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A TABLEAU SYSTEM FOR A FIRST-ORDER HYBRID LOGIC

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Abstract. In this paper a first-order version of hybrid logic is presented. The language is obtained by adding nominals, satisfaction operators and the down-arrow binder to classical first-order modal logic (including constants and function symbols). The satisfaction operators are applied to both formulas and terms. Moreover adding the universal modality is discussed.

This first-order hybrid language is interpreted over varying domains and a sound and complete, fully internalized tableau system for this logic is given.

Keywords: Hybrid logic, first-order modal logic, first-order hybrid logic, tableau systems.

1. Introduction

First-order modal logic in philosophy is an old field of study and has been treated extensively. Propositional hybrid logic is also becoming a well-studied field. However the literature on hybrid logic versions of first-order modal logic is still limited. Some few examples are [1, 2, 4, 6, 7].

Though it is known that hybrid versions of first-order modal logic have many advantages compared to classical first-order modal logic. First of all many classical first-order modal logics lacks the interpolations property, but it has been shown in [1] that a hybrid version of first-order modal logic containing satisfaction operators and the down-arrow binder, fixes this problem.

When it comes to the expressiveness of first-order hybrid languages it may come as no surprise that it is a great deal higher than the expressiveness of classical first-order modal languages. It has long been known that first-order modal logic lacks the power to express certain properties related to natural language semantics. See for instance [11]. That first-order hybrid logic is useful in relation to natural language semantics is also well known and discussed in for instance [3]. A kind of a first-order hybrid logic is also used in [8]. Thus from the viewpoint of natural language semantics first-order hybrid logic is a very natural thing.

Furthermore adding predicate abstraction to first-order modal logic as done in [9] does not give any new expressive powers compared to first-order hybrid logic. Since predicate abstraction easily can be simulated in a first-order hybrid logic with the down-arrow binder and satisfaction operators on terms (see for instance [7]).

Additionally as for propositional hybrid logic a wide range of general completeness results are possible, as discussed in for instance [2] and [6], as well as completely internalized proof systems. This article contains such an internalized proof system, namely a completely internalized tableau system for a first-order hybrid logic.

The first-order hybrid language we will present is in a sense just classical first-order logic combined with a propositional hybrid logic containing nominals, satisfaction operators, and the down-arrow binder. However there is a bit more to it since we will also use satisfaction operators on terms. Furthermore we will also discuss adding the universal modality to the language.

2. A first-order hybrid language

In this section a first-order hybrid language (denoted by **FHL**) is presented and a varying domain semantics for the language is given. The language is obtained by combining classical first-order logic with hybrid logic. First we will give the syntax of the language.

2.1. **Syntax for FHL.** As in classical first-order logic the language **FHL** contains a countable infinite set of first-order variables FVAR, a countable infinite set of constants CON, a countable infinite set of function symbols FSYM, and a countable infinite set of relation symbols RSYM. (For any $n \in \mathbb{N}$ there might be function and relation symbols of arity n.) To get a first-order hybrid logic we further need a countable infinite set of nominals NOM and a countable infinite set of state variables SVAR. (State variables will vary over worlds and nominals will appear as world constants.) Besides the logical symbols $\neg, \vee, \exists, =$, (and) of classical first-order logic, there will be the classical modal operator $\Diamond$, a down-arrow binder $\downarrow$, and for every $u \in \mathsf{NOM} \cup \mathsf{SVAR}$ there will be two kinds of satisfaction operators $@_u$ and $u{:}$.[1]

The terms of **FHL** can now be defined.

Definition 1 (**FHL**-terms)**.** *The set of **FHL**-terms (denoted by T^{FHL}) is given by the following grammar:*

$$t \ ::= \ x \ \mid \ c \ \mid \ u{:}t \ \mid \ f(t_1, \ldots, t_n) \ ,$$

where $x \in \mathsf{FVAR}$, $c \in \mathsf{CON}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, and f is an n-ary function symbol of FSYM.

(When no confusion can arise we will referrer to **FHL**-terms as just terms.) As in classical first-order logic variables and constants are terms and function symbols can be used to recursively defining more complex terms. Furthermore new terms can be constructed from a term t by prefixing it with the satisfaction operator $u{:}$ getting the term $u{:}t$. The intuition behind the term $u{:}t$ is that it denotes what t denotes at the world u. This is crucial since we will interpret our constants and function symbols (as well as the relation symbols) non-rigidly, i.e. they might denote different things in different worlds.[2]

Now for the definition of **FHL**-formulas (or just formulas).

