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STOCHASTIC CONTROL THEORY AND SOME OF ITS INDUSTRIAL APPLICATIONS

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ABSTRACT

The purpose of this paper is to give a survey of some results from stochastic control theory and to discuss their application to industrial control problems. The intention is to state the results, discuss them from the point of view of applications and to give some experiences from practical use. The review is not comprehensive.

1. INTRODUCTION

The paper is limited to discrete time systems. The motivation is that digital computers are used to implement the control algorithms. Mathematically the theory becomes much simpler. Some formulas will, however, formally be more complicated than in the continuous time case. Several important problems, like the selection of the sampling interval, are also neglected by this assumption.

The paper is organized as follows. Control of linear systems with a quadratic criterion is discussed in Section 2. The standard problem is formulated and the solution stated. An important special case of the linear theory, which leads to the so-called minimum variance strategy, is also discussed. The linear theory results in a regulator which is a linear, dynamical system. This is a convenient regulator in several practical problems. The major difficulty in applying the linear theory is to obtain suitable models for the process dynamics and the disturbances. Such models can be obtained from plant experiments using system identification techniques. Some applications of the linear theory are also given in Section 3.

A review of nonlinear theory is given in Section 4. By assuming that the process and its environment can be characterized as Markov processes, the optimal strategies are given by the Bellman equation obtained by Dynamic Programming. The equation is unfortunately unpleasant both analytically and numerically. The main obstacle is dimensionality. To carry out the analysis it is necessary to introduce a hyperstate, which is a probability distribution on the original state space of the problem. For this reason it has not yet been possible to solve any realistic problems using these methods. A few problems have, however, been solved numerically.

One particular problem, namely control of a linear system with constant but unknown parameters, is discussed in Section 4. In this case the nonlinear theory gives dual control strategies in the sense of Feldbaum. The insight obtained from the nonlinear theory can be exploited to construct suboptimal strategies which can be implemented. One class of suboptimal regulators, the so-called self-tuning regulators, are discussed in Section 5. These regulators can be used to control a linear system with constant but unknown parameters and will thus make it possible to compensate for lack of process knowledge by more sophisticated control algorithms. As indicated by the nonlinear theory, the regulators will have some limitations. One drawback is that they are not dual controls. This is discussed in connection with some applications of the regulators in Section 5. Some concluding remarks on the applicability of stochastic control theory are finally given in Section 6.

2. LINEAR THEORY

The well-known linear stochastic control theory deals with control of a system which can be described by the stochastic difference equation

$$\begin{aligned} x(t+1) &= A\,x(t) + B\,u(t) + v(t) \\ y(t) &= C\,x(t) + e(t) \end{aligned} \tag{2.1}$$

where $t \in T = \{\ldots -1, 0, 1, \ldots\}$, $x(t)$ is a $n\times 1$ state vector, $u(t)$ a $p\times 1$ vector of control variables and $y(t)$ a $r\times 1$ vector of measured outputs. The stochastic processes $\{v(t), t\in T\}$ and $\{e(t), t\in T\}$, called process disturbances and measurement errors respectively, are sequences of uncorrelated, random variables with zero mean values and the covariances

$$\mathrm{cov}[v(t), v(t)] = R_1$$

$$\mathrm{cov}[v(t), e(t)] = R_{12}$$

$$\mathrm{cov}[e(t), e(t)] = R_2$$

The initial state of (2.1) is assumed to be a random variable with mean m and covariance R_o.

The performance of the control system is described by the scalar loss function

$$\ell = \frac{1}{N}\{x^T(N)Q_o x(N) + \sum_{t=1}^{N-1} [x^T(t)Q_1 x(t) + u^T(t)Q_2 u(t)] \tag{2.2}$$

where the matrices Q_o, Q_1 and Q_2 are symmetric and non-negative. Since ℓ is a random variable, the criterion of the control is taken as to minimize the

expected loss.

The admissible strategies are such that $u(t)$ is measurable with respect to the sigma-algebra $\mathcal{Y}_t$, generated by $\{y(s),\ 0 \leq s \leq t-1\}$, for each $t \in T$.

