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Weakly Fair Allocations and Strategy-Proofness*

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Abstract

This paper investigates the problem of allocating two types of indivisible objects among a group of agents when a priority-order must be respected and when only restricted monetary transfers are allowed. Since the existence of a fair allocation not generally is guaranteed due the the restrictions on the money transfers, the concept of fairness is weakened, and a new concept of fairness is introduced. This concept is called weak fairness. We define an allocation rule that implements weakly fair allocations and demonstrate that it is coalitionally strategy-proof. In fact, it is the only coalitionally strategy-proof allocation rule that implements a weakly fair allocation.

JEL Classification: C71; C78; D63; D71; D78.
Key Words: Indivisibles; fairness; weak fairness; strategy-proofness.

1 Introduction

The problem of allocating a set of indivisible objects among a number of agents was analyzed in the pioneering work of Shapley and Scarf (1974). In their study, private

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ownership of the objects was presumed and the core of the economy was analyzed. More recently, a modification of this allocation problem, where the objects are regarded as a social endowment, has received considerable attention, see, e.g., Abdulkadiroğlu and Sönmez (1999,2003), Ehlers (2002), Pápai (2000) and Svensson (1999). Taking social endowments as a point of departure also reflects many real-life allocation problems, for example, housing allocation among students on college campuses and assignment of public schools to children. In many of these problems, the goods cannot be allocated arbitrarily among the agents due to that a priority or queue-order must be respected. For example, in school choices, students who live closer to a school and/or have siblings attending at a school may be given a higher priority, by state or local laws, see Abdulkadiroğlu and Sönmez (2003).[1] The main reason for adopting a priority-order is to avoid an allotment of the goods under market-like conditions. This is, in particular, true when the goods are regarded as a social endowment. Consider, for example, the problem of allocating public schools to students. In this case, "the education of students is not and should probably not be organized in a market-like institution", as concluded by Abdulkadiroğlu and Sönmez (2003,p.731). A priority-order is respected, at a given allocation, if no agent envies some other agent with a lower priority (for receiving a better object). However, an agent with a lower priority may envy an agent with a higher priority. Hence, envy is the cost of accepting a priority-order and not organize the allocation on a market. In this paper, we investigate how, at least, some of this envy can be reduced when a priority-order must be respected, but when restricted monetary compensations are allowed.

We investigate as, e.g., Ohseto (2006), an allocation problem with a finite number of agents and two types of objects. There are more than one object of each type, but exactly as many objects as there are agents. The basic problem is to allocate the objects among the agents under the restrictions that each agent must consume exactly one object, each object must be consumed by exactly one agent and the resulting allocation must be consistent with a given priority-order. This may, for example, be a problem of allocating public schools to children in a small town with only two schools, where, e.g., priority is given to children with siblings attending at a specific school.

If monetary compensations not are allowed, the existence of a fair (as in envy-free) allocation is not generally guaranteed. If, on the other hand, it is possible to attach an arbitrary monetary compensation to each good, it is well-known that fair allocations

[1]See also Balinski and Sönmez (1999), Ehlers and Klaus (2007), Roth et al. (2004) and Svensson (1994) and additional examples therein.

exists under very general assumptions on individual preferences.[2] The basic idea in this paper is that by allowing for restricted monetary compensations, at least some envy can be reduced. However, we do not wish to consider negative compensations (i.e., prices), because then some agents may be worse-off when the compensations are introduced.[3] That is, the main reason for introducing money transfers is to compensate agents that not are assigned their best object, it is not to punish the agents that are assigned their top-object. For example, if the most preferred public school not is assigned to a child, then his/her family should receive some money as a compensation, but the children that are assigned their top-choices should not be punished by means of a fee.

Hence, in the *first* major step, we restrict the monetary compensations to be non-negative. This defines lower limits on the compensations. However, by imposing such a bound, the set of fair allocations that satisfy this lower compensation limit may be empty. For this reason, we need to weaken the concept of fairness. We first require that the objects are allotted efficiently according to the priority-order, i.e., that there is no way to reduce envy by reallocating only the objects and at the same time respect the priority-order. Recall now that the main reason for introducing money in the assignment model is to reduce envy. Given this observation, we then require that agent i can never envy some other agent k with a higher priority (for receiving a better object), at an efficient allocation, when the compensation for the object that is assigned to agent k exceeds zero, because if this is that case, there is no reason to transfer a positive amount of money to agent k, since this will only increase the envy further. Hence, a natural weakening of the concept of fairness, when a priority-order and a lower compensation limit must be respected, is the following; an allocation is said to be weakly fair if the objects are allocated efficiently, the compensations do not fall short of the minimum compensation limits and where no agent envies any other agent except, possibly, in the bounding case when (a) the money compensation of each top-object of the agent equals the lower compensation limit and (b) each top-object is assigned to some other agent with a higher priority. Note first that in the case when the compensation for all objects exceeds the lower compensation limits, then the weakly fair allocation is also a fair allocation. Hence, the main difference between the

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[2]See, e.g., Alkan et al. (1991), Maskin (1987), Svensson (1983) and Tadenuma and Thomson (1991).

[3]Non-negativity is not a crucial assumption. In fact, all results are valid even if this assumption is relaxed. If, for example, one of the goods is a "bad object" (e.g., dangerous missions or toxic waste), it is possible to allow for a negative compensation. That is, agents who receive the "good object" may pay a price for not receiving the "bad object".

concepts of fairness and weak fairness is properties (a) and (b), i.e., the weakening of the no-envy condition. In the bounding case when no monetary transfers are allowed, so the compensations equal the lower limits, an allocation is weakly fair if the objects are allocated efficiently and the priority-order is respected.

In the *second* major step, the mechanism designer specifies the budget that can be used in order to reduce envy. More specifically, the mechanism designer specifies a maximal compensation that can be attached to each good. This compensation may, for example, correspond to the travel costs for a child that is assigned a public school that not is located in the neighborhood. This defines, in addition to the lower non-negative limit, an upper limit on the compensations. Given these restrictions on the compensations, we demonstrate that the set of restricted compensation vectors, which are consistent with weak fairness, is non-empty. Moreover, there exists a unique compensation vector in this set that is optimal from the perspective of the agents, in the sense that the compensation for each good in this vector is weakly higher than the compensation for each good in an arbitrary compensation vector in this set. This compensation vector is called the weakly fair and optimal compensation vector. By regarding this compensation vector as mechanism for allocating the indivisible goods, it is demonstrated that it is not possible for any agent or any coalition of agents to manipulate the outcome by reporting false preferences. Such an allocation rule is said to be coalitionally strategy-proof. In the problem of assigning public schools to children, for example, this property means that children (and their parents) do not need to worry about receiving a less attractive school seat as a consequence of reporting their preferences over the schools truthfully. Given a mild regularity condition, we also demonstrate that this allocation rule is, in fact, the only strategy-proof allocation rule that implements a weakly fair allocation.

To the best of our knowledge, this is the first paper that analyzes an allocation problem where only restricted compensations are allowed, i.e., where the allowable monetary compensations are restricted by both an upper and a lower limit. The investigated allocation rule is very appealing, because it always eliminates envy when possible and if it not is possible to eliminate envy, it reduces the envy, by means of monetary compensations. In the seminal article by Demange and Gale (1985) there are also upper and lower limits, given by the sellers and the buyers reservation prices. Consequently, in their multi-item auction model, the upper and lower limits are determined endogenously by the preferences of the sellers and buyers. In our model, the restrictions are not related to the preferences of the agents.

