point in S_k that is closest to $\hat{x}$ in the $||\ ||_{p-1}$ -norm

This idea means that points inside C are considered as likely values of the unknown vector m, and points outside the ellipsoid are considered as not likely. Moreover, the closer to $\hat{x}$ a point is situated the more likely it is considered.

The idea of the algorithm is illustrated in two figures for the case n = 2. In figure 2.2 the sets C and S_k have no intersection and thus the test gives that there are no common factors of the two polynomials. In figure 2.3 the sets intersect and the "best" point $\hat{m}$ is also given in the figure.

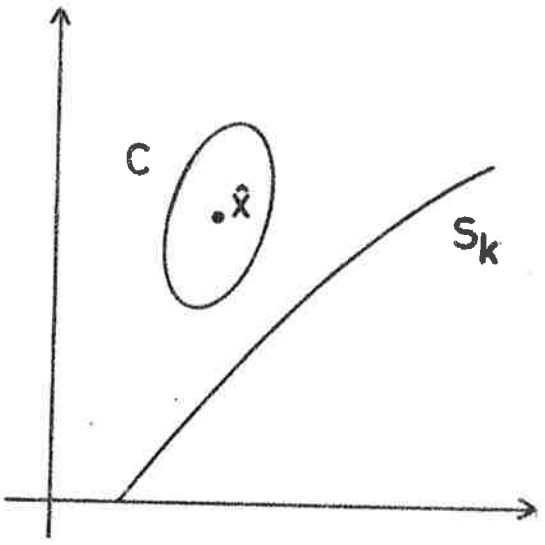

Figure 2.2

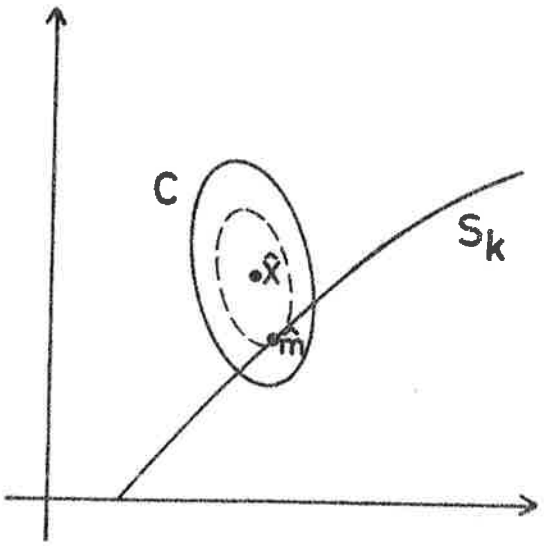

Figure 2.3

In order to apply the algorithm, it is necessary to formulate it in an analytical way, which is easy to do. The analytical version of the algorithm is as follows:

Find the global minimum point $\hat{m}$ of

$$F(x) = (x-\hat{x})^T P^{-1} (x-\hat{x}) \qquad (2.1)$$

under the constraint $x \in S_k$. The polynomials are considered to have k common factors if $F(\hat{m})$ is smaller than γ_n.

It is also possible to give some statistical interpretations of the algorithm. Consider the hypothesis H_o: $m \in S_k$. It is to be tested against H_1: $m \notin S_k$. The test is performed such that H_o is rejected if and only if $F(\hat{m})$ is larger than γ_n. This way of testing the complicated hypothesis seems to be natural.

It is possible to somewhat analyze the significance level of the test. Assume especially that $\hat{x}$ is gaussian distributed. Then the significance level α fulfills $\alpha = P$ (reject H_o, if H_o is true) $= P\ (F(\hat{m}) > \gamma_n$ when $m \in S_k) \leq P\ (F(m) > \gamma_n$ when $m \in S_k) = \alpha_o$ where α_o is given by the equation

$$\gamma_n = \chi^2_{\alpha_o}(n)$$

Moreover, provided that H_o is true and that $\hat{x}$ is gaussian, the estimate $\hat{m}$ is the maximum likelihood estimate of m. This assertion follows from the fact that in this case the likelihood function fulfill

$$-\log L(m) = \text{constant} + F(m)$$

It can be noted that in several practical situations concerning process identification the true covariance matrix P is not known. However, several well-known identification

methods like the least squares method, the generalized least squares method, Clarke (1967) and the maximum likelihood method, Åström-Bohlin (1966) produce an estimated covariance matrix. Of course, this can be used as well as the true covariance matrix, but the statistical interpretations are not so straightforward for this case. It can also be noted that the mentioned identification methods under mild conditions give asymptotically gaussian distributed estimates. This fact justifies the assumption of gaussian distribution of x in the statistical interpretations.

III. IMPLEMENTATIONS OF THE ALGORITHM

The algorithm proposed in the foregoing chapter is merely a precise problem formulation. Three different methods will be treated. It will be assumed that $\hat{a}_0$ and $\hat{b}_0$ may be known and that all other coefficients of $\hat{A}(z)$ and $\hat{B}(z)$ are not known exactly. This is also the most common situation in practice.

The given problem means that a quadratic loss function is to be minimized under a constraint of a complicated form. Moreover, it is a hard task to express the constraint explicitly when the polynomials are of general degrees. This implies that it is not pertinent to try to solve the proposed problem just straightforward.

One way to solve the problem will be to use a reparameterization. When the n-dimensional vector $x \in S_k$, the "degrees of freedom" is $n - k$. It is in fact possible to find a new vector y of dimension $n - k$ and a transformation f such that $x = f(y)$ when $x \in S_k$. This function $x = f(y)$ is such that when y varies over R^{n-k} then x varies over the set S_k or expressed in a stringent way, $f(R^{n-k}) = S_k$.

A second way of solving the problem will be to compute the zeros of the two polynomials. Let the vector y contain these zeros. Also this way means that a transformation of variable is done and that this transformation changes the loss function into a more complicated form. For this method the constraint becomes linear in y, since for example $y_1 = y_7$, $y_2 = y_6$ is to be tested. For this method it will be proposed that linearization of the transformation is suitable in order to get a new problem of a more uncomplicated structure. It will in fact make the loss function quadratic in a new variable y.

The third way discussed for solving the problem will be to utilize the Euclidean algorithm. It means that succesive polynomial divisions are performed. Let y denote the vector consisting of the coefficients of the remainder polynomial at some step. It is to be tested if $y = 0$ or not. The vector y is a function of the vector x, and thus this case also means that the original problem is reformulated using transformations of variables.

Also for this method it will be proposed that linearization of the transformations is suitable.

Since all three ways (called methods in the following) mean that a transformation of variables is used, it would be valuable to analyze the effect of a general transformation. However, the transformations that will be considered are of essentially two different kinds and it will be more pertinent to make simple analysis for each type of transformation. In order to further explain the different methods, it will be especially considered how they work applied to the following simple example.

Example.: Consider the case given by $x = [\hat{a}_1\ \hat{a}_2\ \hat{b}_1\ \hat{b}_2]^T$ where $\hat{A}(z) = z^2 + \hat{a}_1 z + \hat{a}_2$ and $\hat{B}(z) = \hat{b}_1 z + \hat{b}_2$ ($\hat{b}_0 = 0$ for convenience only) and suppose that the covariance matrix of x is P. For this simple example it is easy to formulate the original problem in an explicit form. The loss function is given by (2.1). The constraint means that A(z) has a zero in $-b_2/b_1$, which gives the equation

$$b_2^2 - a_1 b_1 b_2 + a_2 b_1^2 = 0 \qquad (3.1)$$

for the constraint.

Method 1:

As mentioned above, it is possible to parameterize the vector x. Introduce the three polynomials $\tilde{A}(z)$, $\tilde{B}(z)$ and L(z)

$$\tilde{A}(z) = \tilde{a}_0 z^{n_a-k} + \tilde{a}_1 z^{n_a-k-1} + \dots \tilde{a}_{n_a-k}$$

$$\tilde{B}(z) = \tilde{b}_0 z^{n_b-k} + \tilde{b}_1 z^{n_b-k-1} + \dots \tilde{b}_{n_b-k}$$

$$L(z) = z^k + \ell_1 z^{k-1} + \dots \ell_k$$

If $\hat{a}_o$ ($\hat{b}_0$) is known, then put $\tilde{a}_0 = \hat{a}_0$ ($\tilde{b}_0 = \hat{b}_0$). The condition $x \in S_k$ can equivalently be expressed as

$$A(z) = \tilde{A}(z)\, L(z)$$

$$B(z) = \tilde{B}(z)\, L(z) \qquad (3.2)$$

for arbitrary values of the coefficients of $\tilde{A}(z)$, $\tilde{B}(z)$, and $L(z)$. Collect these coefficients (with possible exclusion of $\tilde{a}_0$ and/or $\tilde{b}_0$) in a new vector y, which must have dimension n - k. Then x is parameterized in the elements of y when x belongs to the set S_k. The function F(x) can easily be expressed as a function of y. In this way the constraint is avoided completely. The price for this is a more complicated structure of the loss function F.