The problem can be solved under two sets of assumptions. In one case the controls are linear functions of the measurements. It is then sufficient to assume second order properties of the random processes. If the random processes are assumed to be jointly gaussian, it is no longer necessary to assume that the controls are linear. The solution is formally the same in both cases. It is given by the separation theorem, which says that the optimal control is

$$u(t) = -L\,\hat{x}(t|t-1)$$

where $\hat{x}(t|t-1)$ is the best estimate of the state $x(t)$ given $y(t), y(t-1), \ldots$. The estimate is given by the Kalman filter

$$\hat{x}(t+1|t) = A\,\hat{x}(t|t-1) + b\,u(t) + K[y(t) - \hat{x}(t|t-1)]$$

$$\hat{x}(0) = m$$

The matrices K and L are obtained from solutions of Riccati equations. The formulas first derived by Kalman are standard textbook material. See e.g. Kwakernaak and Sivan (1972) and Åström (1970), where references to original work as well as interpretations are given. A critical review of the linear problem is found in Witsenhausen (1971). A special issue of IEEE transactions, Athans (1971), is also devoted to the linear, quadratic problem. This issue contains many valuable papers covering both theory and applications.

There are several trivial but useful extensions that can be made. The process disturbance and the measurement errors may have unknown mean values. This is taken care of by augmenting the state vector with the vector $z = E\,v(t)$ with the associated state equation

$$z(t+1) = z(t)$$

Correlated disturbances with rational spectral densities can be handled in a similar way by state augmentation.

The solution of the linear, stochastic control problem thus gives a regulator in terms of a linear, dynamical system. The dynamics of the regulator arises from the Kalman filter.

Linear control theory is attractive for control engineers because it gives a regulator which is a linear, dynamical system; a structure which has been known to be useful for a long time. The complexity of the regulator directly reflects the complexity of the model and the criterion. The matrices K and L will in general be time varying. However, if the system has constant parameters and if $N \to \infty$, the matrices K and L will become constant matrices under reasonable conditions. This solution, referred to as steady state control, is frequently used in applications.

Minimum Variance Control

The special case of linear control theory, obtained when the output is a scalar and there is no penalty on the control, is particularly simple. By choosing an innovation's representation of the disturbances and changing the basis in the state space the model (2.1) can be transformed into

$$x(t+1) = \begin{bmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ & & & & \\ -a_{n-1} & 0 & 0 & \ldots & 1 \\ -a_n & 0 & 0 & \ldots & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \\ b_{n-1} \\ b_n \end{bmatrix} u(t) + \begin{bmatrix} k_1 \\ k_2 \\ \\ k_{n-1} \\ k_n \end{bmatrix} \varepsilon(t)$$

$$y(t) = x(t) + \varepsilon(t) \qquad (2.1)$$

Elimination of the state variables gives the following input output relation

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) = b_1 u)t-1) + \ldots + b_n u(t-n) + \varepsilon(t) + c_1 \varepsilon(t-1) + \ldots + c_n \varepsilon(t-n)$$

By introducing the polynomials

$$A(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

$$B(z) = b_1 z^{n-1} + \ldots + b_n$$

$$C(z) = z^n + c_1 z^{n-1} + \ldots + c_n$$

and the forward shift operator q the model can also be written as

$$A(q)y(t) = b(q)u(t) + C(q)\varepsilon(t) \qquad (2.2)$$

The control law which minimizes the variance of the output i.e. the criterion

$$\ell_1 = E\,y^2(t) \qquad (2.3)$$

can of course easily be found from the general theory. The strategy can, however, also be found directly by a very simple argument as shown in Åström (1970). If the polynomial $B(z)$ has no zeros outside the unit circle, the optimal strategy is

$$u(t) = \frac{A(q) - C(q)}{B(q)}\,y(t) \qquad (2.4)$$

Notice that the strategy is obtained without solving a Riccati equation or without spectral factorization. When the polynomial $B(z)$ has zeros outside the unit circle, the Riccati equation will have several solutions. One solution corresponds to (2.4).