Definition 2 (**FHL**-formulas)**.** *The set of **FHL**-formulas (denoted by F^{FHL}) is given by the following grammar:*

$$\varphi \ ::= \ R(t_1, \ldots, t_n) \ \mid \ t_1 = t_2 \ \mid \ u \ \mid \ \neg\psi \ \mid \ (\psi_1 \vee \psi_2) \ \mid \ \Diamond\psi \ \mid \ (\exists x)\psi \ \mid \ @_u\psi \ \mid \ \downarrow v.\psi \ ,$$

where $R \in \mathsf{RSYM}$ is n-ary, $t_1, t_2, \ldots, t_n \in \mathsf{T}^{FHL}$, $u \in \mathsf{NOM} \cup \mathsf{SVAR}$, $x \in \mathsf{FVAR}$, and $v \in \mathsf{SVAR}$.

When we in the following are talking about variables, and nothing else is mentioned, we will be talking about elements of $\mathsf{FVAR} \cup \mathsf{SVAR}$. Free occurrences of first-order variables are defined as in classical first-order logic and the free occurrences of state variables are defined in a similar manner, noting that only the $\downarrow$-binder can bind state variables. A sentence is a formula in which all variables are bound.

2.2. **Semantics for FHL.** Only a varying domain semantics is presented for **FHL**, since constant domain semantics can be seen as a special case of varying domain semantics. If $\langle W, \mathcal{R} \rangle$ is an ordinary modal frame, $\mathbb{D}$ a non-empty set, and $\mathcal{D}$ a function on W such that it assigns a non-empty set $\mathcal{D}(w) \subseteq \mathbb{D}$ to every $w \in W$, then the tuple $\langle W, \mathcal{R}, \mathbb{D}, \mathcal{D} \rangle$ is called a *skeleton*. A *model* is a tuple $\mathcal{M} = \langle W, \mathcal{R}, \mathbb{D}, \mathcal{D}, \ell \rangle$, where $\langle W, \mathcal{R}, \mathbb{D}, \mathcal{D} \rangle$ is a skeleton and $\ell = (\ell_w)_{w \in W}$ is an *interpretation*. The interpretation ℓ interprets the constants, function symbols, and relation symbols non-rigidly, thus an interpretation $\ell = (\ell_w)_{w \in W}$ is such that for all $c \in \mathsf{CON}$: $\ell_w(c) \in \mathbb{D}$; for all n-ary $f \in \mathsf{FSYM}$: $\ell_w(f) : \mathbb{D}^n \to \mathbb{D}$; for all n-ary $R \in \mathsf{RSYM}$: $\ell_w(R) \subseteq \mathbb{D}^n$ (for all $w \in W$). Given a model $\mathcal{M} = \langle W, \mathcal{R}, \mathbb{D}, \mathcal{D}, \ell \rangle$, we will denote $\ell_w(c)$ by $c_w^{\mathcal{M}}$, $\ell_w(f)$ by $f_w^{\mathcal{M}}$, and $\ell_w(R)$ by $R_w^{\mathcal{M}}$.[3] For all nominals $i \in \mathsf{NOM}$ the interpretation ℓ assigns an element of W, i.e. $\ell(i) \in W$, thus the interpretation of nominals does not depend on worlds.

Given a model $\mathcal{M}$, a *valuation* ν in $\mathcal{M}$ is a function $\nu : (\mathsf{FVAR} \cup \mathsf{SVAR}) \to (\mathbb{D} \cup W)$, such that $\nu(x) \in \mathbb{D}$ for all $x \in \mathsf{FVAR}$, and $\nu(u) \in W$ for all $u \in \mathsf{SVAR}$. Given valuations ν and ν' and a variable z, we say that ν' is a *z-variant* of ν if $\nu'(y) = \nu(y)$ for all $y \in \mathsf{FVAR} \cup \mathsf{SVAR}$ with $y \neq z$. For a $w \in W$ and a $z \in \mathsf{FVAR}$, ν' is a *z-variant of ν* ***in*** *w* if ν' is a z-variant of ν and $\nu'(z) \in \mathcal{D}(w)$.

[1]The reason for using two different satisfaction operators is that satisfaction operators will be applied both to terms and formulas. So to avoid confusion two different operators will be used.

[2]The first-order hybrid logics of [1, 2, 4, 6, 7] all have a limited notion of terms. For instance taking terms only to be first-order variables, constants (interpreted non-rigidly) and of the form $u{:}c$ for a $u \in \mathsf{NOM} \cup \mathsf{SVAR}$ and c a constant. However the author sees no reason not to allow terms of arbitrary complexity as given by definition 1. Of course some tableau rules are needed to deal with these terms, however these rules are not that complicated.

[3]Note that for a constant c, $c_w^{\mathcal{M}}$ does not need to be in the domain of the world w, i.e. in $\mathcal{D}(w)$. Furthermore there might be an object $a \in \mathbb{D}$ that does not exists in any domain for any world in W. In other words it is not required that $\mathbb{D} = \cup_{w \in W} \mathcal{D}(w)$ as is done in for instance [9].

