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Projective area-invariants as an extension of the cross-ratio

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1. Introduction

Plane projective geometry treats properties of geometric figures that are invariant under certain transformations, the "projectivities". The classical theorems mainly deal with incidences of points and lines. Metrical concepts, such as distance and area, have less natural sites in projective geometry. One well known exception is the cross-ratio, relating distances between collinear points. On the other hand, area-invariants have been discovered in connection with robot control and vehicle guidance [5-7]. There the problem was to find objects suited as sign posts or marking symbols. For reasons of error robustness and existing hardware, *area* measurements were preferable. Figure 1 shows one object that features area-invariants, and a possible image of it.

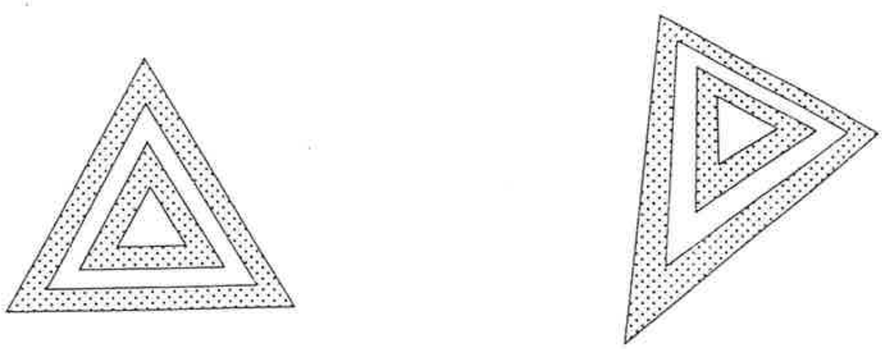

Figure 1. Two views of an object that features area-invariants

The existence of these new area-invariants pose some basic questions, since they indicate that areas in a certain sense have a place in projective geometry. The topic of this paper is an investigation into such fundamental matters, and one major result is that the area-invariants are conceptually justified. Another major result is that both the classical cross-ratio on the line and the area-invariants in the plane can be embedded in a wider formulation where they turn up as two special cases.

The presentation is structured as follows. The concepts of area and invariance are discussed in Section 2. Section 3 treats generalizations of the cross-ratio on the projective line. When passing to the projective plane, it can be done in two main directions. Two points on a line determine an interval of a specific length. In the same way three points in the plane form a triangle of a specific area. The cross-ratio is an invariant relation between lengths of intervals on the line, and a natural extension to the plane is then invariant relations between areas of triangles. This is the topic of Section 4. Section 5 treats another way of extending results from the line to the plane by to rotating the line i.e. to look for invariant relations between areas of conical sections as circles and ellipses. The findings are summarized in Section 6.

2. Areas and Invariants

By definition, *distance* between points is a property dealt with in Euclidean geometry. On the contrary, in affine geometry it is not possible to compare distances between points, unless they all lie on the same line. It is therefore somewhat remarkable that the concept of *area* makes sense in affine geometry, despite the fact that in elementary geometry it is usually defined in terms of distances.

For projective geometry the question arises to what extent the concept of distance and area in an affine space can be transferred to a projective space, claiming the existence of area-relations that are invariant under projectivities. For $n = 1$ the cross-ratio makes an example. Analogous expressions were also studied for triangles and tetrahedrons in Möbius "Der barycentrische Calcul" (1829) [4]. Else, although a natural problem, it seems to have been little studied. An effort is made in this paper.

Areas

We start with an example. Let A_0, A_1, A_2 be three points in a plane π in the Euclidean space $\mathbb{E}^3$. The points are represented by their coordinates. Suppose that the origin $O \notin \pi$. Then

$\det(A_0, A_1, A_2)$ = the volume, with signs depending on the orientation, of the parallelepiped spanned by the vectors $\overline{OA_0}, \overline{OA_1}, \overline{OA_2}$ =

6·(the volume of the tetrahedron with vertices in (O, A_0, A_1, A_2))=

3·(the area of the triangle with vertices (A_0, A_1, A_2)) · (the distance between O and π).

In other words, apart from a factor of proportionality, $\det(A_0, A_1, A_2)$ measures the area of the triangle in π, having vertices in A_0, A_1, A_2. □

We now turn to *projective spaces*, where the notion of area/volume has no a priori meaning. A definition will be made, inspired by the above affine considerations in the case $\pi : \Sigma_0^n x_i = 1$. This plane augmented with the plane at infinity $\Sigma_0^n x_i = 0$, will then serve as a model for $\mathbb{P}^n$.

For $X = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, put

$$\sigma(X) = \sum_0^n x_k \tag{1}$$

Further, for $X_0, \ldots, X_n \in \mathbb{R}^{n+1}$, put

$$\delta(X_0, \ldots, X_n) = \begin{cases} \frac{\det(X_0,\ldots,X_n)}{\sigma(X_0)\ldots\sigma(X_n)} & \text{if } \sigma(X_0)\ldots\sigma(X_n) \neq 0 \\ 0 & \text{if } \det(X_0, \ldots, X_n) = 0 \\ \infty & \text{if } \det(X_0, \ldots, X_n) \neq 0, \\ & \quad \sigma(X_0)\ldots\sigma(X_n) = 0 \end{cases} \tag{2}$$

Here one notices that, by homogeneity, δ is in fact a function on $\mathbb{P}^n \times \ldots \times \mathbb{P}^n$ ($n+1$ times). This fact alone does not qualify it to be a meaningful object in projective geometry. For this also some sort of projectivity invariance is needed. Clearly δ standing for itself does not have such a property. However, there are equations involving several δ-expressions which have (cf. Theorems 1 and 3). Thus, when appearing together with others in such an equation, δ gets a projective meaning. Here it can in fact also be interpreted in terms of affine areas.

The particular choice of $\pi : \sum_0^n x_i = 1$ may seem somewhat arbitrary at first sight. However, the discussion above may also be formulated in terms of coordinate changes instead of mappings. (Both operations correspond to premultiplication by a matrix.) In the new coordinate system the role played by π will be played by another plane. (In particular, premultiplication by a diagonal matrix will correspond to a perspectivity between the two planes.) The invariance equations mentioned above will relate, in the old and new planes, the volumes associated to a given set of $(n+1)$-tuples. When dealing with invariancy, it is thus no restriction to consider the particular plane π only.

The following definitions will be used. (The postfix "ad" in the first one is borrowed from Veblen-Young.) Cf. [8] vol II p. 55 or [4] p. 266 ff. for Definition 3.