Strategy-proofness has been investigated earlier by, e.g., Abdulkadiroğlu and Sönmez

(1999,2003), Ehlers (2002), Pápai (2000) and Svensson (1999), when no monetary transfers are allowed, and by, e.g., Demange and Gale (1985) and Tadenuma and Thomson (1995), when monetary transfers are allowed, but when no exogenous restrictions are imposed on the monetary compensations. In recent papers by Anderson and Svensson (2006), Sun and Yang (2003) and Svensson (2006), an allocation problem, where the monetary transfers are restricted by an upper compensation limit is analyzed. There are, however, three major differences between this paper and the papers mentioned above. First, they do not have a lower bound on the monetary compensations. Second, fair allocations always exists in their models, so there is no need to weaken the concept of fairness. Last, they do not have a priority-order and even if a priority-order is introduced, it would be of no importance, because fair allocations always exists and a fair allocation always satisfy any given priority-order, by definition. In our allocation model, the priority-order is very important, because we need it when it is impossible to avoid an unenvious outcome due to the restrictions on the monetary compensations. Finally, Ohseto (2006) considers an allocation model with money and two types of indivisible goods (like the model in this paper) and identifies the Pareto undominated subset in the set of strategy-proof and fair allocation rules. Even if it is not explicitly stated, one can see that the derived compensation functions also satisfy an exogenous upper bound. However, no lower bound is introduced.

The paper is outlined as follows. In Section 2, we describe the basic model and introduce some useful notation. The concept of weak fairness is introduced in Section 3. Section 4 presents the weakly fair and optimal allocation rule and prove that it is coalitionally strategy-proof. In Section 5, it is shown that the introduced upper limits on the compensations are necessary for coalitionally strategy-proofness. Some of the proofs are collected in the Appendix.

2 The model and basic definitions

Let $N = \{1, ..., n\}$ be the set of agents and $n \geq 2$. There are two types of indivisible objects, denoted by 0 and 1, but no initial property rights. We assume that there are $h_0 \in \{1, ..., n-1\}$ units of object 0 and $h_1 = n - h_0$ units of object 1. There is also some maximal amount of money that can be assigned to the agents, $\overline{m} \in \mathbb{R}_+$. When monetary transfers not are allowed, we let $\overline{m} = 0$. A consumption bundle is a pair $(j, m) \in \{0, 1\} \times \mathbb{R}_+$, and it is assumed that each agent consumes exactly one object together with some non-negative amount of money. An allocation (a, x) is a list of n

consumption bundles, where $a : N \to \{0,1\}$ is a mapping assigning an object a_i to agent $i \in N$, and where $x \in \mathbb{R}^2_+$ distributes the non-negative amount x_j of money to objects of type $j \in \{0,1\}$.[4] An allocation (a, x) is said to be feasible if $\Sigma_{i \in N} a_i = h_1$ and $\Sigma_{i \in N} x_{a_i} \leq \overline{m}$. The latter condition is a resource constraint. If no monetary transfers are allowed, then $x_{a_i} = \overline{m} = 0$ for all $i \in N$, so it is trivially satisfied. The set of feasible allocations is denoted by $\mathcal{A}$.

Each agent $i \in N$ has preferences over consumption bundles, represented by a continuous utility function $u_i : \{0,1\} \times \mathbb{R}^2 \to \mathbb{R}$. Here, $u_i(j, x)$ is the utility agent $i \in N$ obtains, at distribution x, if he is allocated an object of type $j \in \{0,1\}$ and the quantity x_j of money. The utility function is assumed to be strictly increasing in money, i.e., $u_i(j, x) > u_i(j, y)$ if $x_j > y_j$. A list of utility functions $u = (u_1, ..., u_n)$ is a (preference) profile. We also adopt the notational convention of writing $u = (u_C, u_{-C})$ for $C \subset N$. The set of profiles with utility functions having the above properties is denoted by $\mathcal{U}$.

An allocation rule is a non-empty correspondence φ, that, for each profile $u \in \mathcal{U}$, selects a set of allocations, $\varphi(u) \subset \mathcal{A}$, such that $u_i(b_i, y) = u_i(a_i, x)$ for all $i \in N$ if $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(u)$. Hence, the various outcomes in the set $\varphi(u)$ are utility equivalent, and such a correspondence is called essentially single-valued (ESV). Working with correspondences instead of functions, ESV is a necessary assumption for an allocation rule to be strategy-proof (as defined in Sections 4 and 5). This does, however, not exclude that the allocation rule is single-valued.

Finally, let $\pi : N \to N$ be a bijection that determines a priority-order among the agents. The agent $i \in N$ with $\pi(i) = 1$ has the highest priority. For simplicity, and without loss of generality, we suppose that $\pi(i) = i$ for all $i \in N$. The priority-order is respected, at a given allocation (a, x), if and only if $u_i(a_i, x) \geq u_i(a_k, x)$ for all $i, k \in N$ where $k > i$.

3 Weak fairness

If no monetary compensations are allowed and the objects are allotted according to the priority-order, an envy-free outcome can not be guaranteed for all profiles $u \in \mathcal{U}$.

[4] A more general formulation of an allocation (a, x) is obtained if $x \in \mathbb{R}^n$. In that case, agents that are allocated the same type of objects can be allocated different (and even negative) amounts of money. However, such a more general approach would not add anything to the results of this paper, see Ohseto (2006,p.114).

In this sense, envy is the cost of not allocating the objects on a market. The basic idea in this paper is that, at least, some of this envy can be reduced when restricted monetary compensations are allowed. However, we do not wish to punish agents, by introducing monetary compensations. For this reason, we restricted the compensation vector to be non-negative in Section 2. This restriction guarantees that no agent will be worst-off when money is introduced in the model. The compensation vector $(0, 0)$ will be denoted by $\underline{x}$, henceforth.[5]

In our analysis, the set of envy-free allocations will be of primary importance. Formally, for a given profile $u \in \mathcal{U}$, an allocation (a, x) is said to be fair if it is envy-free, i.e., if $u_i(a_i, x) \geq u_i(a_j, x)$ for all $i, j \in N$. A distribution x is said to be fair if there is an assignment a such that allocation (a, x) is a fair allocation. For a given profile $u \in \mathcal{U}$, the set of fair allocations is denoted by $\Phi(u)$ and the corresponding set of fair distributions is denoted by $F(u)$, i.e.:

$$F(u) = \{x \in \mathbb{R}^2 \mid (a, x) \in \Phi(u) \text{ for some assignment } a\}.$$

Note that for some profiles u, the the set $F(u) \cap \mathbb{R}^2_+$ may not have any feasible distributions. For this reason, we need to weaken the concept of fairness. To achieve this objective, we first require that the objects are allotted efficiently according to the priority-order, i.e., that there is no way to reduce envy, by reallocating only the objects, and at the same time respect the priority-order. This property is known as object-efficiency, see, e.g., Svensson and Larsson (2002). Recall now that the main reason for introducing money in the assignment model is to reduce envy. Hence, it is also natural to require that agent i can never envy agent k, at an efficient allocation (a, x), when $x_{a_k} > 0$, because if agent i envies agent k, there is no reason to transfer a positive amount of money to agent k. This will only increase the envy further. Thus, we only allow for envy when the compensation for an object equals the minimum compensation limit. Formally, our weakening of the no-envy criterion is defined as follows.

Definition 1 *For a given profile $u \in \mathcal{U}$, an allocation (a, x) is said to be weakly fair* (WF) *if $x \geq \underline{x}$, the priority-order is respected and:*

(i) $u_i(b_i, x) \geq u_i(a_i, x)$ *for all* $i \in N$ *for some assignment* b *imply that* $u_i(b_i, x) = u_i(a_i, x)$ *for all* $i \in N$,

(ii) $u_i(a_i, x) \geq u_i(a_k, x)$ *for all* $i \in N$ *if* $x_{a_k} > 0$.

[5] This is not a crucial assumption. Our results hold for an arbitrary vector $\underline{x} \in \mathbb{R}^2$.