Now the application of the method to the special example is considered. For this example the vector y is given by

$$y = [\tilde{a}_1 \;\; \tilde{b}_1 \;\; \ell_1]$$

and the loss function can be written as

$$F(x) = F(f(y)) = \Delta y^T P^{-1} \Delta y$$

$$\Delta y = [\tilde{a}_1 + \ell_1 - \hat{\tilde{a}}_1 \quad \tilde{a}_1 \cdot \ell_1 - \hat{a}_1 \quad \tilde{b}_1 - \hat{b}_1 \quad \tilde{b}_1 \cdot \ell_1 - \hat{b}_2]^T$$

Some transformation of variables and linearization of them

In method 2 and 3 another type of transformation of variables is used. Instead of expressing x as a function of a new vector y, a vector y, which contains the new variables, is expressed as a function of x. If y has smaller dimension than x, which often is the case, the difference is not trivial.

Introduce now a transformation given by $y = f(x)$ with $\dim y \leq \dim x$. If strict inequality holds, introduce also some slack variable of dimension $\dim x - \dim y$ through $z = g(x)$. The functions are assumed to satisfy

$$\begin{bmatrix} f'(x) \\ g'(x) \end{bmatrix}$$

non singular close to $x = \hat{x}$.

The constraint will for method 2 and 3 be of the form

$$S\,y = 0 \qquad (3.3)$$

and (of course) z arbitrary. S is a s x n ($s \leq n$) matrix of rank s. The case $s = n$, which in fact is used in method 3, means that the constraint is degenerated to $y = 0$. This means that no optimization had to be performed at all. In order to get an uncomplicated problem, linearization of the transformations is proposed. Since it is assumed that $\hat{m}$ is close to $\hat{x}$ and only values of x close to $\hat{m}$ are considered, it may be appropriate to linearize the transformations around $\hat{x}$. Introduce the points $\hat{y} = f(\hat{x})$ and $\hat{z} = g(\hat{x})$. The linearization is given by

$$\begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} - \begin{bmatrix} f(\hat{x}) \\ g(\hat{x}) \end{bmatrix} \approx \begin{bmatrix} f'(\hat{x}) \\ g'(\hat{x}) \end{bmatrix} (x-\hat{x}) \qquad (3.4)$$

With use of this linearization the loss function can after trivial calculations be expressed as

$$F(x) = \begin{bmatrix} y^T-\hat{y}^T & z^T-\hat{z}^T \end{bmatrix} \begin{bmatrix} f'(\hat{x})Pf'(\hat{x})^T & f'(\hat{x})Pg'(\hat{x})^T \\ g'(\hat{x})Pf'(\hat{x})^T & g'(\hat{x})Pg'(\hat{x})^T \end{bmatrix}^{-1} \begin{bmatrix} y-\hat{y} \\ z-\hat{z} \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} y^T-\hat{y}^T & z^T-\hat{z}^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}^{-1} \begin{bmatrix} y-\hat{y} \\ z-\hat{z} \end{bmatrix} \qquad (3.5)$$

Minimization of F with respect to the free slack variable z gives

$$z = \hat{z} - Q_{22}^{-1} Q_{12}^T (y-\hat{y})$$

and the minimum value of the loss function becomes

$$\tilde{F}(y) = (y-\hat{y})^T (Q_{11}-Q_{12} Q_{22}^{-1} Q_{12}^T) (y-\hat{y}) =$$

$$= (y-\hat{y})^T [f'(\hat{x}) Pf'(\hat{x})^T]^{-1} (y-\hat{y}) \equiv$$

$$\equiv (y-\hat{y})^T P_y^{-1} (y-\hat{y}) \qquad (3.6)$$

Thus the new optimization problem means that (3.6) is to be minimized subject to the constraint (3.3). This problem is of a linear-quadratic type. It is easy to obtain the solution e.g. using the concept of pseudoinverses, see Kalman-Englar (1966) or Lagrange multipliers. The global minimum point is

$$m_y = \hat{y} - P_y S^T (SP_y S^T)^{-1} S \hat{y} \qquad (3.7)$$

The linearization can be interpreted in a geometric way. The transformation means that the sets C and S_k are transformed from the x-space to the y-space. The transformation is chosen such that the constraint becomes a plane through the origin in the y-space, but the set C is not transformed into a new ellipsoid. The linearization means that the transformation of C is approximated to a new ellipsoid. This geometric interpretation is illustrated in figure 3.1.

According to the approximation formulas given in Hald (1952), the linearization can be interpreted statistically as well. First note that the loss function can be written as

$$F(x) = (x - E(\xi) (Var(\xi)^{-1} (x - E(\xi)) \qquad (3.8)$$

Define now the new stochastic variable η through $\eta = f(\xi)$. Then Hald's formulas give

$$E(\eta) = E\, f(\xi) \approx f(\hat{x}) = \hat{y}$$

$$Var(\eta) \approx f'(\hat{x})\, P\, f'(\hat{x})^T = P_y$$

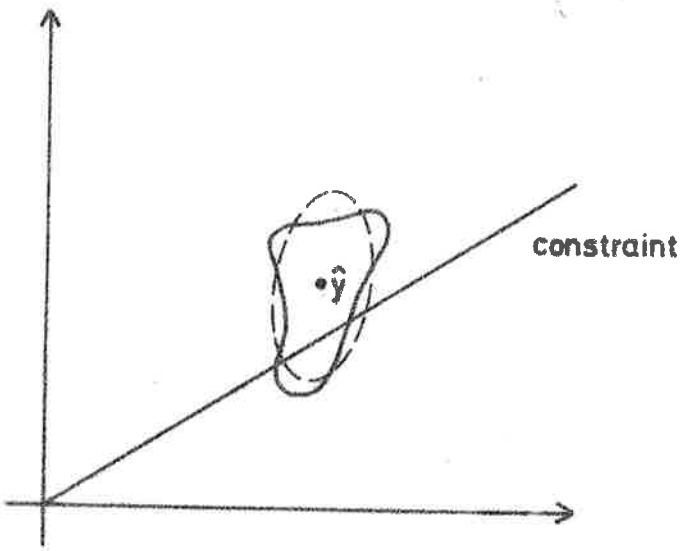

Figure 3.1 Geometric interpretation of the linearization of the transformation of variables. The unbroken line gives the true transformation of the set C. The dashed line gives the approximated transformation of C into a new ellipsoid.

Thus the approximated loss function (3.6) can be written

$$\tilde{F}(y) = (y - E(\eta))^T (Var(\eta))^{-1} (y - E(\eta)) \qquad (3.9)$$

which has the same structure as (3.8).

Method 2.

For this method new variables are introduced by defining the vector y by

$$y = [p_1 \ \dots \ p_{n_a} \ z_1 \ \dots \ z_{n_b}]$$

where $\{p_i\}$ are the zeros of A(z) and $\{z_i\}$ the zeros of B(z). If e.g. $\hat{b}_0$ is not known, also a slack variable is used as discussed above. Using this method the search over x in S_k is not performed in one step. A typical test hypothesis may be $p_1 = z_2$, $p_3 = z_1$, $p_5 = z_4$. Several other, different combinations can be made. Clearly this type of hypothesis can

be expressed in the form (3.3). A row of S consists of one +1, one -1, while the remaining elements of the row are zero.

In order to apply this method, the Jacobian f'(x) is needed. In Wilkinson (1963) expressions for the partial derivatives in f'(y) are given. Let a denote the vector $[a_1 \ \dots \ a_m]$ and p the vector $[p_1 \ \dots \ p_m]$ defined by

$$A(z) = z^m + a_1 z^{m-1} + \dots + a_m = \prod_{i=1}^{m} (z - p_i)$$

The Jacobian dp/da exists if and only if A(z) has no multiple zeros. The matrix holds

$$\frac{dp}{da} = \begin{bmatrix} -\frac{p_1^{m-1}}{A'(p_1)} & -\frac{p_1^{m-2}}{A'(p_1)} & \cdots & -\frac{1}{A'(p_1)} \\ \vdots & \vdots & & \vdots \\ -\frac{p_m^{m-1}}{A'(p_m)} & -\frac{p_1^{m-2}}{A'(p_1)} & \cdots & -\frac{1}{A'(p_1)} \end{bmatrix}$$

This attempt of solution is in fact a bit more complicated than outlined here. It is necessary to treat complex factors and real ones in different ways. The proposed method is examined in detail in Burström (1973), where also numerical experience is reported.