DEFINITION 1

By a *polyad* in $\mathbb{P}^n$ is meant a non-degenerate ordered $(n+1)$-tuple of points in $\mathbb{P}^n$ $\mathcal{A} = (A_0, \ldots, A_n)$. If n=1,2,3 also the terms *dyad*, *triad*, and *tetrad* will be used. □

DEFINITION 2

The *volume* of the polyad is defined by eqs. (2.1) and (2.2). For dyads and triads the terms *length* and *area* will also be used. □

DEFINITION 3

For given points X, Y and a polyad $\mathcal{A}$ in $\mathbb{P}^n$, the following collection of *cross-ratios* is formed for $i, j = 0, \ldots, n$, $i \neq j$,

$$k_{ij} = k_{ij}(X, Y; \mathcal{A}) = k_{ij}(X, Y; A_0, \ldots, A_n) =$$

$$= \frac{\delta(A_0, \ldots, \overset{i}{X}, \ldots, A_n)}{\delta(A_0, \ldots, \underset{j}{X}, \ldots, A_n)} \Big/ \frac{\delta(A_0, \ldots, \overset{i}{Y}, \ldots, A_n)}{\delta(A_0, \ldots, \underset{j}{Y}, \ldots, A_n)} \tag{3}$$

In particular, if $n = 1$, we recognize k_{10} as the ordinary cross-ratio, also denoted

$$CR(X, Y; A_0, A_1) = k_{10} = \frac{\delta(A_0, X)}{\delta(X, A_1)} \Big/ \frac{\delta(A_0, Y)}{\delta(Y, A_1)} \tag{4}$$

The cases when the values 0 or ∞ appear somewhere are treated by the natural limit conventions. □

THEOREM 1

The cross-ratios k_{ij} are invariant under projectivities. □

Proof: To fix the ideas let $n = 2$ and consider k_{12}. Let $T \in PGL(2)$ and $X, Y, A_0, A_1, A_2 \in \mathbb{P}^2$. Suppose at first that no three of the points are collinear and that none of them or their images under T is a point at infinity. Fixing representatives of the points and the projectivity, one has

$$k_{12}(TX, TY; TA_0, TA_1, TA_2) = \frac{\delta(TA_0, TX, TA_2)}{\delta(TA_0, TA_1, TX)} \Big/ \frac{\delta(TA_0, TY, TA_2)}{\delta(TA_0, TA_1, TY)} =$$

$$\frac{\frac{\det T \det(A_0, X, A_2)}{\sigma(TA_0)\sigma(TX)\sigma(TA_2)}}{\frac{\det T \det(A_0, A_1, X)}{\sigma(TA_0)\sigma(TA_1)\sigma(TX)}} \Big/ \frac{\frac{\det T \det(A_0, Y, A_2)}{\sigma(TA_0)\sigma(TY)\sigma(TA_2)}}{\frac{\det T \det(A_0, A_1, Y)}{\sigma(TA_0)\sigma(TA_1)\sigma(TY)}} = \frac{\det(A_0, X, A_2)}{\det(A_0, A_1, X)} \Big/ \frac{\det(A_0, Y, A_2)}{\det(A_0, A_1, Y)}$$

The last expression also equals $k_{12}(X, Y; A_0, A_1, A_2)$, which is shown in the same way. This proves the theorem under the imposed extra assumptions. By limit considerations the result is proved for general projectivities. □

3. The Projective Line

In this section we derive certain projectivity invariant relations between the lengths of certain intervals on the line, where the cross-ratio will be one special case. Relying on a geometric picture, we treat perspectivities by themselves.

Perspectivities

Let ℓ, ℓ' be two lines and $O \notin \ell \cup \ell'$ a point in the affine plane, cf. Figure 2. Augmented with points at infinity, the lines may be thought of as models for $\mathbb{P}^1$. Let P', A', B', with $A' \neq B'$, be three points on ℓ'. P' may coincide with A' or B'. Under the perspectivity with center O the points P',A',B' are mapped on P,A,B respectively. Let p be the point on ℓ corresponding to the point at infinity on ℓ', i.e. such that Op is parallel to ℓ'.

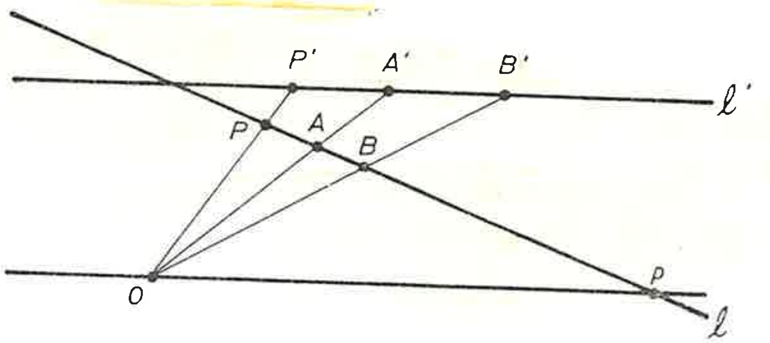

Figure 2.

By a *dilation* on ℓ' with center P' and scale t is meant a mapping

$$H'_{P'} : X' \to X'_t \quad \text{where} \quad \overline{P'X'_t} = t\overline{P'X'} \tag{5}$$

([1] and [2] serve as general references on the transformations appearing, where [2] emphasizes the group theoretic point of view.) This can also be expressed as $\overline{OX'_t} = (1-t)\overline{OP'} + t\overline{OX'}$. In particular

$$\overline{OA'_t} = (1-t)\overline{OP'} + t\overline{OA'}, \quad \overline{OB'_t} = (1-t)\overline{OP'} + t\overline{OB'}$$

Then e.g. $A'_0 = P'$, $A'_1 = A'$. Let A_t, B_t be the corresponding points on ℓ. Our aim is to derive, for a set of values of t, certain relations between the lengths of the dyads (A_t, B_t). These relations shall be valid for *any* perspective image of ℓ'.

On ℓ', let $\overline{OP'} = \lambda\overline{OA'} + \mu\overline{OB'}$, $\lambda + \mu = 1$. Then

$$\overline{OA'_t} = ((1-t)\lambda + t)\overline{OA'} + (1-t)\mu\overline{OB'}, \quad \overline{OB'_t} = (1-t)\lambda\overline{OA'} + ((1-t)\mu + t)\overline{OB'}$$

Now fix a coordinate system O, $\overline{OA} = \frac{1}{\alpha}\overline{OA'}$, $\overline{OB} = \frac{1}{\beta}\overline{OB'}$, in the plane, cf. Figure 2. The line ℓ then has the equation $x + y = 1$. In the corresponding homogeneous coordinates holds

$$A_t = ((1-t)\lambda\alpha + t\alpha, (1-t)\mu\beta), \quad B_t = ((1-t)\lambda\alpha, (1-t)\mu\beta + t\beta)$$

Application of δ (Definition 2) yields

$$\delta(A_t, B_t) = \frac{1}{((1-t)(\lambda\alpha+\mu\beta)+t\alpha)((1-t)(\lambda\alpha+\mu\beta)+t\beta)} \cdot$$

$$\cdot \begin{vmatrix} (1-t)\lambda\alpha + t\alpha & (1-t)\lambda\alpha \\ (1-t)\mu\beta & (1-t)\mu\beta + t\beta \end{vmatrix} \delta(A_1, B_1)$$

Here the unit $\delta(A_1, B_1)$ is included for homogeneity reasons. To achieve homogeneity in t also, we replace A_1, B_1 with A_{t_0}, B_{t_0} and substitute t/t_0 for t. An algebraic computation gives

$$((t_0-t)^2\tilde{S}_1^2 + t(t_0-t)S_1\tilde{S}_1 + t^2 S_2) \cdot \delta(A_t, B_t) = S_2 t t_0 \cdot \delta(A_{t_0}, B_{t_0}) \tag{6}$$

where

$$S_1 = \alpha + \beta, \quad S_2 = \alpha\beta, \quad \tilde{S}_1 = \lambda\alpha + \mu\beta \tag{7}$$

Equivalently we may write

$$(a_t\alpha^2 + b_t\alpha\beta + c_t\beta^2) \cdot \delta(A_t, B_t) = t t_0 \alpha\beta \cdot \delta(A_{t_0}, B_{t_0}) \tag{8}$$

where

$$a_t = \lambda(t_0-t)((t_0-t)\lambda + t),\ b_t = 2(t_0-t)^2\lambda\mu + t t_0,\ c_t = \mu(t_0-t)((t_0-t)\mu + t) \tag{9}$$