Note also that in the special case when no compensation is at the minimum level, the WF allocation is also a fair allocation. In that case, the priority-order is of no importance, because it is always respected, by definition. In the bounding case when $x = \underline{x} = (0, 0)$, i.e., when no monetary transfers are allowed, an allocation is weakly fair if the objects are allocated efficiently and the priority-order is respected.

For a given profile $u \in \mathcal{U}$, the set of WF allocations is denoted by $\Psi(u)$, and the corresponding set of WF distributions is denoted by $WF(u)$, i.e.:

$$WF(u) = \{x \in \mathbb{R}^2_+ \mid (a, x) \in \Psi(u) \text{ for some assignment } a\}.$$

We next illustrate the set $WF(u)$ in a simple economy with three agents, that all have quasi-linear utility functions. Suppose also that there is one object of type 0 and two objects of type 1. In Figure 1, the curve $\widehat{u}_i$ represents all distributions where agent $i \in \{1, 2, 3\}$ is indifferent between receiving object 0 and 1. Because utility is increasing in money, it is clear that agent i strictly prefers object 0 (object 1) at any distribution that is located below (above) the indifference curve. If no monetary compensations are allowed, so the distribution is given by $\underline{x} = (0, 0)$, object 0 is the most preferred object for all agents, but agent 1 is assigned object 0, by the priority-order. Hence, the allocation will not be envy-free. If arbitrary monetary compensations are allowed, any distribution that is located on or between the two indifference curves $\widehat{u}_1$ and $\widehat{u}_2$ is fair, since two of the agents must be assigned an object of type 1, so if in addition $x \geq \underline{x}$, then the distribution belongs to $WF(u)$. However, the distributions that are located on the line segment $z\underline{x}$ also belong to $WF(u)$. To see this, note first that $z \in F(u)$. Note next that if we pick an arbitrary distribution on the line $z\underline{x}$ (not equal to z), at least two agents strictly prefer object 0 over object 1. But then we can identify a WF allocation by assigning object 0 to the agent with the highest priority, i.e., agent 1. The resulting allocation is weakly fair, by Definition 1, since $x_0 = 0$ along this line segment, but it will not be envy-free. However, envy is reduced due to that agents 2 and 3 receive a monetary compensation. Hence, the set of weakly fair distributions, $WF(u)$, is given by the line segment $z\underline{x}$ plus the area that is bounded by $x \geq \underline{x}$ and the two indifference curves $\widehat{u}_1$ and $\widehat{u}_2$.

[FIGURE 1 ABOUT HERE]

4 Weakly fair and optimal allocations

The first main objective of this paper is to weaken the concept of fairness when restricted monetary compensations are allowed and when a priority-order must be respected. The second main objective of this paper is to analyze strategy-proof allocation rules that implement WF allocations. Since the concept of fairness was weakened in the previous section, we next focus on strategy-proof implementation. Suppose now that the mechanism designer have $\overline{m}$ monetary units that (at most) can be used to reduce envy and that he specifies a maximal compensation that can be attached to each good. This maximum amount is denoted by $\overline{x}_j$ for $j \in \{0,1\}$ and we suppose that $\overline{x} > \underline{x}$. Note that the maximum compensations can be chosen arbitrarily as long as $h_0\overline{x}_0 + h_1\overline{x}_1 \leq \overline{m}$. This maximum compensations may, for example, correspond to the travel costs for a child that is assigned a public school that not is located in the neighborhood. The vector $\overline{x} = (\overline{x}_0, \overline{x}_1)$ defines, in addition to the lower limit, an upper limit on the compensations.

We are primarily interested in a compensation vector that is optimal for the agents in the sense that the compensation for each good in this vector is weakly higher than the compensation for each good in an arbitrary compensation vector in the set of weakly fair distributions, $WF(u)$. The reason for this is that if the compensation vector is optimal for the agents, in the above sense, then envy is reduced as much as possible. This compensation vector is said to be WF and optimal[6] with respect to $\overline{x}$ (w.r.t. $\overline{x}$, henceforth).

Definition 2 *For a given profile $u \in \mathcal{U}$, a distribution $x \in \mathbb{R}^2_+$ is said to be weakly fair and optimal w.r.t. $\overline{x}$ if:*

(i) *$x \leq \overline{x}$ and $x \in WF(u)$,*

(ii) *$x \leq y \leq \overline{x}$ and $y \in WF(u)$ imply that $y = x$.*

We also say that an allocation (a, x) is weakly fair and optimal w.r.t. $\overline{x}$ if the distribution x is weakly fair and optimal w.r.t. $\overline{x}$.

The first part of the definition requires that the distribution x is WF and does not exceed the given upper bound. The second part of the definition is a (constrained) efficiency requirement. It means that there is no WF distribution that dominates x and

[6]This concept of optimality was introduced by Sun and Yang (2003) for fair allocations.

at the same time satisfies the upper bound on the compensation vector. The definition does not, á priori, exclude the possibility of none or several optimal distributions. But our next proposition will show the existence, as well as, the uniqueness of a distribution that is WF and optimal w.r.t. $\overline{x}$. Furthermore, efficiency is satisfied to that extent that at least one of the object types achieve maximum compensation.

Proposition 1 *For each vector $\overline{x} \in \mathbb{R}^2_{++}$ and for each profile $u \in \mathcal{U}$,* (i) *there exists a distribution x that is weakly fair and optimal w.r.t. $\overline{x}$,* (ii) *for a distribution x that is weakly fair and optimal w.r.t. $\overline{x}$, it holds that $x_k = \overline{x}_k$ for some $k \in \{0,1\}$ and* (iii) *the distribution that is weakly fair and optimal w.r.t. $\overline{x}$ is unique.*

Proof. (i) Let $R = \{x \in \mathbb{R}^2_+ \mid \underline{x} \leq x \leq \overline{x}\}$. Obviously, the set $R \cap WF(u)$ is compact, since preferences are continuous, and it is non-empty, since $\underline{x} \in R \cap WF(u)$. Then there is a maximal element $x \in R \cap WF(u)$ with respect to the order relation $\geq$. The allocation (a, x) is WF and optimal if the assignment a is defined in the following way. Let $C = \{i \in N \mid u_i(0, x) = u_i(1, x)\}$, where x is the maximal element. Then let agents $i \in N - C$ first choose objects and the associated compensations in accordance with the priority-order, and then assign the remaining objects to C arbitrarily.

(ii) Suppose that an allocation (a, x) is WF and optimal, but that $x_k < \overline{x}_k$ for all $k \in \{0,1\}$. If $x \in F(u)$, then there is arbitrarily small $\varepsilon > 0$ such that $(x_0+\varepsilon_0, x_1+\varepsilon_1)$ is fair, by the perturbation lemma in Alkan et al. (1991,p.1029). But this is a contradiction to optimality. Suppose instead that $x \notin F(u)$. Then object-efficiency requires that:

$$u_i(l, x) > u_i(k, x) \text{ for all } i \in N \text{ with } a_i = l \neq k \text{ when } x_l = 0.$$

To see this, note that since $x \notin F(u)$, there is $j \in N$ such that $a_j = k$ and $u_j(l, x) > u_j(k, x)$, and if the inequality above not is satisfied, there is a Pareto improving change of the assignment. Let now $y \in \mathbb{R}^2_+$ for $y_l = 0$ and $y_k = x_k + \varepsilon$. From the inequality above, it is clear that (a, y) is WF for a sufficiently small $\varepsilon > 0$ when $x_k < \overline{x}_k$, which contradicts that x is WF and optimal w.r.t. $\overline{x}$. Hence, $x_k = \overline{x}_k$.