The control strategy which minimizes ℓ_1 will also minimize

$$\ell_2 = E\left[\frac{1}{N}\sum_{t=1}^{N} y^2(t)\right] \qquad (2.5)$$

Notice that the strategy which minimizes the criterion

$$\ell_3 = E[y^2(t) + Q_2 u^2(t)]$$

is also very easy to calculate. This strategy is, however, not the same as the strategy which minimizes

$$\ell_4 = E \frac{1}{N} \sum_1^N [y^2(t) + Q_2 u^2(t)]$$

It is easy to find cases where the strategy which minimizes ℓ_3 gives an unstable, closed loop system.

The following simple example will be used.

Example 3.1

The minimum variance strategy for the system

$$y(t+1) + a\, y(t) = b\, u(t) + e(t+1) + c\, e(t) \qquad (3.6)$$

is given by

$$u(t) = \frac{a - c}{b}\, y(t) \qquad (2.7)$$

□

Review of Assumptions

The fundamental assumptions made in the linear theory are

- o the process dynamics is linear
- o the disturbances are stochastic processes, which are gaussian or of second order
- o the criterion is to minimize a quadratic function
- o the models for the process and the disturbances are known.

The assumption of linearity is quite reasonable when steady state control of industrial processes are considered. The purpose of the control is to maintain the process variables close to steady state values. It is difficult to verify the assumption on the disturbances. When attempts to model disturbances are made, it is frequently found that a stochastic process model is adequate provided that the model allows for drifting disturbances. Such processes can be modeled by integrators driven by processes with rational spectral densities, for example the ARIMA process introduced by Box and Jenkins (1970). In practice it is also necessary to take into account that there may occasionally be big disturbances.

The quadratic criterion is often controversial. In the special case when $Q_2 = 0$ and $Q_1 = 1$ the criterion (2.2) is simply the minimization of the variance of the output. This can be justified as a criterion for steady state control using the following argument. Through general trade rules, limits are imposed on important quality variables. The customer accepts a batch if his quality control procedures show that the limits are exceeded by a given fraction of the product. Due to the fluctuations in the process the manufacturer selects the set points of the regulators for the quality variables inside the test limits to ensure that his product is accepted. By reducing the variations in the quality variables it is possible to move the set points closer to the limits without changing the probability for acceptance. By doing this there is a gain which can be exploited in many ways as increased production, reduction of raw material or energy consumption. The reduction of fluctuations in quality variables is also of value by itself. It is, however, often very difficult to express this in monetary terms.

In some cases the more general quadratic criterion can be motivated from physical arguments. One good example is in the design of autopilots for ships where the average breaking force due to rudder actions and angle of attack can be given approximatively as a quadratic form. See Koyama (1967). In most cases it is, however, difficult to assign penalties to different combinations of state and control variables. The criterion (2.2) is therefore not always realistic.

The assumption that models for the process and the disturbances are available is frequently the major difficulty in applications. Even if a reasonable model for the process dynamics is available, the characteristics of the disturbances are seldom known. To apply the linear theory it is therefore a necessity to have methods to determine the desired models from process experiments using system identification techniques. This presents little difficulty for single output systems. In this case a canonical form for the model can easily be given, and the parameters of this model can then be estimated. See Åström and Eykhoff (1971) and Eykhoff (1974). The problem is more difficult because of the lack of unique canonical forms. If the observability indices or the controllability indices are known, it is straightforward to obtain a canonical form. Lacking information about the indices there are however many alternatives to explore. See e.g. Kalman (1972) and Mayne (1974).

In many cases the difficulties mentioned above can be overcome; and the linear stochastic control theory, in combination with suitable system identification techniques, is a useful tool for the control engineer. This is clearly witnessed by the increasing number of applications. See Athans (1971).

There are several applications of minimum variance control which have covered all steps from process experiments to on-line control. Applications to paper machine control are given in Åström (1967) and Fjeld and Grimnes (1974). Control of interior climate is discussed in Jensen (1974). An application to design of autopilots for super tankers is done by Källström (1973). Control of a pilot distillation plant is described in Binder and Calvillo (1974). The experience from all these applications have been that closed loop performance can be predicted very accurately from the models obtained from plant experiments and system identification. The plant experiments and the system identification are quite time consuming, and they will also require skilled personnel.