Now given a model $\mathcal{M}$ and a valuation ν in $\mathcal{M}$, a term evaluation function $(\cdot)_{(\cdot)}^{\mathcal{M},\nu} : \mathsf{T}^{FHL} \to \mathbb{D}$ is defined by

– If x is a variable and $w \in W$, then $(x)_w^{\mathcal{M},\nu} = \nu(x)$.
– If c is a constant and $w \in W$, then $(c)_w^{\mathcal{M},\nu} = c_w^{\mathcal{M}}$.
– If i is a nominal, t a term, and $w \in W$, then $(i{:}t)_w^{\mathcal{M},\nu} = (t)_{\ell(i)}^{\mathcal{M},\nu}$.
– If u is a state variable, t a term, and $w \in W$, then $(u{:}t)_w^{\mathcal{M},\nu} = (t)_{\nu(u)}^{\mathcal{M},\nu}$.
– If f is an n-ary function symbol, $t_1, \ldots, t_n$ are terms, and $w \in W$, then $\big(f(t_1, \ldots, t_n)\big)_w^{\mathcal{M},\nu} = f_w^{\mathcal{M}}\big((t_1)_w^{\mathcal{M},\nu}, \ldots, (t_n)_w^{\mathcal{M},\nu}\big)$.

The definition of the semantic relation $\mathcal{M}, w \models_\nu \varphi$ can now be defined by

$\mathcal{M}, w \models_\nu R(t_1, \ldots, t_n)$	*iff*	$\big((t_1)_w^{\mathcal{M},\nu}, \ldots, (t_n)_w^{\mathcal{M},\nu}\big) \in R_w^{\mathcal{M}}$
$\mathcal{M}, w \models_\nu t_1 = t_2$	*iff*	$(t_1)_w^{\mathcal{M},\nu} = (t_2)_w^{\mathcal{M},\nu}$
$\mathcal{M}, w \models_\nu i$	*iff*	$\ell(i) = w$
$\mathcal{M}, w \models_\nu u$	*iff*	$\nu(u) = w$
$\mathcal{M}, w \models_\nu \neg\varphi$	*iff*	$\mathcal{M}, w \not\models_\nu \varphi$
$\mathcal{M}, w \models_\nu \varphi \vee \psi$	*iff*	$\mathcal{M}, w \models_\nu \varphi$ or $\mathcal{M}, w \models_\nu \psi$
$\mathcal{M}, w \models_\nu \Diamond\varphi$	*iff*	there is a $w' \in W$ s.t. $\mathcal{R}(w, w')$ and $\mathcal{M}, w' \models_\nu \varphi$
$\mathcal{M}, w \models_\nu (\exists x)\varphi$	*iff*	there is an x-variant ν' of ν in w s.t. $\mathcal{M}, w \models_{\nu'} \varphi$
$\mathcal{M}, w \models_\nu @_i\varphi$	*iff*	$\mathcal{M}, \ell(i) \models_\nu \varphi$.
$\mathcal{M}, w \models_\nu @_u\varphi$	*iff*	$\mathcal{M}, \nu(u) \models_\nu \varphi$.
$\mathcal{M}, w \models_\nu {\downarrow} u.\varphi$	*iff*	there is an u-variant ν' of ν s.t. $\nu'(u) = w$ and $\mathcal{M}, w \models_{\nu'} \varphi$.

The notion of *satisfiability* and *validity* is defined in the usual manner. Note that if φ is a sentence, wherever $\mathcal{M}, w \models_\nu \varphi$ does not depend on the valuation ν.

3. A tableau system for **FHL**

In this section a tableau system for **FHL** interpreted over varying domains is presented. Tableau proofs will only be of **FHL**-sentences. The tableau system here given is inspired by [9], however introducing hybrid machinery into the language makes it possible to internalise the tableau system completely. The $@_i$ operators will play the role of prefixes and the term operators $i{:}$ will be used instead of the grounding mechanism on terms need for varying domain tableaux in [9].[4]

When doing tableaux for first-order logic, we need something to instantiate quantifiers as in $(\exists x)\varphi$. To make things simpler a new countable infinite set $\mathsf{PAR} = \{p, q, ...\}$ of parameters is introduced. These will behave like constants and is only used to instantiate quantifiers.[5] The language obtained by adding the new set PAR of constants to the language **FHL** will be referred to as the extended language. Hence a extended term or a formula of the extended language is just like a **FHL**-term or a **FHL**-formula except that they might contain parameters. Note that since parameters appears as constants they cannot be bound by quantifies.