Lemma 1

With the notation introduced above, for any perspective image of ℓ' hold the relations:

(i) If $\lambda = 1$, $\mu = 0$ (i.e. $P = A = A_t$, $t \in \mathbb{R}$), then

$$\frac{t_0(t_1-t_2)}{\delta(A, B_{t_0})} + \frac{t_1(t_2-t_0)}{\delta(A, B_{t_1})} + \frac{t_2(t_0-t_1)}{\delta(A, B_{t_2})} = 0 \tag{10}$$

(ii) If $\lambda = \mu = 1/2$, then

$$\frac{t_0(t_1^2-t_2^2)}{\delta(A_{t_0}, B_{t_0})} + \frac{t_1(t_2^2-t_0^2)}{\delta(A_{t_1}, B_{t_1})} + \frac{t_2(t_0^2-t_1^2)}{\delta(A_{t_2}, B_{t_2})} = 0 \tag{11}$$

(iii) For general λ, μ holds, using the notation $\delta_{t_i} = \delta(A_{t_i}, B_{t_i})$,

$$\begin{vmatrix} a_{t_1} & b_{t_1}/\delta_{t_0} - t_0t_1/\delta_{t_1} \\ a_{t_2} & b_{t_2}/\delta_{t_0} - t_0t_2/\delta_{t_2} \end{vmatrix} \begin{vmatrix} b_{t_1}/\delta_{t_0} - t_0t_1/\delta_{t_1} & c_{t_1} \\ b_{t_2}/\delta_{t_0} - t_0t_2/\delta_{t_2} & c_{t_2} \end{vmatrix} - \frac{1}{\delta_{t_0}^2} \begin{vmatrix} a_{t_1} & c_{t_1} \\ a_{t_2} & c_{t_2} \end{vmatrix}^2 = 0 \tag{12}$$

□

Proof: The proof relies on (6) or equivalently (8). Since the perspectivity is uniquely determined by the non-zero numbers α and β, "invariants under perspectives" must be independent of α, β. Writing $m_t = \delta(A_t, B_t)/\delta(A_{t_0}, B_{t_0})$ the basic formula (6) becomes

$$(t_0-t)^2\tilde{S}_1^2 + t(t_0-t)S_1\tilde{S}_1 + t(t - \frac{t_0}{m_t})S_2 = 0 \tag{13}$$

Although (i) and (ii) are special cases of (iii), we prefer to treat them separately.

(i) If $\lambda = 1, \mu = 0$, then (13) simplifies into $(t_0 - t)\alpha + t(1 - \frac{1}{m_t})\beta = 0$. Putting together two such equations, corresponding to $t = t_1$ and $t = t_2$, one gets a homogeneous system of linear equations in the unknowns α and β. This system is known to have a nontrivial solution, determined by the geometrical construction above. Hence the determinant of the system is zero, i.e.

$$\begin{vmatrix} t_0 - t_1 & t_1(1 - \frac{1}{m_{t_1}}) \\ t_0 - t_2 & t_2(1 - \frac{1}{m_{t_2}}) \end{vmatrix} = 0$$

Expansion of the determinant directly gives (10).

(ii) If $\lambda = \mu = 1/2$ then $\tilde{S}_1 = S_1/2$ which simplifies (13) into $(t_0^2 - t^2)S_1^2 + 4t(t - \frac{t_0}{m_t})S_2 = 0$. By the same argument as above, combination of two such equations yields

$$\begin{vmatrix} t_0^2 - t_1^2 & 4t_1(t_1 - \frac{t_0}{m_{t_1}}) \\ t_0^2 - t_2^2 & 4t_2(t_2 - \frac{t_0}{m_{t_2}}) \end{vmatrix} = 0$$

Expansion of the determinant gives (11).

(iii) Combination of two equations (8), corresponding to t_1 and t_2, gives a system of two homogeneous polynomial equations of second order in α, β. This system is known to have a non-trivial solution. A well known result from elimination theory, cf. [9] Ch XI or [3] Ch IV, says that this happens if and only if the resultant of the system vanishes. But here, in the case of two variables, the resultant can be written down explicitly as

$$\begin{vmatrix} a_{t_1} & b_{t_1} - t_0 t_1 \delta_{t_0}/\delta_{t_1} & c_{t_1} & 0 \\ 0 & a_{t_1} & b_{t_1} - t_0 t_1 \delta_{t_0}/\delta_{t_1} & c_{t_1} \\ a_{t_2} & b_{t_2} - t_0 t_2 \delta_{t_0}/\delta_{t_2} & c_{t_2} & 0 \\ 0 & a_{t_2} & b_{t_2} - t_0 t_2 \delta_{t_0}/\delta_{t_2} & c_{t_2} \end{vmatrix} = 0 \tag{14}$$

This determinant is easily rewritten as (12). □

Remark 1. The case (i) is in fact the ordinary *cross-ratio* relation for CR (cf. Definition 3 and Theorem 1). Note that the sum of the nominators in (10) is zero. Hence

$$t_1(t_2 - t_0)(\frac{1}{\delta(P, B_{t_1})} - \frac{1}{\delta(P, B_{t_0})}) + t_2(t_0 - t_1)(\frac{1}{\delta(P, B_{t_2})} - \frac{1}{\delta(P, B_{t_0})}) = 0$$

or, equivalently (cf. Figure 3)

$$\frac{\delta(P, B_{t_0}) - \delta(P, B_{t_2})}{\delta(P, B_{t_2})} / \frac{\delta(P, B_{t_0}) - \delta(P, B_{t_1})}{\delta(P, B_{t_1})} = \frac{t_0 - t_2}{t_2} / \frac{t_0 - t_1}{t_1} \tag{15}$$

But here $\delta(P, B_{t_0}) - \delta(P, B_{t_1}) = \delta(B_{t_1}, B_{t_0})$, $\delta(P, B_{t_0}) - \delta(P, B_{t_2}) = \delta(B_{t_2}, B_{t_0})$ by the geometric interpretation of δ. The left hand side in (15) is thus the cross-ratio (4), and we have reproved that it is invariant under perspectivities. (The way of expressing the invariance of cross-ratios by a formula like (10) was known already by Möbius [4], "Von der metrischen Relationen im Gebiete der Lineal-Geometrie" (1829).) □

Remark 2. The case (ii) relates the lengths of perspective images of dyads with a common center. For reasons that will be apparent in the next chapter (cf. the remark after Theorem 3), we will call this the *polar* case. The polar case (ii) and the cross-ratio case (i) are not as independent as they may seem. In fact (i) can be derived from (ii)

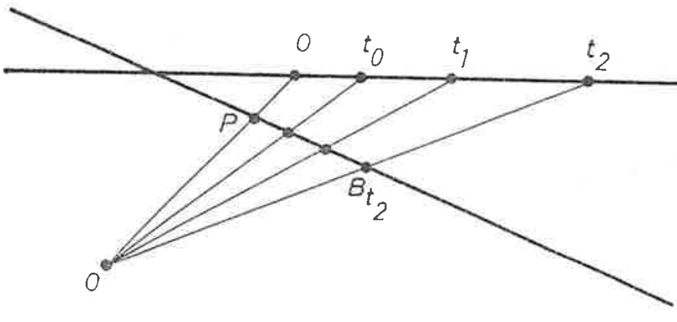

Figure 3.