The effect of the method applied to the example is now illustrated. The zeros of $A(z) = z^2 + a_1 z + a_2$ are

$$p_1 = -\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_2} \quad \text{and} \quad p_2 = -\frac{a_1}{2} - \sqrt{\frac{a_1^2}{2} - a_2}$$

The zero z_1 of B(z) is $z_1 = -b_2/b_1$. Then the vector y is given by $y = [p_1 \ p_2 \ z_1]^T$ for the specific example. Consider the hypothesis $p_1 = z_1$. Then S is a row vector (s=1), namely

$$S = [1 \quad 0 \quad -1]$$

Moreover the Jacobian $f'(\hat{y})$ can be shown to be

$$f'(\hat{y}) = \begin{bmatrix} -1 & -1 & 0 & 0 \\ \hat{p}_2 & \hat{p}_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Method 3. In order to explain this method, which is the Euclidean algorithm, assume for simplicity that $n_a = m$, $n_b = m$, $a_0 = 1$ and $b_0 = 0$. From $\hat{A}(z)$ and $\hat{B}(z)$ the polynomial $D(z) = d_1 z^{m-2} + \ldots + d_{m-1}$ (the first remainder polynomial) is computed from the equation

$$\hat{A}(z) = \hat{B}(z) \left(\frac{1}{\hat{b}_1} z + \frac{\hat{a}_1\hat{b}_1 - \hat{b}_2}{\hat{b}_1^2} \right) + D(z) \qquad (3.10)$$

or explicitly

$$d_1 = \hat{a}_2 - \hat{b}_2 \frac{\hat{a}_1\hat{b}_1 - \hat{b}_2}{\hat{b}_1^2} - \hat{b}_3 \frac{1}{\hat{b}_1}$$

$$d_2 = \hat{a}_3 - \hat{b}_3 \frac{\hat{a}_1\hat{b}_1 - \hat{b}_2}{\hat{b}_1^2} - \hat{b}_4 \frac{1}{\hat{b}_1}$$

$$\vdots$$

$$d_{m-2} = \hat{a}_{m-1} - \hat{b}_{m-1} \frac{\hat{a}_1\hat{b}_1 - \hat{b}_2}{\hat{b}_1^2} - \hat{b}_m \frac{1}{\hat{b}_1}$$

$$d_{m-1} = \hat{a}_m - \hat{b}_m \frac{\hat{a}_1\hat{b}_1 - \hat{b}_2}{\hat{b}_1^2}$$

As a first step it is to be tested if $d_i = 0$ $i = 1,\ldots (m-1)$. If this is not true, a new remainder polynomial $D_1(z)$ of degree $m - 3$ is computed from the equation

$$\hat{B}(z) = D(z)\left(\frac{\hat{b}_1}{d_1} z + \frac{\hat{b}_2 d_1 - b_1 d_2}{d_1^2}\right) + D_1(z)$$

In this way the divisions and the tests are repeated until a remainder polynomial is considered as zero or the remainder polynomial is a constant. In the latter case the result of the procedure is that the original polynomials have no common factors.

Clearly the coefficients of an arbitrary remainder polynomial is a function of the vector x. This function is hard to express explicitly for the general case. For this reason it is proposed that linearization should be utilizied. Linearization must be made in every step. The constraint means that all the coefficients of the present remainder polynomial are zero, and thus S = I in (3.3) for this case. This implies that no optimization at all has to be performed for this method.

It is easy to express the coefficients of D(z) in x and to linearize this transformation. In the next step the coefficients of $D_1(z)$ are expressed as functions of the coefficients of $\hat{B}(z)$ and D(z) and this transformation is linearized, and so on. This means that it is sufficient to consider the Jacobian of the first transformation from $(\hat{a}_1 \dots \hat{a}_m \hat{b}_1 \dots \hat{b}_m)$ to $(\hat{b}_1 \dots \hat{b}_m d_1 \dots d_{m-1})$. It has the structure

$$\begin{bmatrix} 0 & I_m \\ J_1 & J_2 \end{bmatrix}$$

where J_1 is the following matrix of order (m-1) x m

$$J_1 = \begin{bmatrix} -\frac{\hat{b}_2}{\hat{b}_1} & 1 & & \\ & & \ddots & \\ -\frac{\hat{b}_m}{\hat{b}_1} & 0 & & 1 \end{bmatrix}$$

and J_2 is the following matrix of order (m-1) x m

$$J_2 = \begin{bmatrix} \hat{b}_2 & \frac{\hat{a}_1\hat{b}_1-2\hat{b}_2}{\hat{b}_1^2} + \frac{\hat{b}_3}{\hat{b}_1^2} & -\frac{\hat{a}_1\hat{b}_1-2\hat{b}_2}{\hat{b}_1^2} - \frac{1}{\hat{b}_1} & & \\ & & \frac{\hat{b}_3}{\hat{b}_1} & \frac{\hat{b}_2-\hat{a}_1\hat{b}_1}{\hat{b}_1^2} & \cdot & \mathbf{0} \\ \cdot & \cdot & & & \cdot & \\ & & & & \cdot & \cdot \\ \hat{b}_{m-1} & \frac{\hat{a}_1\hat{b}_1-2\hat{b}_2}{\hat{b}_1^2} + \frac{\hat{b}_m}{\hat{b}_1^2} & \cdot & & \cdot & -\frac{1}{\hat{b}_1} \\ \hat{b}_m & \frac{\hat{a}_1\hat{b}_1-2\hat{b}_2}{\hat{b}_1^2} & \frac{\hat{b}_m}{\hat{b}_1} & \mathbf{0} & & \frac{\hat{b}_2-\hat{a}_1\hat{b}_1}{\hat{b}_1^2} \end{bmatrix}$$

In the special example D(z) becomes a scalar

$$d_1 = a_2 - \frac{b_2}{b_1}(a_1b_1-b_2)$$

and the Jacobian is

$$\begin{bmatrix} \frac{\partial b_1}{\partial a_1} & \frac{\partial b_1}{\partial a_2} & \frac{\partial b_1}{\partial b_1} & \frac{\partial b_1}{\partial b_2} \\ \frac{\partial b_2}{\partial a_1} & \frac{\partial b_2}{\partial a_2} & \frac{\partial b_2}{\partial b_1} & \frac{\partial b_2}{\partial b_2} \\ \frac{\partial d_1}{\partial a_1} & \frac{\partial d_1}{\partial a_2} & \frac{\partial d_1}{\partial b_1} & \frac{\partial d_1}{\partial b_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & 1 \\ -\frac{b_2}{b_1} & 1 & b_2 & \frac{a_1b_1-2b_2}{b_1^2} - \frac{a_1b_1-2b_2}{b_1^2} \end{bmatrix}$$

There is another variant of the Euclidean algorithm. It differs from the described one only when the problem is treated in a statistical framework.

The equations (3.10) can be scaled. Since a scale factor will depend on x, the Jacobian will change more than only by the scale factor. Thus the test quantity will depend on the choice of scale factor. This is an unpleasant property of the method. However, it is not the only one. Let

$$A(z) = \prod_{i=1}^{m} (z - p_i), \quad B(z) = b_1 \prod_{i=1}^{m-1} (z - z_i).$$

Assume that p_1 is to be considered equal to z_1 but no other poles and zeros are equal. The remainder polynomial computed by the Euclidean algorithm will be of the form

$$K \prod_{i=1}^{m} \prod_{j=1}^{m-1} (p_i - z_j)$$

K involves the dependence of the scale factor. If it is wanted to test the hypothesis $p_1 = z_1$, then the remaining factors $(p_i - z_j)$ will influence the result since they contaminate the test quantity.

The use of the Euclidean algorithm in the outlined ways is examined in Burström (1973). It is shown by numerical examples that the test quantities obtained with this method may vary considerably due to the choice of scaling. A conclusion is therefore that method 1 or method 2 is to be preferred.

IV. APPLICATION TO LEAST SQUARES IDENTIFICATION

In this chapter the proposed algorithm for testing of common factors will be examined in application to analysis of models obtained by least squares (LS) identification. It will be shown that the new combined algorithm is just the same as the generalized least squares method proposed by Clarke (1967).

The least squares method means that the following model is obtained

$$(1+\hat{a}_1 q^{-1} + \ldots \hat{a}_n q^{-n})y(t) = (\hat{b}_1 q^{-1} + \ldots \hat{b}_n q^{-n})u(t)+\varepsilon(t)$$

by minimization of

$$V = \sum_{t=n+1}^{N} \varepsilon^2(t)$$

The input of the process is denoted u(t) and the measured output by y(t). The notation q^{-1} is the backward shift operator and N denotes the number of data. It is straightforward to extend the following analysis to the case of $n_a \neq n_b$.