(iii) If $\overline{x} \in WF(u)$, then $\overline{x}$ is fair and the only distribution that is WF and optimal. Suppose now that allocations (a, x) and (b, y) both are WF and optimal w.r.t. $\overline{x}$ and that $x \neq \overline{x}$, $y \neq \overline{x}$ and $x \neq y$. According to part (ii) above, we can, without loss of generality, assume that (a) $x_1 = \overline{x}_1$ and $y_0 = \overline{x}_0$ or (b) $x_1 = y_1 = \overline{x}_1$ and $y_0 > x_0$.

In case (a), for all $i \in N$ with $a_i = 0$:

$$u_i(0, x) \geq u_i(1, x) \Rightarrow u_i(0, y) > u_i(1, y) \Rightarrow b_i = 0.$$

The first inequality follows from the definition of WF since $x_1 = \overline{x}_1 > 0$. The second inequality follows from monotonicity. Further, $b_i = 0$, by WF since $y_0 = \overline{x}_0 > 0$. But then $b_i = 0$ if $a_i = 0$. Hence, $a = b$. But then allocation $(a, \overline{x})$ is fair, which is a contradiction.

In case (b) $y_0 > x_0 \geq 0$, so y is fair. But then x cannot be optimal, which is a contradiction.

In conclusion, $x = y$ must be the case. ■

We next illustrate the results in Proposition 1 with the aid of two simple examples. Again, we assume that there are three agents with quasi-linear preferences and that there is one object of type 0 and two objects of type 1. We must now introduce the maximum compensation limit, $\overline{x}$. In Figure 2, we see that all distributions in the region $\underline{x}zstx^*z$ satisfy the first requirement in Definition 2, i.e., they are all WF and none of them exceed the maximum compensation limit. The unique distribution that is WF and optimal w.r.t. $\overline{x}$ is given by x^*. Note also that $x_1 = \overline{x}_1$ at this distribution, so one of the compensations is given by the maximal amount, as predicted in Proposition 1(ii). Finally, in this example, envy is eliminated, at cost $x_0^* h_0 + x_1^* h_1 < \overline{m}$. However, the lowest cost at which the envy is eliminated is given by $z_0 h_0 + z_1 h_1$, but as we later demonstrate, the reason for selecting distribution x^* is that it is the only distribution that guarantees a strategy-proof outcome, given the upper limit $\overline{x}$. In Figure 3, a somewhat different case is illustrated. Here, the only distributions that satisfy the first criterion of Definition 2, are the distributions that are located along the line segment $\underline{x}x^*$. Note that none of these distributions are fair, and among these non-fair distributions, the unique distribution that is WF and optimal w.r.t. $\overline{x}$ is given by x^*, and again $x_1 = \overline{x}_1$. In this case, it is not possible to eliminate all envy, it will only be reduced, because the envious agents (i.e., agents 2 and 3) will obtain a monetary compensation.

[FIGURE 2 ABOUT HERE]
[FIGURE 2 ABOUT HERE]

In accordance with our definition of a WF and optimal allocation, we define an allocation rule φ to be WF and optimal w.r.t. $\overline{x}$ if for each profile $u \in \mathcal{U}$ and for all $(a, x) \in \varphi(u)$, allocation (a, x) is WF and optimal w.r.t. $\overline{x}$. If we let:

$$\varphi(u) = \{(a, x) \in \mathcal{A} \mid (a, x) \text{ is WF and optimal w.r.t. } \overline{x}\},$$

Part (i) of Proposition 1 shows that the set $\varphi(u)$ is non-empty for all profiles u, so there exists a WF and optimal allocation rule. Note also that φ defined in this way is an allocation rule, because φ is essentially single-valued. Moreover, from Part (iii) of Proposition 1, we know that each profile $u \in \mathcal{U}$ is uniquely mapped on a distribution by a WF and optimal allocation rule. This part of an allocation rule is called the distribution rule. Hence, if φ is WF and optimal w.r.t. $\overline{x}$ and if f is the corresponding distribution rule, then $f(u) = x$ precisely when $(a, x) \in \varphi(u)$ for some assignment a. Our first main theorem establishes that no agent or no coalition of agents can obtain a higher utility by manipulating an allocation rule that is WF and optimal w.r.t. $\overline{x}$. The following definition of strategy-proofness will be employed.

Definition 3 *An allocation rule φ is manipulable at a given profile $u \in \mathcal{U}$ by a coalition $C \subset N$ if there is a profile $v \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_C, u_{-C})$ such that $u_i(b_i, y) > u(a_i, x)$ for all $i \in C$. If the allocation rule is not manipulable by any coalition at any profile, it is said to be coalitionally strategy-proof* (CSP).

This means that in order to manipulate, it is sufficient that there is some profile $u \in \mathcal{U}$ and some coalition C such that by reporting preferences v_i for all $i \in C$, instead of u_i, the coalition can find some allocation $(b, y) \in \varphi(v_C, u_{-C})$ that makes all of its members better off.

Theorem 1 *An allocation rule φ that is weakly fair and optimal w.r.t. $\overline{x}$ is coalitionally strategy-proof.*

Proof. Let φ be an allocation rule that is WF and optimal w.r.t. $\overline{x}$, and suppose that it is manipulable by a coalition $C \subseteq N$ at a profile $u \in \mathcal{U}$. Then there is a profile $v \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_C, u_{-C})$, such that:

$$u_i(b_i, y) > u_i(a_i, x) \text{ for all } i \in C. \tag{1}$$

Now $x_k = \overline{x}_k$ for $k = 0$ or $k = 1$, according to Part (ii) of Proposition 1. Suppose that $x_1 = \overline{x}_1$. If $x = \overline{x}$, then x is fair and condition (1) cannot be satisfied. Hence, $x_0 < \overline{x}_0$ must be the case.

We first show that $b_i = 0$ if $i \in C$. Suppose that $b_i = 1$ for some $i \in C$. Then by condition (1) and WF, $u_i(1, y) > u_i(a_i, x) \geq u_i(1, x)$, since $x_1 = \overline{x}_1 > 0$. This is a contradiction to monotonicity since $y_1 \leq x_1 = \overline{x}_1$. Hence, $b_i = 0$ if $i \in C$.

We next demonstrate that $y_0 > x_0$. Suppose that $y_0 \leq x_0$. If $x_0 > 0$, then x is fair and, according to condition (1), fairness and monotonicity:

$$u_i(0, y) > u_i(a_i, x) \geq u_i(0, x) \geq u_i(0, y) \text{ for all } i \in C,$$

which is a contradiction. If $x_0 = 0$, then $y_0 = 0$. Hence, $y = x = (0, \overline{x}_1)$. But this is not consistent with condition (1) since the priority-order is exogenously given and then the utility distribution is unique, by the optimality requirement. Hence, $y_0 > x_0$.

We will now show that $a = b$. Note first that for all $i \in C$ with $a_i = 0$, it is also true that $b_i = 0$, by the above conclusion. Consider now some $i \in N - C$ with $a_i = 0$. Then by monotonicity and WF:

$$u_i(0, y) > u_i(0, x) \geq u_i(1, x) \geq u_i(1, y),$$

since $x_1 = \overline{x}_1 > 0$. But $y \in F(v_C, u_{-C})$, so the inequality above implies that $b_i = 0$, because $y_0 > 0$. Thus, for all $i \in N$ with $a_i = 0$, it holds that $b_i = 0$. Hence, $a = b$.