The tableau rules are given in figure 1. If t is a term of the extended language, t is called closed if it contains no first-order or state variables. A tableau branch is closed if it contains both $@_i\varphi$ and $@_i\neg\varphi$ for some $i \in \mathsf{NOM}$ and some extended formula φ. A tableau is called closed if all its branches are closed. A tableau proof of a **FHL**-sentence φ is a closed tableau starting with the formula $@_i\neg\varphi$ for some nominal i not occurring in φ.

[4]The only other tableau system for a hybrid version of first-order modal logic, know to the author, is the system introduced in [4], where only rules for constant domains are given, and some limitation on terms are imposed.

[5]In [9], [10], and other literature on first-order tableau systems, parameters are a new kind of variables. This is essential in [9] where variables are assigned values rigidly and constants non-rigidly. However this problem is here dealt with by instantiating the quantified variable x by $i{:}p$ instead of just p, for a parameter p. This works since $i{:}p$ is a rigid term, which at the same time carries the information of which domain there has been quantified over, in the sense that we will think of $i{:}p$ as belonging to the domain of the world i. This will become much clearer in the completeness proof.

Propositional rules:

$$\frac{@_i(\varphi \vee \psi)}{@_i\varphi \mid @_i\psi}\,(\vee) \qquad \frac{@_i\neg(\varphi \vee \psi)}{\begin{array}{c}@_i\neg\varphi\\ @_i\neg\psi\end{array}}\,(\neg\vee) \qquad \frac{@_i\neg\neg\varphi}{@_i\varphi}\,(\neg\neg)$$

Modal rules:

$$\frac{@_i\Diamond\varphi}{\begin{array}{c}@_i\Diamond j\\ @_j\varphi\end{array}}\,(\Diamond)^1 \qquad \frac{@_i\neg\Diamond\varphi \quad @_i\Diamond j}{@_j\neg\varphi}\,(\neg\Diamond)$$

Quantifier rules:

$$\frac{@_i(\exists x)\varphi}{@_i\varphi[i{:}p/x]}\,(\exists)^2 \qquad \frac{@_i\neg(\exists x)\varphi}{@_i\neg\varphi[i{:}p/x]}\,(\neg\exists)^3$$

Equality rules:

$$\frac{}{@_i j{:}t = j{:}t}\,\textbf{(ref)}^4 \qquad \frac{@_i j{:}t = k{:}s \quad @_i\varphi}{@_i\varphi[k{:}s//j{:}t]}\,\textbf{(sub)}^5$$

@ rules:

$$\frac{@_i@_j\varphi}{@_j\varphi}\,\textbf{(@)} \qquad \frac{@_i\neg@_j\varphi}{@_j\neg\varphi}\,(\neg\ \textbf{@})$$

$$\frac{}{@_i i}\,\textbf{(nom ref)} \qquad \frac{@_i j \quad @_i\varphi}{@_j\varphi}\,\textbf{(nom)} \qquad \frac{@_i j \quad @_k\Diamond i}{@_k\Diamond j}\,\textbf{(bridge)}$$

Downarrow rules:

$$\frac{@_i \downarrow w.\varphi}{@_i\varphi[i/w]}\,(\downarrow) \qquad \frac{@_i\neg \downarrow w.\varphi}{@_i\neg\varphi[i/w]}\,(\neg\downarrow)$$

Term rules:

$$\frac{@_i k_1{:}t = k_2{:}s}{@_j k_1{:}t = k_2{:}s}\,\textbf{(:1)} \qquad \frac{@_i j}{@_k i{:}t = j{:}t}\,\textbf{(:2)}^4 \qquad \frac{}{@_i k{:}j{:}t = j{:}t}\,\textbf{(:3)}^4$$

$$\frac{@_i R(t_1, ..., t_n)}{@_i R(i{:}t_1, ..., i{:}t_n)}\,\textbf{(:fix 1)} \qquad \frac{@_i\neg R(t_1, ..., t_n)}{@_i\neg R(i{:}t_1, ..., i{:}t_n)}\,\textbf{(:fix 2)}$$

$$\frac{@_i t = s}{@_i i{:}t = i{:}s}\,\textbf{(:fix 3)} \qquad \frac{@_i\neg t = s}{@_i\neg i{:}t = i{:}s}\,\textbf{(:fix 4)} \qquad \frac{}{@_i f(t_1, ..., t_n) = f(i{:}t_1, ..., i{:}t_n)}\,\textbf{(:func)}^7$$

[1] The nominal j is new to the branch. [2] Where p is a parameter and $i{:}p$ is new to the branch. [3] Where p is any parameter. [4] Where t is a closed term. [5] $\varphi[k{:}s//j{:}t]$ is the formula φ where some of the occurrences of $j{:}t$ have been replaced by $k{:}s$. [7] Where f is a n-ary function symbol and $t_1, ..., t_n$ are all closed terms.

