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Control of Preferences in Social Networks

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We consider the problem of deriving optimal marketing policies for the spread of innovations in a social network. We seek to compute policies that account for i) endogenous network influences, ii) the presence of competitive firms, that also wish to influence the network, and iii) possible uncertainties in the network model. Contrary to prior work in optimal advertising, which also accounts for network influences, we assume a dynamical model of preferences and we compute optimal policies for either a finite or infinite horizon. The optimal policies are related to and extend priorly introduced notions of centrality measures usually considered in sociology. We also compute robust optimal policies for the case of misspecified dynamics or uncertainties which can be modeled as external disturbances of the nominal dynamics. We show that the optimization exhibits a certainty equivalence property, i.e., the optimal values of the control variables are the same as if there were no uncertainty. Finally, we investigate the scenario where a competitive firm also tries to influence the network. In this case, robust optimal solutions are computed in the form of i) Nash and Stackelberg solutions, and ii) max-min solutions.

Key words: Dynamic programming/optimal control: Applications, Marketing: Advertising and media,
Games/group decisions: Differential, Networks/graphs: Applications

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1. Introduction

This paper is concerned with the derivation of optimal marketing strategies in a social network of customers whose preferences are affected by both their neighbors' preferences and the incentives provided through advertising. Similar questions appear in different formulations, for example, the adoption of dominant strategies in a network of strategic players Ellison (1993), Young (2001), Jackson and Watts (2002), the convergence of beliefs in a social network Golub and Jackson (2007) or the influence of word-of-mouth communication in the adoption of new products Alkemade and Castaldi (2005), Dubey et al. (2006). In all these formulations, the question remains the same,

that is: *what is the group of agents that we should target so that the maximum cascade of further influences results?*

This work is closely related to the literature on optimal advertising starting with Vidale and Wolfe (1957) in a monopoly framework and it has been extended to differential games in oligopolies, a detailed survey of which can be found in Jørgensen and Zaccour (2004). The main objective of this line of work, as very well stated in Sethi (1977), is to *set up an optimal control problem to determine the optimal rate of advertising expenditures over time in a way that maximizes the net profit of the firm*. To this end, prior work has focused on i) the derivation of *dynamic models* which capture the sales response to advertising, and ii) the computation of an *optimal policy* of advertising as a function of the sales.

Those models which capture the effect of advertising on sales are usually described by means of a differential or difference equation which describes the evolution of the state (*sales rate* or *market share*) as a function of the state and the advertising expenditures. We will assume that firms have some way of knowing or estimating the dynamics of sales response to advertising. The estimation of these dynamics will not be part of this work. Moreover, several sales-to-advertising models are also a function of other properties, such as price or quality, which will not be considered here.

Prior sales-to-advertising models usually capture the following phenomena: i) advertising effects persist over the current period but diminish with time Vidale and Wolfe (1957), ii) marginal advertising effects diminish or remain constant with the size of advertising Leitmann and Schmitendorf (1978), iii) advertising effects diminish with the size of sales Vidale and Wolfe (1957), Case (1979), Deal (1979), iv) advertising effects diminish with the size of competitive advertising Deal and Zions (1973), Case (1979), Erickson (1985, 1992), Chintagunta and Vilcassim (1992), Fruchter and Kalish (1997), and v) advertising effects are affected by *word-of-mouth communication* (or *excess advertising*) Jørgensen (1982).

Depending on the formulation of sales response to advertising, models have also been categorized in: i) *sales response models* (where the state is the rate of sales) Vidale and Wolfe (1957), ii) *market share models* (where the state is the share of the market) Case (1979), iii) *diffusion models* (which

capture the market growth) Bass (1969), and iv) *goodwill models* (which capture the evolution of advertising capital) Nerlove and Arrow (1962).

Our model is also related to those models. It exhibits *diminishing returns with time* in the absence of advertising effort, *constant marginal returns with the size of advertising*, and *diminishing returns with the size of competitive advertising*. It extends traditional advertising models by also considering the effect of word-of-mouth communication through a *network of interactions* similarly to Alkemaded and Castaldi (2005), Dubey et al. (2006). However, the analysis here is not restricted to the equilibrium state of the evolution of preferences. Instead, the dynamics of network effects become part of the optimization. Using this model, we are able to derive analytically optimal advertising strategies which are related to and extend priorly introduced notions of centrality measures usually considered in sociology Bonacich (1987).

Due to the inclusion of network interactions in the derivation of the optimal marketing strategy, this work is also related to the *diffusion of innovations* and *cascading phenomena in social networks* Domingos and Richardson (2001), Richardson and Domingos (2002), Goldenberg et al. (2001), Kempe et al. (2003). In such models, each customer may purchase the marketed product with a probability that depends on the neighbors' probabilities of purchasing the product. Then, the optimal marketing plan can be derived based on the expected increase in profit that this marketing plan incurs. Of course, the computation of such optimal marketing plan will depend on the amount of influence each customer has on its neighbors, a notion that is usually termed as the *network value* of a customer Richardson and Domingos (2002). Several other models of interactions have been proposed including the *linear threshold model* motivated by Schelling (1978), Granovetter (1978), where nodes become activated if the number of activated neighbors exceeds a threshold. Another model of interactions is the *independent cascade model* of Goldenberg et al. (2001), where, once a node is activated, it is given the chance to activate its neighbors, while its success depends on a probability distribution which is independent for each node. One characteristic of these models is the computational complexity of computing the set of nodes that, if targeted, will incur the largest possible influence in the network of customers. Furthermore, it is assumed that there is a *unique*

seller who is trying to influence the network, ignoring this way the potential effect that a competitive seller may have on its sales.

Due to the complexity resulting from the inclusion of network interactions, the computation of the optimal policy of a firm might be challenging. For example, Kempe et al. (2003) deals with the algorithmic question posed by Domingos and Richardson (2001) on how we should select the set of nodes that will cause the largest possible influence in the population. In fact, an approximation algorithm is proposed, based on the submodularity property of the influence function and in the context of the *linear threshold model* of Schelling (1978), Granovetter (1978) and the *independent cascade model* of Goldenberg et al. (2001), that computes the optimal set of nodes with a performance guarantee of 63%. A similar algorithmic approximation is derived by Bharathi et al. (2007) for the computation of best responses in the presence of multiple firms (innovations) under the framework of Goldenberg et al. (2001).

For the study of competition when multiple firms are present, a game theoretic model is proposed by Goyal and Kearns (2011). According to this model, two firms are competing for the diffusion of innovations in a given network, where diffusion follows a form of threshold dynamics similar to Granovetter (1978). Goyal and Kearns (2011) is dealing with the computation of upper bounds of the *price of anarchy*, and how network structure may amplify the initial budget differences. A similar network diffusion model has also been considered by Fazeli and Jadbabaie (2012), where nodes update their preferences upon arrival of a Poisson clock and according to the payoffs received by playing a coordination game with their neighbors. Furthermore, Nash equilibria are computed for the strategic interaction between the two firms assuming the smallest possible adoption for each strategy.

This paper is also concerned with the computation of optimal marketing policies in the presence of word-of-mouth communication (due to the network structure) and multiple firms. Its contribution, which distinguishes it from prior literature, lies in the combination of three important factors i) *dynamic network effects* in the formation of preferences which are included in the optimization,

ii) *misspecifications/uncertainties* in the assumed model of evolution of preferences, and iii) *uncertainty* in the intentions of a competitive firm that also tries to influence the network. Although network influences in the formation of preferences are present in several models, the optimization is usually performed at steady-state, e.g., Goyal and Kearns (2011) or Fazeli and Jadbabaie (2012). Here, instead the dynamics of preferences become part of the optimization. Furthermore, although uncertainties due to the presence of a competitive firm might be taken into account in several models, we would like to also compute optimal marketing policies under the presence of uncertainties in the network structure. Usually stochastic extensions of existing models have been considered, e.g., Sethi (1983), Prasad and Sethi (2004). In this paper, we would like to consider uncertainties that can incorporate possible unmodeled dynamics. Under these perturbed dynamics, we formulate a max-min optimization to compute an optimal policy which is robust to a class of norm-bounded uncertainties. We show that the optimization exhibits a *certainty equivalence property*, that is, the optimal values of the control variables are the same as if there were no uncertainty.

Finally, we investigate the possibility that a competitive firm also tries to influence the network, introducing a second form of uncertainty. In this case, and when the objective of the competitive firm is to maximize its sales, the strategy of the competitive firm may not be known. We will either assume that i) *the competitive firm is a competitive fridge* which tries to enter the market, introducing a notion of sequential optimization (expressed by a Stackelberg solution), or ii) both firms have the ability of simultaneous play (expressed by a Nash solution). Under these scenarios, we provide a complete characterization of open-loop Nash and Stackelberg solutions. These solutions are also a subset of closed-loop (or Markovian) Nash solutions. A complete characterization of the set of closed-loop Nash solutions is going beyond the scope of this paper, since it is highly case-dependent, i.e., it depends on the class of policies which will be considered reasonable for the scenario of interest. However, the proposed framework can easily be utilized to provide closed-loop Nash solutions for a specified class of policies. Finally, we investigate the scenario where firms are also uncertain about the objectives of the competitor, which can be formulated as a max-min optimization.