3. NONLINEAR THEORY

The obvious nonlinear extension of the linear problem discussed in the previous section is obtained by replacing the linear difference equation (2.1) by the nonlinear equation

$$x(t+1) = f(x(t), u(t), v(t)) \qquad (3.1)$$
$$y(t) = g(x(t), u(t), e(t))$$

and by replacing the quadratic loss (2.2) by the general loss function

$$\ell = \frac{1}{N} \sum_{1}^{N} h(x(t), u(T)) \qquad (3.2)$$

If it is assumed that the disturbances are such that the stochastic process $\{x(t), y(t); t\in T\}$ is a Markov process, the control problem can be formulated precisely and existence theorems can be given in some cases. The theory is often formulated by giving the transition probabilities of the process directly instead of giving the nonlinear equations. The theory is presented in the Dynamic Programming framework. To do this the hyperstate

$$p_t(A) = P\{x(t)\in A \mid \mathcal{Y}_t\} \qquad (3.3)$$

where $\mathcal{Y}_t$ is the sigma-algebra generated by $\{y(s), 1 \leq s \leq t\}$. By introducing the function

$$V_t(p) = \min E\{ \sum_{k=t}^{N} h(x(t), u(t) \mid \mathcal{Y}_t\} \qquad (3.4)$$

the following functional equation for V_t is obtained by standard Dynamic Programming arguments

$$V_t(p) = \min_u\{\int h(x,u)p_t(dx) + E[V_{t+1}(p_{t+1}) \mid \mathcal{Y}_t]\} \qquad (3.5)$$

This equation, which is called the Bellman equation after Bellman (1961), is the starting point for many investigations. To have the right hand side of (3.5) well defined it is necessary to have an equation for updating the conditional probability distribution (3.3). Such equations were given by Stratonovich (1960), Kushner (1967) and Wonham (1964). Updating the conditional distribution is equivalent to a nonlinear filtering problem, which has been studied extensively. Numerical aspects of this problem are recently given by Levieux (1974). The functional equation (3.5) has been investigated in Florentin (1962), Åström (1964), Mortensen (1966), and Stratonovich (1963).

The numerical solution of the functional equation (3.5) is very difficult because of the dimensionality of state. For example, when $x\in R^3$, the hyperstate p is a probability distribution over R^3. The conceptually simple problem of storing the function V thus becomes a major difficulty. This difficulty is the main reason why the intensive interest in nonlinear stochastic control in the sixties ended in dispair. The situation is expressively stated in Wonham (1968):

"To summarize: the only cure for dynamic disturbances is tight feedback (high rates of information processing) and large control forces. With fixed constraints on computation capacity and control force, little more can be achieved by subtle changes in control logic."

The nonlinear theory has thus not had any practical applications. In spite of this it will be shown in the next section that the nonlinear theory will give some insight which indirectly may be of some practical significance.

4. LINEAR SYSTEMS WITH CONSTANT BUT UNKNOWN PARAMETERS

When the linear theory was discussed in Section 2, it was remarked that it was a severe restriction from the practical point of view to assume that the parameters of the models for the process dynamics and the disturbances were known. The lacking knowledge had to be supplied by a combination of experimentation and statistics. An alternative to this would be to formulate the problem in such a way that the lacking knowledge of the process parameters becomes part of the problem statement. The linear quadratic problem discussed in Section 2 will thus be considered, but the parameters of the models are assumed to be unknown.

This problem can be regarded as a special case of the nonlinear problem by introducing the unknown parameters as new state variables. If the parameters are constants, the associated state equations are simply given by

$$\theta(t+1) = \theta(t)$$

It turns out that the analysis to follow can equally well be performed when the parameters are given by

$$\theta(t+1) = D \;\; \theta(t) + w(t) \qquad (4.1)$$

where θ is a vector of unknown parameters, D is a known matrix and $\{w(t), t\in T\}$ a white noise process. When D equals the identity matrix, the parameters are simply Wiener processes. If in addition $w = 0$ the parameters are constants. The model (4.1) is therefore somewhat inconsistently referred to as the case of "drifting parameters".