The LS method can easily be formulated using matrices as well. Introduce

$$Y = \begin{bmatrix} y(n+1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y(N) \end{bmatrix} \qquad \Phi = \begin{bmatrix} -y(n) & \cdots & -y(1) & u(n) & \cdots & u(1) \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ -y(N-1) & \cdots & -y(N-n) & u(N-1) & \cdots & u(N-n) \end{bmatrix}$$

$$\hat{\varphi} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \vdots \\ \hat{b}_n \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon(n+1) \\ \vdots \\ \varepsilon(N) \end{bmatrix}$$

With these notations the loss function V can be written as

$$V(\hat{\varphi}) = \varepsilon^T\varepsilon = Y^TY - 2Y^T\Phi\hat{\varphi} + \hat{\varphi}^T\Phi^T\Phi\hat{\varphi}$$

The LS estimate $\hat{\varphi}_{LS}$ is easily obtained by minimization. It is

$$\hat{\varphi}_{LS} = (\Phi^T\Phi)^{-1}\Phi^TY$$

and the corresponding minimum value of the loss function is

$$V(\hat{\varphi}_{LS}) = Y^TY - Y^T\Phi(\Phi^T\Phi)^{-1}\Phi^TY$$

An estimated variance matrix of $\hat{\varphi}_{LS}$ can be obtained as well, see e.g. Åström (1968). It is

$$P_{\hat{\varphi}_{LS}} = V(\hat{\varphi}_{LS}) \cdot (\Phi^T\Phi)^{-1}/N$$

It is now possible to examine the function F for this case. Straightforward calculations give

$$F(\hat{\varphi})\frac{V(\hat{\varphi}_{LS})}{N} = (\hat{\varphi}-\hat{\varphi}_{LS})^T(\phi^T\phi)(\hat{\varphi}-\hat{\varphi}_{LS}) =$$

$$= \hat{\varphi}^T\phi^T\phi\hat{\varphi} - 2\hat{\varphi}^T\phi^T\phi\hat{\varphi}_{LS} + \hat{\varphi}_{LS}^T\phi^T\phi\hat{\varphi}_{LS} =$$

$$= \hat{\varphi}^T\phi^T\phi\hat{\varphi} - 2\hat{\varphi}^T\phi^TY + Y^T\phi(\phi^T\phi)^{-1}\phi^TY =$$

$$= V(\hat{\varphi}) - V(\hat{\varphi}_{LS})$$

which means

$$F(\hat{\varphi}) = \frac{V(\hat{\varphi}) - V(\hat{\varphi}_{LS})}{V(\hat{\varphi}_{LS})} N \qquad (4.1)$$

Thus it is proved that minimization of $F(\hat{\varphi})$ is equivalent to minimization of $V(\hat{\varphi})$. Introduce again the parameterization used in chapter III. The constraint $\hat{\varphi}\in S_k$ can be formulated as

$$\hat{A}(z) = \tilde{A}(z)\tilde{C}(z) \qquad \hat{B}(z) = \tilde{B}(z)\tilde{C}(z) \qquad (4.2)$$

where

$$\tilde{A}(z) = 1 + \tilde{a}_1 z + \ldots + \tilde{a}_{n-k} z^{n-k}$$

$$\tilde{B}(z) = \tilde{b}_1 z + \ldots + \tilde{b}_{n-k} z^{n-k}$$

$$\tilde{C}(z) = 1 + \tilde{c}_1 z + \ldots + \tilde{c}_k z^k$$

Form a new vector $\hat{\theta}$ by

$$\hat{\theta} = [\tilde{a}_1 \ldots \tilde{a}_{n-k}\ \tilde{b}_1 \ldots \tilde{b}_{n-k}\ \tilde{c}_1 \ldots \tilde{c}_k]^T$$

In this way the vector $\hat{\varphi}$ is expressed as a function of the vector $\hat{\theta}$ through (4.2). It turns out that the loss function $V(\hat{\varphi}(\hat{\theta}))$ can be expressed using polynomial operators as

$$V(\hat{\varphi}(\hat{\theta})) = \sum_{t=n+1}^{N} \varepsilon^2(t)$$

$$\varepsilon(t) = \tilde{A}(q^{-1})\,\tilde{C}(q^{-1})\,y(t) - \tilde{B}(q^{-1})\,\tilde{C}(q^{-1})\,u(t) \qquad (4.3)$$

However, this is exactly the loss function for the generalized least squares (GLS) estimation of the model (4.3), see Clarke (1967) and Söderström (1972).

The established fact that (4.1) holds and the interpretation given above have several important implications.

1. It is shown in Söderström (1972) that the GLS method can be interpreted as an optimization of $V(\hat{\varphi}(\hat{\theta}))$ and that the involved minimization method gives slow (only linear) convergence. The calculations made above lead to the suggestion of a more efficient way for computing the GLS estimates. As a first step $\hat{\varphi}_{LS}$ and $P_{\hat{\varphi}_{LS}}$ are computed. In the second step $F(\hat{\varphi}(\hat{\theta}))$ is minimized. Note that in this minimization the data are <u>not</u> needed explicitly which means that the iterations are rather fast. If a fast minimization method is chosen, then the number of iterations needed may be small. For some simple first-order systems this new algorithm for computing the GLS model has reduced the computing time to about 1/10. This implication is mentioned also in Söderström (1973).

2. The properties of the asymptotic loss function $W(\hat{\theta}) = \lim_{N\to\infty} \frac{1}{N} V(\hat{\varphi}(\hat{\theta}))$ are analyzed in Söderström (1972). It is proved that the number of local minimum points of $W(\hat{\theta})$ depends on the signal to noise ratio. An immediate implication of this is that the minimization is no trivial operation. It is of large importance to get good start values for the minimization in order to provide convergence to the global minimum point of $F(\hat{\varphi}(\hat{\theta}))$. A method for computing start values is discussed in the appendix.

3. Another observation can be made from (4.1). The expression is of the same structure as test quantities for testing order of models, see Åström (1968) and Bohlin (1970). This fact indicates but does not prove that a suitable value of γ_n would be $\chi^2_\alpha(k)$.

Summarizing the chapter, it has been shown that the least squares method combined with the proposed algorithm for testing common factors is nothing but the generalized least squares algorithm. It constitutes in fact a very efficient way to compute the GLS model. The close connection between the GLS method and the LS method with a high order model and some tests of common factors were pointed out earlier in Åström-Eykhoff (1971) and Söderström (1972). Some similar ideas were briefly discussed by Wieslander (1969).

V. NUMERICAL EXAMPLES

The numerical examples given in this chapter are all concerning the GLS method, discussed in chapter IV.

Examples 1 and 2.

Asymtotic values of $\hat{\varphi}_{LS}$ and $P_{\hat{\varphi}_{LS}}$ were computed using theoretical expressions. The number of data, N, was assumed to be 1000. The systems are general first-order systems,

Example 1 $a = -0.5$ $b = 1.0$ $c = 0.5$

Example 2 $a = -0.8$ $b = 1.0$ $c = 0.7$

$$y(t) + a\,y(t-1) = b\,u(t-1) + e(t) + ce(t-1) \qquad (5.1)$$

where e(t) is white noise with variance 1.0 and independent of the input. The input is also assumed to be white noise with variance 1.0. The loss function $F(\hat{\varphi})$ is minimized numerically using method 1. The order of the LS model was chosen as n = 1 ... 5. It turned out, in accordance with lemma A.2, that when $k < n - 1$ then the system of equations (A.6) became singular, which indicates a too small value of k.

In order to evaluate the models, the following variables have been computed.

i) the poles, the zeros and the static gain for the true system (5.1) the LS model of order n and the reduced, GLS, model with $k = n - 1$ $(n > 1)$.

ii) The deviation in the impulse response for the LS and the GLS models. Let $\{g_i\}$ denote the true impulse response and $\{\hat{g}_i\}$ the impulse response obtained from a model. The quantity

$$\Delta G = \sum_{i=1}^{\infty} (\hat{g}_i - g_i)^2$$

is computed.

iii) The step responses for the system and the two models.

iv) The test quantity $F(\hat{m})$.