The remainder of the paper is organized as follows. Section 2 describes the problem under consideration. Section 3 discusses some necessary background on dynamic programming. Section 4 derives finite- and infinite-horizon optimal policies in a monopoly under unperturbed and perturbed preferences' update. Section 5 computes Stackelberg and Nash solutions in a duopoly. Finally, Section 6 presents concluding remarks.

Notation: For any vector $x \in \mathbb{R}^n$, where x_i is its i th entry,

- $|x|$ denotes its Euclidean norm,
- $|x|_\infty \triangleq \max\{|x_1|, \dots, |x_n|\}$,
- $\max_1^+(x) \triangleq \max\{0, x_1, x_2, \dots, x_n\}$,
- $\max_i^+(x) \triangleq \max\left\{\{0, x_1, x_2, \dots, x_n\} \setminus \bigcup_{k=1}^{i-1} \max_k^+(x)\right\}$, $i > 1$,
- for some $\alpha > 0$, $\text{sat}(x; \alpha) \triangleq (y_1, y_2, \dots, y_n)$ such that

$$y_i = \begin{cases} \alpha & x_i \geq \alpha \\ x_i & 0 < x_i < \alpha, \\ 0 & x_i \leq 0 \end{cases} \quad i = 1, 2, \dots, n.$$

2. Problem Description

2.1. Evolution of preferences

The problem considers a pair of firms $\mathcal{L} = \{a, b\}$ and a finite set of customers or nodes $\mathcal{I} = \{1, 2, \dots, n\}$.¹ We will denote a firm by $\ell \in \mathcal{L}$ and a customer by $i \in \mathcal{I}$. Although we assume that nodes represent customers, we may also consider the case where a node $i \in \mathcal{I}$ represents a group of customers with similar characteristics. Nodes are connected through a directed network whose links are described by a row stochastic matrix W .² The matrix W captures how nodes' proclivities towards the product of either firm a or firm b are affected by its neighbors.

Let $x_{i,k}^\ell \geq 0$ be the proclivity of node i towards buying the product of firm $\ell \in \{a, b\}$ at time k , and $x_k^\ell \triangleq (x_{1,k}^\ell, x_{2,k}^\ell, \dots, x_{n,k}^\ell) \in \mathbb{R}_+^n$ be the vector of proclivities over the whole network. We will refer to this vector as the *state* of firm ℓ and we will denote by $\mathcal{S}^\ell \subset \mathbb{R}_+^n$ the corresponding set of states.

Firm $\ell \in \mathcal{L}$ is able to influence the proclivity of node $i \in \mathcal{I}$ towards its product by marketing its product to node i , e.g., by offering discounts or warranties. Let $u_{i,k}^\ell \geq 0$ denote the amount of funds

that firm ℓ spends on marketing its product to node i at time k , and $u_k^\ell \triangleq (u_{1,k}^\ell, u_{2,k}^\ell, \dots, u_{n,k}^\ell) \in \mathbb{R}_+^n$ be the vector of funds firm ℓ spends over the set of nodes \mathcal{I} . We will refer to this quantity as the *control* of firm ℓ . We will assume that the amount of funds each firm can spend at any given time cannot be larger than M^ℓ , i.e.,

$$\sum_{i \in \mathcal{I}} u_{i,k}^\ell \leq M^\ell \quad \text{for all } k = 0, 1, \dots \quad (1)$$

Let $\mathcal{C}^\ell \subset \mathbb{R}_+^n$ denote the resulting constraint set of controls.

The specific relation between the controls and the states is motivated by the work of Dubey et al. (2006), Friedkin (2001) on social influence network theory and it is described by the following difference equation:

$$x_{k+1}^\ell = \Theta W x_k^\ell + (I - \Theta) \varphi(u_k^\ell, u_k^{-\ell}) \quad (2)$$

which provides the proclivity of node i at time $k+1$ as a convex combination of i) a weighted average of the proclivities of the neighbors and ii) the external influence caused by both own and competitive advertising. The notation $-\ell$ denotes the complementary set $\mathcal{L} \setminus \ell$. The matrix Θ satisfies:

$$\Theta = \text{diag}\{\theta_1, \theta_2, \dots, \theta_n\}, \quad 0 \leq \theta_i < 1, \quad \forall i \in \mathcal{I}. \quad (3)$$

The constraint (3) has a natural interpretation since it implies that there is no node that completely ignores external influence. Furthermore, in the absence of external influence, it also models diminishing returns with time. We will simplify notation by rewriting the dynamics in the form:

$$x_{k+1}^\ell = A x_k^\ell + B \varphi(u_k^\ell, u_k^{-\ell}), \quad (4)$$

where $A \triangleq \Theta W$ and $B \triangleq I - \Theta$. Variations of this nominal model will also be considered later on in this paper when firms are uncertain about the accuracy of the model.

The function $\varphi: \mathcal{C}^\ell \times \mathcal{C}^{-\ell} \rightarrow [0, \alpha_1] \times \dots \times [0, \alpha_n]$, for some $\alpha_i > 0$, $i \in \mathcal{I}$, maps the control vectors of both firms to a vector of influences over the set of nodes \mathcal{I} . It is assumed to be nonnegative and bounded above. We will refer to this function as the *influence function*. We would like function φ to also satisfy:

ASSUMPTION 1. *The influence function $\varphi : \mathcal{C}^\ell \times \mathcal{C}^{-\ell} \rightarrow [0, \alpha_1] \times \dots \times [0, \alpha_n]$, for some $\alpha_i > 0$, $i \in \mathcal{I}$, is such that:*

1. $\varphi_i(u_k^\ell, u_k^{-\ell}) \geq 0$, if $u_{i,k}^\ell \geq u_{i,k}^{-\ell}$;
2. $\varphi_i(u_k^\ell, u_k^{-\ell}) = 0$, if $u_{i,k}^\ell < u_{i,k}^{-\ell}$.

That is, a customer would be influenced towards either one of the firms depending on the relative size of their advertising. One candidate function which satisfies the above property is:

$$\varphi_i(u_k^\ell, u_k^{-\ell}) \triangleq \text{sat}(u_{i,k}^\ell - u_{i,k}^{-\ell}; \alpha_i) \quad (5)$$

for some $\alpha_i > 0$, $i = 1, 2, \dots, n$.

We will refer to the above model as **duopoly**. When, instead, $u_{i,k}^{-\ell} \equiv 0$ for all $i \in \mathcal{I}$ and $k = 0, 1, \dots$, we will refer to this model as **monopoly**.

The proposed update of preferences exhibits constant marginal returns with the size of own advertising and diminishing returns with the size of competitive advertising, which is due to the definition of the influence function. It also exhibits diminishing returns with time, due to the definition of the matrix Θ . Finally, it models the effect of word-of-mouth (or excess) advertising due to the assumed network of connections.

2.2. Objective

The *utility* of firm $\ell \in \mathcal{L}$ at time k is defined as:

$$g(x_k^\ell, u_k^\ell) = V(x_k^\ell) - C(u_k^\ell) \quad (6)$$

where we assume that the *reward* is linear with the proclivities of the nodes, i.e., $V(x_k^\ell) = v^\top x_k^\ell$, for some vector $v \in \mathbb{R}_+^n$, and the *cost* is linear with the funds spent on advertising, i.e., $C(u_k^\ell) = c^\top u_k^\ell$, for some $c \in \mathbb{R}_+^n$.

For some discount factor $\beta \in (0, 1)$, the *objective* of firm ℓ has the following form

$$\max_{\pi^\ell \in \Pi^\ell} \left\{ J_{\pi^\ell}(x) \triangleq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k g(x_k^\ell, \mu_k^\ell(x_k^\ell)) \right\} \quad (7)$$

over the set of infinite sequences of policies Π^ℓ with elements $\pi^\ell = (\mu_0^\ell, \mu_1^\ell, \dots)$ where μ_k^ℓ is a function from the set of states \mathcal{S} to the set of controls \mathcal{C} . The above optimization is subject to the dynamics (4). Later on, we are also going to consider variations of this optimization, especially when dynamics (4) are perturbed and robust optimal policies need to be derived.

For the remainder of the paper, the proposed advertising model characterized by the dynamics (4) and the utility function (6) will be denoted by \mathfrak{M} .

2.3. Assumptions and preliminaries

For the remainder of the paper, we are going to consider the following assumptions:

ASSUMPTION 2. $\beta v^T B - c^T > 0$.

That is, $\beta v_i(1 - \theta_i) - c_i > 0$, $i = 1, 2, \dots, n$, i.e., for every unit of advertising effort, *the discounted return of each node is strictly greater than the corresponding cost*. This is a reasonable assumption and it is also related to the existence of a non-degenerate solution to the optimization problems considered herein.