FIGURE 1. Tableau rules for **FHL**.

The classical rules are standard rules that can be found in many texts on first-order modal logic. The hybrid rules for @ and down-arrow are also standard and can for instance be found in [4]. The term rules are new rules added to deal with the : operator on terms.[6]

A *@-formula* (or *@-sentence*) is a formula (or sentence) in the extended language on the form $@_i\varphi$, for some formula (or sentence) φ of the extended language and some nominal i. Note that since quantifiers are instantiated by parameters prefix a nominal, no free first-order variables will occurs after an application of the rules **(∃)** or **(¬∃)**. Similar no new free state variables occurs after applications of the rules **(↓)** or **(¬ ↓)**. These considerations and the restriction on the rules **(ref)**, **(: 2)**, and **(: 3)**, ensures that all formulas occurring on a tableau for a **FHL**-sentence, will all be @-sentences. At the same time the use of parameters and nominals in the rules **(∃)**, **(¬∃)**, **(↓)**, and **(¬ ↓)** ensures that no accidental binding of any free variables happens. Note also that if the formula $@_i t = s$ occurs on a branch, t and s will be closed terms, and thus no accidental binding of free variables can happen in the use of the rule **(sub)**.

3.1. **Soundness and completeness.** Soundness is not hard to prove. It is done in the same way as in [9]. The proof of the tableau system being complete is in a sense also standard. It is shown that if a **FHL**-sentence φ does not have a tableau proof then $\neg\varphi$ is satisfiable, and thus φ is not valid. The idea behind the proof is taken from [10] and uses a variant of a standard Lindenbaum-Henkin construction.

Before the proof of completeness some terminology is needed. If S is a finite set of @-sentences we may construct a tableau for this set by simply putting all the sentences of S on one tableau branch, and then use the given tableau rules on this branch.[7] Note that if a finite set S of @-sentences has a closed tableau, then any finite set $S' \supseteq S$ also has a closed tableau. Now the notion of consistency can be defined. A set S of @-sentences is *inconsistent* if there is a closed tableau for some finite subset of S. A set of @-sentences is *consistent* if it is not inconsistent. A set S of @-formulas is *◊-complete* if;

$$@_i\Diamond\varphi \in S \quad\Longrightarrow\quad @_i\Diamond j, @_j\varphi \in S, \text{ for some nominal } j,$$

and S is *∃-complete* if;

$$@_i(\exists x)\varphi \in S \quad\Longrightarrow\quad @_i\varphi[i{:}p/x] \in S, \text{ for some parameter } p.$$

Further a set S of @-formulas *omits infinitely many nominals* if there are infinitely many nominals in NOM that does not occur in S, and similar S *omits infinitely many parameters* if there infinitely many parameters not in S.

Lemma 3. *If S is a consistent set of @-sentences that omits infinitely many nominals and parameters, then S can be extended to a maximally consistent set S' of @-sentences that is both ◊-complete and ∃-complete.*

Proof: First enumerate the countable many @-sentences of the extended language: $@_{i_1}\varphi_i, @_{i_2}\varphi_2, \ldots$ Then for all $n \in \mathbb{N}$ define S_n recursively by:

$$S_1 = S,$$

$$S_{n+1} = \begin{cases} S_n \cup \{@_{i_n}\varphi_n\}, & \text{if } \varphi_n \text{ is not of the form } \Diamond\psi \text{ or } (\exists x)\psi, \\ & \text{and the set } S_n \cup \{@_{i_n}\varphi_n\} \text{ is consistent.} \\ S_n \cup \{@_{i_n}\varphi_n, @_{i_n}\Diamond j, @_j\psi\}, & \text{if } \varphi_n \text{ is of the form } \Diamond\psi,\ j \text{ is a new nominal not occurring} \\ & \text{in } S_n \text{ or } @_{i_n}\varphi_n, \text{ and the set } S_n \cup \{@_{i_n}\varphi_n\} \text{ is consistent.} \\ S_n \cup \{@_{i_n}\varphi_n, @_{i_n}\psi[i{:}p/x]\}, & \text{if } \varphi_n \text{ is of the form } (\exists x)\psi,\ p \text{ a new parameter not occurring} \\ & \text{in } S_n \text{ or } @_{i_n}\varphi_n, \text{ and the set } S_n \cup \{@_{i_n}\varphi_n\} \text{ is consistent.} \\ S_n, & \text{otherwise.} \end{cases}$$

This definition works since by the assumption on S, S_n will omits infinitely many nominals and parameters, for all $n \in \mathbb{N}$.

