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TRAVELING WAVES FOR THE WHITHAM EQUATION

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Abstract. The existence of traveling waves for the original Whitham equation is investigated. This equation combines a generic nonlinear quadratic term with the exact linear dispersion relation of surface water waves on finite depth. It is found that there exist small-amplitude periodic traveling waves with sub-critical speeds. As the period of these traveling waves tends to infinity, their velocities approach the limiting long-wave speed c_0, and the waves approach a solitary wave. It is also shown that there can be no solitary waves with velocities much greater than c_0. Finally, numerical approximations of some periodic traveling waves are presented.

1. Introduction

The study of waves on the surface of a fluid has been a source of intriguing mathematical problems for a long time. When studying such waves, viscosity is often neglected, so that the governing equations are the nonlinear Euler equations, supplemented by a set of nonlinear boundary conditions at the unknown fluid surface. This set of equations is commonly known as the water-wave problem. Of special interest is the study of permanent progressive waves, such as solitary or traveling periodic waves. These waves which are also called steady waves propagate without changing their shape over time.

An early highlight in the study of such steady waves was the discovery by Gerstner [16] of a family of exact solutions of the two-dimensional Euler equations in the form of periodic traveling waves. A special feature of this family of solutions is that it includes surface profiles that are not smooth, but have a cusp [11, 9]. While Gerstner's wave has non-zero vorticity, most studies of steady surface waves have been pursued in the case when the flow is irrotational. Starting with the seminal work of Stokes [30] in the mid 1800's, periodic wave trains on the surface of a fluid have attracted a great deal of attention. Stokes made the conjecture that the highest wave has a sharp crest [31], and a great deal of work has been directed towards understanding this phenomenon, including the mathematical proof of the fact that this highest wave exists. For an overview of results in this direction, the reader may consult the surveys by Toland [33] and Groves [17], and the book by Okamoto and Shoji [25]. While Gerstner's wave is an exact solution only for infinite depth, Stokes waves have been shown to exist for any depth.

A different line of research was initiated by the discovery of the solitary wave by John Scott Russell [28]. His observations and experiments gave an impetus to finding

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a mathematical formulation capable of describing such waves. The Korteweg-de Vries (KdV) equation

$$\eta_t + c_0\,\eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x + \tfrac{1}{6}c_0 h_0^2\,\eta_{xxx} = 0 \tag{1}$$

is a simplified model equation for waves at the water surface which includes the essential effects of nonlinearity and dispersion [7, 22]. Balancing these two effects is the basic mechanism behind the existence of both solitary-wave solutions and periodic traveling waves. Equation (1) is given in dimensional form, and $c_0 := \sqrt{gh_0}$ is the limiting long-wave speed, h_0 denotes the undisturbed water depth (assuming a flat bottom), and g is the gravitational constant of acceleration. The function $\eta(t,x)$ describes the deflection of the fluid surface from the rest position at a point x at time t. The equation is a valid approximation describing the evolution of surface water waves in the case when the waves are long compared to the undisturbed depth h_0 of the fluid, and the average amplitude of the waves is small when compared to h_0 [18]. In addition, transverse effects are assumed to be weak.

The success of the KdV equation in describing steady waves and the discovery of its completely integrable Hamiltonian structure has led to an intense study of this equation for the last four decades. The mathematical theory for the KdV equation has reached a very advanced level, with a solid theory of well-posedness in place, and a sound understanding of the stability properties of solitary and traveling waves [1, 2, 4, 5, 20, 26]. However, as a model for water waves, the KdV equation may not be the best choice for a number of reasons. Most importantly, it has some shortcomings concerning the propagation of shorter waves. The linear wave speed in the KdV equation is given by

$$c(\xi) = c_0 - \tfrac{1}{6}c_0 h_0^2\xi^2, \tag{2}$$

where $\xi = \frac{2\pi}{\lambda}$ is the wave number, and λ is the wavelength. This is a second-order approximation to the wave speed

$$c(\xi) = \tfrac{\omega}{\xi} = \sqrt{\tfrac{g\tanh\xi h_0}{\xi}}, \tag{3}$$

of the linearized water-wave problem. The latter expression for $c(\xi)$ appears when the full water-wave problem is linearized around the vanishing (irrotational) solution, and solutions of the form $\exp(ix\xi - i\omega t)$ are sought [18, 35]. However, as noted in [12, 13], the dispersion relation takes a different form in the presence of vorticity. A comparison of the two expressions (2) and (3) for $c(\xi)$ is presented in Figure 1. As can be seen, the linearized KdV equation does not give a faithful representation of the full dispersion relation even for intermediate values of the wave number ξ. This problem with the KdV equation as a model for water waves was recognized early on, and has been remedied somewhat by the introduction of the regularized long-wave equation

$$\eta_t + c_0\,\eta_x + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x - \tfrac{1}{6}h_0^2\,\eta_{xxt} = 0, \tag{4}$$

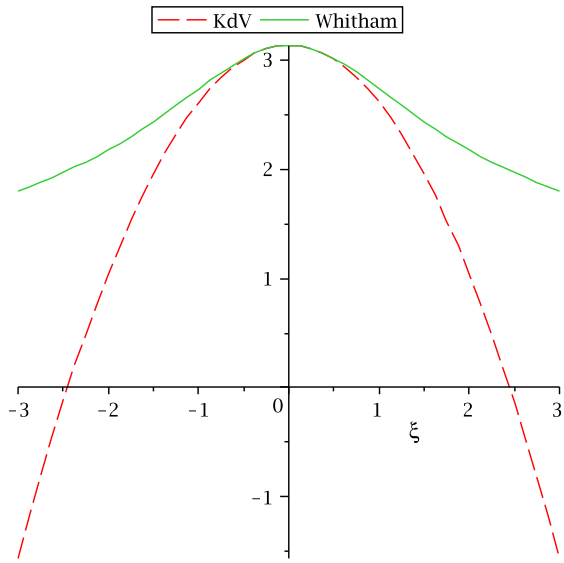

Figure 1. Comparison of linear wave speeds $c = \omega/\xi$ for the KdV and Whitham equations. Here $g = 9.81$ and $h_0 = 1$. The maximum of both graphs is at $c_0 = \sqrt{gh_0}$.

by Peregrine [27] and Benjamin, Bona and Mahoney [3]. The linear wave speed of (4) is given by

$$c(\xi) = \frac{c_0}{1 + \frac{1}{6}h_0^2\xi}, \tag{5}$$

which is qualitatively closer to (3) than (2). A comprehensive review of these modeling issues was given in [3].