Two different systems were tried. The results are given in Table 5.1-5.2 and figures 5.1-5.2.

n	Original model				$\hat{F}(m)$	Reduced (GLS) model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
1	0.64	-	2.75	0.081	-			
2	0.46 ± i 0.24	0.42	1.67	0.020	10.81	0.484	1.927	0.000 921
3	0.19 ± i 0.42 0.60	0.24 ± i 0.38	2.17	0.005	2.20	0.522	2.106	0.001 854
4	-0.01 ± i 0.47 0.51 ± i 0.17	0.02 ± i 0.47 0.46	1.92	0.001	0.64	0.499	1.994	0.000 006
5	-0.15 ± i 0.46 0.36 ± i 0.33 0.58	-0.13 ± i 0.46 0.38 ± i 0.27	2.04	0.000 3	0.16	0.501	2.007	0.000 007

Table 5.1 The true system is given by a = -0.5, b = 1.0 and c = 0.5. This means that it has a pole in 0.5 and that the static gain is 2.0.

n	Original model				$\hat{F}(m)$	Reduced (GLS) model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
1	0.87	-	7.68	0.31	-			
2	0.64 ± i 0.10	0.49	3.76	0.101	5.1	0.75	3.87	0.081
3	0.28 ± i 0.47 0.84	0.30 ± i 0.45	5.88	0.039	1.1	0.83	6.00	0.050
4	0.03 ± i 0.59 0.70 ± i 0.06	0.05 ± i 0.59 0.56	4.45	0.018	1.2	0.78	4.55	0.012
5	-0.15 ± i 0.60 0.48 ± i 0.37 0.83	-0.14 ± i 0.60 0.48 ± i 0.33	5.40	0.008	0.5	0.81	5.41	0.009

Table 5.2 The true system is given by a = -0.8, b = 1.0 and c = 0.7. This means that it has a pole in 0.8 and that the static gain is 5.0

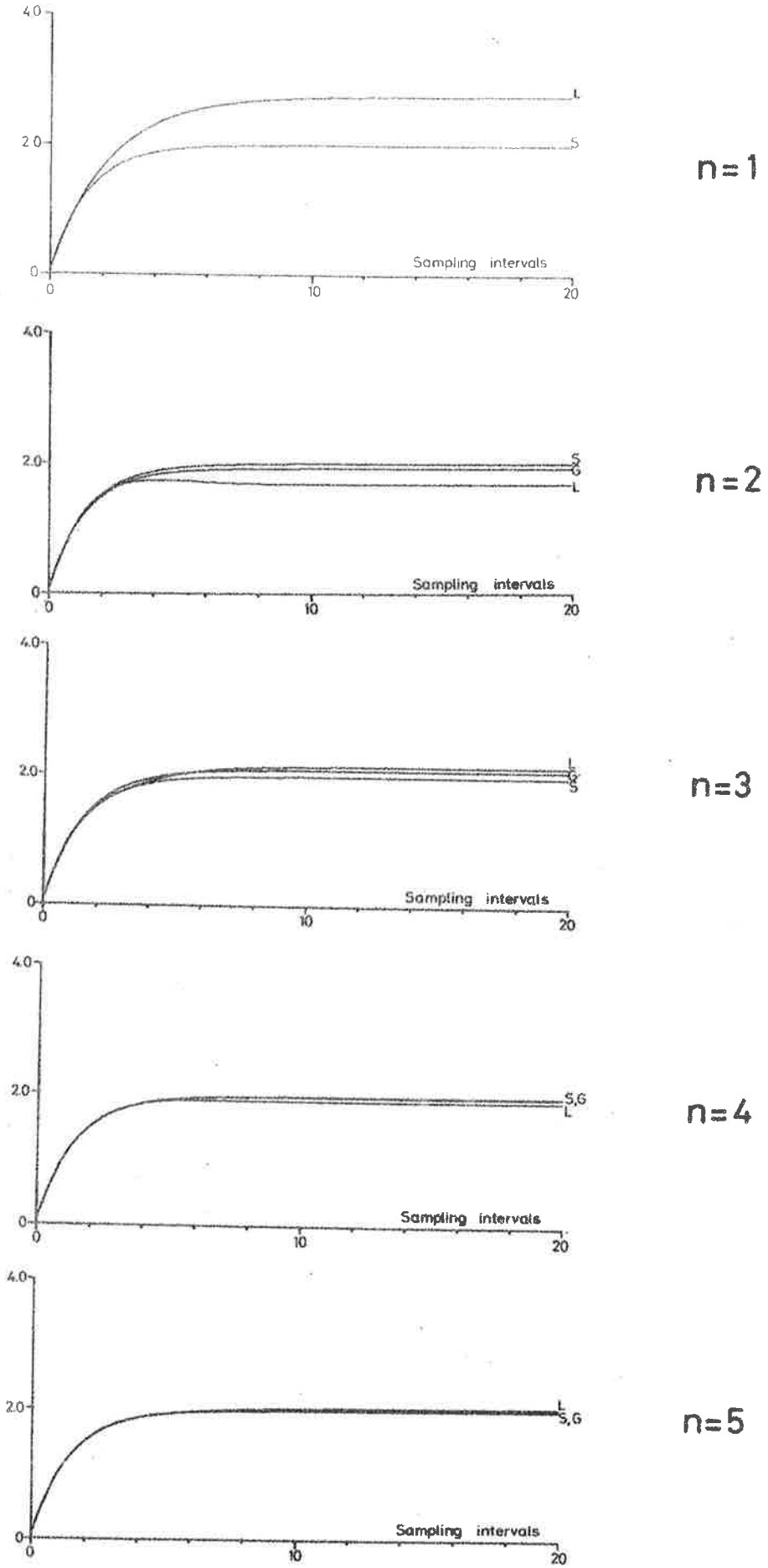

Figure 5.1. Example 1. Step responses of the true system, (S), the original model obtained by least squares identification, (L), and the reduced (GLS) modél, (G).

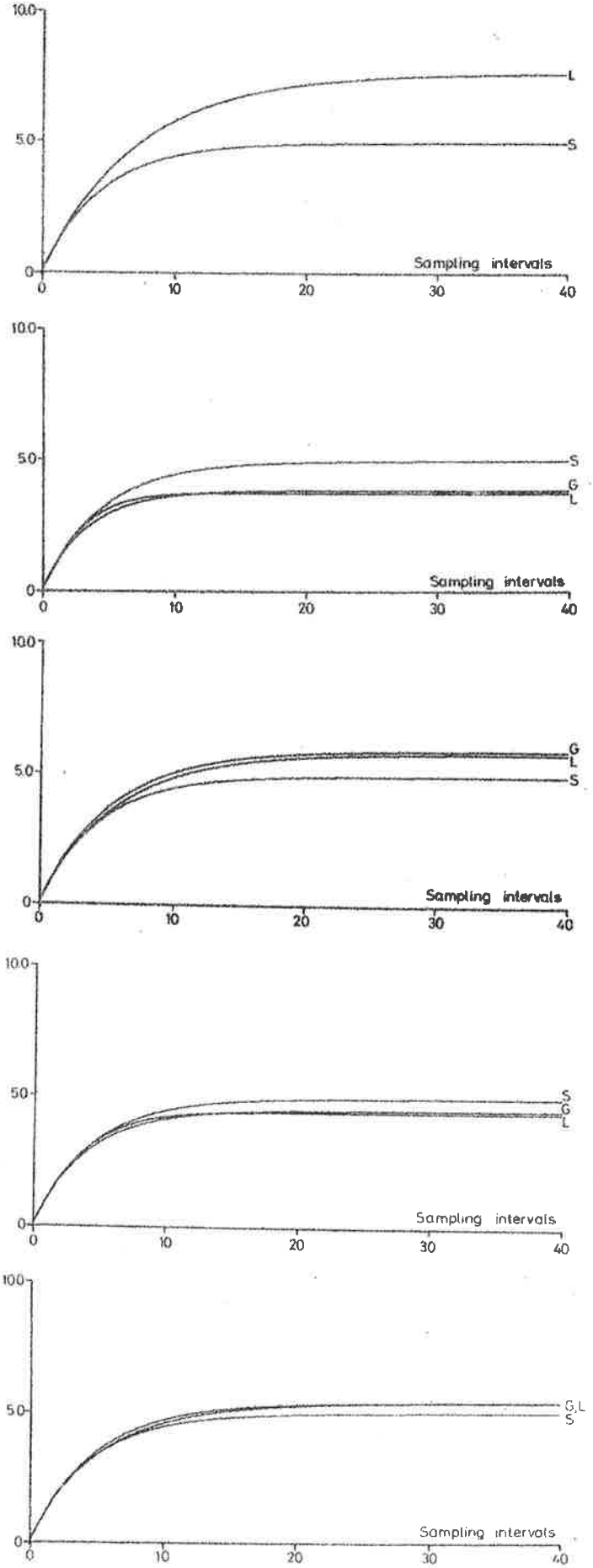

Figure 5.2. Example 2. Step responses of the true system, (S), the original model obtained by least squares identification, (L), and the reduced (GLS) model, (G).

Examples 3 and 4.

In order to further evaluate the validity of using the theoretical asymptotic expressions for $\hat{\varphi}_{LS}$ and $P_{\varphi_{LS}}$, 10 different realizations using 1000 simulated data were used for the systems given in example 1 and 2. The case n = 5, k = 4 was chosen. The results are presented in Tables 5.3-5.4 and Figures 5.3-5.4.

The conclusions which can be made from the examples are the following. The simplified models are as good as the original ones. No clear difference between the two kinds of models have been detected. An advantage of the reduced model, however, is that it contains fewer parameters.