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[6]The term rules are inspired by the @ rules. For instance **(:2)** plays the role of **(nom)** and **(:3)** plays the roles of **(@)** and **(¬ @)**. The rules **(:fix1)** - **(:fix4)** and **(:func)** are included to the deal with the semantics of $i{:}t$. Note that a general substitution rule of the form

$$\frac{@_i\varphi}{@_i\varphi[i{:}t//t]}$$

is not sound. The term $i{:}t$ cannot be substituted for t in the formula $@_j k{:}t = k{:}t$ in a sound way. Thus all the rules **(:fix1)** - **(:fix4)** and **(:func)** are needed to secure that the substitution only take place at the top level.

[7]So if φ is a **FHL**-sentence, then a tableau proof for φ is the same as a closed tableau for the finite set $\{@_i\neg\varphi\}$ (for some nominal i not occurring in φ).

Each S_n is consistent. This is easily proven by induction on $n \in \mathbb{N}$, using the fact that if φ_n is on the form $\Diamond\psi$ or $(\exists x)\psi$, then the consistency of $S_n \cup \{@_{i_n}\varphi_n\}$ implies the consistency of $S_n \cup \{@_{i_n}\varphi_n, @_{i_n}\Diamond j, @_j\psi\}$ and $S_n \cup \{@_{i_n}\varphi_n, @_{i_n}\psi[i{:}p/x]\}$, where j and p are new.

Now define S' by

$$S' = \bigcup_{n \in \mathbb{N}} S_n.$$

Since S_n is consistent for all $n \in \mathbb{N}$ it easily follows that also S' is consistent.

To show that S' is $\Diamond$-complete, assume that $@_i\Diamond\varphi \in S'$. Let $n \in \mathbb{N}$ be such that $@_{i_n}\varphi_n$ is the formula $@_i\Diamond\varphi$. Then since $@_{i_n}\varphi_n \in S'$ and S' is consistent, $S_n \cup \{@_{i_n}\varphi_n\}$ is also consistent. But then, by the construction of S_{n+1}, $@_{i_n}\Diamond j, @_j\psi \in S_{n+1} \subseteq S'$, for some new nominal j. That S' also is $\exists$-complete is proved in the same way.

That S' is maximal consistent is clear, since all formulas that can be added without destroying consistency have been added in the construction of S'. $\square$

Lemma 4. *Let S be a maximal consistent set of @-sentences, which is $\Diamond$-complete and $\exists$-complete. Then S obeys the tableau rules, i.e. if the premises of a rule are in S then the conclusion is also in S.*

For instance if the @-sentences $@_ij$ and $@_i\varphi$ are in S then so is $@_j\varphi$.[8]

To prove completeness assume that the **FHL**-sentence φ does not have a tableau proof, i.e. there is no closed tableau starting with $@_i\neg\varphi$ (for a $i \in \mathsf{NOM}$ not in φ). But then $\{@_i\neg\varphi\}$ is consistent. Since φ only contains finitely many nominals and no parameters, φ also omits infinitely many nominals and parameters. Thus by lemma 3 there is a maximal consistent set S of @-sentences that contains $@_i\neg\varphi$ and is $\Diamond$-complete and $\exists$-complete. Using this maximal consistent set S a model $\mathcal{M} = \langle W, \mathcal{R}, \mathbb{D}, \mathcal{D}, \ell\rangle$ can be constructed such that it satisfies $@_i\neg\varphi$. Now for the construction of the model $\mathcal{M}$:

First define the relation $\sim$ on the set NOM by:

$$i \sim j \iff @_j i \in S.$$

$\sim$ is a equivalence relation on the set NOM, which is seen using lemma 4 and the rules **(nom ref)** and **(nom)**.

The set of worlds W is then defined as the set of $\sim$-equivalence classes:

$$W = \mathsf{NOM}/_\sim.$$

The members of W will be denoted by $[i]$. The accessibility relation $\mathcal{R}$ on W is defined by

$$[i]\mathcal{R}[j] \iff @_i\Diamond j \in S.$$

That this is well-defined follows from lemma 4 and the rules **(nom)** and **(bridge)**.

To define the domain $\mathbb{D}$ of the model, first let D be the set defined by

$$D = \{i{:}t \mid \text{for some } i \in \mathsf{NOM} \text{ and some closed extended term } t\}.$$

Now define a relation $\equiv$ on D by

$$i{:}t \equiv j{:}s \iff @_k i{:}t = j{:}s \in S \text{ for some } k \in \mathsf{NOM}.$$

This relation is also easily seen to be a equivalence relation on the set D. It follows by lemma 4 applied to the rules **(ref)**, **(sub)**, and **(:1)**. The domain of the model is now defined by

$$\mathbb{D} = D/_\equiv.$$

The elements of $\mathbb{D}$ will be denoted by $\overline{i{:}t}$. For all $[i] \in W$ define

$$\mathcal{D}([i]) = \{\overline{j{:}p} \mid j \in [i] \text{ and } p \text{ is a parameter}\}.$$

Note that $\mathcal{D}([i]) \subseteq \mathbb{D}$ for all $i \in \mathsf{NOM}$, since p is a closed term.