as a limit case $P \to A$, $p \to A$. Likewise (ii) can be derived from (i). We prove the latter statement using a process that also works in the plane, cf. quadrangles in the next section. On ℓ' in Figure 4, change the notation B', B'_t into B'^+, B'^+_t, and introduce corresponding points B'^-, B'^-_t symmetrically spaced around A'. With obvious notations on ℓ, the equation (13) for case (i) may be written (it suffices to consider the case $t_0 = 1$)

$$\frac{\delta(A, B_t^+)}{\delta(A, B^+)} = \frac{\alpha\beta^+ t}{(1-t)\alpha + t\beta^+}, \qquad \frac{\delta(A, B_t^-)}{\delta(A, B^-)} = \frac{\alpha\beta^- t}{(1-t)\alpha + t\beta^-}$$

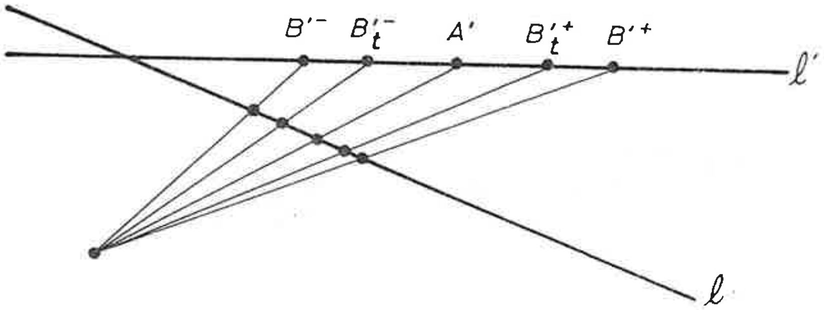

Figure 4.

The fact that $\delta(A', B'^+) = -\delta(A', B'^-)$ yields

$$\alpha\beta^+\delta(A, B^+) = -\alpha\beta^-\delta(A, B^-), \qquad \beta^+ + \beta^- = 2\alpha$$

Denoting the common value of the members of the first equation by c we get

$$\delta(B_t^-, B_t^+) = \delta(A, B_t^+) - \delta(A, B_t^-) =$$
$$ct\left(\frac{1}{(1-t)\alpha + t\beta^+} + \frac{1}{(1-t)\alpha + t\beta^-}\right) = c\frac{4S_1 t}{(1-t^2)S_1^2 + 4t^2 S_2}$$

with $S_1 = \beta^+ + \beta^-$, $S_2 = \beta^+\beta^-$. Insertion of $t = 1$ gives $c = S_2\delta(B^-, B^+)/S_1$, which in turn gives eq (13) for case (ii). □

Remark 3. The cases (i) and (ii) are the only situations where (iii) reduces to a linear relation in $1/\delta_i$. This happens if and only if completion of squares in (12), as a quadratic form in $1/\delta_i$, gives only two quadratic expressions of different signs. This in turn happens if and only if either the first term in (12) is a square in itself or the second term cancels. In both cases the condition is that

$$\begin{vmatrix} a_{t_1} & c_{t_1} \\ a_{t_2} & c_{t_2} \end{vmatrix} = 0$$

for all t_1, t_2. We obtain $\lambda = 0$ or $\mu = 0$ or $\lambda = \mu$. □

Remark 4. Two points A, B divide the projective line into two "intervals". One of these, the one that not contains the point at infinity, may be called the "finite" one. Under a perspectivity a finite interval may be mapped onto a non-finite one. This situation is reflected by a change of signs in δ, but does not alter the validity of the lemma. It is in order to avoid such considerations, irrelevant for the invariants, that we talk about dyads instead of intervals. These aspects are still more accentuated in higher dimensions. □

Projectivities. Homologies The formulas of Lemma 1 remain true when a projectivity is applied to all appearing points A_t, B_t. The proof is a simpler version of the proof for the plane given in the next section and is omitted here.

THEOREM 2
The cases (i), (ii), and (iii) of Lemma 1 describe invariants under projectivities (i.e. the equations remain valid when applying a projectivity to all points involved). □

Remark. One could also consider *translations*, corresponding to $P' = \infty$. By a perspectivity $\ell' \to \ell$, every translation on ℓ' is transferred to a projectivity on ℓ. This projectivity has a single fixed point and is thus a *parabolic projectivity*. It is also associated to the *elations* in the plane case. By means of a suitable limit process every such elation may be parameterized by the same τ as was used on ℓ'. However, since the result does not hold for the full group of projectivities (but only for a group of elations), we do not develop this case any further. □

4. The Projective Plane

This section is devoted to projectivity invariant relations between the areas of certain triangles in the plane. We will also consider quadrangles. As in the previous section we start with perspectivities.

Let π, π' be two distinct planes and $O \notin \pi \cup \pi'$ a point in the three-dimensional affine space. Augmented with lines at infinity, the planes may be thought of as models for $\mathbb{P}^2$. Let P', A', B', C', be four points in π' with A', B', C' non-colinear. Under the perspectivity from π' to π with center O, the points P', A', B', C' are mapped on P, A, B, C respectively. Let p denote the line in π such that Op is parallel to π'. It thus corresponds to the line at infinity in π'.

Our construction of invariants is based on the same ideas as in Section 3. A *dilation* on π' with center P' and scale t is as before defined by

$$H'_{P'} : X' \to X'_t \quad \text{where} \quad \overline{P'X'_t} = t\overline{P'X'} \tag{16}$$

It follows that

$$\overline{OA'_t} = (1-t)\overline{OP'} + t\overline{OA'},\ \overline{OB'_t} = (1-t)\overline{OP'} + t\overline{OB'},\ \overline{OC'_t} = (1-t)\overline{OP'} + t\overline{OC'}$$

On π', there exist λ, μ, ν (barycentric coordinates) such that

$$\overline{OP'} = \lambda\overline{OA'} + \mu\overline{OB'} + \nu\overline{OC'}, \qquad \lambda + \mu + \nu = 1 \tag{17}$$

Then

$$\begin{aligned}
\overline{OA'_t} &= ((1-t)\lambda + t)\,\overline{OA'} + (1-t)\mu\overline{OB'} + (1-t)\nu\overline{OC'} \\
\overline{OB'_t} &= (1-t)\lambda\overline{OA'} + ((1-t)\mu + t)\,\overline{OB'} + (1-t)\nu\overline{OC'} \\
\overline{OC'_t} &= (1-t)\lambda\overline{OA'} + (1-t)\mu\overline{OB'} + ((1-t)\nu + t)\,\overline{OC'}
\end{aligned}$$

Let A_t, B_t, C_t be the points in π corresponding to A'_t, B'_t, C'_t in π'. Fix the coordinate system O, $\overline{OA} = \frac{1}{\alpha}\overline{OA'}$, $\overline{OB} = \frac{1}{\beta}\overline{OB'}$, $\overline{OC} = \frac{1}{\gamma}\overline{OC'}$, for the space. The plane π then has the equation $x + y + z = 1$. In the corresponding homogeneous coordinates holds

$$\begin{aligned}
A_t &= (((1-t)\lambda + t)\,\alpha, (1-t)\mu\beta, (1-t)\nu\gamma) \\
B_t &= ((1-t)\lambda\alpha, ((1-t)\mu + t)\,\beta, (1-t)\nu\gamma) \\
C_t &= ((1-t)\lambda\alpha, (1-t)\mu\beta, ((1-t)\nu + t)\,\gamma)
\end{aligned}$$

Application of δ and introduction of t_0 and $\delta(A_{t_0}, B_{t_0}, C_{t_0})$ yields, in the same way as in Section 3,

$$\begin{aligned}
((t_0 - t)^3\tilde{S}_1^3 + t(t_0 - t)^2\tilde{S}_1^2 S_1 + t^2(t_0 - t)\tilde{S}_1 S_2 + t^3 S_3) \cdot \delta(A_t, B_t, C_t) &= \\
t^2 t_0 S_3 \delta(A_{t_0}, B_{t_0}, C_{t_0})&
\end{aligned} \tag{18}$$

where $S_1 = \alpha + \beta + \gamma, \tilde{S}_1 = \lambda\alpha + \mu\beta + \nu\gamma, S_2 = \alpha\beta + \beta\gamma + \gamma\alpha, S_3 = \alpha\beta\gamma$.