Also recognizing the problems of the KdV equation as a model equation for water waves, Whitham introduced what is now called the Whitham equation [34]. The idea was to use the exact form of the wave speed (3) instead of a second-order approximation like (2) or (5). The equation proposed by Whitham has the form

$$\eta_t + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta\,\eta_x + K_{h_0} * \eta_x = 0, \tag{6}$$

where the convolution is in the x-variable. The equation is written in dimensional variables, with $\eta(t, x)$ representing the deflection of the surface from rest, just as in the KdV equation. The convolution kernel is given by

$$K_{h_0} := \mathcal{F}^{-1}\left(\sqrt{\tfrac{g\tanh h_0\xi}{\xi}}\right), \tag{7}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform to be defined by (9) in Section 2. Often, instead of the kernel K_{h_0}, the kernel

$$\tfrac{\pi}{4}\exp\left(-\tfrac{\pi}{2}|x|\right) \tag{8}$$

is used. This kernel matches the asymptotic behavior of K_{h_0} [35], and has certain mathematical advantages over (7), such as not having a singularity at the origin. Moreover,

this approximation gives rise to a differential equation, the so-called Burgers-Poisson equation [14]. The properties of (8) were exploited by Seliger [29], who showed that for this simplified kernel wave breaking is possible.

Even though there does not exist a formal asymptotic expansion or a rigorous proof of convergence of solutions of (6) to solutions of the water-wave problem, the Whitham equation remains a source of intriguing problems. The monograph by Naumkin and Shishmarev [24] is devoted entirely to equations like (6). In particular, some questions of Whitham concerning breaking and peaking of waves described by generalizations of (6) are answered. However, the work of Naumkin and Shishmarev is mainly focused on problems of time evolution. Steady solutions of the equation (6) with the kernel (8) were studied in [36]. However, for the original Whitham equation the literature is rather sparse. The inherently non-local character of (6) makes things much more intricate. In particular, it is still not known whether the proper Whitham equation (with the kernel K_{h_0}) admits a nontrivial solitary-wave solution.

The present article is a study of steady waves for the non-local Whitham equation with its original kernel. In Section 3 we make use of the Crandall-Rabinowitz local bifurcation theorem to prove the existence of small-amplitude periodic traveling waves. A similar treatment was outlined by Gabov in [15], but for the exact kernel (7) no proof was given. In Section 4 we prove *a priori* continuity and compactness properties of bounded traveling-wave solutions. These properties imply convergence of periodic solutions to solitary-wave solutions. Section 5 is on non-existence. It is shown that for large velocities there can be no continuous solitary-wave solutions of the steady Whitham equation. In Section 6 we compute numerical approximations of both traveling and solitary waves. It is worth mentioning that the Whitham equation has excited interest precisely for the reason that it features wave breaking and peaking. This was indicated already by Whitham [34], and investigated at length by Naumkin and Shishmarev in the monograph [24]. According to this theory, there is a highest wave, which will have a cusp at the center. Some computations in this direction are carried out in Section 6.

2. Preliminaries

In this article, the standard notation of mathematical analysis is used. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the set of measurable real-valued functions of a real variable whose p^{th} powers are Lebesgue integrable over a subset $\Omega \subseteq \mathbb{R}$. If $f \in L^p(\Omega)$, its norm is given by $\|f\|^p_{L^p(\Omega)} := \int_\Omega |f|^P \, dx$. The space $L^\infty(\Omega)$ consists of all measurable, essentially bounded functions with norm $\|f\|_{L^\infty(\Omega)} := ess \ \sup_{x\in\Omega} |f(x)|$. We define the Fourier transform $\mathcal{F}$ of a function $f \in L^1(\mathbb{R})$ by

$$\mathcal{F}f(\xi) := \int_{-\infty}^{\infty} f(x) \exp(-ix\xi) \, dx,$$

and the inverse Fourier transform $\mathcal{F}^{-1}$ by

$$\mathcal{F}^{-1}f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \exp(ix\xi) \, d\xi, \tag{9}$$

for any $\hat{f} \in L^1(\mathbb{R})$. We shall also use the notation $\hat{f} := \mathcal{F}f$. The Fourier coefficients of $2L$-periodic functions on $\mathbb{R}$ are defined by

$$\hat{f}_k := \int_{-L}^{L} f(x) \exp\left(-ix\tfrac{k\pi}{L}\right) dx.$$

We write

$$f(x) \sim \frac{1}{2L} \sum_{k\in\mathbb{Z}} \hat{f}_k \exp\left(ix\tfrac{k\pi}{L}\right)$$

to indicate that under certain conditions on f, this infinite trigonometric series converges to f pointwise, uniform, or in norm. For example, if $f \in L^p((-L, L))$, $p > 1$, then the Carleson–Hunt theorem [19] guarantees that the series converges to $f(x)$ almost everywhere. If in addition $f(x)$ is an even function, the series can be written as

$$f(x) \sim \frac{1}{2L}\hat{f}_0 + \frac{1}{L}\sum_{k=1}^{\infty} \hat{f}_k \cos\left(ix\tfrac{k\pi}{L}\right) = \frac{1}{L}\sum_{k=0}^{\infty}{}' \hat{f}_k \cos\left(ix\tfrac{k\pi}{L}\right),$$

where the prime indicates that the first term of the sum is multiplied by $1/2$.
Next we turn to recording some elementary properties of the Whitham kernel, K_{h_0}, and its Fourier transform. It is immediate that the function $\sqrt{g(\tanh h_0\xi)/\xi}$ is even and strictly decreasing on $(0, \infty)$. It is in fact real analytic since in a neighborhood of the origin

$$\frac{\tanh \xi}{\xi} = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right) B_{2n}\xi^{2(n-1)}}{(2n)!} > 0,$$

by using the Taylor series expansion for tanh (B_n are the Bernoulli numbers). Moreover, $\sqrt{g(\tanh h_0\xi)/\xi}$ takes the following limits:

$$\lim_{\xi\to 0} \sqrt{\tfrac{g\tanh h_0\xi}{\xi}} = \sqrt{gh_0}, \qquad \lim_{\xi\to\infty} \sqrt{\tfrac{g\tanh h_0\xi}{\xi}} = 0.$$

Consequently, $\int_{-\infty}^{\infty} K_{h_0}(x)\, dx = \sqrt{gh_0}$, and

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \sqrt{gh_0} \left\| \mathcal{F}^{-1}\left(\sqrt{\tfrac{\tanh\xi}{\xi}}\right) \right\|_{L^1(\mathbb{R})}. \tag{10}$$

Thus it can be shown that $K_{h_0} \in L^1(\mathbb{R})$ in the following way. The substitution of variables $y = x\xi$ and partial integration shows that the growth of $\mathcal{F}^{-1}\left(\sqrt{(\tanh\xi)/\xi}\right)$ is of order $x^{-1/2}$ as $x \to 0$ (for a rigorous proof of this fact, cf. Section 4). Since the function $\sqrt{(\tanh\xi)/\xi}$ is analytic, the inverse Fourier transform has rapid decay. Thus splitting the integral according to

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \int_{|x|\le 1} |K_{h_0}(x)|\, dx + \int_{|x|\ge 1} |K_{h_0}(x)|\, dx,$$

it is plain that K_{h_0} has finite $L^1(\mathbb{R})$-norm. In fact, this argument establishes more generally that $K_{h_0} \in L^p(\mathbb{R})$ for $1 \le p < 2$.