Finally, we show that x is not optimal. Let $x' = (x_0 + \varepsilon, \overline{x}_1)$ for $\varepsilon > 0$ and small. Then $x' \in F(u)$ because:

$$\text{for all } i \in N \text{ with } a_i = 0 : u_i(0, x') > u_i(0, x) \geq u_i(1, x) = u_i(1, x'),$$

by WF, since $x_1 > 0$. Moreover:

$$\text{for all } i \in N \text{ with } a_i = 1 : u_i(1, x) \geq u_i(1, y) \geq u_i(0, y) > u_i(0, x),$$

by WF and monotonicity, since $x_1 = \overline{x}_1 \geq y_1$ and $y_0 > x_0 \geq 0$. But then, $u_i(1, x') > u_i(0, x')$ for all $i \in N$ with $a_i = 1$, for a sufficiently small $\varepsilon > 0$. Thus, $x' \in F(u)$. This means that x cannot be optimal, which is a contradiction. Hence, φ cannot be manipulable. ■

5 The class of weakly fair and coalitionally strategy-proof allocation rules

Theorem 1 shows the existence of a coalitionally strategy-proof allocation rule that implements weakly fair allocations. A more general problem is, of course, to characterize all weakly fair and coalitionally strategy-proof allocation rules that are available to a mechanism designer when the monetary compensations are restricted. In this section, this class of allocation rules is characterized, given a mild regularity condition. We demonstrate that all weakly fair and coalitionally strategy-proof allocation rules, in fact, are optimal w.r.t. some quantity restriction $\overline{x}$.

Our point of departure is a finite amount of money and a regularity condition that implies that all available money is distributed by the allocation rule for some preference profile (this is not a crucial assumption, see footnote 7). Moreover, we restrict the manipulation possibilities of the agents, by only considering quasi-linear utility functions. One could expect that by narrowing the domain of the allocation rule, the class of strategy-proof allocation rules should increase. As Theorems 2 and 3 (below) demonstrate, this is, however, not the case. Formally, we restrict our attention to a class of utility functions where the utility of agent $i \in N$ when he is assigned object $j \in \{0, 1\}$ and x_j units of money is given by: $u_{ij} + x_j$. For notational simplicity, we will often describe the (quasi-linear) preferences for agent $i \in N$ as: $u_i = (u_{i0}, u_{i1}) \in \mathbb{R}^2$. The set of profiles with quasi-linear utility functions is denoted by $\mathcal{U}^q \subset \mathcal{U}$.

In Theorem 2, we investigate allocation rules that are Pareto indifferent (PI). This means that if an allocation belongs to the outcome of the allocation rule, at some profile, then every allocation that all agents are indifferent between, also belong to the outcome of the allocation rule. An example of such a rule is the WF and optimal allocation rule, defined in the previous section:

$$\varphi(u) = \{(a, x) \in \mathcal{A} \mid (a, x) \text{ is WF and optimal w.r.t. } \overline{x}\}.$$

PI correspondences are obviously interesting from a welfarist point of view, because every allocation that all agents are indifferent between are treated equally. If, on the other hand, the allocation rule is used to choose exactly one particular outcome, a tie-breaking rule is necessary. Such a rule may influence the strategic behavior of the agents. In Theorem 3 we, therefore, drop the PI assumption and characterize the class of essentially single-valued (ESV) allocation rules, a class that also includes single-valued allocation rules (recall also that an allocation rule is ESV, by definition). We restrict, however, our attention to allocation rules that are said to be fairness selective. This condition is in the spirit of point (ii) in the definition of weak fairness, i.e., the allocation rule selects fair allocations when possible. It means that if the outcome x is fair at a profile and an agent changes his preferences so that x is still fair, then the outcome at the new profile, not necessarily equal to x, is fair. The formal definition is given below. The theorems are given in this section while the proofs are delegated to the Appendix. In both theorems, we assume that all available money is distributed among the agents for some profile in $\mathcal{U}^q$.[7] Such an allocation rule is called regular.

[7]We need not assume that all available money is distributed among the agents for some profile. To prove Theorems 2 and 3, it suffices that there is a finite upper bound on how much money that can be distributed by a particular distribution rule, and that this bound is attained for some profile in $\mathcal{U}^q$. This bound need not equal $\overline{m}$.

Definition 4 *An allocation rule $\varphi : \mathcal{U}^q \to 2^{\mathcal{A}}$ is:*

(i) Pareto indifferent (PI) *if $(a,x) \in \varphi(u)$ and $u_{b_i} + y_{b_i} = u_{a_i} + x_{a_i}$ for all $i \in N$ imply that $(b,y) \in \varphi(u)$,*

(ii) Fairness selective (FS) *if for all $u, v \in \mathcal{U}^q$ and $i \in N$, $f(u) \in F(u) \cap F(v_i, u_{-i})$ implies that $f(v_i, u_{-i}) \in F(v_i, u_{-i})$,*

(iii) Regular *if for some $u \in \mathcal{U}^q$ there is an allocation $(a,x) \in \varphi(u)$ where $h_0 x_0 + h_1 x_1 = \overline{m}$.*

Theorem 2 *A weakly fair, regular and Pareto indifferent allocation rule $\varphi : \mathcal{U}^q \to 2^{\mathcal{A}}$ is coalitionally strategy-proof if and only if it is optimal w.r.t. some distribution $\overline{x} \in \mathbb{R}^2_{++}$ where $h_0\overline{x}_0 + h_1\overline{x}_1 = \overline{m}$.*

Note that a weakly fair and optimal allocation rule is fairness selective (see Lemma 3 in the Appendix), so, in this case, fairness selection is an implication of the other conditions.

We next consider the case that includes single-valued allocation rules. That means that the assumption of Pareto indifference has to be dropped. But let us first introduce a weaker form of strategy-proofness.

Definition 5 *An allocation rule φ is strongly manipulable at a given profile $u \in \mathcal{U}$ by a coalition $C \subset N$ if there is a profile $v \in \mathcal{U}$ such that $u_i(b_i, y) > u(a_i, x)$ for all $i \in C$ for all allocations $(a,x) \in \varphi(u)$ and $(b,y) \in \varphi(v_C, u_{-C})$. If the allocation rule is not strongly manipulable by any coalition at any profile, it is said to be weakly coalitionally strategy-proof* (WCSP).

Note that this definition of manipulation is rather demanding. To manipulate, it is necessary that there is some profile u and some coalition C such that by reporting preferences v_i for all $i \in C$, instead of u_i, each allocation $(b,y) \in \varphi(v_C, u_{-C})$ is better for all members of the coalition C than any allocation $(a,x) \in \varphi(u)$. Of course, any CSP allocation rule is also WCSP. However, even if the set of WCSP in general is larger than the set of CSP allocation rules, there are cases when the two concepts coincide. If the allocation rule is single-valued this is obviously the case. We next prove that this also is the case when the allocation rule is ESV and weakly fair.

Proposition 2 *A weakly fair and essentially single-valued allocation rule $\varphi : \mathcal{U}^q \to 2^{\mathcal{A}}$ is coalitionally strategy-proof if it is weakly coalitionally strategy-proof and fairness selective.*

A close consequence of Proposition 2 and Theorem 2 is our last theorem.

Theorem 3 *A fairness selective, weakly fair, essentially single-valued and regular allocation rule* $\varphi : \mathcal{U}^q \to 2^{\mathcal{A}}$ *is coalitionally strategy-proof if and only if it is optimal w.r.t. some distribution* $\overline{x} \in \mathbb{R}^2_{++}$ *where* $h_0\overline{x}_0 + h_1\overline{x}_1 = \overline{m}$.

A Appendix: Proofs

A.1 The proof of Theorem 2

We first note that the "if" part of the theorem follows directly from Theorem 1. In order to prove the "only if" part of the theorem, we specify and prove a series of lemmas. In these lemmas, we suppose that the assumptions from Theorem 2 hold. The following lemma will be useful in the analysis.[8]

Lemma 1 *Suppose that* $u \in \mathcal{U}^q$*, and let* $x, y \in F(u)$ *be two fair distributions and* $(a, x) \in \Phi(u)$ *a fair allocation. Then* $(a, y) \in \Phi(u)$ *is also a fair allocation.*

Let now $\overline{u} \in \mathcal{U}^q$ and $(a, \overline{x}) \in \varphi(\overline{u})$ for some assignment a and some distribution $\overline{x} \in \mathbb{R}^2_{++}$ where $h_0\overline{x}_0 + h_1\overline{x}_1 = \overline{m}$. Such a profile exists since φ is regular. The existence of a profile at which the outcome of the allocation rule is efficient is shown in the following lemma.