In the following lemma a number of special cases for λ, μ, ν, single out naturally. Introduce first the notation

$$\begin{aligned}
h_3(t_1, t_2, t_3) &= t_1^2 t_2^3 - t_1^3 t_2^2 + t_2^2 t_3^3 - t_2^3 t_3^2 + t_3^2 t_1^3 - t_3^3 t_1^2 \\
&= (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 t_2 + t_2 t_3 + t_3 t_1)
\end{aligned}$$

(Here the subscript 3 refers to triads, cf. quadrangles below)

Lemma 2

With the notation introduced above, for any perspective image of the configuration in π' hold the relations (cf. Figure 5):

(i) If $\lambda = 1$, $\mu = \nu = 0$ (i.e. $P = A = A_t$, $t \in \mathbb{R}$), then

$$\begin{aligned}
&\frac{t_0^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)}{\delta(A, B_{t_0}, C_{t_0})} - \frac{t_1^2(t_2 - t_3)(t_3 - t_0)(t_0 - t_2)}{\delta(A, B_{t_1}, C_{t_1})} + \\
&\frac{t_2^2(t_3 - t_0)(t_0 - t_1)(t_1 - t_3)}{\delta(A, B_{t_2}, C_{t_2})} - \frac{t_3^2(t_0 - t_1)(t_1 - t_2)(t_2 - t_0)}{\delta(A, B_{t_3}, C_{t_3})} = 0
\end{aligned} \tag{19}$$

(ii) If $\lambda = \mu = \nu = 1/3$ (i.e. P' = the center of A', B', C') then

$$\begin{aligned}
&\frac{t_0^2 h_3(t_1, t_2, t_3)}{\delta(A_{t_0}, B_{t_0}, C_{t_0})} - \frac{t_1^2 h_3(t_2, t_3, t_0)}{\delta(A_{t_1}, B_{t_1}, C_{t_1})} + \\
&\frac{t_2^2 h_3(t_3, t_0, t_1)}{\delta(A_{t_2}, B_{t_2}, C_{t_2})} - \frac{t_3^2 h_3(t_0, t_1, t_2)}{\delta(A_{t_3}, B_{t_3}, C_{t_3})} = 0
\end{aligned} \tag{20}$$

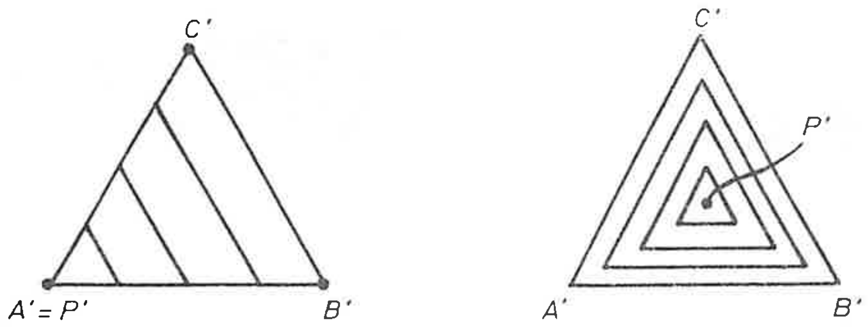

Figure 5.

(iii) For general λ, μ, ν holds

$$R(\overline{\lambda}, \overline{t}, 1/\overline{\delta}) = 0 \tag{21}$$

with $\overline{\lambda} = (\lambda, \mu, \nu)$, $\overline{t} = (t_0, t_1, t_2, t_3)$, $1/\overline{\delta} = (1/\delta_{t_0}, 1/\delta_{t_1}, 1/\delta_{t_2}, 1/\delta_{t_3})$, where R is the resultant of the system of three equations (18) corresponding to t_1, t_2, t_3. R is homogeneous of degree 6 in $1/\overline{\delta}$. □

Proof: The proof follows the same steps as in Theorem 2, and is omitted here. The proof for $\lambda = \mu = \nu = 1/3$ has also been given in [6]. □

Projectivities. Homologies

We note that the δ:s in Lemma 2 represent dignities (certain areas) of the plane π itself, inherited from its affine structure, without reference to the particular perspectivity used in the proof. It remains to characterize also the parameters t and λ, μ, ν appearing in the nominators by means of intrinsic properties of π only.

We recapitulate some notions from plane projective geometry. As a model for $\mathbb{P}^2$ we use the augmented plane π in the three-dimensional affine space. A subclass of the projectivities will play a particular role in the sequel. Thus, let P be a point and p a line in π. By a ***perspective collineation*** with ***center*** P and ***axis*** p is meant a projectivity leaving fixed every point on p and every line on (=through) P. In particular, if $P \notin p$ the collineation is called a ***homology***, and if $P \in p$ an ***elation***.

It is well-known that a homology is uniquely determined by its center P and axis p, together with one point Q and its image $\tilde{Q}$, cf. [1] p 53. For later reference we repeat the proof, beginning with the uniqueness. Here and in the sequel we denote the intersection of the lines a and b by $a \cdot b$.

For any Y, by the invariance of the lines on P, the image $\tilde{Y}$ lies on the line PY, cf. Figure 6. On the other hand, if $Y \notin Q\tilde{Q}$ then by the invariance of $Y_p = QY \cdot p$, the line $QY = Y_pY$ is mapped onto the line $Y_p\tilde{Q}$. It follows that $\tilde{Y} = PY \cdot Y_p\tilde{Q}$, uniquely. The case $Y \in Q\tilde{Q}$ is treated by repeated use of this argument, first using the known property $Q \to \tilde{Q}$ to construct a pair $Z \to \tilde{Z}$ with $Z \notin Q\tilde{Q}$, then using the property $Z \to \tilde{Z}$ to construct $Y \to \tilde{Y}$.

The existence of such a homology, and a bit more, can be proved by means of a perspectivity $\pi \to \pi'$. Choose O and π' so that p corresponds to the line at infinity in π'. Let Q and $\tilde{Q}$ in π correspond to Q' and $\tilde{Q}'$ in π'. Then it is possible to find a value of t such that, cf. (16), $H^t_{P'} : Q' \to \tilde{Q}'$. Letting "persp" stand for "perspectivity with

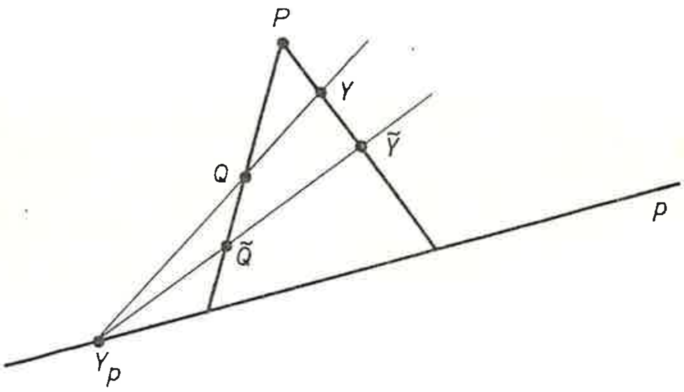

Figure 6.

center O", a mapping $X \to X_t$ in π is defined by the diagram

$$\begin{array}{ccc} \pi' & \xrightarrow{H^t_{P'}} & \pi' \\ \uparrow \text{persp} & & \downarrow \text{persp} \\ \pi & \longrightarrow & \pi \end{array}$$

Since the dilation $H^t_{P'}$ on π' may be described as a homology with center P' and with the line at infinity as axis, the composite map in the diagram is a homology with center P and axis p. Moreover it maps Q onto $\tilde{Q}$. Hence, the existence of a homology with the stated properties is established.