Since the existence of traveling waves is in view, we make the usual ansatz $\eta(t,x) = \phi(x-ct)$, with $c>0$ being the propagation speed of a right-going steady wave. Using this form, the equation (6) transforms into

$$-c\,\phi' + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\phi\,\phi' + K_{h_0} * \phi' = 0,$$

which may be integrated to

$$-c\,\phi + \tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi^2 + K_{h_0} * \phi = B, \tag{11}$$

for some real constant B. For solutions $\phi \in L^2(\mathbb{R})$, it appears that the convolution $K_{h_0} * \phi$ is in $L^2(\mathbb{R})$ since K_{h_0} is in $L^1(\mathbb{R})$. Therefore, the left-hand side must vanish as $|x| \to \infty$, and we shall consider here only the case when $B=0$. The scaling

$$\phi \mapsto \tfrac{3}{4}\tfrac{c_0}{h_0\,c}\phi$$

then yields the normalized problem

$$\phi = \phi^2 + \tfrac{1}{c}K_{h_0} * \phi. \tag{12}$$

3. Existence of periodic traveling waves

Theorem 3.1. *For a given $L>0$ and a given depth $h_0>0$, there exists a local bifurcation curve of steady, $2L$-periodic, even and continuous solutions of the Whitham equation. Those solutions are perturbations of $C\cos(\pi x/L)$, $C\in\mathbb{R}$, and their wave speed at the bifurcation point is determined by the full dispersion relation*

$$c^* = \sqrt{\frac{gL\tanh\left(h_0\pi/L\right)}{\pi}}. \tag{13}$$

In particular, as $L\to\infty$ we have $c^*\to\sqrt{gh_0}$.

We shall make use of the Crandall-Rabinowitz bifurcation theorem [21, Section I.5], which we state in a form suitable for our purposes. Here and elsewhere D_c is the Fréchet derivative with respect to c.

Lemma 3.2. *Let W be a Banach space, $c\in I := (0,\sqrt{gh_0})$ a parameter, and let $\mathcal{L}: W\to W$ be the Fréchet derivative at 0 with respect to u of the Whitham map*

$$u \mapsto u - \tfrac{1}{c}K_{h_0} * u - u^2. \tag{14}$$

Suppose that $\mathcal{L}$ and $D_c\,\mathcal{L}$ exist and are continuous $W\to W$, and that for some $c^*\in I$ the following conditions hold:

i) $\dim\ker(\mathcal{L}) = 1$,
ii) $W = \ker(\mathcal{L}) \oplus \operatorname{ran}(\mathcal{L})$,
iii) $(D_c\,\mathcal{L})\ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = 0$.

Then there exist $\varepsilon>0$ and a continuous bifurcation curve $\{(c_s,\phi_s)\colon |s|<\varepsilon\}$ with $c_s|_{s=0} = c^*$, such that ϕ_0 is the vanishing solution of (12), and $\{\phi_s\}_s$ is a family of nontrivial solutions with corresponding wave speeds $\{c_s\}_s$. Moreover, we have

$$\operatorname{dist}(\phi_s, \ker(\mathcal{L})) = o(s) \quad \text{in } W.$$

Remark 3.3. We remark that our method works equally well for the generalized Whitham equation

$$\eta_t + \tfrac{3}{2}\tfrac{c_0}{h_0}\,\eta^p\,\eta_x + K_{h_0} * \eta_x = 0,$$

whenever $1 \le p \in \mathbb{Z}$. In that case the Whitham map becomes $u \mapsto u - \frac{1}{c}K_{h_0} * u - u^p$. Since the linearization around the vanishing solution is the same for this map as for (14), all that is needed to check is the continuity of the full map in W. As we shall see in the proof of Theorem 3.1, our choice of W is an algebra, so that continuity is evident.

Remark 3.4. It can be seen from the proof of Theorem 3.1 that for wave speeds $c \neq \sqrt{gh_0}$ and different from (13), the linear Whitham map $\mathcal{L}$ is a continuous bijection $W \to W$. It then follows from the implicit function theorem [21, Thm I.1.1] that in a neighborhood of the trivial flows, there are no other solutions in W of the Whitham equation.

Before we turn to the proof, let us explain how the convolution operator $K_{h_0}*$ acts on periodic functions. Suppose then that $f \in L^\infty(\mathbb{R})$ is periodic. Since K_{h_0} is in $L^1(\mathbb{R})$, we can write the integral

$$\begin{aligned}\int_{-\infty}^{\infty} K_{h_0}(x-y)f(y)\,dy &= \sum_{k=-\infty}^{\infty}\int_{-L}^{L} K_{h_0}(x-y+2kL)f(y)\,dy\\ &= \int_{-L}^{L}\left(\sum_{k=-\infty}^{\infty} K_{h_0}(x-y+2kL)\right)f(y)\,dy \equiv \int_{-L}^{L} A(x-y)f(y)\,dy.\end{aligned}$$

Inspection of the the definition of $A(x)$ shows that it is $2L$-periodic, even, and continuous on $[-L,L]\setminus\{0\}$. Moreover, a straightforward proof using Minkowski's inequality shows that $A(x)$ belongs to $L^p(-L,L)$, for $1 \le p < 2$. Therefore, according to the Carleson–Hunt theorem [19], $A(x)$ can be approximated pointwise by its Fourier series. Thus we have

$$A(x) = {\sum_{k=0}^{\infty}}' \hat{A}_k \cos\left(\tfrac{k\pi x}{L}\right), a.e.,$$

where the Fourier coefficients of A are given by

$$\begin{aligned}\hat{A}_k &= \int_{-L}^{L}\sum_{j=-\infty}^{\infty} K_{h_0}(x+2jL)\exp\left(-\tfrac{ixk\pi}{L}\right)dx\\ &= \sum_{j=-\infty}^{\infty}\int_{-L}^{L} K_{h_0}(x+2jL)\exp\left(-\tfrac{i(x+2jL)k\pi}{L}\right)dx\\ &= \int_{-\infty}^{\infty} K_{h_0}(x)\exp\left(-\tfrac{ixk\pi}{L}\right)dx = \hat{K}_{h_0}(\tfrac{k\pi}{L}).\end{aligned} \tag{15}$$

Thus it appears that the periodic problem is given by the same multiplier as the problem on the line, and we have the representation

$$K_{h_0} * f(x) = \frac{1}{L}{\sum_{k=0}^{\infty}}' \hat{f}_k\hat{A}_k \cos\left(\tfrac{k\pi x}{L}\right) = \frac{1}{L}{\sum_{k=0}^{\infty}}' \hat{f}_k\hat{K}_{h_0}(\tfrac{k\pi}{L}) \cos\left(\tfrac{k\pi x}{L}\right). \tag{16}$$