ASSUMPTION 3. $\alpha_i^\ell \geq M_i^\ell$ for all $i \in \mathcal{I}$ and $\ell \in \mathcal{L}$.

This assumption implies that each node's capacity of getting influenced through advertising is larger than the advertising power of each firm. This is not a necessary assumption for the existence of solutions, however, it simplifies the following analysis. The derivation of the corresponding solutions in case Assumption 3 does not hold is also straightforward and qualitatively remains identical.

In the presentation of the model, we have implicitly assumed that the evolution of preferences is governed by identical dynamics for both firms. This assumption allows for a cleaner presentation of the analysis, however, as it will become obvious later, it does not change qualitatively the solutions.

We also assume that the utility functions of both firms are of the same form. This implies that benefits and costs are materialized as a function of the proclivities and investments similarly for both firms. This is a reasonable assumption, however, the following analysis can be easily modified to include the case of different utility functions.

Note, finally, that the proposed dynamics (4) constitute a linear time-invariant system with bounded inputs. It is straightforward to show that the above system is *input-output stable* in the sense that there exist nonnegative constants ζ, θ such that the solution to the difference equation, denoted $x(k, x_0, u)$, satisfies $|x(k, x_0, u)| \leq \zeta + \theta \|u\|_\infty$, where $\|u\|_\infty \triangleq \sup\{|u_k| : k \in \mathbb{Z}_+\}$. This is due to the fact that W is a row stochastic matrix and Θ satisfies the constraint (3). The constraint (3) on matrix Θ also implies the *controllability* (cf., Kailath (1980)) of the system (A, B) , simply because $\text{rank}(B) = \text{rank}(I - \Theta) = n$.

2.4. Alternative models and discussion

The dynamics (4) are based on the assumption that agents are bounded rational, since their preferences are a weighted average of neighbors' preferences. Full rationality instead may not necessarily lead to better models due to the resulting computational complexity. A similar model in the context of evolution of preferences without external influence has also been considered by Friedkin and Johnsen (1999), Golub and Jackson (2007) to study the diffusion of innovations and norms in a social network. This model has also been related to alternative measures of centrality Bonacich (1987), Friedkin (1991).

In this paper, we modified the model used by Friedkin and Johnsen (1999), Golub and Jackson (2007) to include the possibility of an external control influence (4), e.g., due to advertising effects. The proposed model bears similarities with several previously introduced advertising models, e.g., the goodwill models of Nerlove and Arrow (1962), new product diffusion models Bass (1969) or extensions of the Vidale-Wolfe model Vidale and Wolfe (1957). In the following subsections we discuss some of the similarities and differences between these models with the proposed \mathfrak{M} .

2.4.1. Comparison with goodwill models *Advertising goodwill* models (see, e.g., (Jørgensen and Zaccour 2004, Section 3.5)) capture the evolution of the advertising capital. For example, the advertising goodwill model introduced in the seminal paper Nerlove and Arrow (1962) assumes the following dynamics

$$\dot{G}(t) = u(t) - \delta G(t), \quad (8a)$$

where $G(t)$ represents the advertising capital. The main difference with the proposed model \mathfrak{M} is that the latter includes directly the interpersonal influences through the assumed communication network, thus modeling a form of word-of-mouth communication. Note also that the control input or advertising effort u influences directly the advertising capital. Similar is the assumption in \mathfrak{M} , where the advertising effort directly influences the preferences of all nodes. This is not necessarily the case in other advertising models, where the advertising effort only applies to the *undecided* part of the population.

The dynamics (8a) can also be modified to include the possibility of multiple firms, e.g., the models in Fershtman (1984), Chintagunta (1993). For example, the model considered in Chintagunta (1993) assumes

$$\dot{G}_i(t) = \sqrt{u_i(t)} - \delta G_i(t), \quad G_i(0) = G_{i0} > 0, \quad i \in \{1, 2\}, \quad (8b)$$

and the sales rate x_i (similarly to the proposed vector of proclivities) depends on the advertising capital of both firms, i.e., $x_i = x_i(G_1, G_2)$, where $\partial x_i / \partial G_i > 0$ and $\partial x_i / \partial G_j < 0$ for $i \neq j$.

Note that the square root of the control input in (8b), which has also been used in other advertising models (see, e.g., Case (1979)), captures *diminishing marginal returns with the size of advertising effort*. Alternatively, diminishing marginal returns can also be modeled indirectly by considering a squared cost in the utility function. For example, in Deal (1979) the term u_i^2 is considered instead in the cost function, or in Gould (1970) more general non-linear functions of u_i are considered which are convex increasing. In \mathfrak{M} , instead, diminishing/constant marginal returns with the advertising effort are modeled indirectly by assuming the saturation effect in the influence function.

A squared cost term in the utility function could also have been included in the proposed model \mathfrak{M} . For example, an alternative utility function that incorporates diminishing marginal returns with the size of advertising could be:

$$g(x_k^\ell, u_k^\ell) = v^\top x_k^\ell - (u_k^\ell)^\top C u_k^\ell \quad (9)$$

where $C \triangleq \text{diag}(c)$, i.e., C is a diagonal matrix where the diagonal entries coincide with the entries of

the vector c . Some of the nice analytical properties of \mathfrak{M} are also shared by the above quadratic cost function (9), such as the forthcoming analytical solution of the monopoly optimization problem.

2.4.2. Comparison with market-share response models The goodwill advertising models and the proposed model \mathfrak{M} differ from market-share response models emanating from the model of Vidale-Wolfe, Vidale and Wolfe (1957). An extension of this model to a duopoly has been considered by Deal and Zions (1973):

$$\dot{x}_i = (1 - x_i - x_j)u_i - \delta_i x_i, \quad x_i(0) = x_{i0}, \quad (10a)$$

for all $i, j \in \{1, 2\}$, $i \neq j$, and for some constants $\{\delta_i\}$. A small modification Deal et al. (1979) can also account for *excess advertising* effects due to word-of-mouth influences in the population, e.g.,

$$\dot{x}_i = (1 - x_i - x_j)u_i - \delta_i x_i + e_i(u_i - u_j)(x_i + x_j), \quad x_i(0) = x_{i0}, \quad (10b)$$

for all $i, j \in \{1, 2\}$, $i \neq j$, and for constants $\{e_i\}$, where the last term represents the word-of-mouth switching from j to i .

Contrary to both \mathfrak{M} and the goodwill advertising models, where the advertising effort applies directly to the whole population, in the market-share response generalizations of Vidale-Wolfe's model Vidale and Wolfe (1957), the control applies only to the *undecided* part of the population. The last term of the dynamics (10b), which models excess advertising, applies to the *decided* part of the market and models transfers due to excess of advertising. This term also resembles the influence function φ considered in \mathfrak{M} , where the influence on a node depends only on the excess part of the advertising efforts at that node.

Note, however, that a small modification of \mathfrak{M} can account for behaviors that are present in the market-share models Vidale and Wolfe (1957). For example, if we instead consider the influence function:

$$\varphi_i(u_k^\ell, u_k^{-\ell}) \triangleq \text{diag}(\alpha^\ell \mathbf{1} - x_k^{-\ell}) u^\ell - \text{diag}(\alpha^\ell \mathbf{1} - x_k^\ell) u^{-\ell}, \quad (11)$$

then the advertising efforts of either firm applies only on the part of the market which is either

undecided or has different preferences. When we assume the alternative dynamics with the influence function (11), then an analytical derivation of a closed-form solution, even for the monopoly framework, is not feasible any more. In the forthcoming analysis, we will only consider the initially proposed influence function which provides closed-form solutions, however future work may include alternative forms of the influence function that may accept only numerical solutions.

Similar remarks also hold for the models emanating from the Lanchester model of combat, such as the models of Kimball (1957), Erickson (1985, 1992), Chintagunta and Vilcassim (1992), Fruchter and Kalish (1997). The main difference of Lanchester models with the Vidale-Wolfe models is that in the latter ones the effect of competitive advertising onto the market share is indirectly included (through the undecided portion of the market). Instead, in the Lanchester models, the effect of competitive advertising is directly included in the dynamics of market share.

This discussion reveals the flexibility of the proposed model \mathfrak{M} to incorporate alternative behaviors or modeling ideas which have already been discussed in prior literature. In several cases though, it is desirable that a sales-to-advertising model also provides closed-form solutions. The proposed model \mathfrak{M} and its extensions herein exhibit most of the observed phenomena of sales-to-advertising models and, as we will discuss later, it provides attractive closed-form expressions of optimal strategies under several scenarios.

3. Dynamic Programming Background

The notation and part of the analysis in this section follows Bertsekas and Shreve (1978).

3.1. The dynamic programming algorithm

Denote by \mathcal{J} the set of all extended real-valued functions of the form $J: \mathcal{S} \rightarrow \mathbb{R}^*$, defined on the state space \mathcal{S} and taking values on the extended real line $\mathbb{R}^* = [-\infty, +\infty]$.