Now for the definition of the interpretation ℓ. For all constants $a \in \mathsf{CON} \cup \mathsf{PAR}$ and $[i] \in W$ define

$$a^{\mathcal{M}}_{[i]} = \overline{i{:}a},$$

[8] That this is so can be seen the following way: Assume that $@_ij, @_i\varphi \in S$. Now if $@_j\varphi \notin S$ then $S \cup \{@_j\varphi\}$ must be inconsistent by the maximallity of S. So there is a finite subset $A \subseteq S \cup \{@_j\varphi\}$ such that A has a closed tableau. Then we can construct a closed tableau for the finite set $(A \setminus \{@_j\varphi\}) \cup \{@_ij, @_i\varphi\}$ $(\subseteq S)$ using the **(nom)** rule. But this contradict the consistency of S, hence $@_j\varphi \in S$. The other cases, except the rules $(\Diamond)$ and $(\exists)$, are similar. For the rules $(\Diamond)$ and $(\exists)$, the lemma follows from the $\Diamond$-completeness and $\exists$-completeness of S.

which is well-defined by lemma 4 and **(:2)**. For a n-ary relation symbol R and $[i] \in W$, define $R^{\mathcal{M}}_{[i]}$ by

$$R^{\mathcal{M}}_{[i]}(\overline{i_1{:}t_1}, ..., \overline{i_n{:}t_n}) \iff @_i R(i_1{:}t_1, ..., i_n{:}t_n) \in S, \tag{1}$$

for all $\overline{i_1{:}t_1}, ..., \overline{i_n{:}t_n} \in \mathbb{D}$. This is well-defined by lemma 4 and the rules **(nom)**, **(sub)**, and **(:1)**. For a n-ary function symbol f and $[i] \in W$, define $f^{\mathcal{M}}_{[i]} : \mathbb{D}^n \to \mathbb{D}$ by

$$f^{\mathcal{M}}_{[i]}(\overline{i_1{:}t_1}, ..., \overline{i_n{:}t_n}) = \overline{i{:}f(i_1{:}t_1, ..., i_n{:}t_n)} \tag{2}$$

for all $\overline{i_1{:}t_1}, ..., \overline{i_n{:}t_n} \in \mathbb{D}$. That this is well-defined follows from lemma 4 and the rules **(:1)**, **(:2)**, and **(sub)**. Finally for nominals $i \in \mathsf{NOM}$ let $\ell(i) = [i]$. Now for the central lemma:

Lemma 5 (Truth lemma)**.** *For all @-sentences $@_i\varphi$,*

$$@_i\varphi \in S \implies \mathcal{M}, [i] \models_\nu \varphi \text{ , for some (all) valuations } \nu.$$

$$@_i\neg\varphi \in S \implies \mathcal{M}, [i] \not\models_\nu \varphi \text{ , for some (all) valuations } \nu.$$

The completeness of the tableau system follows from this lemma. Since $@_i\neg\varphi \in S$ by the definition of S, $\mathcal{M}, [i] \not\models_\nu \varphi$. Thus $\mathcal{M}$ is a model that falsifies the sentence φ at the world $[i]$, and it follows that φ cannot be a valid sentence. Now the proof of the Truth lemma requires the following extra lemma:

Lemma 6. *If t is a term of the extended language that contains no variables, and i is a nominal, then*

$$t^{\mathcal{M},\nu}_{[i]} = \overline{i{:}t},$$

for all valuations ν in $\mathcal{M}$.

Proof: The proof goes by induction on the construction of t. t cannot be a first-order variable by assumption, and if t is a constant $a \in \mathsf{CON} \cup \mathsf{PAR}$, then $a^{\mathcal{M},\nu}_{[i]} = a^{\mathcal{M}}_{[i]} = \overline{i{:}a}$. If t is on the form $u{:}s$, then $u \in \mathsf{NOM}$ since t contains no variables. Further s cannot contain any variables either, and thus

$$(u{:}s)^{\mathcal{M},\nu}_{[i]} = s^{\mathcal{M},\nu}_{\ell(u)} = s^{\mathcal{M},\nu}_{[u]} \overset{(*)}{=} \overline{u{:}s} \overset{(**)}{=} \overline{i{:}u{:}s},$$

where $(*)$ follows by the induction hypothesis, and $(**)$ from lemma 4 and the rule **(:3)**.