Proof of Theorem 3.1. Looking for a steady solution we consider first the linearized equation

$$\mathcal{L}\,\psi := \psi - \tfrac{1}{c}K_{h_0} * \psi = 0.$$

For $\psi \in L^\infty(\mathbb{R})$ we see that

$$\hat{\psi}\left(1 - \tfrac{1}{c}\sqrt{\tfrac{g\tanh h\xi}{\xi}}\right) = 0.$$

This makes sense in the setting of distributions. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing functions (see [32]). Then $\frac{1}{c}\widehat{K_{h_0} * \psi}$, $\hat{\psi}$ and $\frac{1}{c}\hat{K}_{h_0}$ all exist in $\mathcal{S}'(\mathbb{R})$. Since $1 - \sqrt{g\tanh(h\xi)/\xi}$ is in $L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$, the product of $\hat{\psi}$ and this function is well-defined acting on functions in $\mathcal{S}(\mathbb{R})$. The convolution theorem [32, Section 4.3] then implies that $\frac{1}{c}\widehat{K_{h_0} * \psi}(v) = \frac{1}{c}(\hat{\psi}\hat{K}_{h_0})(v)$ for any $v \in \mathcal{S}(\mathbb{R})$. Now, if $c < \sqrt{gh_0}$ the support of $\hat{\psi}$ is contained in $\{\pm\xi_0\}$, where $\xi_0 := \xi_0(c, h_0)$ is the unique positive root of $g\tanh h_0\xi = c^2\xi$; if $c = \sqrt{gh_0}$ then $\operatorname{supp}(\hat{\psi}) \subset \{0\}$; and if $c > \sqrt{gh_0}$ it follows that $\hat{\psi}(\xi) = 0$ for all ξ. The non-trivial solutions of the linear problem are thus given by

$$\begin{cases} \psi(x) = C, & c = \sqrt{gh_0}, \\ \psi(x) = C\cos(\xi_0 x), & c < \sqrt{gh_0}, \end{cases} \tag{17}$$

where $C \in \mathbb{R}$ can be any constant. Note that the constant solutions different from zero are non-physical, and therefore discarded in this analysis. We want to find even periodic small amplitude solutions by bifurcating from a curve of trivial flows. For this purpose, fix the depth h_0 and the half wavelength $L > 0$. The speed $c > 0$ shall be our bifurcation parameter. It is clear from (17) that, in any real linear space of $2L$-periodic functions,

$$\dim\ker(\mathcal{L}) = 1,$$

if and only if $\xi_0 = k\pi/L$, $k \in \mathbb{Z}^+$. Settling for the lowest mode, $k = 1$, gives a unique c as in (13), which from now on will be presupposed as our candidate for c^* as in Lemma 3.2.

Looking for even, continuous, and periodic solutions, we introduce the commuting Banach algebra

$$W := \left\{ u(x) = \frac{1}{L}\sum_{k=0}^{\infty}{}' \hat{u}_k \cos\left(\tfrac{k\pi x}{L}\right) \,\middle|\, \|u\| := \frac{1}{L}\sum_{k=0}^{\infty}{}' |\hat{u}_k| < \infty \right\},$$

which is a suitable subalgebra of the Wiener algebra (cf. [6]). This follows since for even functions the complex Fourier coefficients satisfy $\hat{u}_k = \hat{u}_{-k}$, so that our norm is equivalent to the classical norm for the Wiener algebra. Note that each member of W is uniformly continuous on all of $\mathbb{R}$. We shall consider the Whitham equation as the map (14) from W, and it will be shown that it is a continuous map into W. As shown in (16) the periodic problem is given by the same multiplier as the problem on the line. In effect,

$$\mathcal{L}\,u \sim \frac{1}{L}\sum_{k=0}^{\infty}{}' \hat{u}(k)\left(1 - \tfrac{1}{c}\hat{A}(k)\right)\cos\left(\tfrac{k\pi x}{L}\right) \tag{18}$$

holds *a.e.* on $[-L, L]$. By the Riemann-Lebesgue lemma [23, p.133] $\hat{A}(k) \to 0$ as $k \to \infty$, so the right-hand side is in W, hence continuous, and

$$\| \mathcal{L}\, u\| \leq \left(1 + \tfrac{1}{c} \max_k \{\hat{A}(k)\}\right) \|u\|,$$

so that $\mathcal{L}\colon W \to W$ is continuous. Since also the left-hand side is continuous, (18) is an equality, which in its turn implies that the full non-linear Whitham map $u \mapsto \mathcal{L}\, u - u^2$ is a continuous endomorphism on W, since this is an algebra. The fact that $\ker(\mathcal{L}) = \operatorname{span}_{\mathbb{R}}\,(\cos(\pi x/L))$ yields

$$\hat{A}(1) = c, \qquad \text{and} \qquad \hat{A}(k) \neq c, \quad k \neq 1. \tag{19}$$

To show that $\operatorname{codim} \operatorname{ran}(\mathcal{L})$ is one-dimensional, consider a given $u \in W$. Take $u^{\perp} \in W$ with $\widehat{u^{\perp}}(1) = 0$. Then the function

$$v(x) := \frac{1}{L} {\sum_{k=0}^{\infty}}' \frac{\widehat{u^{\perp}}(k)}{1 - \frac{1}{c}\hat{A}(k)} \cos\left(\tfrac{k\pi x}{L}\right)$$

is well-defined and belongs to W (this can be seen from (15), but it also follows from the Riemann-Lebesgue lemma in combination with (19)). Indeed

$$v(x) = \mathcal{L}^{-1}\, u^{\perp}(x).$$

Consequently,

$$u(x) = \mathcal{L}\, v + \tfrac{\hat{u}(1)}{L} \cos\left(\tfrac{k\pi x}{L}\right),$$

so that $W = \ker(\mathcal{L}) \oplus \operatorname{ran}(\mathcal{L})$. The derivative with respect to the bifurcation parameter c is

$$(D_c\, \mathcal{L})u = -(D_c \tfrac{1}{c} K_{h_0}) * u = \frac{1}{c^2} K_{h_0} * u.$$

Hence—by exactly the same arguments as above—we have that

$$(D_c\, \mathcal{L})u = \frac{1}{L c^2} {\sum_{k=0}^{\infty}}' \hat{u}(k)\hat{A}(k) \cos\left(\tfrac{k\pi x}{L}\right)$$

is bounded as a map on W. In particular

$$(D_c\, \mathcal{L}) \ker(\mathcal{L}) = \ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = 0.$$

□

4. Continuity and compactness of bounded solutions

We present here a regularity and a compactness result for traveling solutions of the Whitham equation. This casts light on the relation between L-periodic solutions and solitary wave solutions.