For some time horizon $N \in \mathbb{N}$, consider the generic finite-horizon optimization problem:

$$\max_{\pi \in \Pi} \left\{ J_{N,\pi}(x_0) \triangleq E \left\{ g(x_N) + \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k, w_k) \right\} \right\} \quad (12)$$

over any admissible policy $\pi = \{\mu_0, \mu_1, \dots, \mu_{N-1}\} \in \Pi$, where $\mu_k \in \mathcal{M}$ for all k , and \mathcal{M} is the set of functions from the set of states \mathcal{S} to the set of controls \mathcal{C} . Furthermore, $g(x_N)$ defines the cost at the final stage, which depends only on the final state x_N .

The above optimization is subject to the system dynamics $x_{k+1} = f(x_k, u_k, w_k)$, where $\{w_k\}$ denotes a noise sequence taking values in a measurable space $(\mathcal{W}, \mathfrak{F})$. Denote $J_N^*(x)$ the optimal value of the N -stage objective function. Finally, assume that $|g(x, u, w)| < \infty$, for all $x \in \mathcal{S}$, $u \in \mathcal{C}$, and $w \in \mathcal{W}$.

For any function $J \in \mathcal{J}$, define the following function

$$(TJ)(x) \triangleq \max_{u \in \mathcal{C}(x)} E\{g(x, u, w) + \beta J(f(x, u, w))\}, \quad x \in \mathcal{S}.$$

Note that $(TJ)(\cdot)$ is the *optimal value function* for the one stage problem that has stage cost g and terminal cost βJ .

Also, we will denote by T^k the composition of the mapping T with itself k times; i.e., for all $k = 1, 2, \dots$, we write

$$(T^k J)(x) = (T(T^{k-1} J))(x), \quad x \in \mathcal{S}.$$

For convenience, we also write $(T^0 J)(x) = J(x)$.

Similarly, for any function $J \in \mathcal{J}$ and any policy $\mu : \mathcal{S} \rightarrow \mathcal{C}$, we denote:

$$(T_\mu J)(x) \triangleq E\{g(x, \mu(x), w) + \beta J(f(x, \mu(x), w))\}. \quad (13)$$

Again, $T_\mu J$ may be viewed as the cost function associated with the policy μ for the one-stage problem that has stage cost g and terminal cost βJ .

The **dynamic programming algorithm** (DP) is the following algorithm; for any $k = 1, \dots, N$ compute

$$J_k(x) = (TJ_{k-1})(x), \quad (14)$$

with initial condition $J_0(x) = g(x)$. The last step of the DP algorithm provides the N -stage value, $J_N(x)$, $x \in \mathcal{S}$.

Define

$$H(x, u, J) \triangleq E \{g(x, u, w) + \beta J(f(x, u, w))\}. \quad (15)$$

ASSUMPTION 4. *The above sequence $\{J_k\} \subset \mathcal{J}$ is a non-decreasing sequence satisfying $H(x, u, J_1) < \infty$, and*

$$\lim_{k \rightarrow \infty} H(x, u, J_k) = H(x, u, \lim_{k \rightarrow \infty} J_k),$$

for all $x \in \mathcal{S}$ and $u \in \mathcal{C}$.

The above assumption excludes problems where exchangeability of expectation with the limit is not possible. This assumption is satisfied when we consider a monotonously increasing sequence of functions $\{J_k\}$ in \mathcal{J} and also the functions J_k are measurable with respect to the probability measure under consideration. This will be due to the Lebesgue's Increasing Convergence Theorem (cf., Jones (1993)).

PROPOSITION 1 (**Optimality of DP**). *Let Assumption 4 hold, and assume that $J_{k,\pi}(x) < \infty$ for all $x \in \mathcal{S}$, $\pi \in \Pi$, and $k = 1, 2, \dots, N$. Then, $J_N^* = T^N(J_0)$.*

Proof. See Proposition 3.1 in Bertsekas and Shreve (1978). \square

3.2. Infinite horizon problems

Consider now the infinite horizon optimization problem:

$$\max_{\pi \in \Pi} \left\{ J_\pi(x_0) = \lim_{N \rightarrow \infty} E \left\{ \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k(x_k), w_k) \right\} \right\}, \quad (16)$$

over any admissible infinite policy $\pi = \{\mu_0, \mu_1, \dots\}$ and subject to the system dynamics $x_{k+1} = f(x, u, w)$. Let also define the optimal value of this problem as $J^*(x) \triangleq \sup_{\pi \in \Pi} J_\pi(x)$.

The following is a condition on the optimal stationary policy.

PROPOSITION 2 (**Optimal stationary policy**). *Consider the infinite horizon optimization problem of (16) and assume that $J_0(x) \leq H(x, u, J_0)$ for all $x \in \mathcal{S}$ and $u \in \mathcal{C}$ where $J_0(x) = g(x)$. Then, the optimal value of the infinite horizon optimization problem is $J^*(x) = \lim_{N \rightarrow \infty} J_N(x)$, where $J_N(x)$*

is the N -th stage value of the dynamic programming algorithm. Let also Assumption 4 hold. Then, a stationary policy $\pi^* = (\mu^*, \mu^*, \dots) \in \Pi$ is optimal if and only if

$$T_{\mu^*}(J_{\pi^*}) = T(J_{\pi^*}). \quad (17)$$

Proof. See Proposition 5.5 in Bertsekas and Shreve (1978). \square

4. Optimal Policy in Monopoly

In this section, we compute the optimal policy of a firm when there is no competitive firm, and also the dynamics are either a) unperturbed, or b) perturbed. Since we consider a single firm, we will skip the superscript ℓ for the remainder of this section.

4.1. Unperturbed dynamics

The dynamics we consider in this section are described by (4) with $u_k^{-\ell} \equiv 0$, i.e.,

$$x_{k+1} = Ax_k + B\varphi(u_k) \triangleq f(x_k, u_k). \quad (18)$$

In the remainder of this section, we compute the optimal policy for the 1) finite-horizon, and 2) infinite-horizon optimization problem.

First, define: $\tilde{A}_k \triangleq \sum_{j=0}^k \beta^j A^j$ and $h_{k+1}^T \triangleq \beta v^T \tilde{A}_k B - c^T$, for $k = 0, 1, \dots$. Note that $\tilde{A}_0 = I$ and $h_1^T = \beta v^T B - c^T$.

Before computing the solutions to the finite- and infinite-horizon optimization problems, note that:

CLAIM 1. $v^T \tilde{A}_{k+1} \geq v^T \tilde{A}_k$ for all $k = 0, 1, \dots$

Proof. First note that

$$\begin{aligned} v^T \tilde{A}_{k+1} &= v^T \sum_{j=0}^{k+1} \beta^j A^j \\ &= v^T \sum_{j=0}^k \beta^j A^j + v^T \beta^{k+1} A^{k+1} \geq v^T \tilde{A}_k. \end{aligned}$$

where the last inequality results from the fact that all the entries of matrix A are nonnegative. \square

4.1.1. Finite-horizon optimization We first consider the finite-horizon optimization

$$\max_{\pi \in \Pi} \left\{ J_{\pi}(x_0) \triangleq g(x_N) + \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k(x_k)) \right\}. \quad (19)$$

where $g(x) \triangleq v^T x$ defines the utility at the last stage.

PROPOSITION 3 (*N*-th stage optimal policy). *Consider the finite horizon optimization problem (19) under the dynamics (18). The *N*th stage optimal value of the dynamic programming iteration, is*

$$J_N^*(x) = v^T \tilde{A}_N x + \sum_{k=0}^{N-1} \beta^k h_{N-k}^T u_{N-k}^*. \quad (20)$$

The optimal control at time k , for $k = 0, 1, \dots, N-1$, is $u_{N-k}^* = (u_{1,N-k}^*, \dots, u_{n,N-k}^*)$, where

$$u_{i,N-k}^* = \begin{cases} M & i = \arg \max_1^+ (h_{N-k}) \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

Proof. We are going to show the statement by induction. According to the dynamic programming algorithm, the k -th stage optimal value is

$$J_k(x) = \max_{u_k \in \mathcal{C}(x)} \{g(x, u_k) + \beta J_{k-1}(f(x, u_k))\}$$

where $J_0(x) = g(x) = v^T x$. By applying the operator T to J_0 , we get the optimal value for the first stage, which is

$$\begin{aligned} J_1(x) &= \max_{u_1 \in \mathcal{C}(x)} \{g(x, u_1) + \beta J_0(f(x, u_1))\} \\ &= \max_{u_1 \in \mathcal{C}(x)} \{(v^T + \beta v^T A)x + (\beta v^T B - c^T)u_1\} \\ &= v^T \tilde{A}_1 x + h_1^T u_1^*. \end{aligned}$$

where the optimal stage control is $u_1^* = (u_{1,1}^*, \dots, u_{n,1}^*)$ such that

$$u_{i,1}^* = \begin{cases} M & i = \arg \max_1^+ (h_1) \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

Note that the value $J_1(\cdot)$ is given by expression (20) if we set $N = 1$ and the optimal stage control u_1^* is given by expression (21) if we set $N = 1$ and $k = 0$.