The assumption (4.1) is not particularly appealing from the point of view of applications because it means that the model has randomly varying parameters, but that the statistical properties of the parameters are known. This is of course not a very realistic situation.

From the theoretical point of view it is, however, interesting to observe that the case of constant parameters is as difficult as the case of "drifting parameters".

It will now be discussed how far the nonlinear theory can be exploited in this particular case. To use Dynamic Programming and the functional equation (3.5) it is necessary to introduce a hyperstate which is a probability distribution over a space whose dimension is the sum of n in (2.1) or (3.1) and the number of unknown parameters. It is clear that even a numerical solution of the functional equation (3.5) is out of the question for any

problem of reasonable size. It is therefore of interest to consider further simplifications which will make it possible to reduce the dimension of the hyperstate. Two examples of such simplifications are given below.

Example 4.1

Consider the model (2.1) with C = I and e = 0. All state variables are thus measurable without error. Furthermore assume that the unknown parameters are parameters of the matrices A and B and that the stochastic process $\{w(t), t\in T\}$ is gaussian. The conditional distribution of the parameters given the sigma-algebra generated by $\{y(s), 1 \leq s \leq t\}$ is then gaussian and can be characterized by the conditional mean and two conditional covariances. See Farison (1967).

□

Example 4.2

Consider the model (3.1). Assume that the parameters c_i are all zero, that the unknown parameters are $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and that $\{\varepsilon(t), t\in T\}$ are gaussian. Then the conditional distribution of the parameters is gaussian. See Mayne (1963).

□

The second example, which corresponds to minimum variance control, will now be explored in some detail. The analysis follows Åström and Wittenmark (1971).

Introduce the notations

$$\theta(t)=\mathrm{col}[a_1(t)a_2(t)\ldots a_n(t)\ b_1(t)b_2(t)\ldots b_n(t)]$$

$$\varphi(t-1)=[-y(t-1)\ -y(t-2)\ldots -y(t-n)u(t-1)u(t-2)\ldots \ldots u(t-n)]$$

$$\varphi_o(t-1)=[-y(t-1)\ -y(t-2)\ldots -y(t-n)\ 0\ u(t-2)\ldots \ldots u(t-n)]$$

The process model

$$y(t) + a_1(t-1)y(t-1)+\ldots+a_n(t-n)y(t-n) = = b_1(t-1)u(t-1)+\ldots+b_n(t-1)u(t-n) + \varepsilon(t) \qquad (4.2)$$

can then be written as

$$y(t+1) = \varphi(t)\theta(t) + \varepsilon(t) = b(t)u(t) + + \varphi_o(t)\theta(t) + \varepsilon(t) \qquad (4.2)$$

where the parameters are assumed given by (4.1). Recall that the case of unknown but constant parameters corresponds to D = I and v = 0. To apply the nonlinear theory the hyperstate (3.3) should first be determined. This appears difficult because the stochastic process $\{y(t), t\in T\}$ is not gaussian. It was, however, observed by Mayne (1963) that the hyperstate (3.3) is gaussian. See also Bohlin (1970). This means a tremendous reduction of the dimensionality of the problem because the hyperstate can be represented by the conditional mean and the conditional covariance

$$\hat{\theta}(t) = E\{\theta(t)|\mathcal{Y}_t\}$$

$$P(t) = E\{[\theta(t)-\hat{\theta}(t)][\theta(t)-\hat{\theta}(t)]^T|\mathcal{Y}_t\}$$

where $\mathcal{Y}_t$ is the sigma-algebra generated by y(t), y(t-1),... .