Theorem 4.1. *Let ϕ be a solution of* (12) *such that $\|\phi\|_\infty < 1/2$. Then ϕ is continuous.*

Proof. Without loss of generality we pursue the analysis for $k(\xi) := \sqrt{\tanh(\xi)/\xi}$. In view of that $D_\xi \tanh \xi = 1 - \tanh^2 \xi \in \mathcal{S}(\mathbb{R})$, it follows from the Leibniz rule that

$$D_\xi^n k(\xi) \in \mathcal{O}\left(\xi^{-1/2-n}\right) \quad \text{as} \quad |\xi| \to \infty.$$

Hence we use partial integration to rewrite

$$K(x) := \frac{1}{2\pi} \int k(\xi) \exp(ix\xi)\, d\xi = \frac{1}{(-ix)^n} \int k^{(n)}(\xi) \exp(ix\xi)\, d\xi,$$

for any $x \neq 0$, $n \in \mathbb{Z}^+$. Consequently, we have well-defined derivatives of all orders away from the origin,

$$D_x^j \int k(\xi) \exp(ix\xi)\, d\xi \in \mathcal{O}(x^{j-n}) \quad \text{as} \quad |x| \to \infty.$$

For any fixed j, we may choose n as large as required to obtain that $K(x)$ is smooth away from the origin, and all its derivatives have rapid decay at infinity. Consider then

$$K * \phi(x) = I_1(x) + I_2(x) := \int\limits_{|x-z|\leq 1} K(x-z)\phi(z)\, dz + \int\limits_{|x-z|\geq 1} K(x-z)\phi(z)\, dz.$$

Since $|K(x-z)| \leq C|x-z|^{-2}$ and ϕ is bounded, it follows from an application of the dominated convergence theorem that $I_2(x)$ is continuous. By the change of variables $\xi \mapsto s := (x-z)\xi$ we have that

$$\begin{aligned} I_1(x) &= \frac{1}{2\pi} \iint\limits_{|z-x|\leq 1} k(\xi) \exp\left(i(x-z)\xi\right)\phi(z)\, d\xi\, dz \\ &= \frac{1}{2\pi} \int\limits_{|z-x|\leq 1} \frac{x-z}{|x-z|^{3/2}} \left(\int \sqrt{\frac{\tanh\left(s|x-z|^{-1}\right)}{s}} \exp\left(is\right) ds \right) \phi(z)\, dz. \end{aligned}$$

Likewise, the inner integral can be divided into two parts,

$$\begin{aligned} &\int \sqrt{\frac{\tanh sy}{s}} \exp\left(is\right) ds = I_i(y) + I_{ii}(y) \\ &:= \int\limits_{|s|\leq 1} \sqrt{\frac{\tanh sy}{s}} \exp\left(is\right) ds + \int\limits_{|s|\geq 1} \sqrt{\frac{\tanh sy}{s}} \exp\left(is\right) ds, \end{aligned}$$

where we have used the shorthand $y := |x-z|^{-1}$. It is clear that $|\tanh sy| \leq 1$, so that $I_i(y)$ is well-defined. Its integrand is furthermore bounded by $|s|^{-1/2}$, uniformly for all y.

Now to I_{ii}. Using partial integration, we obtain that

$$\int\limits_{|s|\geq 1} \sqrt{\frac{\tanh(sy)}{s}} \exp(is)\, ds = -2\sin(1)\sqrt{\tanh(y)}$$
$$+ \frac{1}{2i} \int\limits_{|s|\geq 1} \frac{\left(sy\tanh^2(sy) - sy + \tanh(sy)\right)}{s^{3/2}\sqrt{\tanh(|s|y)}} \exp(is)\, ds$$
$$= -2\sin(1)\sqrt{\tanh(y)} + \frac{1}{2i} \int\limits_{|s|\geq 1} \frac{f(sy)}{s^{3/2}} \exp(is)\, ds,$$

where $f(\tau) := (\tau\tanh^2\tau - \tau + \tanh\tau)/\sqrt{|\tanh\tau|}$. It is immediate that the boundary term is bounded by 2, and it can be seen that f is uniformly bounded with $\|f\|_\infty = 1$. This implies that if $x_n \to x$, then there is a uniform integrable bound, $C|x-z|^{-1/2}(|s|^{1/2} + |s|^{3/2})^{-1}$, for the integrands of $I_1(x_n)$. Just as for $I_2(x)$ it follows from dominated convergence that $I_1(x)$ is continuous, and hence $K * \phi(x)$ is. Using (12), we see that

$$|\phi(x) - \phi(y)| = \frac{|K*\phi(x) - K*\phi(y)|}{1 - \phi(x) - \phi(y)} \leq \frac{|K*\phi(x) - K*\phi(y)|}{1 - 2\|\phi\|_\infty}, \tag{20}$$

and hence ϕ is continuous. Here we have used the assumption that $\|\phi\|_\infty < 1/2$. □

Corollary 4.2. *Let $r < 1/2$. The set of solutions of the steady Whitham equation* (12) *contained in the closed ball $\|\phi\|_\infty \leq r$ is compact in $L^\infty_{loc}(\mathbb{R})$.*

Proof. Pick any sequence $(\phi_n)_n$ of solutions of (12) that fulfill $\|\phi_n\|_\infty \leq r$. By Theorem 4.1 those are continuous on $\mathbb{R}$. Moreover, it can be seen from the proof of Theorem 4.1 that the continuity of $\phi * K(x)$ is uniform with respect to $\|\phi\|_\infty$. It then follows from (20) that ϕ_n are equicontinuous. The Arzela-Ascoli theorem thus yields the existence of a subsequence $(\phi_{n_k})_k \subseteq (\phi_n)_n$ and a continuous function ϕ, such that ϕ_{n_k} converges to ϕ in $L^\infty_{loc}(\mathbb{R})$.

To prove that ϕ is a solution of the Whitham equation, let $v \in C_0(\mathbb{R})$ be any continuous function with compact support. Then

$$\int \left(\phi_n(x) - \phi_n^2(x) - \int \tfrac{1}{c} K_{h_0}(y-x)\phi_n(y)\, dy\right) v(x)\, dx = 0.$$

Since $\phi_n(x)$ converges pointwise to $\phi(x)$, the functions $v, \frac{1}{c}K_{h_0} \in L^1$, and $\|\phi_n\|_\infty \leq r$, it follows from the Lebesgue bounded convergence theorem that

$$\int (\phi - \phi^2 - \tfrac{1}{c}K_{h_0} * \phi) v\, dx = 0.$$

In view of that v is arbitrary this implies that ϕ fulfills (12) almost everywhere. The fact that ϕ is continuous implies that it is indeed a solution of the steady Whitham equation in the pointwise sense. □

Remark 4.3. In Section 3 we find periodic solutions for any period $L > 0$. Under the conditions of Corollary 4.2, any such sequence of solutions converges to traveling-wave solution on the line as $L \to \infty$. This will be illustrated numerically in Section 6.