Assume that the value iteration for the N -step optimization horizon gives (20), i.e.,

$$J_N(x) = v^T \tilde{A}_N x + \sum_{k=0}^{N-1} \beta^k h_{N-k}^T u_{N-k}^* \quad (23)$$

where $u_{N-k}^* = (u_{1,N-k}^*, \dots, u_{n,N-k}^*)$ is such that

$$u_{i,N-k}^* = \begin{cases} M & i = \arg \max_1^+ (h_{N-k}) \\ 0 & \text{otherwise,} \end{cases}$$

for $k = 0, 1, \dots, N-1$.

Consider now an $(N+1)$ -step optimization horizon. The value at $(N+1)$ is:

$$\begin{aligned} J_{N+1}(x) &= (TJ_N)(x) \\ &= \max_{u_{N+1} \in \mathcal{C}} \{g(x, u_{N+1}) + \beta J_N(f(x, u_{N+1}))\} \\ &= v^T (I + \beta \tilde{A}_N A) x + \max_{u_{N+1} \in \mathcal{C}} h_{N+1}^T u_{N+1} + \beta \sum_{k=0}^{N-1} \beta^k h_{N-k}^T u_{N-k}^* \\ &= v^T (I + \beta \tilde{A}_N A) x + h_{N+1}^T u_{N+1}^* + \beta \sum_{k=0}^{N-1} \beta^k h_{N-k}^T u_{N-k}^* \\ &= v^T \tilde{A}_{k+1} x + \sum_{i=1}^{k+1} \beta^i (\beta v^T B \tilde{A}_{k-i+1} - c^T) u_{k-i+1}^* \end{aligned} \quad (24)$$

where $u_{N+1}^* = (u_{1,N+1}^*, \dots, u_{n,N+1}^*)$ is such that

$$u_{i,N+1}^* = \begin{cases} M & i = \arg \max_1^+ (h_{N+1}) \\ 0 & \text{otherwise,} \end{cases} \quad (25)$$

for $i = 1, 2, \dots, n$. Thus, we showed that the values of the dynamic programming iteration are provided by equation (20).

Finally, to show optimality of the dynamic programming iteration, subtract equations (23) from (24) to get:

$$\begin{aligned} J_{N+1}(x) - J_N(x) &= v^T (\tilde{A}_{N+1} - \tilde{A}_N) x + \\ &\quad \sum_{k=0}^{N-1} \beta^k (h_{N+1-k}^T u_{N+1-k}^* - h_{N-k}^T u_{N-k}^*) + \beta^N h_1^T u_1^*. \end{aligned}$$

By Claim 1, we have that $v^T (\tilde{A}_{N+1} - \tilde{A}_N) x \geq 0$ for all $x \in \mathcal{S}$. By Assumption 2 and (25), we also have

$$h_{N+1}^T u_{N+1}^* \geq h_N^T u_N^* \geq \dots \geq h_1^T u_1^* > 0.$$

Therefore, $J_{N+1}(x) \geq J_N(x)$ for all $x \in \mathcal{S}$ and Assumption 4 is satisfied. Then, by Proposition 1, the dynamic programming iteration provides the optimal value of the finite-horizon optimization (19). \square

The optimal marketing strategy given by (21) is a consequence of Assumption 3. As already pointed out, the corresponding optimal strategy when Assumption 3 does not hold qualitatively remains identical. In particular, it is straightforward to check that, in this case, the optimal control at time k will suggest that we should split the marketing resources among the largest entries of h_k , i.e., the maximum entry receives the largest share, the second maximum entry receives the largest share out of the remaining resources and so forth.

4.1.2. Infinite-horizon optimization We would like to solve the following optimization problem:

$$\max_{\pi \in \Pi} \left\{ J_\pi(x_0) \triangleq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k(x_k)) \right\} \quad (26)$$

subject to the discrete-time dynamics (18). First recall the definition of $H(x, u, J)$ from (15). Given also that $J_0(x) = v^T x$, it is straightforward to show, under Assumption 2, that:

CLAIM 2. $J_0(x) \leq H(x, u, J_0)$, for all $x \in \mathcal{S}$ and $u \in \mathcal{C}(x)$.

Note also that:

LEMMA 1. *The matrix $(I - \beta A)$ is non-singular for any $\beta \in (0, 1)$.*

Proof. Note that, by construction, $(I - \beta A)$ is strictly diagonally dominant,³ since the magnitude of its i -th diagonal entry $1 - \beta \theta_i w_{ii}$ satisfies

$$\begin{aligned} 1 - \beta \theta_i w_{ii} &= 1 - \beta \theta_i (1 - \sum_{j \neq i} w_{ij}) \\ &= 1 - \beta \theta_i + \beta \sum_{j \neq i} \theta_i w_{ij} > \beta \sum_{j \neq i} \theta_i w_{ij}, \end{aligned}$$

i.e., it is strictly larger than the sum of magnitudes of all non-diagonal entries of the i th row. By Levy-Desplanques theorem (cf., Horn and Johnson (1985)) the matrix $(I - \beta A)$ is non-singular. \square

LEMMA 2. Let $\beta \in (0, 1)$ and $A \in \mathbb{R}^{n \times n}$ such that $(I - \beta A)$ is non-singular. Then

$$\tilde{A}_k = \sum_{j=0}^k \beta^j A^j = (I - \beta A)^{-1} (I - \beta^{k+1} A^{k+1}), \quad (27)$$

$k = 0, 1, \dots$. Furthermore, if $\lim_{k \rightarrow \infty} A^k$ exists, then $\tilde{A}_\infty \triangleq \sum_{j=0}^{\infty} \beta^j A^j = (I - \beta A)^{-1}$.

Proof. To show the first statement, simply multiply from the left with $(I - \beta A)$. The second statement is a direct consequence of (27) if we take the limit as $k \rightarrow \infty$. \square

Define also: $h_\infty^\top \triangleq \beta v^\top \tilde{A}_\infty B - c^\top$.

PROPOSITION 4 (**Optimal Stationary Policy**). Consider the infinite horizon optimization problem (26) under the deterministic and unperturbed dynamics (18). Then, the stationary policy $\pi^* = (\mu^*, \mu^*, \dots) \in \Pi$, such that $\mu^*(x) = (\mu_1^*, \mu_2^*, \dots, \mu_n^*)$ with

$$\mu_i^* = \begin{cases} M & i = \arg \max_1^+ (h_\infty) \\ 0 & \text{otherwise} \end{cases} \quad (28)$$

for $i \in \mathcal{I}$, is an optimal policy for the infinite horizon optimization problem. Furthermore, the optimal infinite value is

$$J^* = v^\top \tilde{A}_\infty x + \frac{M}{1 - \beta} \max_1^+ (h_\infty). \quad (29)$$

Proof. Due to Claim 2, we have $J_0(x) \leq H(x, u, J_0)$ for all $x \in \mathcal{S}$ and $u \in \mathcal{C}(x)$. Also, as we showed in the proof of Proposition 3, due to Claim 1 and Assumption 2, $J_{k+1}(x) \geq J_k(x)$ for every $x \in \mathcal{S}$. Thus, Assumption 4 is satisfied and, according to Proposition 2, in order to show that the stationary policy $\pi^* = (\mu^*, \mu^*, \dots)$ is optimal, it suffices to show that $T_{\mu^*}(J_{\pi^*}) = T(J_{\pi^*})$.

First, we compute $J_{\pi^*}(x)$: Similarly to Proposition 3 and taking into account (27), the stationary policy π^* establishes the following sequence of values

$$\begin{aligned} J_{N, \pi^*} &= v^\top \tilde{A}_N x + \sum_{k=0}^{N-1} \beta^k h_{N-k}^\top \mu^* \\ &= v^\top \tilde{A}_\infty (I - \beta^{N+1} A^{N+1}) x + \\ &\quad \sum_{k=0}^{N-1} \beta^k \left(\beta v^\top \tilde{A}_\infty (I - \beta^{N-k} A^{N-k}) B - c^\top \right) \mu^* \\ &= v^\top \tilde{A}_\infty x + \sum_{k=0}^{N-1} \beta^k h_\infty^\top \mu^* - \beta^{N+1} v^\top \tilde{A}_\infty A^{N+1} x - \end{aligned}$$

$$\beta^{N+1} v^T \tilde{A}_\infty \sum_{k=0}^{N-1} A^{N-k} B \mu^*.$$

Note that

$$\sum_{k=0}^{N-1} A^{N-k} B \mu^* = \sum_{k=1}^N A^k B \mu^* = \sum_{k=1}^N W^k \Theta^k (I - \Theta) \mu^*.$$

Since the diagonal entries of Θ satisfy $0 \leq \theta_i < 1$ for every $i \in \mathcal{I}$ and μ^* is bounded, the above series is convergent. Therefore, we have

$$J_{\pi^*} \triangleq \lim_{k \rightarrow \infty} J_{k, \pi^*} = v^T \tilde{A}_\infty x + \frac{1}{1 - \beta} h_\infty^T \mu^*.$$