The proof also indicates the possibility to parameterize the set of homologies with a given center and axis. In fact, letting ∞' denote the line at infinity in π', $H^t_{P'}$ is characterized by

$$CR(X'_t, X'; P', \infty' \cdot P'X') = t$$

The invariance of cross-ratios under perspectives then legitimates the following alternate definition of homologies.

DEFINITION 4
By the *homology* in π with *center* P, *axis* p, *scale* t, is meant the mapping $H^t_{P,p} : X \to X_t$, where X_t is the unique point on PX determined by $CR(X_t, X; P, p \cdot PX) = t$. For a given triad (A, B, C) the set of all triads (A_t, B_t, C_t) is called the *homological range* of (A, B, C) and is denoted by $\mathcal{H}_{P,p}(A, B, C)$. □

Given A', B', C' in π', the barycentric coordinates of P' are obtained by solving (17) for λ, μ, ν. Cramer's rule gives

$$\lambda = \frac{det(P', B', C')}{det(A', B', C')}, \quad \mu = \frac{det(A', P', C')}{det(A', B', C')}, \quad \nu = \frac{det(A', B', P')}{det(A', B', C')}$$

In terms of the cross-ratios of Definition 3 and by the conventions for treating points at infinity, one checks that

$$\lambda = k_{01}(P', A'; A'P' \cdot \infty', B', C')$$

$$\mu = k_{12}(P', B'; A', B'P' \cdot \infty', C')$$

$$\nu = k_{20}(P', C'; A', B', C'P' \cdot \infty')$$

The invariance of cross-ratios under perspectivities leads to

DEFINITION 5
By the *configuration coefficients* of $(P, p; A, B, C)$ are meant

$$\lambda = k_{01}(P, A; AP \cdot p, B, C)$$

$$\mu = k_{12}(P, B; A, BP \cdot p, c)$$

$$\nu = k_{20}(P, C; A, B, CP \cdot p)$$

□

(Note that, contrary to π', in π the configuration coefficients have no interpretation as barycentric coordinates.) By means of the uniqueness of homologies (stated above) and another reference to Theorem 1, we obtain

LEMMA 3
Let T be a projectivity on π with $\tilde{P} = TP$, $\tilde{p} = Tp$. Then the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{T} & \tilde{X} \\ \Big\downarrow{\scriptstyle H^t_{P,p}} & & \Big\downarrow{\scriptstyle H^t_{\tilde{P},\tilde{p}}} \\ X_t & \xrightarrow{T} & \tilde{X}_t \end{array}$$

If $TA = \tilde{A}$, $TB = \tilde{B}$, $TC = \tilde{C}$ it follows that $(P, p; A, B, C)$ and $(\tilde{P}, \tilde{p}; \tilde{A}, \tilde{B}, \tilde{C})$ have the same configuration coefficients and that $T : \mathcal{H}_{P,p}(A, B, C) \to \mathcal{H}_{\tilde{P},\tilde{p}}(\tilde{A}, \tilde{B}, \tilde{C})$, $T : (A_t, B_t, C_t) \to (\tilde{A}_t, \tilde{B}_t, \tilde{C}_t)$. □

Summing up, Lemma 2 gives homogeneous relations between the areas of triads belonging to a particular homological range $\mathcal{H}_{P,p}(A, B, C)$ on π, with configuration coefficients λ, μ, ν. Lemma 3 says that projectivities on π transfer homological ranges onto homological ranges, without altering t and λ, μ, ν. We have thus proved

THEOREM 3
Let $(A_{t_i}, B_{t_i}, C_{t_i}) \in \mathcal{H}_{P,p}(A, B, C)$, $i = 0, 1, 2, 3$, and let λ, μ, ν be the configuration coefficients of $(P, p; A, B, C)$. Then the cases (i), (ii), and (iii) of Lemma 2 describe invariants under projectivities. □

Remark. The polar case. The case (ii) $\lambda = \mu = \nu = 1/3$ has some special features. Let P be a point and (A, B, C) a triangle in a plane π. A new triangle (A_1, B_1, C_1) is defined by $A_1 = PA \cdot BC$, $B_1 = PB \cdot CA$, $C_1 = PC \cdot AB$. The triangles (A, B, C) and (A_1, B_1, C_1) are then perspective from P. By Desargue's theorem this happens if and only if they also are perspective from a line p. (This means that the points of intersection $AB \cdot A_1B_1$, $BC \cdot B_1C_1$ and $CA \cdot C_1A_1$ all lie on p.) The situation is described by saying that P and p are *pole* and *polar* with respect to (A, B, C), cf. Figure 7 and [1]

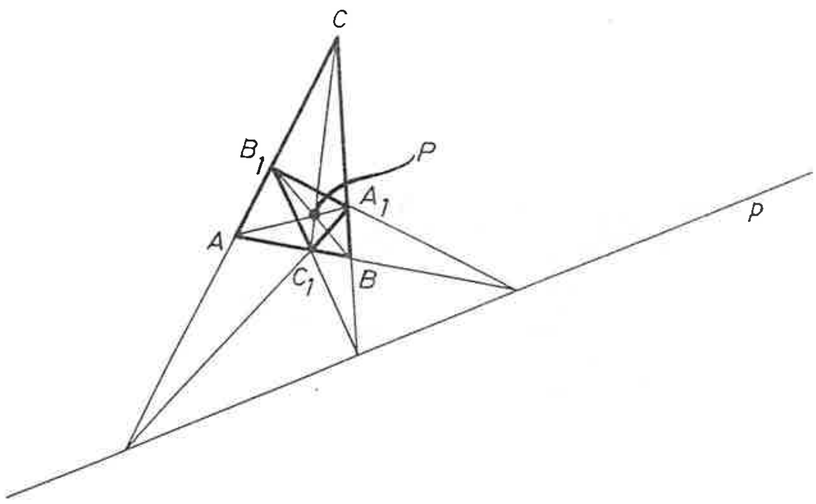

Figure 7.

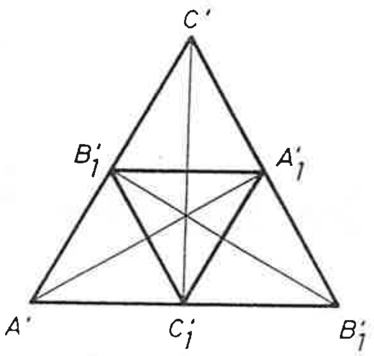

Figure 8.

p 29. The corresponding λ, μ, ν are found by means of the invariance of the pole/polar property under perspectivities. The perspectivity $\pi \to \pi'$ maps the polar p onto the line at infinity, $P \to P'$, $(A, B, C) \to (A', B', C')$, and $(A_1, B_1, C_1) \to (A_1', B_1', C_1')$, (cf. Figure 8). Hence P' is pole and the line at infinity is polar with respect to the triangle (A', B', C'). By means of similar triangles and medians one finds that P' is the center of the triangle (A, B, C) i.e. $OP' = (OA' + OB' + OC')/3$. This shows that $\lambda = \mu = \nu = 1/3$. For this reason we refer to (ii) as the *polar case*. $\square$

Quadrangles

Generally speaking, by a *simple k-point* in $\mathbb{P}^n$ is meant an ordered k-tuple of points in $\mathbb{P}^n$. If $k = n + 1$ it is a polyad, and if $k < n + 1$ it may be considered as a polyad in a k-dimensional projective subspace of $\mathbb{P}^n$. Since polyads, here triads, are treated in the main line of this work, only the case $k > n + 1$ remains to be studied. By means of Definition 2, in a natural way one associates an "area" to every simple k-point in $\mathbb{P}^2$ by

$$\Delta(X_1, \ldots, X_k) = \delta(X_1, X_2, X_3) + \delta(X_1, X_3, X_4) + \ldots + \delta(X_1, X_{k-1}, X_k)$$

(cf. [8] vol II, p 104 for the affine case). For triads $k = 3$ we know from Theorem 3 that there exist area-invariants. The natural question arises whether this is true for $k > 3$.

