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The motion of electrons in the field of a homogeneously winded toroid

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The motion of electrons in the field of a homogeneously wound toroid

By SVEN GÖSTA NILSSON

With 3 figures in the text

The focusing properties of the toroid are investigated for the use of it as a beta-spectroscope or as a device for space focusing in a more complicated spectroscope.

We assume a winding carrying an electric current to be distributed homogeneously along the whole surface of a toroid. Expressed in cylindrical coordinates in accordance with Fig. 1 the field inside the toroid can be easily determined:

$$B = B_\varphi = \frac{\text{const}}{r} = \frac{B_0 r_0}{r}.$$

The field is thus constant in the φ - and z -direction as long as we stay inside the toroid.

We further assume a β -radioactive point source to be placed inside the toroid at $r = r_0$, $\varphi = 0$, $z = 0$. The limits for the angles of injection will be determined by an entrance slit. Possible directions inside this entrance slit will be described by the parameters ξ and η , where ξ and η both measure deviations from the tangent at the point of injection, ξ in the z -direction, η in the r -direction. (Fig. 2.)

The relativistic equation of motion reads

$$\frac{d}{dt} \left(\frac{m_0 \bar{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = -e \bar{v} \times \bar{B}$$

where \bar{v} is the velocity vector and $-e$ the charge of the electron. From this equation it is immediately apparent that $|\bar{v}| = \text{const.}$ is a solution. (The Lorentz force performs no work.) The relativistic mass $\frac{m_0}{\sqrt{1 - v^2/c^2}} = m$ can thus

be brought outside the differential operator $\frac{d}{dt}$, and the classical formulae are formally regained.

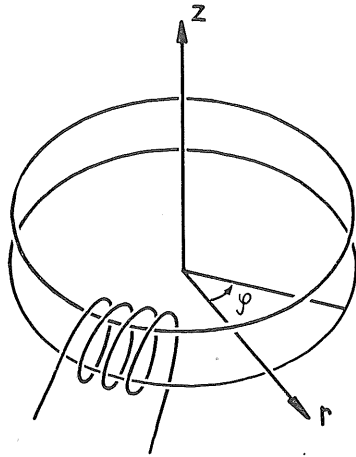


Fig. 1. The orientation of the cylindrical coordinate system in relation to the toroid.

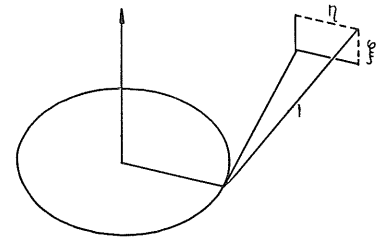


Fig. 2. The significance of the parameters ξ and η as determining the direction of the initial ray.

Introducing new variables

$$\varrho = \frac{r}{r_0}, \quad x = \frac{z}{z_0}, \quad \tau = \frac{e B_0}{m} \cdot t, \quad \alpha = \frac{m v_0}{e B_0 r_0}$$

the equation (resolved into its components) can be rewritten

$$(1) \quad \ddot{\varrho} - \varrho \dot{\varphi}^2 = \frac{1}{\varrho} \dot{x}$$

$$(2) \quad \frac{1}{\varrho} \frac{d}{d\tau} (\varrho^2 \dot{\varphi}) = 0$$

$$(3) \quad \ddot{x} = -\frac{\dot{\varrho}}{\varrho} \quad \text{where } \dot{\varrho} = \frac{d\varrho}{d\tau} \text{ etc.}$$

The initial conditions on those equations are

$$\begin{aligned} \varrho(0) &= 1 & \dot{\varrho}(0) &= \alpha \eta \\ \varphi(0) &= 0 & \dot{\varphi}(0) &= \beta = \alpha \sqrt{1 - \xi^2 - \eta^2} \\ x(0) &= 0 & \dot{x}(0) &= \alpha \xi. \end{aligned}$$

An equation containing only ϱ and its derivatives with respect to τ , can be obtained by integrating (2) and (3) once and substituting into (1)

$$(4) \quad \ddot{\varrho} = \frac{\beta^2}{\varrho^2} + \frac{\xi \alpha - \log \varrho}{\varrho}.$$

As the right hand side of this equation is the derivative of a function having a single maximum and tending monotonously towards $-\infty$ on both sides of this maximum, it is easily found that the solutions of this equation are *periodic* in τ .

The approximate solution of the differential equation (4).

The periodicity of the solutions proved, we now proceed to look for approximate solutions of the equation. We put

$$(5) \quad \varrho = 1 + f,$$

and assume the field to be strong enough so that $|f| \ll 1$. This condition is equivalent to α being sufficiently small.

We substitute (5) into (4), expand in powers of f , eliminate the constant term in the expansion by a new substitution $f = y + a$, and obtain, admitting terms up to the order y^3 only:

$$(6) \quad \ddot{y} + w^2 y = b y^2 + d y^3$$

where w^2 , b and d are all of order 1 if $|\xi|_{\max} |\eta|_{\max}$ and α are assumed small of order 0.1.

This equation (6) is now tackled by a modification of the Lindstedt-Poincaré method.