Realization	Original model				$F(\hat{m})$	Reduced model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
1	-0.21 ± i 0.50 0.34 ± i 0.42 0.73	-0.16 ± i 0.52 0.42 ± i 0.30	2.36	0.022	7.1	0.46	1.82	0.006
2	-0.22 ± i 0.48 0.41 ± i 0.43 0.58	-0.28 ± i 0.57 0.52 ± i 0.45	2.21	0.016	12.2	0.49	1.99	0.000 2
3	-0.19 ± i 0.49 0.41 ± i 0.34 0.54	-0.16 ± i 0.50 0.39 ± i 0.23	2.01	0.005	1.4	0.53	2.13	0.003
4	-0.26 ± i 0.50 0.43 ± i 0.45 0.64	-0.26 ± i 0.53 0.49 ± i 0.38	2.23	0.007	3.0	0.52	2.14	0.003
5	-0.16 ± i 0.50 0.37 ± i 0.33 0.64	-0.16 ± i 0.53 0.43 ± i 0.28	2.25	0.006	1.5	0.51	2.06	0.006

Table 5.3 The true system is given by a = -0.5, b = 1.0 and c = 0.5. This means that it has a pole in 0.5 and that the static gain is 2.0

(cont'd)

Realization	Original model				$\hat{F}(m)$	Reduced model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
6	-0.14 ± i 0.51 0.52 ± i 0.27 0.27	-0.14 ± i 0.54 0.40 ± i 0.15	1.92	0.0021	4.7	0.51	2.16	0.007
7	-0.17 ± i 0.51 0.41 ± i 0.29 0.53	-0.18 ± i 0.52 0.41 ± i 0.28	2.27	0.014	0.3	0.55	2.32	0.016
8	-0.14 ± i 0.52 0.47 ± i 0.33 0.44	-0.15 ± i 0.50 0.43 ± i 0.37	2.26	0.017	0.3	0.52	2.26	0.015
9	-0.15 ± i 0.53 0.34 ± i 0.25 0.53	-0.14 ± i 0.50 0.36 ± i 0.34	2.26	0.012	2.2	0.52	2.13	0.003
10	-0.17 0.01 ± i 0.58 0.56 ± i 0.24	-0.01 0.07 ± i 0.56 0.32	2.06	0.015	6.5	0.51	2.07	0.001

Table 5.3 cont'd

Realization	Original model				$F(\hat{m})$	Reduced model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
1	-0.18 ± i 0.63 0.48 ± i 0.38 0.88	-0.17 ± i 0.64 0.51 ± i 0.36	6.74	0.15	8.0	0.84	6.47	0.10
2	-0.19 ± i 0.61 0.50 ± i 0.45 0.84	-0.24 ± i 0.66 0.57 ± i 0.47	6.11	0.079	11.7	0.83	5.93	0.045
3	-0.19 ± i 0.62 0.50 ± i 0.40 0.84	-0.17 ± i 0.63 0.48 ± i 0.31	5.51	0.016	2.4	0.82	5.71	0.026
4	-0.22 ± i 0.61 0.52 ± i 0.44 0.85	-0.23 ± i 0.63 0.54 ± i 0.40	6.12	0.056	2.4	0.83	6.18	0.070
5	-0.15 ± i 0.62 0.50 ± i 0.34 0.84	-0.15 ± i 0.64 0.51 ± i 0.33	6.03	0.050	0.9	0.83	5.95	0.046

Table 5.4 The true system is given by a = -0.8, b = 1.0 and c = 0.7. This means that it has a pole in 0.8 and that the static gain is 5.0 (cont'd)

Realization	Original model				$\hat{F}(m)$	Reduced model		
	poles	zeros	static gain	ΔG		pole	static gain	ΔG
6	-0.18 ± i 0.62 0.51 ± i 0.40 0.84	-0.15 ± i 0.65 0.49 ± i 0.26	5.55	0.036	5.1	0.82	5.84	0.041
7	-0.16 ± i 0.62 0.50 ± i 0.34 0.82	-0.17 ± i 0.63 0.49 ± i 0.33	5.99	0.063	1.2	0.83	6.09	0.061
8	-0.13 ± i 0.63 0.52 ± i 0.36 0.78	-0.13 ± i 0.61 0.49 ± i 0.39	5.54	0.054	2.0	0.81	5.61	0.026
9	-0.17 ± i 0.64 0.46 ± i 0.36 0.83	-0.16 ± i 0.62 0.46 ± i 0.39	6.10	0.071	1.3	0.83	6.04	0.054
10	-0.09 ± i 0.62 0.43 ± i 0.26 0.77	-0.05 ± i 0.63 0.37 ± i 0.34	5.22	0.030	5.7	0.80	5.18	0.002

Table 5.4 cont'd

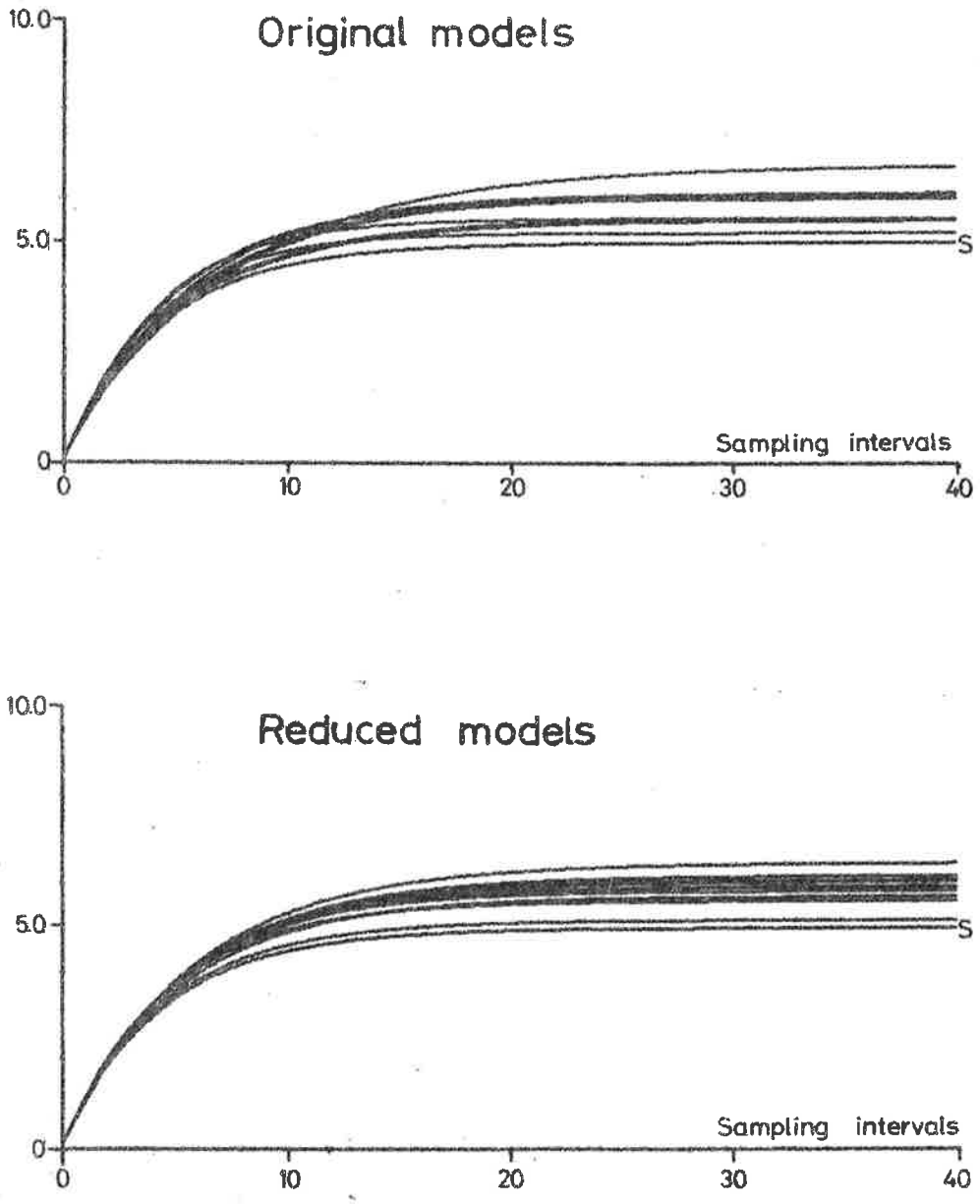

Figure 5.4. Example 3. Step responses of the true system, (S) and 10 models obtained from different realizations.