5. Nonexistence of a class of solitary waves

The Whitham equation was designed to incorporate both breaking and dispersion. However, if the depth $h_0 > 0$ is small when compared to the wave speed, then the dispersion term is small, and moreover, dispersion is very weak. As a result, for large velocities c, there are no traveling waves.

Theorem 5.1. *There are no steady and bounded continuous solutions of the Whitham equation with*

$$c > \kappa\sqrt{gh_0} \quad \text{and} \quad \inf\phi \leq 0 < \sup\phi, \tag{21}$$

where $\kappa = 2\left(\sqrt{2}+1\right)\left\|\mathcal{F}^{-1}\left(\sqrt{(\tanh\xi)/\xi}\right)\right\|_{L^1(\mathbb{R})}$.

Remark 5.2. Note that the condition (21) means that there are no solitary waves with velocities much larger than the critical long wave speed $\sqrt{gh_0}$. Using the estimate

$$1 = \left\|\mathcal{F}\mathcal{F}^{-1}\left(\sqrt{\tfrac{\tanh\xi}{\xi}}\right)\right\|_{L^\infty(\mathbb{R})} \leq \left\|\mathcal{F}^{-1}\left(\sqrt{\tfrac{\tanh\xi}{\xi}}\right)\right\|_{L^1(\mathbb{R})},$$

the value of κ appearing in the statement of the theorem may be estimated below by $2\left(\sqrt{2}+1\right)$.

Proof of Theorem 5.1. The proof proceeds by contradiction. Suppose that there exists a nontrivial bounded solution ϕ to (12). Then the following inequalities must hold.

$$\left(\|\phi\|_{L^\infty(\mathbb{R})} - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\right)\|\phi\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})} \leq \left(\|\phi\|_{L^\infty(\mathbb{R})} + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\right)\|\phi\|_{L^\infty(\mathbb{R})},$$

so that

$$1 - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})} \leq 1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}, \tag{22}$$

in view of that $\sup\phi > 0$. Note first that

$$\phi^2(x) \geq \phi(x) - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})}$$

for all x. This is a simple consequence of (12). For the desired contradiction it is thus enough to show that there is some x, such that

$$\phi^2(x) < \phi(x) - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})},$$

or in other words

$$\phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})} < 0. \tag{23}$$

An application of (22) yields that

$$\phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})} \leq \phi^2(x) - \phi(x) + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}(1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}),$$

and we set out to examine the right hand side,

$$F(\phi) := \phi^2 - \phi + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}(1 + \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}).$$

Observe that $F(\phi)$ is negative whenever

$$\frac{1}{2}\left(1 - \sqrt{2 - (2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2}\right) < \phi < \frac{1}{2}\left(1 + \sqrt{2 - (2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2}\right). \tag{24}$$

The left and right hand sides of (24) are real if $\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \frac{1}{2}(\sqrt{2} - 1)$. Taking the scaling and (10) into consideration, that follows from the requirement (21). Therefore, under that assumption, we have that

$$\frac{1}{2}\left(1 - \sqrt{2 - (2\|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2}\right) < \frac{1}{2} < 1 - \|\tfrac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})},$$

in view of (22). Since ϕ is continuous with $\inf \phi \leq 0$, there is thus an x, such that both inequalities in (24) are satisfied. As a result, we have obtained that (23) holds, reaching the desired contradiction. □

6. Numerical Approximation

For the numerical approximation of periodic traveling waves of the Whitham equation, a spectral projection is used. As above, the undisturbed depth h_0 and the wavelength L are fixed, and the speed c is used as the bifurcation parameter. For the purpose of approximating periodic solutions of (11), a Fourier method is optimal. To define the Fourier-collocation projection, define the subspace

$$S_N = \operatorname{span}_{\mathbb{C}} \left\{ \exp(ikx) \;\middle|\; k \in \mathbb{Z},\ -N/2 \leq k \leq N/2 - 1 \right\}$$

of $L^2((0, 2\pi))$. The collocation points are defined to be $x_j = \frac{2\pi j}{N}$ for $j = 0, 1, ...N - 1$. Let I_N be the interpolation operator from $C^\infty_{per}([0, 2\pi])$ onto S_N. As explained in [10], this operator is defined in the following way. Given $u \in C^\infty_{per}([0, 2\pi])$, $I_N u$ is the unique element of S_N that coincides with u at the collocation points x_j. For the spectral projection, we use the equation (11) with $B = 0$. According to Theorem 3.1, the equation is defined on the interval $[-L, L]$, whereas the discrete Fourier transform to be used is most conveniently defined on $[-\pi, \pi]$. Therefore, the scaling $\phi(x) \to \phi(ax)$ is used, where $a = \frac{L}{\pi}$. Special attention has to be paid to the operator K_{h_0}. A straightforward calculation shows that

$$\big(K_{h_0} * u\big)(ax) = \sqrt{a} K_{h_0/a} * \big(u(a\,\cdot)\big)(x). \tag{25}$$

Therefore, the rescaled equation for 2π-periodic solutions is

$$-c\,\phi + \tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi^2 + \sqrt{a} K_{h_0/a} * \phi = 0.$$

The discretized form of this equation is

$$-\,c\,\phi_N + \tfrac{3}{4}\tfrac{c_0}{h_0}\,\phi_N^2 + \sqrt{a}\left[K_{h_0/a}\right]_N \phi_N = 0, \tag{26}$$

which is enforced at the collocation points x_j. If ϕ_N is written in terms of its discrete Fourier coefficients $\tilde{\phi}_N(k)$ as

$$\phi_N(x) = \sum_{-N/2 \le k \le N/2-1} \tilde{\phi}_N(k) \exp(ikx),$$

the operator $\left[K_{h_0/a}\right]_N$ can be evaluated using the formula

$$\left[K_{h_0/a}\right]_N \phi_N(x) = \sqrt{\tfrac{gh_0}{a}}\, \tilde{\phi}_N(0) \; + \sum_{\substack{1-N/2 \le k \le N/2-1 \\ k \neq 0}} \sqrt{\tfrac{g}{k} \tanh k(h_0/a)}\; \tilde{\phi}_N(k) \exp(ikx).$$

Thus the operator $\left[K_{h_0/a}\right]_N$ is the truncation at the $N/2$-st Fourier mode of the operator given by the periodic convolution with $K_{h_0/a}$. Note that this formulation includes the truncation of the Fourier mode $\tilde{\phi}_N(-N/2)$ which otherwise can lead to instabilities in the computation. The equation (26) is treated pseudospectrally. That is, multiplication is carried out in physical space, while the term involving $K_{h_0/a}$ is evaluated using the discrete Fourier transform.