One-Step and Certainty Equivalence Control

Having obtained the hyperstate it is not straightforward to do the minimization. Since

$$E\ y^2(t+1) = E\ E\{y^2(t+1)|\mathcal{Y}_t\} = E\ E\{[b\ u(t) + + \varphi_o(t)\theta(t) + e(t)]^2|\mathcal{Y}_t\} \qquad (4.3)$$

is a quadratic function of u(t), it is straightforward to show that the optimal control is

$$u(t) = -\frac{\hat{b}\varphi_o\hat{\theta} + \varphi_o p_{\theta b}}{\hat{b}^2 + p_{bb}} \qquad (4.4)$$

where

$$p_{\theta b} = E[\theta-\hat{\theta}][b-b]$$

The control law (4.4) is called the one-step control because it minimizes the loss function (4.3), which is the expected loss over one step only. It is also called the cautious control.

Notice that if $p_{bb} = 0$, then the control reduces to

$$u(t) = -\frac{\varphi_o\hat{\theta}}{\hat{b}}(t) \qquad (4.5)$$

In the case of constant parameters the minimum variance control is given by (2.4) i.e.

$$u(t) = -\frac{\varphi_o\theta}{b}(t)$$

Notice that C(z) = 1 in the model (4.2). The control (4.5) can be interpreted as the control obtained by computing the optimal control under the condition that the parameters are known and then simply substituting the true parameters by their estimates. The control (4.5) is therefore called the certainty equivalence control.

Example 4.3

For the system (3.6) of Example 3.1 the one-step control (4.3)

$$u(t) = \frac{\hat{a}\hat{b} + p_{ab}}{\hat{b}_2 + p_{bb}} \qquad (4.6)$$

and the certainty equivalence control (4.4) becomes

$$u(t) = \frac{\hat{a}}{\hat{b}}y(t) \qquad (4.7)$$

□

The control laws (4.4) and (4.5) have empirically been found to be useful, at least when they are

applied to non-minimum phase systems with constant parameters. The controllers will, however, have difficulties when they are applied to systems where the parameter b changes its sign. The control laws are very similar when $p_{bb} << \hat{b}^2$ but very different when $p_{bb} >> \hat{b}^2$. In particular, the gain of the certainty equivalence controller becomes very large while the gain of the one-step control becomes very small for small values of $\hat{b}$. The certainty equivalence controller will thus give very large control signals for small b. The one-step controller will, on the other hand, exhibit the so-called turn-off effect, which means that the control signal will be zero for long periods of time. Intuitively this can be explained as follows. When the estimate of b is poor i.e. $p_{bb} >> \hat{b}^2$, then the gain of the regulator (4.4) will be small. The control signal applied to the plant will then also be small. The estimate of b will at the same time be even worse, the gain lower, etc. This phenomena was observed in Åström-Wittenmark (1970). It is also discussed in Hughes and Jacobs (1974).

Dual Control

When the parameters of the system (3.2) are known, the control strategy (3.4) will minimize both the criterion (3.3) and the criterion (3.5). This is unfortunately not true in the case of unknown parameters. While it is fairly simple to find the one-step control for the system (4.2), it is a much more difficult task to find the control which minimizes the criterion (3.5). In principle the solution is given by the functional equation (3.5). Introducing

$$V(\hat{\theta}(t),P(t),\varphi_o(t),t) = \text{Min } E\{\sum_{k=t}^{N} y^2(t+1)|\mathcal{Y}_t\} \quad (4.8)$$

The Bellman equation (3.5) then becomes

$$V(\hat{\theta}(t),P(t)\varphi_o(t),t) =$$

$$\underset{u(t)}{\text{Min}}\{[\varphi(t)\hat{\theta}(t)]^2 + \varphi(t)P(t)\varphi^T(t) + \sigma^2$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V(\hat{\theta}(t+1),P(t+1),\varphi_o(t+1),t+1)e^{-s^2/2}\,ds\} \quad (4.9)$$

where

$$\hat{\theta}(t+1) = D\,\hat{\theta}(t) + K(t)\,\sigma^2 + \varphi(t)P(t)\varphi^T(t)\,s$$

$$K(t) = D\,P(t)\varphi^T(t)[\sigma^2 + \varphi(t)P(t)\varphi^T(t)]^{-1}$$

$$\varphi_{o1}(t+1) = -\varphi(t)\hat{\theta}(t) + \sigma^2 + \varphi(t)P(t)\varphi^T(t)\,s$$