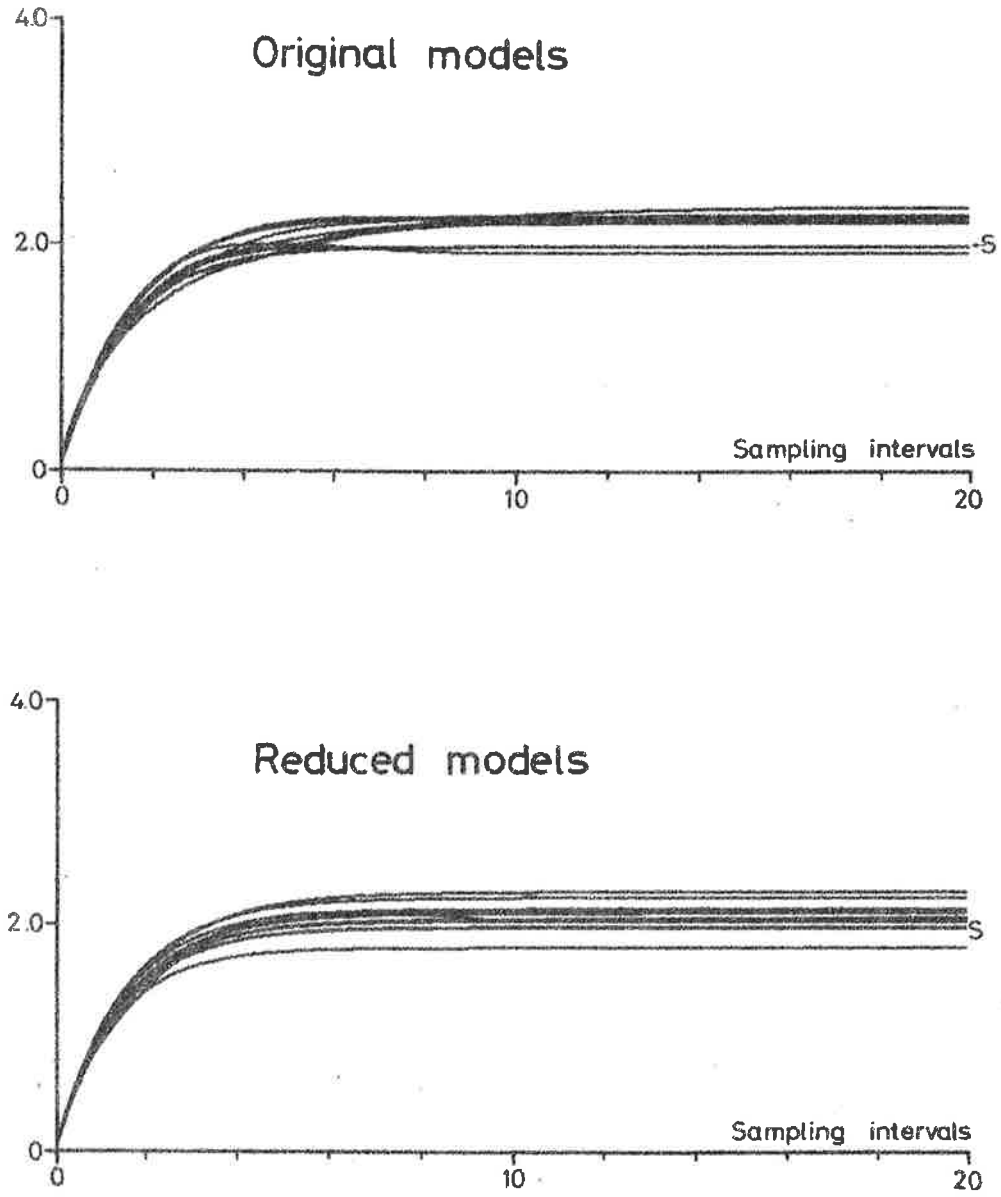

Figure 5.3. Example 4. Step responses of the true system, (S) and 10 models obtained from different realizations.

Comparisons between the three methods

Methods 2 and 3 have been examined by Burström (1973). To some extent the same examples were used as the ones considered here. The main conclusion of Burström's work is that method 2 is superior to method 3.

Method 2 was tried on example 1. The reduced models were considerably worse for method 2 than for method 1. For instance, the value of the pole was 0.54 for n = 3, 0.60 for n = 4 and 0.59 for n = 5. These values should be compared with the values given in table 5.1 for method 1.

A principal drawback of method 1 is that the loss function can have more than one local minimum point, Söderström (1972). However, in the considered examples no numerical problems occur, perhaps because of good initial values for the minimizations. This drawback does not appear in the other two methods since the loss function is approximated in these cases.

A small advantage of method 3 was observed by Burström (1973). When the input signal is white noise, it turned out in examples that the reduced model gave consistent estimates. It is in fact possible to prove this theoretically, see e.g. Söderström (1973b). The basic tool needed for a proof is lemma A.1. However, such a proof is not included here, since the method has also a large drawback. The experience of Burström was, in accordance with the discussion in chapter III, that the test quantities seem to be quite irrelevant.

VI. SUMMARY AND CONCLUSIONS

- A basic idea for testing common factors of two polynomials has been proposed. The covariance matrix of the estimates of the coefficients is assumed to be given and is used in the algorithm. The algorithm can be formulated as a minimization of a quadratic loss function under a complicated constraint.

- Three different ways of implementing the algorithm were discussed. The first way is a reparameterization which gives a new optimization problem without constraints. The second way is to compare the zeros of the two polynomials. The third way is based on the Euclidean algorithm. The experience is that the first method is also the most preferable one. One drawback of this method is that the new optimization problem can have several local extrema. Thus it is important to have good initial values for the optimization. A way of constructing initial values is discussed. The second way of implementing the algorithm is in general superior to the third one. On the whole the third way can not be recommended for use without special cautions since the test quantities seem to be quite irrelevant in this case.

- Applied to models obtained by least squares identification it is proved that the method is nothing but a new and much more efficient method to obtain the generalized least squares model.

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APPENDIX. COMPUTATION OF INITIAL VALUES.

It was mentioned in chapter IV that the initial values for the minimization can have influence not only on the number of required iterations but also on the result of the minimization. The reason for this is that the loss function can have several local minimum points. In this appendix is discussed a way of obtaining initial values for the minimization. Under certain conditions it will be proved that they are consistent estimates $\{a_i\}$ and $\{b_i\}$.

Assume that the true system is governed by the equation

$$A(q^{-1})y(t) = B(q^{-1})u(t) + v(t) \qquad (A.1)$$

where u(t) and v(t) are independent and v(t) can have an arbitrary covariance function. The polynomials of (A.1) are

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_o} q^{-n_o}$$

$$B(q^{-1}) = \quad b_1 q^{-1} + \ldots + b_{n_o} q^{-n_o}$$

For the test of common factors the following LS model is available

$$\hat{A}(q^{-1})y(t) = \hat{B}(q^{-1})u(t) + \varepsilon(t) \qquad (A.2)$$

where

$$\hat{A}(q^{-1}) = 1 + \hat{a}_1 q^{-1} + \ldots + \hat{a}_n q^{-n}$$

$$\hat{B}(q^{-1}) = \quad \hat{b}_1 q^{-1} + \ldots + \hat{b}_n q^{-n}$$

Lemma A.1. Assume that u(t) is white noise. Then asymptotically the impulse responses corresponding to the transfer functions

$\frac{B(q^{-1})}{A(q^{-1})}$ and $\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})}$ coincide in the first n points.

Remark: The result of the lemma can be expressed as

$$\frac{B(z)}{A(z)} - \frac{\hat{B}(z)}{\hat{A}(z)} = O(z^{n+1}), \quad z \to 0 \tag{A.3}$$

or, equivalently

$$B(z)\hat{A}(z) - \hat{B}(z)A(z) = O(z^{n+1}), \quad z \to 0 \tag{A.4}$$

Proof: The asymptotic equations for the LS estimates give, see e.g. Åström (1968),

$$Eu(t-i)\,[-y(t-1) \ \ldots \ -y(t-n)u(t-1) \ \ldots \ u(t-1)] \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_n \\ \hat{b}_1 \\ \vdots \\ \hat{b}_n \end{bmatrix}$$

$$= Eu(t-i)y(t)$$

for $i = 1, \ldots n$

This can be written as

$$0 = Eu(t-i)\,[\hat{A}(q^{-1})y(t) - \hat{B}(q^{-1})u(t)] =$$

$$= Eu(t-i)\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})}\,u(t) + \frac{\hat{A}(q^{-1})}{A(q^{-1})}\,v(t)\right] =$$

$$= Eu(t-i)\left[\frac{\hat{A}(q^{-1})B(q^{-1}) - A(q^{-1})\hat{B}(q^{-1})}{A(q^{-1})}\,u(t)\right] =$$

$$= Eu(t-i)\left[\sum_{j=1}^{\infty} h_j u(t-j)\right]$$

Since u(t) is white noise this equation gives

$$h_i = 0 \quad i = 1, \ldots n$$

which is nothing but (A.3) or (A.4) expressed in an alternate form. □

Based on this lemma the following algorithm is proposed. The initial values of $\tilde{A}(q^{-1})$ and $\tilde{B}(q^{-1})$ are computed from

$$\tilde{A}(z)\hat{B}(z) - \hat{A}(z)\tilde{B}(z) = 0\ (z^{2n-2k+1}), \qquad z \to 0 \tag{A.5}$$

The algorithm can be implemented as a system of linear equations, namely

$$\begin{bmatrix} 0 & & -1 & & \\ & \mathbf{0} & & \mathbf{0} & \\ \hat{b}_1 & & -\hat{a}_1 & & \\ \cdot & & \cdot & & \\ \cdot & & \cdot & & \\ & \hat{b}_1 & & & -\hat{a}_1 \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \hat{b}_{2n-2k-1} & \hat{b}_{n-k} & -\hat{a}_{2n-2k-1} & \cdot & -\hat{a}_{n-k} \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \cdot \\ \cdot \\ \tilde{a}_{n-k} \\ \tilde{b}_1 \\ \cdot \\ \cdot \\ \tilde{b}_{n-k} \end{bmatrix} = \begin{bmatrix} -\hat{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -\hat{b}_{2n-2k} \end{bmatrix} \tag{A.6}$$

where $\hat{b}_i$ and $\hat{a}_i$ are replaced with zero if $i > n$.