Lemma 2 *If* $v \in \mathcal{U}^q$ *and* $v_i = -\overline{x}$ *for all* $i \in N$*, then* $f(v) = \overline{x}$*.*

Proof. Let $v \in \mathcal{U}^q$ and $v_i = -\overline{x}$ for all $i \in N$, and suppose that $(a, x) \in \varphi(v)$ and $f(v) = x \neq \overline{x}$. If $x < \overline{x}$, then the entire coalition N can manipulate and achieve $\overline{x}$, by reporting the profile $\overline{u}$. This is not consistent with CSP. Hence, by the resource constraint, we need only consider the case when $x_k \geq \overline{x}_k$ and $x_l < \overline{x}_l$ for $l, k \in \{0, 1\}$ and $l \neq k$. However:

$$v_{ik} + x_k = -\overline{x}_k + x_k \geq 0 > -\overline{x}_l + x_l = v_{il} + x_l \text{ for all } i \in N,$$

which contradicts that (a, x) is WF, i.e., all agents with $a_i = l$ envies the agents with $a_i = k$, and this cannot be the case because $x_k > 0$. Thus, $x = \overline{x}$. ■

Our next lemma shows that if there, for a certain profile, is a fair distribution satisfying the upper bound and lower limit, the outcome of the allocation rule must also be a fair

[8] The proof can be found in, e.g., Svensson (2006).

distribution. Hence, a consequence of strategy-proofness is that fairness has a priority to non-fairness in the definition of weak fairness.

Lemma 3 *Let $u \in \mathcal{U}^q$ and $R = \{x \in \mathbb{R}^2_+ \mid \underline{x} \leq x \leq \overline{x}\}$. If $R \cap F(u) \neq \varnothing$, then $f(u) \in F(u)$.*

Proof. Suppose that $R \cap F(u) \neq \varnothing$, but $x = f(u) \notin F(u)$. In this case, $x_k \geq \overline{x}_k$ for some $k \in \{0, 1\}$, because $x < \overline{x}$ is not consistent with CSP (see the proof of Lemma 2). Moreover, $x_l = 0$ for $l \neq k$, since $x \notin F(u)$. Let now $(a, x) \in \varphi(u)$ and $C = \{i \in N \mid a_i = l\}$. Since $x \notin F(u)$, it follows from the definition of WF that: $u_{il} + x_l > u_{ik} + x_k$ for some $j \in N - C$. Moreover, $u_{il} + x_l > u_{ik} + x_k$ for all $i \in C$, by object-efficiency, otherwise there is a Pareto improving change of the assignment. Because $R \cap F(u) \neq \varnothing$, there is a distribution $y \in R \cap F(u)$ and, consequently, an assignment b such that $(b, y) \in \Phi(u)$. This observation, monotonicity and WF yield:

$$u_{il} + y_l \geq u_{il} + 0 = u_{il} + x_l > u_{ik} + x_k \geq u_{ik} + \overline{x}_k \geq u_{ik} + y_k \text{ for all } i \in C.$$

Hence, $b_i = a_i$ if $i \in C$, because $(b, y) \in \Phi(u)$, and, as a consequence, $b = a$. Moreover:

$$u_{ik} + x_k \geq u_{ik} + \overline{x}_k \geq u_{ik} + y_k \geq u_{il} + y_l \geq u_{il} + 0 = u_{il} + x_l \text{ for all } i \in N - C.$$

by monotonicity and fairness. But this contradicts that there is an agent $j \in N - C$, as defined above. Hence, $x = f(u) \in F(u)$. ■

We next establish that if the distribution is fair, then the compensation cannot exceed the upper bound and, moreover, the compensation for (at least) one of the object types must equal the upper bound.

Lemma 4 *Let $u \in \mathcal{U}^q$. If $f(u) \in F(u)$, then $f(u) \leq \overline{x}$ and $f_k(u) = \overline{x}_k$ for some $k \in \{0, 1\}$.*

Proof. Suppose that $x = f(u) \in F(u)$ and, consequently, that $(a, x) \in \varphi(u)$ and $(a, x) \in \Phi(u)$ for some assignment a. If $x = \overline{x}$, we are done, and $x < \overline{x}$ is not consistent with CSP (see the proof of Lemma 2). For this reason, we assume that $x_0 < \overline{x}_0$ and $x_1 \geq \overline{x}_1$. Hence, it remains only to prove that $x_1 = \overline{x}_1$.

Let now $v \in \mathcal{U}^q$ be defined as in Lemma 2 and $C = \{i \in N \mid a_i = 0\}$. Since $(a, x) \in \Phi(u)$, it follows from monotonicity that:

$$u_{i0} + \overline{x}_0 > u_{i0} + x_0 \geq u_{i1} + x_1 \geq u_{i1} + \overline{x}_1 \text{ for all } i \in C,$$
$$v_{i1} + \overline{x}_1 = v_{i0} + \overline{x}_0 = 0 \text{ for all } i \in N - C.$$

From these inequalities, it is clear that $(a, \overline{x}) \in \Phi(u_C, v_{-C})$. Hence, if we can prove that $(a, \overline{x}) \in \varphi(u_C, v_{-C})$, we are done, because if this is the case, then $(a, x) \in \varphi(u)$ and $(a, \overline{x}) \in \varphi(u_C, v_{-C})$, so $v_{i1} + \overline{x}_1 \geq v_{i1} + x_1$ or, equivalently, $x_1 \leq \overline{x}_1$, by CSP. The latter inequality, in combination with the assumption $x_1 \geq \overline{x}_1$, gives $x_1 = \overline{x}_1$, as desired. The remaining part of the proof demonstrates that $(a, \overline{x}) \in \varphi(u_C, v_{-C})$.

Suppose now that $(b, y) \in \varphi(u_C, v_{-C})$ and note that $\overline{x} \in R \cap F(u_C, v_{-C})$ because $(a, \overline{x}) \in \Phi(u_C, v_{-C})$. Consequently, $y \in F(u_C, v_{-C})$, by Lemma 3, and $(a, y) \in \Phi(u_C, v_{-C})$, by Lemma 1. But φ is PI and, therefore, $(a, y) \in \varphi(u_C, v_{-C})$, so it remains only to prove that $y = \overline{x}$. Because $v \in \mathcal{U}^q$ is defined as in Lemma 2, it is clear that $(a, \overline{x}) \in \varphi(v)$. Then, since φ is CSP, there are agents $i, j \in C$ such that:

$$u_{i0} + y_0 \geq u_{i0} + \overline{x}_0 \text{ and } v_{j0} + \overline{x}_0 \geq v_{j0} + y_0.$$

These inequalities give $y_0 = \overline{x}_0$. Thus, $y_1 \leq \overline{x}_1$, by the resource constraint. Suppose now that $y_1 < \overline{x}_1$. In this case:

$$\begin{aligned} u_{i0} + y_0 = u_{i0} + \overline{x}_0 \geq u_{i1} + \overline{x}_1 > u_{i1} + y_1 \text{ for all } i \in C, \\ v_{i0} + y_0 = v_{i0} + \overline{x}_0 = v_{i1} + \overline{x}_1 > v_{i1} + y_1 \text{ for all } i \in N - C \end{aligned}$$

But this is a contradiction to $(a, y) \in \Phi(u_C, v_{-C})$. Hence, $y_1 = \overline{x}_1$ and, therefore, $y = \overline{x}$, which concludes the proof. ■

We next demonstrate that if the distribution is WF, but not fair, then the compensation that is associated with one of the objects must be given by the minimum compensation limit and, moreover, that the compensation for the other object must equal the upper bound.