The method referred to, is applicable to an equation of the form $\ddot{y} + w^2 y = b y^2$, where, however, $|b|$ is to be $\ll 1$. Apart from the existence of a second term on the right side of the equation the main difficulty is the condition $|b| \ll 1$. This can be fulfilled by an artifice. We can prove $|y|$ to be small of the order α^2 and substitute $y = x q$, where $q \sim \alpha^2 \ll 1$. (6) now takes the form

$$(7) \quad \ddot{x} + w^2 x = b q x^2 + d q^2 x^3.$$

We then write, expanding in powers of q

$$\begin{aligned} x &= x_0 + q x_1 + q^2 x_2 + \dots \\ w^2 &= w_0^2 + q w_1^2 + q^2 w_2^2 + \dots \end{aligned}$$

The rapidity with which we obtain satisfactory values on w_0^2 and x depends on the smallness of q .

We admit terms only to the order α^2 . The real power of the method, however, is apparent first when higher approximations are aquired.

Determination of the focusing angle Φ .

We use expression (2), integrate once and expand in powers of f , insert the expression found above, and finally integrate over a whole period $\tau = \frac{2\pi}{w_0}$. For the integrated expression, we use the notation Φ_τ .

The ray corresponding to the central ray in ordinary lenses is not the "tangential" ray with initial conditions $\xi = 0, \eta = 0$. Instead $\xi = -\alpha, \eta = 0$ gives the solution $\rho = \text{const} = 1, x = \text{const} \cdot \tau$.

Here the Lorentz force compensates the "centrifugal force". We define the Φ_τ corresponding to this latter ray as the focusing angle Φ .

$$(7) \quad \Phi = 2\pi\alpha \left(1 - \frac{3}{2}\alpha^2\right).$$

The magnitude of the field or the value of α thus essentially determines the magnitude of Φ .

The form of the image in a plane through the z -axis at the angle Φ is then derived.

ρ and x are determined by Taylor expansions from points on the respective rays corresponding to the respective Φ_τ 's

$$\rho_\Phi = \rho_{\Phi_\tau} + \dot{\rho} \Delta\tau + \dots$$

where $\Delta\tau$ is determined from

$$\Phi - \Phi_\tau = \dot{\varphi}_{\Phi_\tau} \Delta\tau + \frac{1}{2} \ddot{\varphi}_{\Phi_\tau} (\Delta\tau)^2$$

x is determined analogously. The final expression for x is:

$$(8) \quad x_\Phi = \pi\alpha [-2\alpha + \alpha^3 + \xi'(\xi'^2 + \eta^2 - 2\alpha^2)], \text{ where } \xi' = \xi + \alpha.$$

At this point it is to be noticed that beside the sine-oscillation in x there is a constant drift, to the second order independent of the initial x -velocity. This drift is opposite in direction for electrons and positrons — a property which makes the apparatus useful for instance for the study of γ -rays via pair creation.

Form of entrance slit. The resolving power without regard to the finite dimensions of the source and the counter slit.

x is now considered as function of the parameters ξ and η . The form of the entrance slit determines the limits of variation of the latter. The slit form is chosen to make $T/\Delta x$ maximum, where T signifies the transmission and Δx the total variation in x .

The best form of the entrance slit is shown by the egg-shaped line in Fig. 3. The dotted curves are the locus of rays exactly at focus in Φ . The signs + and - signify the "landscaping" of the $x(\xi, \eta)$ -surface.

Under these conditions the transmission $T = \alpha^2 \%$, and the "base spread"

$$\mu_0 = \frac{\Delta \rho_0}{\rho} \left[\text{or } \frac{\Delta(H\rho)}{H\rho} \right] \simeq \frac{1}{2} \alpha^2.$$

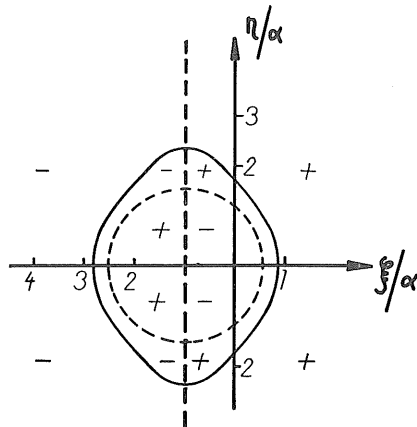


Fig. 3. The eggshaped curve shows the optimal form of the entrance slit as function of the parameters of injection. The dotted curves signify the ξ, η -locus of rays exactly confocal with the central ray. The signs + and - denote the level of x in relation to x_0 of the central ray.

The "spread" μ or the relative half width of the line is of the order $1/2 \mu_0$, and the resolving power $R = \frac{1}{\mu}$.

Table I.

| α | T | μ | R | Φ |
|------------------|-----|-------|-----|--------|
| 0.1 | 1 % | 1/2 % | 400 | 35° |
| 0.2 | 4 % | 2 % | 100 | 68° |
| 0.2 ¹ | 1 % | 1 % | 200 | 68° |

Calculations are in progress on the problem of combining the toroid with a spectroscope of the flat type for obtaining double focusing of a high order.

My thanks are due to Professor Kai Siegbahn for kindly suggesting the problem and supervising the work.

¹ Signifies another entrance slit then the optimal one.

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