We will consider the case $k = 4$ in a particular situation, reminding of the polar case (ii) in Theorem 3. Starting as usual in an affine plane π', let A', B', C', D' be a parallelogram and let P' be the intersection of its diagonals (cf. Figure 9).

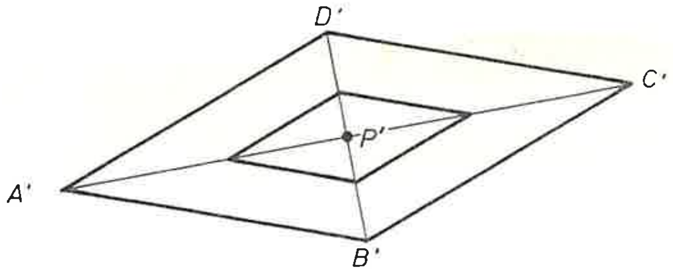

Figure 9.

By a dilation with center P', scale t, the points A'_t, B'_t, C'_t, D'_t are constructed. After a perspectivity $\pi' \to \pi$ one obtains the situation of Figure 10.

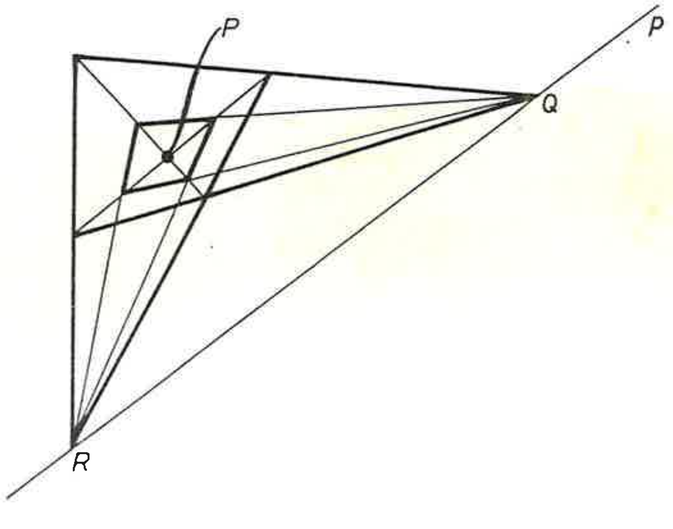

Figure 10.

Let A_t, B_t, C_t, D_t be the images of A'_t, B'_t, C'_t, D'_t, and let the line p in π correspond to the line at infinity in π'.

By considering the homological ranges $\mathcal{H}_{P,p}(A, B, C)$ and $\mathcal{H}_{P,p}(A, D, C)$ separately, the whole quadrangle may be treated. In both cases the configuration coefficients are $(1/2, 0, 1/2)$. Let

$$\overline{OA} = \frac{1}{\alpha}\overline{OA'}, \quad \overline{OB} = \frac{1}{\beta^+}\overline{OB'}, \quad \overline{OC} = \frac{1}{\gamma}\overline{OC'}, \quad \overline{OD} = \frac{1}{\beta^-}\overline{OD'}$$

Put $S_3^+ = \alpha\beta^+\gamma, \quad S_3^- = \alpha\beta^-\gamma$. Then

$$\Delta(A_t, B_t, C_t, D_t) = \delta(A_t, B_t, C_t) - \delta(A_t, D_t, C_t)$$

Computation of δ, as in (18), gives

$$\Delta(A_t, B_t, C_t, D_t) = \frac{t^2 S_3^+ \delta(A,B,C)}{\sigma(A_t)\sigma(B_t)\sigma(C_t)} - \frac{t^2 S_3^- \delta(A,D,C)}{\sigma(A_t)\sigma(D_t)\sigma(C_t)}$$

The facts that $\delta(A',B',C') = -\delta(A',D',C')$ and that P' is the midpoint of $A'C'$ and $B'D'$ yield

$$S_3^+\delta(A,B,C) = -S_3^-\delta(A,D,C), \quad \alpha + \gamma = \beta^+ + \beta^-$$

Denote by c the common value of the members of the first equation. Let s_1 be the common value of the second. Put $s_2 = \alpha\gamma + \beta^+\beta^-$, $s_4 = \alpha\gamma\beta^+\beta^-$. Then

$$\begin{aligned}\Delta(A_t, B_t, C_t, D_t) &= ct^2\left(\frac{1}{\sigma(A_t)\sigma(B_t)\sigma(C_t)} - \frac{1}{\sigma(A_t)\sigma(D_t)\sigma(C_t)}\right) \\ &= \frac{ct^2 s_1}{\sigma(A_t)\sigma(B_t)\sigma(C_t)\sigma(D_t)}\end{aligned}$$

Insertion of $t = 1$ gives $cs_1 = s_4\Delta(A,B,C,D)$. After a straightforward calculation we obtain the analogue of (18):

$$((1-t^2)^2 s_1^4 + 4t^2(1-t^2)s_1^2 s_2 + 16t^4 s_4)\Delta(A_t,B_t,C_t,D_t) = 16t^2 s_4 \Delta(A,B,C,D)$$

In analogy with h_3 above we define

$$h_4(t_1,t_2,t_3) = t_1^2t_2^4 - t_1^4t_2^2 + t_2^2t_3^4 - t_2^4t_3^2 + t_3^2t_1^4 - t_3^4t_1^2 = (t_1^2-t_2^2)(t_2^2-t_3^2)(t_3^2-t_1^2)$$

By a now familiar argument we obtain

LEMMA 4

For any perspective image of the configuration in π' holds the relation

$$\frac{t_0^2 h_4(t_1,t_2,t_3)}{\Delta_{t_0}} - \frac{t_1^2 h_4(t_2,t_3,t_0)}{\Delta_{t_1}} + \frac{t_2^2 h_4(t_3,t_0,t_1)}{\Delta_{t_2}} - \frac{t_3^2 h_4(t_0,t_1,t_2)}{\Delta_{t_3}} = 0 \qquad (22)$$

□

This is in fact an invariant under general projectivities. To prove this, one needs some invariant configuration property, replacing the configuration coefficients in Lemma 3. To this end one notices that P is a vertex and p the opposite side of the diagonal triangle P,Q,R of the complete quadrangle defined by A,B,C,D (cf. [1] Ch 2). Let us in this case say that $(P,p;A,B,C,D)$ is a *diagonal configuration.* This property is preserved under projectivities. Defining in a natural way the homological range $\mathcal{H}_{P,p}(A,B,C,D)$, an analogue of Lemma 3 holds true in this particular case. We obtain

THEOREM 4

Suppose that $(P,p;A,B,C,D)$ is a diagonal configuration. Let $(A_{t_i},B_{t_i},C_{t_i},D_{t_i}) \in \mathcal{H}_{P,p}(A,B,C,D)$, $i = 0,1,2,3$. Then the equation (22) is invariant uder projectivities.