The equation (A.5) can also be expressed as

$$\frac{\hat{B}(z)}{\hat{A}(z)} - \frac{\tilde{B}(z)}{\tilde{A}(z)} = 0(z^{2n-2k-1}), \quad z \to 0 \tag{A.7}$$

Introduce the impulse response $\hat{h}_i$ from

$$\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})} = \sum_{i=1}^{\infty} \hat{h}_i q^{-1}$$

Then (A.7) can be implemented as

$$\tilde{A}(z) \sum_{i=1}^{\infty} \hat{h}_i z^i - \tilde{B}(z) = 0(z^{2n-2k+1}), \quad z \to 0$$

or expressed as a system of linear equations

$$\begin{bmatrix} 0 & & & & 0 \\ \hat{h}_1 & 0 & & & \\ \vdots & \ddots & & & \\ & & \hat{h}_1 & & -1 \\ & & \vdots & 0 & \\ & & \hat{h}_{n-k} & & \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_{n-k} \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_{n-k} \end{bmatrix} = \begin{bmatrix} -\hat{h}_1 \\ \vdots \\ \\ \\ \vdots \\ -\hat{h}_{2n-2k} \end{bmatrix} \qquad \text{(A.8)}$$

The equations (A.8) has the disadvantage compared with (A.6) that the impulse response of the model (A.2) must be computed. However, an advantage is that in practice only a system of (n-k) equations must be solved, namely; the last (n-k) equations in (A.8). The $\tilde{b}_i$ -variables are easily computed when the $\tilde{a}_i$ -variables are known.

Something can be said about the effect of the orders of the polynomials.

Lemma A.2 Consider the proposed algorithm and suppose that u(t) is white.

i) If $n \geq 2n_o$ and $k = n - n_o$, then the algorithm will give unique estimates, namely
$\tilde{A}(q^{-1}) = A(q^{-1})$, $\tilde{B}(q^{-1}) = B(q^{-1})$

ii) If $n - k > n_o$ and $k \geq n_o$, then the algorithm will not have a unique solution (if solutions exist, there are infinitely many).

Proof: Part i) follows from lemma A.1 after a comparison of (A.3) and (A.7). Part ii) is proved as follows. Denote the matrices of (A.6) and (A.8) by P_1 and P_2 resp. Consider the vectors P_1x and P_2x where the vector x holds

$$x = \begin{bmatrix} 0 \\ 1 \\ a_1 \\ \cdot \\ \cdot \\ a_{n_o} \\ 0 \\ 0 \\ -b_1 \\ \cdot \\ \cdot \\ -b_{n_o} \end{bmatrix}$$

The condition $n - k > n_o$ guarantees the possible construction of x.

Then $P_1 x = 0$ is equivalent to

$$A(z)\,\hat{B}(z) - \hat{A}(z)\,B(z) = O(z^{n-k+n_o+1}) \quad z \to 0 \qquad (A.9)$$

and $P_2 x = 0$ can be expressed as

$$A(z)\sum_{i=1}^{\infty} \hat{h}_i z^{i} - B(z) = O(z^{n-k+n_o+1}) \quad z \to 0$$

which is equivalent to (A.9). According to (A.4) these expressions are fulfilled provided $n - k + n_o + 1 \leq n + 1$ or $k \geq n_o$. Thus the matrices P_1 and P_2 are singular, which completes the proof. □

Note that the singularity of the matrix in (A.8) can also be examined using theory from Kalman-Arbib-Falb (1969).

The second part of the lemma gives also valuable information concerning the determination of a proper model order when the input signal is white noise. Suppose that k satisfies $n_o \leq k < n - n_o$. It is always possible to find k and n such that these conditions are satisfied for a given value of n_o. When they are fulfilled, the systems of equations (A.6) and (A.8) become singular, thus indicating that the number of common factors is chosen unproperly. A way to estimate the true system order n_o is the following. Consider the equations (A.6) for different values of k. Suppose that the

system becomes singular for some value(s) of k. Increase k until the system becomes non singular. Then $n_o = n - k$. The first part of lemma A.2 gives that the matrix P_1 surely is non singular for this combination.

The inequalities $n_o \leq k < n - n_o$ are illustrated in figure A.1 for the cases $n_o = 1$ and 2. Clearly, $1 \leq k \leq n - 1$ must always hold.

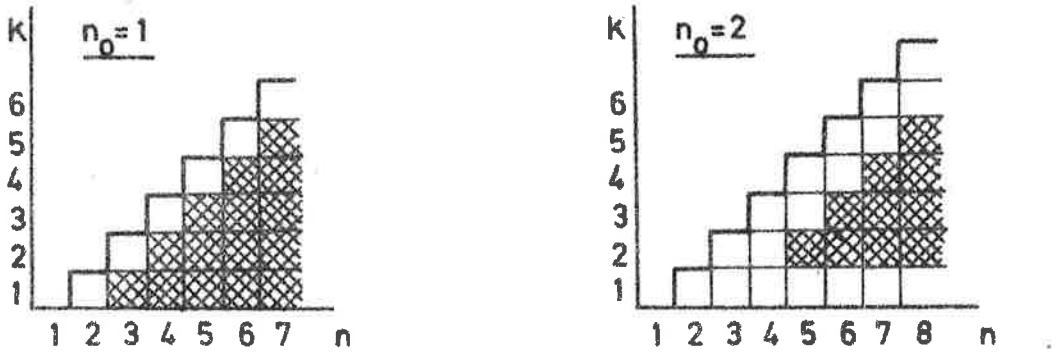

Figure A.1. Illustration of the conditions $n_o \leq k < n - n_o$. The dashed squares are combinations which satisfy the conditions.

Of course, a test of singularity of the matrices P_1 and P_2 is not a trivial task. One way to do it is to compute the condition numbers for different values of k. For considerably large values of this number the matrices may be considered as singular.

The proposed algorithm can in fact be interpreted asymptotically as an instrumental variable (IV) method. With use of the notations of section IV, this class of methods can be described by, cf Wong-Polak (1967) and Young (1970).

$$Z^T\theta\hat{\varphi} \approx Z^T Y \tag{A.10}$$

where Z is a matrix of order (N-n) x 2n such that

i) $\lim \frac{1}{N} Z^T\phi$ is non singular a.s. (A.11)

ii) $\lim \frac{1}{N} Z^T(\phi\varphi_o - Y) = 0$ a.s. (A.12)

where φ_o denotes the true parameter values.

It is easy to prove consistency of such a method (assumed that $n = n_o$).

With the special choice

$$Z = \begin{bmatrix} u(n) & u(n-1) & \cdot & \cdot & u(1-n) \\ \cdot & & & & \cdot \\ & & & & \\ \cdot & & & & \cdot \\ u(N-1) & \cdot & & \cdot & u(N-2n) \end{bmatrix}$$

it is easy to see that (A.12) is fulfilled. Moreover, it turns out that (A.10) can asymptotically be equivalently expressed as (A.8). From lemma A.2 then (A.11) follows.

The specific Z is a special case of a class of Z-matrices treated by Finigan-Rowe (1973). Thus it is possible to obtain the first part of lemma A.2 using their theory as well.

When the initial values of $\tilde{A}(z)$ and $\tilde{B}(z)$ are found, it is easy to compute initial values of $\tilde{C}(z)$. A suitable way to do this is to minimize $F(\hat{\varphi}(\hat{\theta}))$ with respect to the $\tilde{c}_i$ -parameters only and with the initial values of $\tilde{A}(z)$ and $\tilde{B}(z)$ kept fixed. This restricted loss function is quadratic and thus the minimization is trivial.