The resulting system of equations can be solved using any standard nonlinear equation solver. We have chosen to use the Matlab routine *fsolve* which appears to work very efficiently. To make sure that the computed functions are approximate traveling waves for the Whitham equation, we have also used a dynamic integrator for the time-dependent Whitham equation. The equation (6) is translated to the interval $[0, 2\pi]$ by the scaling $\eta(x,t) \to \frac{1}{a}\eta(ax,t)$, where $a = \frac{L}{\pi}$ as before. The discretization is then defined by the following problem. Find a function $\eta_N : [0,T] \to S_N$, such that

$$\left\{ \begin{array}{l} \partial_t \eta_N + \frac{3}{2}\frac{c_0}{h_0}\partial_x I_N(\eta_N^2) + \frac{1}{\sqrt{a}}\left[K_{h_0/a}\right]_N * \partial_x \eta_N = 0, \quad x \in [0, 2\pi], \\ \eta_N(\cdot, 0) = \phi_N. \end{array} \right. \tag{27}$$

In Figure 2, a branch of traveling-wave solutions is shown. Here the wavelength is chosen to be 2π, and the depth is $h_0 = 1$. Note that in this case, the wavenumber is $k = \frac{2\pi}{2L} = 1$, and therefore the phase velocity of a linear wave is given by $\sqrt{g \tanh(h_0)} \sim 2.7334$. In panel (c) shown in Figure 2, it appears that as the amplitude approaches zero, the velocity of the traveling wave approaches the linear wave speed. Note also that not the whole branch is shown in panels (a) and (b). Two periods of the highest wave we were able to compute is shown in panel (d). This solution seems to nearly have a cusp, a fact already noted by Whitham [34] using an asymptotic argument. Since a Fourier-collocation method is used, it is implicitly assumed that the solutions are smooth, and it is not possible to find the very highest wave predicted by Whitham. A possible method for finding the highest wave is outlined in [8], where a scheme based on Lagrange polynomials is used, and the highest point on the wave is treated as a boundary condition. However, the Whitham equation as it appears here was not treated in [8]. In Table 1, we record the numerical errors incurred by the time integration of an approximate traveling wave with velocity $c = 2.7$ propagating for 5 and 50 periods. To find the most advantageous combinations of the number of Fourier modes N and the time step h, we used a computation for one period. We then use this combination, and integrated for

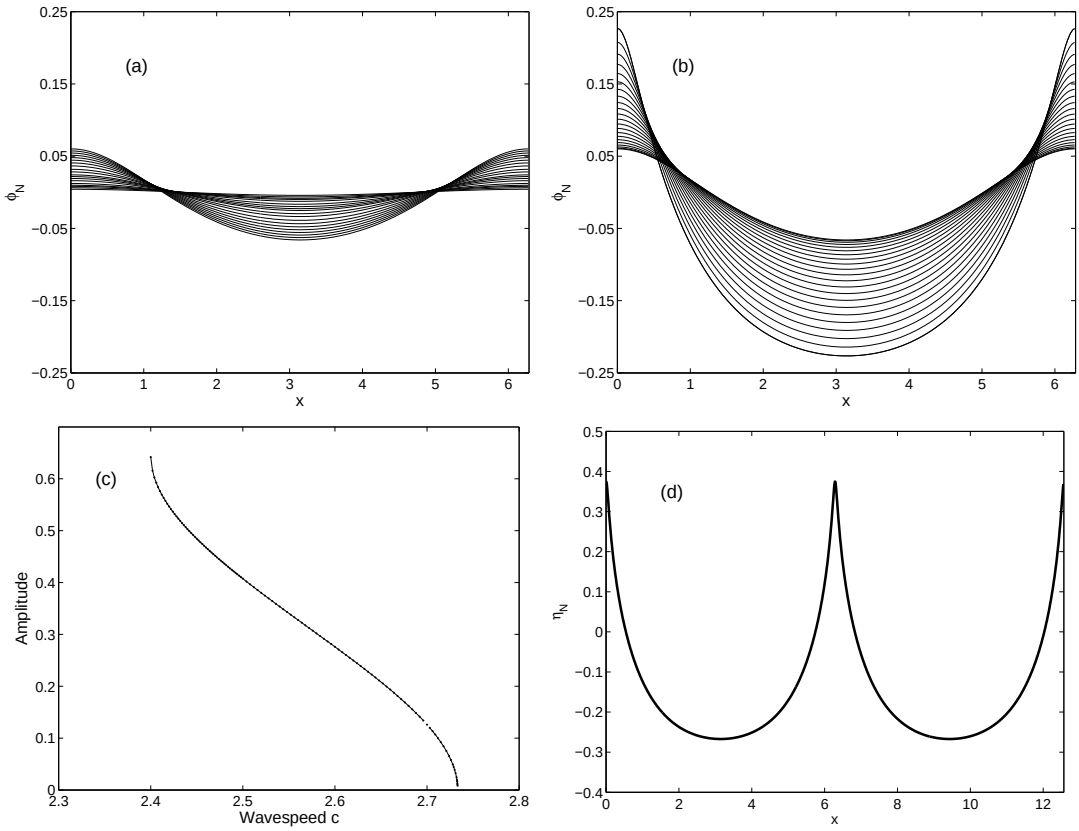

Figure 2. (a) and (b) Part of a branch of solutions of (26) with $h_0 = 1$ and $L = \pi$. Note that the highest wave is not shown here. (c) Amplitude vs. wave speed. (d) Two periods of the (nearly) highest wave.

N	h	L^2-error	$\lvert u\rvert_\infty - \lvert u_N\rvert_\infty$	L^2-error	$\lvert u\rvert_\infty - \lvert u_N\rvert_\infty$
2^5	1.0e-03	7.092e-04	9.927e-04	0.0078	0.0045
2^6	1.0e-03	3.821e-06	3.606e-06	3.316e-05	3.022e-06
2^7	1.0e-04	6.058e-06	1.208e-08	9.899e-07	6.675e-09
2^8	5.0e-06	1.217e-07	2.038e-11	2.198e-07	5.417e-11
		5 periods		50 periods	

Table 1. Error in evolution code after 5 and 50 periods for the traveling wave shown in Figure 3.

5 and 50 periods. The discrete L^2-error, the difference in maximal height between the original wave, and the profile after 5 and 50 periods were computed. As can be seen, the error is decreasing for increasing N and decreasing h. Moreover, the fact that the difference in maximal height is generally smaller than the L^2-error suggests that the error incurred during the time evolution is mostly due to a phase shift of the solution. This can also be observed in Figure 3, where the same traveling wave is shown after time

integration for 10000 periods. These results also suggest that the traveling waves are orbitally stable, but no special investigation of this question has been carried out.