Given $\mu^* = (\mu_1^*, \mu_2^*, \dots, \mu_n^*)$ where μ_i^* is given by (28), we have:

$$h_\infty^T \mu^* = M \cdot \max_1^+(h_\infty). \quad (30)$$

Thus,

$$J_{\pi^*} = v^T \tilde{A}_\infty x + \frac{M}{1 - \beta} \max_1^+(h_\infty).$$

We are ready now to compute $T_{\mu^*}(J_{\pi^*})$ and $T(J_{\pi^*})$. In particular,

$$\begin{aligned} T_{\mu^*}(J_{\pi^*}) &= g(x, \mu^*) + \beta J_{\pi^*}(f(x, \mu^*)) \\ &= v^T \left(I + \beta \tilde{A}_\infty A \right) x + h_\infty^T \mu^* + \frac{\beta M}{1 - \beta} \max_1^+(h_\infty). \end{aligned}$$

Due to condition (30) and the fact that $I + \beta \tilde{A}_\infty A \equiv \tilde{A}_\infty$, we have

$$T_{\mu^*}(J_{\pi^*}) = v^T \tilde{A}_\infty x + \frac{M}{1 - \beta} \max_1^+(h_\infty).$$

Finally,

$$\begin{aligned} T(J_{\pi^*})(x) &= \max_{u \in \mathcal{C}(x)} \{g(x, u) + \beta J_{\pi^*}(f(x, u))\} \\ &= v^T (I + \beta \tilde{A}_\infty A) x + \max_{u \in \mathcal{C}(x)} \{h_\infty^T u\} + \frac{\beta M}{1 - \beta} \max_1^+(h_\infty) \\ &= v^T \tilde{A}_\infty x + M \max_1^+(h_\infty) + \frac{\beta M}{1 - \beta} \max_1^+(h_\infty) \\ &= v^T \tilde{A}_\infty x + \frac{M}{1 - \beta} \max_1^+(h_\infty). \end{aligned}$$

Hence, we showed that $T_{\mu^*}(J_{\pi^*}) = T(J_{\pi^*})$, which implies that π^* is an optimal stationary policy. Also, J_{π^*} provides the optimal value of the infinite-horizon optimization. \square

In other words, according to (28), the firm is going to invest the largest possible amount M to the node which corresponds to the maximum entry of

$$h_{\infty}^T = \beta v^T \tilde{A}_{\infty} B - c^T = \beta v^T (I - \beta A)^{-1} (I - \Theta) - c^T.$$

Note that this decision is affected by the following factors:

1. how easily node i can be influenced by the firm's advertising policy, which is measured by $1 - \theta_i$,
2. how large is the “network value” of node i throughout the optimization horizon, expressed by the i th entry of $\beta v^T (I - \beta A)^{-1}$, which measures the effect of every unit of advertising effort spent in i on the proclivities of all nodes that are connected directly or indirectly to i ,
3. how small is the cost of every unit of advertising effort in node i , expressed by c_i .

Note also that the matrix $(I - \beta A)^{-1}$, which influences the optimal decision, can be interpreted as a measure of the centrality of the nodes. In fact, Bonacich in his work on measures of centrality Bonacich (1987), introduced the following centrality measure $c(\gamma, \beta) \triangleq \gamma (I - \beta A)^{-1} A \mathbf{1}$, where γ is a scaling factor. When $\gamma = 1$, $c(1, \beta)$ has several nice interpretations. To see this, note that the centrality measure, which is equivalently written as $c(1, \beta) = (I + \beta A + \beta^2 A^2 + \dots) A \mathbf{1}$, constitutes a measure of closeness, since it is high for a node which is connected to other nodes with short and highly weighted paths. The parameter β represents the degree of information (benefits in our model) that is transmitted from one node to another node. In our case, where A is a row stochastic matrix, the above centrality measure takes on the following form

$$c(1, \beta) = (I + \beta A + \beta^2 A^2 + \dots) \mathbf{1} = (I - \beta A)^{-1} \mathbf{1}.$$

In the context of our dynamic model, we can say that $c(1, \beta)$ represents a measure of the relative importance of nodes (in terms of benefits) when the initial condition is $x_0 = \mathbf{1}$ and there is no external influence (i.e., there is no control input).

Note that in our model both the initial condition and the control input affect the returns of the advertising firm. Since we are only interested in the computation of the optimal advertising policy, an appropriate centrality (or network value) measure would be $\beta v^T \tilde{A}_\infty B - c^T$. The highest entry of this vector will provide the highest benefits over time. Note that when $\beta = 0$, the control input does not have any implication to the returns. In that case, centrality could be measured by $v^T \tilde{A}_\infty$, since it is only the initial condition that affects the returns.

4.2. Perturbed Dynamics

In this section, we are going to consider a family of perturbations of the nominal model (18), described by

$$x_{k+1} = Ax_k + B\varphi(u_k) + Fq_k, \quad (31)$$

where we have neglected the effect of the second firm. The term q_k corresponds to an unknown signal caused possibly by misspecified system dynamics. The sequence $\{q_k\}$ may feed back in a possibly nonlinear way on the history of x . We will impose the following constraint on the size of any instance of this perturbation sequence:

$$|q_k| \leq \eta, \quad \text{for all } k = 0, 1, \dots, \quad (32)$$

where $\eta > 0$ is a measure of the firm's confidence of the accuracy of the nominal model. Let \mathcal{Q} denote the resulting constraint set of disturbances.

Note that due to the presence of the unknown (but bounded) signal q_k our initial assumption that $\mathcal{S} \subset \mathbb{R}_+^n$ may be violated. As we noted though in Section 2.3, the system is input-output stable, therefore an appropriate shift of the state can always guarantee that the dynamics will evolve within the positive cone. In particular, consider $\bar{x} \in \mathbb{R}_+^n$, such that

$$Fq_k + \bar{x} \geq 0, \quad (33)$$

for all q_k satisfying (32), and define instead the dynamics:

$$x_{k+1} = Ax_k + B\varphi(u_k) + Fq_k + \bar{x} \triangleq f(x_k, u_k, q_k). \quad (34)$$

Note that shifting the dynamics by \bar{x} does not change qualitatively the model, since the state x still describes propensities, but relative to \bar{x} .

For some $F \in \mathbb{R}^{n \times n}$ define the vector $r_{k+1}^T \triangleq \beta v^T \tilde{A}_k F$, for $k = 0, 1, \dots$, with $r_1^T = \beta v^T F$. Let also: $r_\infty^T \triangleq \beta v^T \tilde{A}_\infty F$. We would like to solve the following optimization:

$$\max_{\pi \in \Pi} \min_{\sigma \in \Sigma} \left\{ J_{(\pi, \sigma)}(x_0) \triangleq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k(x_k)) \right\}, \quad (35)$$

subject to the perturbed dynamics (34) and the constraints (32)–(33). Here Σ denotes the set of sequences of policies $\sigma = (\nu_0, \nu_1, \dots)$ of the uncertainty, where ν_k is a function from the set of states \mathcal{S} to \mathcal{Q} . Note also that due to the new shifted dynamics, a utility function of the form $g(x, u) = v^T x - c^T u - \lambda(\bar{x})$ would have been more appropriate. However, in that case, and since the last term is constant, the optimal policy of (35) would have been identical.

PROPOSITION 5 (Optimal policy under uncertainty). *Consider the infinite horizon optimization of (35) under the perturbed dynamics (34) and the constraint (32)–(33). The optimal stationary policy is $\mu^* = (\mu_1^*, \dots, \mu_n^*)$, such that*

$$\mu_i^* = \begin{cases} M & i = \arg \max_1^+ (h_\infty) \\ 0 & \text{otherwise} \end{cases}, \quad i \in \mathcal{I}. \quad (36)$$

Proof. To solve this optimization problem, we implement the dynamic programming iteration. In fact, we recursively implement the operator $T(\cdot)$ defined as

$$(TJ)(x) \triangleq \max_{u \in \mathcal{C}} \min_{q \in \mathcal{Q}} \{g(x, u) + \beta J(f(x, u, q))\}, \quad (37)$$

for any $x \in \mathcal{S}$. The dynamic programming iteration gives:

$$J_N(x) = v^T \tilde{A}_N x + \sum_{k=0}^{N-1} \left[\beta^k h_{N-k}^T u_{N-k}^* + \beta^k r_{N-k}^T q_{N-k}^* + \beta^{k+1} v^T \tilde{A}_{N-k} \bar{x} \right],$$

for all $N = 1, 2, \dots$, where u_k^* and q_k^* denote the sequences of optimal investments and disturbances, respectively. In particular, $u_k^* = (u_{1,k}^*, \dots, u_{n,k}^*)$ and $q_k^* = (q_{1,k}^*, \dots, q_{n,k}^*)$, are such that

$$u_{i,k}^* = \begin{cases} M & i = \arg \max_1^+ (h_k) \\ 0 & \text{otherwise} \end{cases}, \quad i \in \mathcal{I},$$

and $r_k^T q_k^* = -\eta |r_k|_\infty$. In other words, the disturbance places all its weight on the maximum (in absolute value) entry of r_k , or

$$q_{i,k}^* = \begin{cases} -\eta & i = \arg \max_1^+(r_k) \\ 0 & \text{otherwise} \end{cases}, \quad i \in \mathcal{I}.$$

The order of max and min in the definition of the operator $T(\cdot)$ does not change the optimal policies. Note also that:

$$\begin{aligned} H(x, u, q, J_0) &= g(x, u) + \beta J_0(f(x, u, q)) \\ &= J_0(x) + \beta v^T A x + \beta v^T (Fq + \bar{x}) + (\beta v^T B - c^T)u \\ &\geq J_0(x) \end{aligned}$$

for all $x \in \mathcal{S}$, $u \in \mathcal{C}^*$, $q \in \mathcal{Q}^*$ and under condition (33). Thus, from Proposition 2, the dynamic programming iteration provides the optimal infinite value.