$$\varphi_{oi}(t+1) = \varphi_{o(i-1)}(t) \quad i = 2, \ldots, 2n \quad (4.10)$$

See Åström and Wittenmark (1971). The functional equation (4.9), which will give the optimal strategy, can be solved numerically if the number of parameters is sufficiently small, say one or at the most two parameters. Solutions for special cases are given in Florentin (1962), Bohlin (1969), Åström and Wittenmark (1971). The controllers obtained differ significantly from the one-step control and the certainty equivalence control in those cases where the parameter b in the process model (4.2) changes its sign. The control given by (4.9) is a dual control in Feldbaum's sense. Simulations have shown that the dual controller avoids the turn-off effect of the one-step controller and also the very large control signals of the certainty equivalence control. For any practical problem of reasonable size the control given by (4.9) can not be computed. It is therefore a good research problem to find approximations which can be computed. In Tse et.al. (1973) and Tse and Bar-Shalom (1973) approximate solutions to the functional equation (3.5) are given for some examples including interception and soft-landing. The simulations show the superiority of the dual controllers as compared with the certainty equivalence controllers. The computing time for the approximate dual controller is, however, 45 s on a UNIVAC 1108. Other approximations of the dual controller are given in the paper by Bar-Shalom et.al. (1974) at this symposium and by Wittenmark (1974).

5. SELF-TUNING REGULATORS

The properties of the one-step controller and the certainty equivalence controller will now be explored further in the case when the controllers are applied to a system with constant but unknown parameters. It will thus be assumed that D = I and v = 0 in the equation (4.1) which describes the parameter variation. Since the regulators were derived under the assumption that the parameters c_1, c_2, ..., c_n in the model (3.2) are all zero, it can be expected that the regulators will behave well when applied to systems with this property. In this case the parameter estimates will be unbiased. If the closed loop system is stable, the parameter estimates will then converge to the model parameters as time tends to infinity. The control laws obtained in the limit will be the same as the control laws which could be computed if the model parameters were known a priori. Regulators with this property will be called self-tuning controllers. Convergence conditions are given in the paper by Ljung and Wittenmark (1974) at this symposium. Notice that if the one-step controller and the certainty equivalence controller both converge, they will converge to the same regulator because $p_{bb} \to 0$ as $t \to \infty$. The transient behaviour of the algorithms may however differ significantly.

In Åström and Wittenmark (1973) it was shown that both the one-step controller and the certainty equivalence controller are self-tuning when applied to a system described by the model (3.2). This is somewhat surprising because the least squares estimate is biased when the polynomial C(z) in (3.2) is different from unity. The basic idea behind this result can be illustrated by the following simple example.

Example 5.1

Assume that the certainty equivalence controller (4.7) is applied to the system described by the equation (3.6). Let $r_{yu}(\tau)$ denote the sample covariance of the input and the output signals i.e.

$$r_{yu}(\tau) = \frac{1}{t}\sum_{k=1}^{t} y(T)u(t-\tau)$$

The conditional mean values of the parameters are then given by

$$r_{yy}(1) + \hat{a}(t)r_{yy}(0) = \hat{b}(t)r_{yu}(0)$$

$$r_{yu}(1) + \hat{a}(t)r_{yu}(0) = \hat{b}(t)r_{uu}(0)$$

Since the control signals were generated by the control law (4.7), we get

$$r_{yy}(1) = -\frac{1}{t}\sum_{k=1}^{t} [\hat{a}(t)-\hat{b}(t)\hat{a}(k)/\hat{b}(k)]y^2(k)$$

$$r_{yu}(1) = -\frac{1}{t}\sum_{k=1}^{t} [\hat{a}(t)-\hat{b}(t)\hat{a}(k)/\hat{b}(k)]y(k)u(k) \qquad (5.1)$$

If the input and the output signals are bounded and if the controller gain

$$\alpha(t) = \hat{a}(t)/\hat{b}(t)$$

converges as $t \to \infty$, it can be shown that the right hand sides of (5.1) converge to zero as $t \to \infty$. Hence if the closed loop system obtained with a certainty equivalence controller is stable and if the controller gain converges as $t \to \infty$, then the closed loop system is such that

$$\lim_{t\to\infty} r_{yy}(1)$$

$$\lim_{t\to\infty} r_{yu}(1) \qquad (5.2)$$

The certainty equivalence controller thus attempts to drive certain covariances of the input and the output to zero. Assuming that the gain of the controller converges as time increases i.e.