□

Remark. Comparing (20) and 22, where in both cases the center of the figure was used as the center of the homology, one notes at least two common features. First, the number of figures needed were in both cases four, and second, the coefficients h_3 and

h_4 in the invariant formula have the same structure. The problem arises whether this can be generalized to general k-points. The answer is *no*, at least in the sense that the number of figures needed depends on k. This number is highly dependent on the symmetry properties of the figure. Calculations with a symbolic manipulation program have showed that for regular pentagons, $k = 5$, one needs nine and for regular hexagons, $k = 6$, six t-values (i.e. homological images of the reference k-point). □

5. Conical Area-invariants

The derivation of two-dimensional area-relations for regions enclosed by ellipses and, after suitable interpretation, general conic sections will repeated only briefly since the result for perspectivities was given in [6,7]. The interesting point here is the extension to the full projective group. We restrict ourselves to the analogue of the "polar case" in Section 4, i.e. when the center and axis of the homological range are pole and polar of the configurations considered.

Consider quadratic functions

$$q(x) = \frac{1}{2}x^T Qx + a^T x + b, \quad x \in \mathbb{R}^2,\ Q\,\text{symmetric}$$

If Q is non-singular, then

DEFINITION 6
By the *fundamental form* of q is meant

$$\alpha(q) = \begin{cases} \frac{a^T Q^{-1} a - 2b}{\sqrt{\det Q}} & \text{if } Q \text{ is nonsingular} \\ \infty & \text{otherwise} \end{cases}$$

□

The fundamental form $\alpha(q)$ can, apart from a factor, be interpreted as an area. Here α itself changes in an irregular way under projectivities. However, when grouping together a number of α:s in a particular equation, we will see that each of them allows a projectively meaningful interpretation as an area.

Perspectivities

Now consider a non-degenerate cone in the three-dimensional space. Let O be its vertex and let π, π' be two planes with $O \notin \pi \cup \pi'$, $\pi \neq \pi'$. Let $\ell = \pi \cap \pi'$. Two conics C and C' are defined by the intersections of the cone with π and π' respectively. Suppose that C' is an ellipse. Let P' be the center of C'. Let $\bar{e}'$ be the conjugate direction of ℓ with respect to C' (i.e. the direction determined by the locus of all midpoints of chords of C' parallel to ℓ). Choose the length of $\bar{e}'$ so that it can be represented by a directed segment connecting ℓ and P'. Then by classical theory of conics it is possible to choose $\bar{f}'//\ell$ so that, for some t,

$$C' : x'^2 + y'^2 = t^2 \tag{23}$$

For the perspective images of three such conics corresponding to three different t-values the following lemma holds. (See [7] for a proof.)

LEMMA 5

With the notation introduced above, for any perspectivity holds

$$\frac{t_1^2 - t_2^2}{(\frac{\alpha(q_0)}{t_0^2})^{2/3}} + \frac{t_2^2 - t_0^2}{(\frac{\alpha(q_1)}{t_1^2})^{2/3}} + \frac{t_0^2 - t_1^2}{(\frac{\alpha(q_2)}{t_2^2})^{2/3}} = 0 \tag{24}$$

Remark. At first sight this lemma only gives an analytical relation between the fundamental forms of three particular quadratic functions $q_i, i = 0, 1, 2$. However, since $\alpha(q_t)$ has an interpretation as the area connected with $C_t : q_t(x) = 0$, it changes only by a proportionality factor under affine coordinate transformations on π. Because of the homogeneity of (24), one may thus choose for q_i the polynomials defining C_{t_i} in *any* affine basis for π. □

Projectivities

Projectivities map conics onto conics. If C has the equation $q(x) = 0$, then the image under T has the equation $(Tq)(x) = 0$, where

$$(Tq)(x) = q(T^{-1}x)$$

Here it is preferable to work with homogeneous coordinates, since then the calculation of Tq can be done by means of matrix operations.

Equation (23) describes a family of conical sections, obtained from each other by dilations with center P'. After a perspectivity, the situation in π is described by homologies $H^t_{P,p}$, where P, p are the images of P' and the line at infinity, respectively. The *homological range* $\mathcal{H}_{P,p}(C)$ of conical sections is defined as for polyads.

The concepts of *pole* and *polar* are central in projective geometry, cf. e.g. [1] Ch. 8. In π' the center P' of C' and the line at infinity are pole and polar with respect to C'. These properties are preserved under perspectivities, i.e. P and p are pole and polar with respect to C. The situation is unaltered after any projectivity on π. For this particular pole-polar configuration it is thus possible to formulate an analogue of Lemma 3. Together with Lemma 5, cf. also the remark above, it yields:

THEOREM 5

Let $C_{t_i} \in \mathcal{H}_{P,p}(C)$, $i = 0, 1, 2$, where P and p are pole and polar with respect to C. Let C_{t_i} have the equation $q_i(x) = 0$. Then the formula in Lemma 5 is an invariant under projectivities (i.e. when replacing q_i by Tq_i). □

Remark. It is noteworthy that the number of terms in (24) is three, while it earlier in the plane has been at least four (cf. the remark after Theorem 4). □

6. Conclusions

The main contributions of this paper is the definition and justification of area-invariants in projective geometry, and the common frame-work from where the different invariants turn up as special cases. More specifically, there has been a complete characterization of invariants concerning lengths of intervals on the line. Only in two cases, case (i) and (ii) of Lemma 1, are the invariants linear. The first case is the well known cross-ratio, and the second case is what we call the polar case.

The generalization to the plane can be done in different directions. One can either view points (on the line or in the plane) as the basic entity, or one can view the geometric figures (intervals, triangles, circles) as the basic entity involved. The first view was adopted already by Möbius who generalized the cross-ratio, as was recalled in Theorem 1. The second view used here leads to another generalization of the cross-ratio, as in Theorems 3-5, where the invariants are relations between the areas of the separate geometric figures involved. Remarkably enough, we found that these invariants turn out to be linear if the figures involved are related in a pole/polar configuration.

7. References

[1] COXETER, H.S.M. (1974): *Projective Geometry*, University of Toronto Press, Toronto.

[2] HARTSHORNE, R. (1967): *Foundations of Projective Geometry*, Benjamin, New York.

[3] HODGE, W.V.D and PEDOE, D. (1947): *Methods of Algebraic Geometry*, Cambridge University Press, Cambridge.

[4] MÖBIUS, A.F. (1885): *Gesammelte Werke*, Erster Band, Hirzel, Leipzig.

[5] NIELSEN, L. (1988): "Automated Guidance of Vehicles using Vision and Projective Invariant Marking," *Automatica*, **24**, 135-148.

[6] NIELSEN, L. and SPARR, G. (1987): "Perspective area-invariants," 5th SCIA, June 2-5, Stockholm, Sweden.

[7] NIELSEN, L. and SPARR, G. (1987): "Invariants based on areas and volumes in projective spaces," CODEN: LUTFD2/TFRT-7357, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

[8] VEBLEN, O. and YOUNG, J.W. (1910): *Projective Geometry*, Ginn and Company, Boston.

[9] VAN DER WAERDEN, B.L. (1955): *Algebra*, 3rd edition, Springer-Verlag, Berlin-Heidelberg-New York.