Consider the stationary policy (36) for the monopolistic firm and the stationary policy $\sigma^* = (\nu^*, \dots, \nu^*)$ for the disturbance such that $r_\infty^T \nu^* = -\eta |r_\infty|_\infty$. Similarly to the proof of Proposition 4, the corresponding infinite value is

$$\begin{aligned} J_{(\pi^*, \sigma^*)}(x) &= v^T \tilde{A}_\infty x + h_\infty^T \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k \mu^* + r_\infty^T \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k \nu^* + \\ &\quad \beta v^T \tilde{A}_\infty \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k \bar{x} \\ &= v^T \tilde{A}_\infty x + \frac{1}{1-\beta} \left[M \max_1^+(h_\infty) - \eta |r_\infty|_\infty + \beta v^T \tilde{A}_\infty \bar{x} \right]. \end{aligned}$$

By following similar reasoning with the proof of Proposition 4, we can show that

$$T_{(\mu^*, \nu^*)}(J_{(\pi^*, \sigma^*)}) = T(J_{(\pi^*, \sigma^*)}).$$

Therefore, according to Proposition 2, (π^*, σ^*) provides the optimal lower value. It is also straightforward to show that the sequence of policies (π^*, σ^*) also provides the optimal upper value, defining this way a solution to the max-min optimization problem. \square

Note that the robust optimal policy for the perturbed model coincides with the optimal policy for the unperturbed or riskless model, i.e., it exhibits a *certainty equivalence property*.

5. Optimal Policy in Duopoly

5.1. Preliminaries

The previous section computed the optimal robust policy for the problem of monopoly under norm-bounded model uncertainty. In this section, we would also like to include the possibility that a competitive firm tries to influence the preferences of the customers towards buying its own product as described by the more general duopoly model (4).

The presence of a competitive firm introduces a new source of uncertainty. We will either assume that i) the competitive firm has the form of a competitive fringe which tries to enter the market, introducing a notion of sequential optimization (expressed by a Stackelberg solution), or ii) both firms have the ability of simultaneous play (expressed by a Nash solution).

Each firm $\ell \in \mathcal{L}$ solves the following optimization problem:

$$\max_{\pi^\ell \in \Pi^\ell} \left\{ J_{(\pi^\ell, \pi^{-\ell})}(x_0^\ell) \triangleq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k g(x_k^\ell, \mu_k^\ell(x_k^\ell)) \right\} \quad (38)$$

subject to the system dynamics

$$x_{k+1}^\ell = Ax_k^\ell + B\varphi(\mu_k^\ell, \mu_k^{-\ell}) \quad (39)$$

where $\pi^\ell = (\mu_1^\ell, \mu_2^\ell, \dots)$ and $\pi^{-\ell} = (\mu_1^{-\ell}, \mu_2^{-\ell}, \dots)$ are the infinite sequences of policies of the firms ℓ and $-\ell$, respectively.

DEFINITION 1 (STACKELBERG SOLUTION). A Stackelberg solution is a pair of policies $(\pi^{\ell*}, \pi^{-\ell*}) \in \Pi^\ell \times \Pi^{-\ell}$ such that

$$\pi^{-\ell*} \in \text{BR}_{-\ell}(\pi^{\ell*}) \triangleq \arg \max_{\pi^{-\ell}} \{ J_{(\pi^{-\ell}, \pi^{\ell*})}(x_0^{-\ell}) \mid \pi^{\ell*} \}$$

and, $\pi^{\ell*} \in \arg \max_{\pi^\ell \in \Pi^\ell} \{ J_{(\pi^\ell, \pi^{-\ell*})}(x_0^\ell) \mid \pi^{-\ell*} \in \text{BR}_{-\ell}(\pi^\ell) \}$.

In the above definition of a Stackelberg solution, we will refer to firm ℓ as the *leader* and firm $-\ell$ as the *follower*. Note that the definition implies that firm ℓ (or *leader*) announces first its policy, while firm $-\ell$ (or *follower*) reacts to that policy.

DEFINITION 2 (NASH SOLUTION). A pair of policies $(\pi^{\ell*}, \pi^{-\ell*}) \in \Pi^\ell \times \Pi^{-\ell}$ is a Nash solution if $\pi^{-\ell*} \in \text{BR}_{-\ell}(\pi^{\ell*})$ and $\pi^{\ell*} \in \text{BR}_\ell(\pi^{-\ell*})$.

We will also refer to these solutions as *Markovian* or *closed-loop Nash solutions*. If, instead, the maximization in the definition of the Nash solution is restricted to the set of sequences of control inputs in \mathcal{C}^ℓ , then the corresponding solutions will be referred to as *open-loop Nash solutions*. Note that these definitions of Nash solutions implicitly assumes a simultaneous announcement of policies for both firms.

A straightforward implication of the above definitions is that *any Stackelberg solution is also a Nash solution*.

5.2. Open-loop stationary Nash solutions

In this section, we will restrict our attention to open-loop Nash solutions that are also stationary, i.e., time-independent. Before characterizing this family of Nash solutions, define the set of actions $\mathcal{A}^\ell \triangleq \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $\ell \in \mathcal{L}$, such that for each $i \in \{1, 2, \dots, n\}$, $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n})$ where

$$\alpha_{i,j} \triangleq \begin{cases} M & j = \arg \max_i^+(h_\infty), \\ 0 & \text{otherwise,} \end{cases} \quad j = 1, 2, \dots, n.$$

In other words, the action α_i corresponds to investing all available funds to the i th largest non-negative entry of h_∞ . Note that the set of actions define an isomorphic set of stationary policies, i.e., for each action α_i there is a stationary policy $(\alpha_i, \alpha_i, \dots)$. Let us also denote by $J_{(i,j)}(x)$ the corresponding infinite horizon value for initial condition x when one firm applies stationary policy $(\alpha_i, \alpha_i, \dots)$ and the other firm applies $(\alpha_j, \alpha_j, \dots)$. Any other open-loop stationary policy μ^ℓ can be represented as a mixture of actions in \mathcal{A}^ℓ , i.e.,

$$\mu^\ell = \begin{cases} \alpha_1, & \text{with probability } p_1^\ell \\ \dots & \\ \alpha_n, & \text{with probability } p_n^\ell \end{cases}, \quad \ell \in \mathcal{L}, \quad (40)$$

where $p_i^\ell \geq 0$, $i \in \mathcal{I}$, and $\sum_i p_i^\ell = 1$. The corresponding value of the objective function (38) for any open-loop stationary policy is characterized by the following proposition.

PROPOSITION 6 (Payoffs under open-loop policies). *When both firms $\ell \in \mathcal{L}$ apply an open-loop*

stationary policy $\pi^\ell = (\mu^\ell, \mu^\ell, \dots)$ satisfying (40), the infinite value of the objective function $J_{(\pi^\ell, \pi^{-\ell})}$ defined by (38), is $J_{(\pi^\ell, \pi^{-\ell})} = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} J_{(i,j)} p_i^\ell p_j^{-\ell}$, where

$$J_{(i,j)}(x) = \begin{cases} v^T \tilde{A}_\infty x + \frac{1}{1-\beta} [-c^T \alpha_i], & i = j \\ v^T \tilde{A}_\infty x + \frac{1}{1-\beta} [h_\infty^T \alpha_i], & i \neq j \end{cases}, x \in \mathcal{S}^\ell, \ell \in \mathcal{L}. \quad (41)$$