Lemma 5 *Let $u \in \mathcal{U}^q$. If $f(u) \notin F(u)$, then $x_l = 0$ and $x_k = \overline{x}_k$ for some $l, k \in \{0, 1\}$ and $l \neq k$.*

Proof. Suppose that $x = f(u) \notin F(u)$, so $(a, x) \in \varphi(u)$ for some assignment a. Then $x_l = 0$ for some $l \in \{0, 1\}$, say $l = 0$, by the definition of WF. But then $x_1 \geq \overline{x}_1$, because $x_1 < \overline{x}_1$ is not consistent with CSP (see the proof of Lemma 2). Hence, it remains only to prove that $x_1 = \overline{x}_1$. In the remaining part of this proof, we let $C = \{i \in N \mid a_i = 1\}$.

To obtain a contradiction, suppose that $x_1 > \overline{x}_1$. Consider now the profile $v \in \mathcal{U}^q$, where $v_i = -(0, \overline{x}_1)$ for all $i \in N$, and the allocation $(b, y) \in \varphi(v_C, u_{-C})$. We need to demonstrate that $a = b$ and $y = (0, \overline{x}_1)$, because if this is the case, then coalition

C can manipulate at the profile (v_C, u_{-C}) and achieve x_1 instead of $\overline{x}_1$, which is not consistent with CSP, i.e., the assumption $x_1 > \overline{x}_1$ cannot hold.

We first demonstrate that $a = b$. By WF and $x_0 = 0$, it follows that $(0, \overline{x}_1) \in R \cap F(v_C, u_{-C})$, because:

$$v_{i1} + \overline{x}_1 = v_{i0} + 0 = 0 \text{ for all } i \in C,$$
$$u_{i0} + 0 = u_{i0} + x_0 \geq u_{i1} + x_1 > u_{i1} + \overline{x}_1 \text{ for all } i \in N - C.$$

Hence, $R \cap F(v_C, u_{-C}) \neq \varnothing$ and, therefore, $y = f(v_C, u_{-C}) \in F(v_C, u_{-C})$, by Lemma 3, and also $y \leq \overline{x}$ and $y_k = \overline{x}_k$ for some $k \in \{0, 1\}$, by Lemma 4. Moreover, $(b, y) \in \Phi(v_C, u_{-C})$, because $y \in F(v_C, u_{-C})$. These observations, WF and monotonicity then gives:

$$u_{i0} + y_0 \geq u_{i0} + 0 = u_{i0} + x_0 \geq u_{i1} + x_1 > u_{i1} + \overline{x}_1 \geq u_{i1} + y_1 \text{ for all } i \in N - C.$$

Hence, $b_i = a_i$ if $i \in N - C$, because $(b, y) \in \Phi(v_C, u_{-C})$, and, as a consequence, $b = a$.

We next demonstrate that $y = (0, \overline{x}_1)$. From the above observation that $(b, y) \in \Phi(v_C, u_{-C})$, it follows that:

$$v_{i1} + y_1 \geq v_{i0} + y_0 \Rightarrow y_1 - \overline{x}_1 \geq y_0 \text{ for all } i \in C.$$

A consequence of the fact that $y \leq \overline{x}$ is that $y_1 - \overline{x}_1 \leq 0$, so $y_0 \leq y_1 - \overline{x}_1 \leq 0$, by the above inequality. But then $y_0 = 0$, and, as a consequence, $y_1 = \overline{x}_1$. Hence, $y = (0, \overline{x}_1)$. ■

We have now proved two important properties of the WF and CSP allocation rule, the first is valid in the case when the distribution is fair (Lemma 4), and the second is valid in the case when the distribution is WF, but not fair (Lemma 5). Given these results, we can prove Theorem 2. In order to do so, it remains only to prove that the allocation rule is optimal w.r.t. $\overline{x}$. This result is formally provided in the following lemma.

Lemma 6 *Let $u \in \mathcal{U}^q$ and $y \in R \cap WF(u)$. Then $y \leq f(u)$.*

Proof. Suppose that $y \in R \cap WF(u)$ and $(a, x) \in \varphi(u)$. We first consider the case when $x = f(u) \notin F(u)$. By Lemma 5, we can, without loss of generality, assume that $x = (0, \overline{x}_1)$. Suppose now that $y \neq x$, but that $y \leq x$ is not true. Then $y_0 > x_0 = 0$ and $y_1 \leq x_1 = \overline{x}_1$, because $y \in R$. By the definition of WF, it follows that for all $i \in N$ with $a_i = 0$ and some $i \in N$ with $a_i = 1$ it holds that:

$$u_{i0} + y_0 > u_{i0} + 0 \geq u_{i1} + \overline{x}_1 \geq u_{i1} + y_1.$$

But this cannot be true since $y \in WF(u)$ and $y_0 > 0$. Hence, $y \leq x$.

Consider now the case when $x \in F(u)$. Then, according to Lemma 4, we can, without loss of generality, assume that $x_1 = \overline{x}_1$. If $x_0 = \overline{x}_0$ we are done, so let instead $x_0 < \overline{x}_0$. Suppose again that $y \neq x$, but that $y \leq x$ is not true. Then $y_0 > x_0$ and $y_1 \leq x_1 = \overline{x}_1$, because $y \in R$. If $y \notin F(u)$, then $y_1 = 0$ and:

$$u_{i0} + y_0 > u_{i0} + x_0 \geq u_{i1} + x_1 \geq u_{i1} + y_1 \text{ for all } i \in N \text{ with } a_i = 0.$$

But this is a contradiction to WF, since $y \notin F(u)$ and $y_1 = 0$. Hence, the case when $y \notin F(u)$ can be excluded, and we only have to consider the case when $x, y \in F(u)$, $y_0 > x_0 and\ y_1 \leq x_1$. Let now $C = \{i \in N \mid a_i = 0\}$ and $v \in \mathcal{U}^q$, where $v_i = -y$ for all $i \in N$. Then $y \in F(v_C, u_{-C})$ since $v_{i0} + y_0 = v_{i1} + y_1 = 0$ for all $i \in C$. But then $y \in R \cap F(v_C, u_{-C}) \neq \varnothing$. This means that there is a distribution $z = f(v_C, u_{-C}) \in F(v_C, u_{-C})$, by Lemma 3, where, in addition, $z \in R$, by Lemma 4. From PI it then follows that $(a, z) \in \varphi(v_C, u_{-C})$. We next observe that $z_0 > x_0$. To see this, note that $z_j = \overline{x}_j$ for some $j \in \{0, 1\}$, since $z = f(v_C, u_{-C})$. Hence, if $z_0 = \overline{x}_0$, then $z_0 > x_0$, because $x_0 = 0$. Suppose instead that $z_1 = \overline{x}_1$, and recall that $x_1 = \overline{x}_1$. Since $z \in F(v_C, u_{-C})$, it follows that:

$$v_{i0} + z_0 \geq v_{i1} + z_1 \Leftrightarrow z_0 - y_0 \geq z_1 - y_1 \text{ for all } i \in C,$$

and hence:

$$z_0 \geq z_1 + y_0 - y_1 > z_1 + x_0 - x_1 = x_0.$$

We conclude that $z_0 > x_0$. But then the coalition C can manipulate at the profile u and achieve z_0 instead of x_0, which contradicts that φ is CSP. Hence, $y \leq x$ also in this case. ■

A.2 The proof of Proposition 2 and Theorem 3

To prove Proposition 2 the following lemma will be useful.