$$\hat{a}(t)/\hat{b}(t) \to \alpha$$

The closed loop system obtained in the limit then becomes

$$y(t+1) + (a-\alpha b)y(t) = e(t+1) + c\, e(t)$$

The condition (5.2) then gives

$$\alpha = \alpha_1 = \frac{a - c}{b} \qquad (5.3)$$

or

$$\alpha = \alpha_2 = \frac{a - 1/c}{b}$$

The second solution is impossible because it gives an unstable, closed loop system. We thus find that if the certainty equivalence controller converges it will converge to a regulator which is identical to the minimum variance regulator (3.7) for the system (3.6).

Notice that the argument can also be extended to the one-step controller.

□

In the example it is not necessary to assume that the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$ converge; it is only required that their ratior converges.

In the particular case the normal equations which give the estimates may be ill conditioned, and difficulties may arise if care is not taken when solving them. See Nahorski (1974). The difficulty can be avoided by using pseudo inverses or by giving the parameter $\hat{b}$ a fixed value.

Extensive simulations of the certainty equivalence controller and the one-step controller when applied to a system with constant or slowly varying parameters are given in Wieslander-Wittenmark (1971), Wittenmark (1973) and Wouters (1974) in this symposium. Extensions of the algorithm to include penalty on the control are given in Åström and Wittenmark (1974) and to multivariable systems in Peterka and Åström (1973).

A self-tuning regulator will be an attractive solution for a linear stochastic control problem. It substitutes the need for process identification by a more complex control algorithm. The computational requirements both for the one-step control and the certainty equivalence control are very modest.

No practical control problem can be solved by theory alone. There will always be some elements of subjective judgement involved. For the certainty equivalence regulator the sampling interval, the model complexity and the exponential forgetting must be selected. Experience has whown that these factors are in many cases easily determined by engineering judgement. The certainty equivalence regulator has therefore successfully been used in several applications. Applications in the paper industry are described in Cegrell and Hedqvist (1973, 1974), Borisson and Wittenmark (1974). Control of an ore crusher was discussed at this symposium, Borisson and Syding (1974).

It has also been used to control an enthalphy exchanger, Jensen (1974), and as an autopilot for a super tanker, Källström (1974). In all these applications the regulator has been running on-line on a real industrial process. The simulation study by Wouters (1974) at this symposium also indicates the feasibility as a regulator for a stirred tank reactor. Similarly the paper by Vanacek (1974) at this symposium indicates the applicability to air/fuel controllers.

6. CONCLUSIONS

Many process engineers claim that most of the industrial control problems (60% - 70%) are solved sufficiently well using ordinary PID regulators. Even if this statement is true, I believe that there are control loops where more complex control algorithms are profitable. Regulation of important quality variables where there is a significant pay-off in reduction of fluctuations is a typical example. One possibility to obtain such control laws is to use linear, stochastic control theory where the potential of modeling disturbances as random processes are exploited. The major difficulty is to obtain appropriate models for the process dynamics and the disturbances. Such models can be obtained by system identification. Linear stochastic control theory in combination with system identification has been successfylly applied to solve real industrial problems.

The nonlinear stochastic control theory does not yield strategies that are conveniently computed. The theory gives, however, insight into nonlinear problems. This insight can then be used to construct control strategies heuristically. It can also be used to understand the behaviour of control laws designed by other arguments. A typical case is that of adaptive control where the nonlinear theory gives insight into the properties of certain suboptimal regulators. In this way it is possible to obtain self-tuning regulators and dual controllers. Even if much theoretical work remains before these problems are fully understood, some useful applications of available results have already been made.

In summary, while stochastic control theory is a highly technical area, which often is considered as a playground for academics, it is my belief that there are some results that can be profitably used to solve certain industrial control problems.

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