Proof. When the pair of stationary policies $(\pi^\ell, \pi^{-\ell})$ is applied, where $\pi^\ell = (\mu^\ell, \mu^\ell, \dots)$ and $\pi^{-\ell} = (\mu^{-\ell}, \mu^{-\ell}, \dots)$, the corresponding value of the objective function of firm ℓ will be:

$$\begin{aligned} J_{(\pi^\ell, \pi^{-\ell})}(x) &= v^T \tilde{A}_\infty x + \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k [(h_\infty + c)^T \varphi(\mu^\ell(x), \mu^{-\ell}(x)) - c^T \mu^\ell(x)] \\ &= v^T \tilde{A}_\infty x + \frac{1}{1-\beta} [(h_\infty + c)^T \varphi(\mu^\ell(x), \mu^{-\ell}(x)) - c^T \mu^\ell(x)] \end{aligned}$$

for some initial state $x \in \mathcal{S}^\ell$. If $\mu^\ell = \mu^{-\ell} = \alpha_i$, then the corresponding infinite value of the objective function of ℓ , denoted $J_{(i,i)}$, is:

$$J_{(i,i)}(x) = v^T \tilde{A}_\infty x + \frac{1}{1-\beta} [-c^T \alpha_i].$$

If, instead, $\mu^\ell = \alpha_i$ and $\mu^{-\ell} = \alpha_j$, $i \neq j$, the corresponding infinite value of the objective function ℓ , denoted $J_{(i,j)}$, is:

$$J_{(i,j)}(x) = v^T \tilde{A}_\infty x + \frac{1}{1-\beta} [h_\infty^T \alpha_i].$$

Then, the corresponding expected return of firm $\ell \in \mathcal{L}$ is:

$$\begin{aligned} J_{(\pi^\ell, \pi^{-\ell})}(x) &= v^T \tilde{A}_\infty x + \sum_{i,j \in \mathcal{I}} \frac{[(h_\infty + c)^T \varphi(\alpha_i, \alpha_j) - c^T \alpha_i]}{1-\beta} p_i^\ell p_j^{-\ell} \\ &= \sum_{i,j \in \mathcal{I}} \left[v^T \tilde{A}_\infty x + \frac{[(h_\infty + c)^T \varphi(\alpha_i, \alpha_j) - c^T \alpha_i]}{1-\beta} \right] p_i^\ell p_j^{-\ell} \\ &= \sum_{i,j \in \mathcal{I}} J_{(i,j)} p_i^\ell p_j^{-\ell}, \end{aligned}$$

which concludes the proof. \square

Thus, we may define an equivalent one-stage symmetric game of two players, finite set of actions $\mathcal{A}^\ell = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ for each player $\ell \in \mathcal{L}$, and payoff matrix of the row player which is given by Table 1.

A direct consequence of Proposition 6 is the following:

	α_1	α_2	...	α_n
α_1	$J_{(1,1)}$	$J_{(1,2)}$...	$J_{(1,n)}$
α_2	$J_{(2,1)}$	$J_{(2,2)}$...	$J_{(2,n)}$
...
α_n	$J_{(n,1)}$	$J_{(n,2)}$...	$J_{(n,n)}$

Table 1 Equivalent one-shot symmetric game in open-loop stationary policies.

LEMMA 3. *The following hold:*

1. $J_{(i,j)}(x) \geq J_{(i,i)}(x)$ for all $i, j \in \mathcal{I}$ with $i \neq j$;
2. $J_{(i,j)}(x) = J_{(i,j')}(x)$ for all $i, j, j' \in \mathcal{I}$ with $j \neq i$ and $j' \neq i$;
3. $J_{(i,j)}(x) \geq J_{(j,i)}(x)$ for all $i, j \in \mathcal{I}$ with $i > j$.

PROPOSITION 7 (**Stackelberg & Nash solutions**). *Let us consider the optimization problem (38) under the dynamics (39) and the constraints (1) with $M^\ell = M^{-\ell}$, i.e., both firms have identical advertising power. For any $\ell \in \mathcal{L}$, the pair of open-loop stationary policies $\pi^* = (\pi^{\ell*}, \pi^{-\ell*})$ where $\pi^{\ell*} = (\mu^{\ell*}, \mu^{\ell*}, \dots)$ and μ^ℓ is defined by (40) satisfying either*

- (1) $p_1^\ell = p_2^{-\ell} = 1$, or
- (2) $p_1^\ell = p_2^{-\ell} = \frac{J_{(1,2)} - J_{(2,2)}}{J_{(1,2)} - J_{(1,1)} + J_{(2,1)} - J_{(2,2)}}$,

defines an open-loop Nash solution. Furthermore, when $\ell \in \mathcal{L}$ has the opportunity to announce its policy first, the open-loop stationary policy corresponding to (1) defines an open-loop Stackelberg solution.

Proof. The first claim is a direct consequence of Lemma 3 and the fact that any one of the policies corresponding to the cases (1) and (2) defines a Nash solution for the equivalent one-shot symmetric game of Table 1.

Assume now that ℓ has the opportunity to announce its strategy first. In order to show that $(\pi^{\ell*}, \pi^{-\ell*})$ defines a Stackelberg solution, we need to verify that the leader's policy $\pi^{\ell*}$ guarantees maximum return over all possible announced policies. It is straightforward to show that any announced policy that does not allocate all available funds to $\arg \max_1^+(h_\infty)$ will result to a best response of the follower that decreases leader's utility. \square

The conclusions of Proposition 7 do not necessarily hold when we consider different spending powers for the firms, i.e., when $M^\ell \neq M^{-\ell}$. However, extending the conclusions of Proposition 7 to that case is straightforward.

Another straightforward implication of Proposition 7 is summarized in the following corollary.

COROLLARY 1. *The open-loop stationary Nash solutions characterized by Proposition 7 are also closed-loop Nash solutions.*

This is due to the fact that open-loop strategies are a subset of Markovian or state-dependent strategies. A complete characterization of the set of closed-loop Nash solutions is going beyond the scope of this paper, since it is highly case-dependent, i.e., it depends on the class of policies which will be considered reasonable for the application of interest. For example, if we assume that the class of strategies over which the optimization is executed are affine functions of the state, then a new class of closed loop Nash solutions can easily be computed using the framework proposed in this paper.

5.3. Max-min solutions

Computing an optimal strategy which is robust to any possible policy of the competitor can be formulated as a max-min optimization. Consider two firms with different expenditure capabilities. In particular, consider the following two scenarios: a) $M^\ell > M^{-\ell}$, and b) $M^\ell \leq M^{-\ell}$ for any $\ell \in \mathcal{L}$. Then, firm $\ell \in \{a, b\}$ solves the following max-min optimization:

$$\max_{\pi \in \Pi} \min_{\sigma \in \Sigma} \left\{ J_{(\pi, \sigma)}(x_0) \triangleq \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \beta^k g(x_k, \mu_k(x_k)) \right\} \quad (42)$$

over the set Π of infinite sequences of policies (μ_0, μ_1, \dots) and subject to the system dynamics

$$x_{k+1} = Ax_k + B\varphi(\mu_k, \nu_k). \quad (43)$$

The set Σ denotes the collection of infinite sequences of policies (ν_0, ν_1, \dots) of the competitor. In words, the above optimization reflects the situation at which the firm wishes to announce a strategy

which will provide the optimal returns assuming that the competitor acts to minimize these returns. To simplify notation, we have removed the superscript ℓ from the above optimization variables. It is straightforward to show that:

PROPOSITION 8. *Let us consider the optimization problem (42) under the dynamics (43) and the constraints (1). If $M^\ell > M^{-\ell}$, i.e., the advertising power of the firm ℓ is larger than the one of its competitor, then the optimal strategy of the firm will be a stationary policy (μ^*, μ^*, \dots) such that*

$$\mu_i^* = \begin{cases} M & i = \arg \max_1^+(h_\infty) \\ 0 & \text{otherwise} \end{cases}, \quad i \in \mathcal{I}. \quad (44)$$

Note that this is not necessarily the case when the advertising power of the firm is less than the competitor's. In that case, any strategy will be optimal, since the competitor has the power to counteract any announced strategy of the firm.

6. Conclusions

We discussed the problem of deriving optimal advertising strategies in a network of customers or groups of customers. Contrary to prior work, the dynamics of preferences were also affected by an underlying network of interactions which introduces a form of word-of-mouth communication between nodes. The derived optimal policies are related to and extend priorly introduced notions of centrality measures usually considered in sociology. Although the assumed model of evolution of preferences might be the outcome of an identification process, it is likely that we are uncertain about its accuracy. To this end, we also considered a perturbed model which models possible misspecifications or uncertainties of the nominal model, and we derived robust optimal strategies. It was shown that the monopoly model exhibits a certainty equivalence property, i.e., the optimal strategies for the perturbed model coincide with the optimal strategies for the unperturbed or riskless model. Finally, we investigated robust policies in a duopoly framework. In particular, we characterized the set of open-loop Nash solutions. The model can easily be utilized to accommodate scenarios at which more complicated forms of strategies are of interest, leading to new forms of closed-loop Nash solutions. We also characterized the set of max-min solutions in a duopoly framework, when firms makes no assumptions about the utilities of the competitor.

Endnotes

1. An extension of the forthcoming analysis to multiple number of firms will be straightforward.
2. A row stochastic matrix W is a nonnegative matrix which also satisfies $W\mathbf{1} = \mathbf{1}$, i.e., the sum of its entries in any row is equal to 1.
3. A matrix is *strictly diagonally dominant* if in every row of the matrix, the magnitude of the diagonal entry in that row is larger than the sum of the magnitudes of all the other (non-diagonal) entries in that row.

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