Lemma 7 *Let $\varphi : \mathcal{U}^q \to 2^{\mathcal{A}}$ be a weakly fair, fairness selective and weakly coalitionally strategy-proof allocation rule, $u \in \mathcal{U}^q$ and $(a, x) \in \varphi(u)$. Then, for each coalition $C \subset N$ and for any positive number $\varepsilon > 0$, there is a profile $v \in \mathcal{U}^q$ such that for all $(b, y) \in \varphi(v_C, u_{-C})$,*

$$b_i = a_i \text{ and } y_{a_i} \geq x_{a_i} - \varepsilon \text{ for all } i \in C.$$

Proof. Let $(a, x) \in \varphi(u)$ and $C \subset N$. If $x \notin F(u)$ then the assignment a is unique (see the proof of Part (i) of Proposition 1). In that case, let $v = u$ to complete the proof. Consider $x \in F(u)$ and let $\varepsilon > 0$ be an arbitrary positive number. Suppose next that $(b, y) \in \varphi(v_C, u_{-C})$, and define $v \in \mathcal{U}^q$ as:

$$v_{ia_i} = -x_{a_i} + \varepsilon/2 \text{ and } v_{ij} = -x_j \text{ if } j \neq a_i \text{ for all } i \in N.$$

Note first that for all $i \in C$, $f(u) \in F(u) \cap F(v_i, u_{-i})$ and, hence, by FS, $f(v_i, u_{-i}) \in F(v_i, u_{-i})$, i.e., the outcome $f(v_i, u_{-i})$ is fair. If we repeat the substitution of u_i with v_i for all $i \in C$, we obtain $y \in F(v_C, u_{-C})$.

Now $(a, x) \in \varphi(u)$ implies that $(a, x) \in \Phi(v_C, u_{-C})$, by construction of v. If $(a', x) \in \Phi(v_C, u_{-C})$, but $a'_i \neq a_i$ for some $i \in C$, then:

$$v_{ia'_i} + x_{a'_i} \geq v_{ia_i} + x_{a_i},$$

by fairness and, as a consequence, $0 \geq \varepsilon/2$, by construction of v, which is a contradiction. Hence, if $(a', x) \in \Phi(v_C, u_{-C})$, then $a'_i = a_i$ for all $i \in C$. But $(b, y) \in \Phi(v_C, u_{-C})$, because $y \in F(v_C, u_{-C})$, and, therefore, $b_i = a_i$ for all $i \in C$, by Lemma 1. From fairness, it now follows that:

$$v_{ia_i} + y_{a_i} \geq v_{ij} + y_j \text{ for all } j \in \{0, 1\} \text{ and all } i \in C$$

By applying the definition of v, this condition can be rewritten to:

$$y_{a_i} - x_{a_i} + \varepsilon/2 \geq y_j - x_j \text{ for all } j \in \{0, 1\} \text{ and all } i \in C. \tag{2}$$

Moreover, by WCSP, there is an agent $l \in C$ and an allocation $(a', x) \in \varphi(u)$ such that:

$$v_{la_l} + y_{a_l} \geq v_{la'_l} + x_{a'_l}.$$

Then, by construction of v:

$$y_{a_l} - x_{a_l} + \varepsilon/2 \geq 0. \tag{3}$$

From conditions (2) and (3), it now follows that $y_{a_i} - x_{a_i} \geq -\varepsilon$ for all $i \in C$. Hence, $y_{a_i} \geq x_{a_i} - \varepsilon$, as desired. ■

Proof of Proposition 2. WCSP is implied by CSP. To prove the converse suppose that φ is WCSP, but not CSP. Then there are two profiles $u, v \in \mathcal{U}$, a non-empty coalition $C \subset N$, and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(v_C, u_{-C})$, such that

$u_{ib_i} + y_{b_i} > u_{ia_i} + x_{a_i}$ for all $i \in C$. Define now $\varepsilon > 0$ so that: $u_{ib_i} + y_{b_i} > u_{ia_i} + x_{a_i} + \varepsilon$ for all $i \in C$. By Lemma 7 there is a profile $v' \in \mathcal{U}$ such that:

$$\text{if } (c, z) \in \varphi(v'_C, u_{-C}), \text{ then } c_i = b_i \text{ and } z_{b_i} > y_{b_i} - \varepsilon \text{ for all } i \in C.$$

Thus, for each $(c, z) \in \varphi(v'_C, u_{-C})$, the following holds:

$$u_{ic_i} + z_{c_i} = u_{ib_i} + z_{b_i} > u_{ib_i} + y_{b_i} - \varepsilon > u_{ia_i} + x_{a_i} \text{ for all } i \in C.$$

This shows that φ is strongly manipulable, which is a contradiction. ■

Proof of Theorem 3. The first part of the theorem follows from Theorem 1. Suppose now that φ is CSP. Let $\tilde{\varphi}$ be an induced allocation rule that, for all $u \in \mathcal{U}$, is defined as follows:

$$(b, x) \in \tilde{\varphi}(u) \Leftrightarrow \text{there is } (a, x) \in \varphi(u) \text{ and } u_{ib_i} + x_{b_i} = u_{ia_i} + x_{a_i} \text{ for all } i \in N.$$

Note that $\tilde{\varphi}$ is well-defined, because φ is ESV. Clearly, $\tilde{\varphi}$ is a PI and fair allocation rule; the correspondence $\tilde{\varphi}$ is ESV and PI, by definition, and $\tilde{\varphi}$ is WF since φ is WF. The allocation rule $\tilde{\varphi}$ is obviously WCSP, because φ is CSP. Then by Proposition 2, the allocation rule $\tilde{\varphi}$ is also CSP. It is also regular since φ is regular. But then $\tilde{\varphi}$ is optimal w.r.t. some vector $\overline{x} \in \mathbb{R}^2_{++}$, by Theorem 2, and, hence, φ is also optimal w.r.t. some vector $\overline{x} \in \mathbb{R}^2_{++}$. ■

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Figure 1 The set of WF distributions

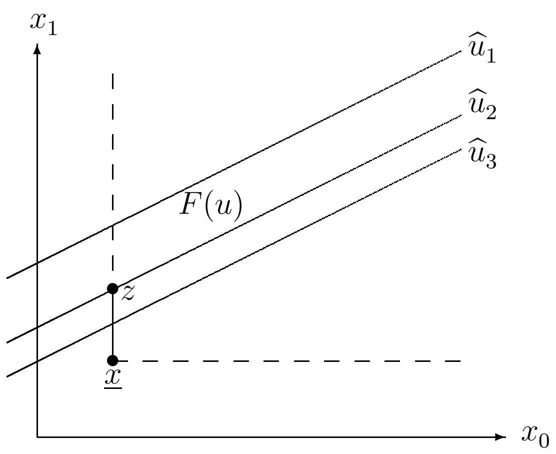

Figure 2 A first example of a distribution that is WF and optimal w.r.t. $\overline{x}$

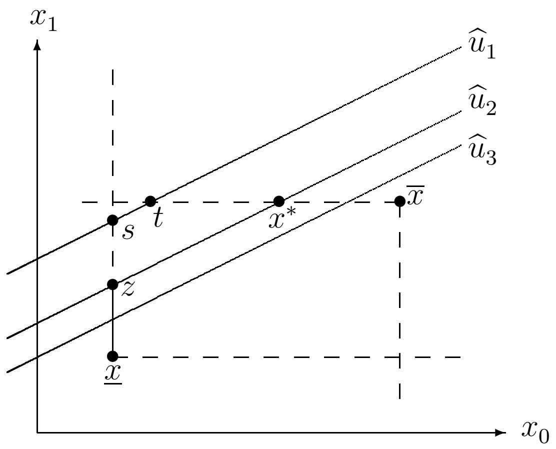

Figure 3 A second example of a distribution that is WF and optimal w.r.t. $\overline{x}$