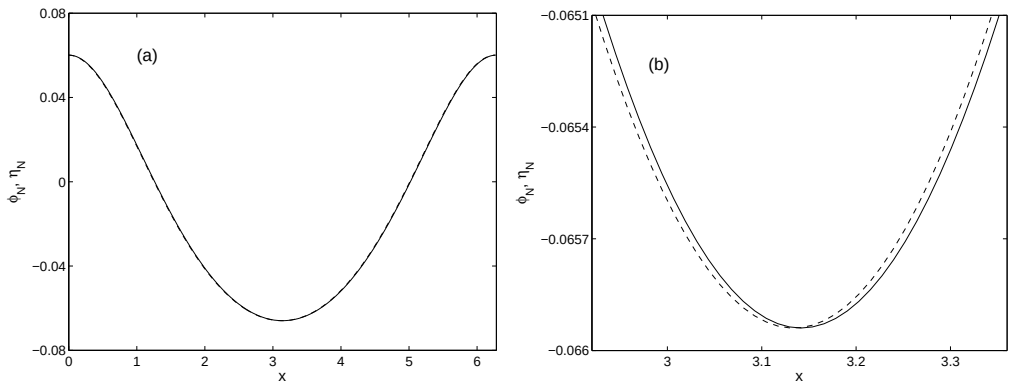

Figure 3. Solid line: approximate traveling wave ϕ_N for the Whitham equation with $h_0 = 1$, $L = \pi$, and $c = 2.7$. Dashed line: η_N after time integration using (27) for 10000 periods. In (a), the difference between ϕ_N and η_N is hardly visible. In this computation, $N = 512$ and $h = 0.0005$. The L^2 error was 0.0021, while the difference in height was $2.2385e-06$. This and the magnification (b) suggests that the the error is mainly due to a phase shift.

In Section 4, a connection between traveling waves with finite period and solitary waves is given. In particular, it is shown that if the amplitude of a family of traveling waves with increasing wavelength L is bounded below $\frac{1}{2}$, then these traveling waves converge to a solitary wave. Here, we want to illustrate this result numerically. In Figure 4, a family of approximate traveling waves is shown in the case when the wavelength L in increasing, while h_0 and c are held constant. Note that amplitude is initially increasing, but seems to level off to an approximate value of 0.145. As Figure 5 shows, even though the wavelength L keeps increasing, the shape of the traveling waves does not change very much if a certain threshold is passed. The numerical evidence suggests that these waves converge to a solitary wave, as was intimated by the proof in Section 4. Generally, a solitary wave is assumed to decay to zero at infinity. For the limiting solitary wave suggested in figures 4 and 5, this can be achieved by a Galilean transformation of the form

$$\phi \to \phi + \gamma \quad \text{and} \quad c \to c + 2\gamma.$$

This introduces a non-zero constant B in equation (11). However, it can be seen that the constant levels off to zero as the amplitudes of the sequence of traveling wave approaches the asymptotic value as shown in Figure 4 (b).

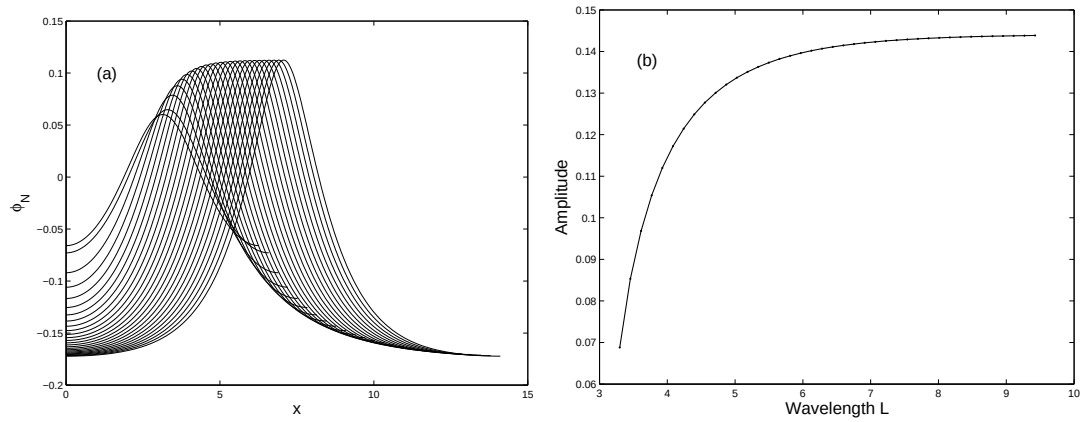

Figure 4. (a) Approximate traveling waves for the Whitham equation with $h_0 = 1$ and $c = 2.733$, and with increasing wavelength. (b) Amplitude as a function of wavelength.

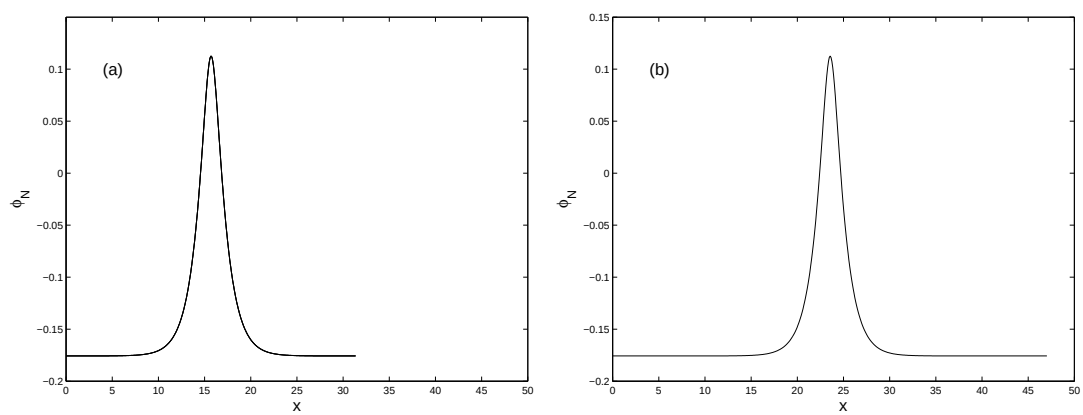

Figure 5. (a) Approximate traveling wave for the Whitham equation with $h_0 = 1$, $c = 2.733$, and $L = 5\pi$. (b) Approximate traveling wave for the Whitham equation with $h_0 = 1$, $c = 2.733$, and $L = 7.5\pi$.

7. Conclusion

We have investigated the existence of traveling-wave solution of the Whitham equation, a nonlinear dispersive integro-differential equation capable of supporting breaking and peaking solutions. It has been found that small-amplitude traveling-wave solutions exist. Moreover, in the limit as the wavelength goes to infinity, these solutions converge to traveling-wave solutions on the real line. Nontrivial bounded and continuous solutions do not exist if the wave speed c is much larger than the limiting long-wave speed c_0. Numerical approximations have been found of various traveling-wave solutions, including small-amplitude and finite-amplitude waves, as well as waves which are near the highest

wave which is known to have a cusp. As the wavelength increases, the traveling waves appear to converge to a solitary wave.

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