Finally assume that t is on the form $f(t_1, ..., t_n)$, and that t contains no variables. Then $t_1, ..., t_n$ contains no variables either. Then by the induction hypothesis

$$f(t_1, ..., t_n)^{\mathcal{M},\nu}_{[i]} = f^{\mathcal{M}}_{[i]}((t_1)^{\mathcal{M},\nu}_{[i]}, ..., (t_n)^{\mathcal{M},\nu}_{[i]}) = f^{\mathcal{M}}_{[i]}(\overline{i{:}t_1}, ..., \overline{i{:}t_n}) = \overline{i{:}f(i{:}t_1, ..., i{:}t_n)} = \overline{i{:}f(t_1, ..., t_n)},$$

where the last equality follows from lemma 4 and the rules **(:func)** and **(:fix3)**. $\square$

Proof of lemma 5: The proof goes by induction on the complexity of φ. If φ is $R(t_1, ..., t_n)$, Then

$$\begin{aligned} @_i R(t_1, ..., t_n) \in S &\implies @_i R(i{:}t_1, ..., i{:}t_n) \in S \\ &\implies R^{\mathcal{M}}_{[i]}(\overline{i{:}t_1}, ..., \overline{i{:}t_n}) \\ &\implies R^{\mathcal{M}}_{[i]}((t_1)^{\mathcal{M},\nu}_{[i]}, ..., (t_n)^{\mathcal{M},\nu}_{[i]}) \\ &\implies \mathcal{M}, [i] \models_\nu R(t_1, ..., t_n), \end{aligned}$$

for all valuations ν. Here the first implication follows from lemma 4 and **(:fix1)**, the second by the definition (1), and the third by lemma 6 (because $R(t_1, ..., t_n)$ is assumed to be a @-sentence it cannot contain any variables). Furthermore if $@_i\neg R(t_1, ..., t_n) \in S$ then by lemma 4 and **(:fix2)** $@_i\neg R(i{:}t_1, ..., i{:}t_n) \in S$, and since S is consistent $@_i R(i{:}t_1, ..., i{:}t_n) \notin S$. Thus by definition $R^{\mathcal{M}}_{[i]}(\overline{i{:}t_1}, ..., \overline{i{:}t_n})$ does not hold, and so neither does $R^{\mathcal{M}}_{[i]}((t_1)^{\mathcal{M},\nu}_{[i]}, ..., (t_n)^{\mathcal{M},\nu}_{[i]})$. It then follows that

$$\mathcal{M}, [i] \not\models_\nu R(t_1, ..., t_n),$$

as required.

The case where φ is on the form $t_1 = t_2$ is similar.

φ cannot be u for a $u \in \mathsf{SVAR}$, since then $@_i\varphi$ is not a @-sentence. If φ is j for $j \in \mathsf{NOM}$, $\ell(j) = [j]$. But then

$$\mathcal{M}, [i] \models_\nu j \iff [j] = [i] \iff j \sim i \iff @_i j \in S,$$

and the claim follows, since $@_i\neg j \in S$ implies that $@_i j \notin S$.

The cases where φ is $\psi_1 \vee \psi_2$ or $\neg\psi$ are easy.

Now assume that φ is on the form $\Diamond\psi$. If $@_i\Diamond\psi \in S$, then by $\Diamond$-completeness, $@_i\Diamond j, @_j\psi \in S$, for some nominal j. Thus by the induction hypothesis it follows that $\mathcal{M}, [j] \models_\nu \psi$. But since $@_i\Diamond j \in S$, $[i]\mathcal{R}[j]$, and thus $\mathcal{M}, [i] \models_\nu \Diamond\psi$. Assume now that $@_i\neg\Diamond\psi \in S$. If $\mathcal{M}, [i] \models_\nu \Diamond\psi$ then there is a $j \in \mathsf{NOM}$ such that $[i]\mathcal{R}[j]$ and $\mathcal{M}, [j] \models_\nu \psi$. But then $@_i\Diamond j \in S$ by the definition of $\mathcal{R}$, and thus using lemma 4 and $(\neg\Diamond)$ it follows that $@_j\neg\psi \in S$. But by induction this implies that $\mathcal{M}, [j] \not\models_\nu \psi$, which is a contradiction, and thus $\mathcal{M}, [i] \not\models_\nu \Diamond\psi$ must be the case.

Assume now that φ is on the form $(\exists x)\psi$. If $@_i(\exists x)\psi \in S$ it follows by the $\exists$-completeness of S that also $@_i\psi[i{:}\,p/x] \in S$, for some parameter p. By the induction hypothesis $\mathcal{M}, [i] \models_\nu \psi[i{:}\,p/x]$. But then also $\mathcal{M}, [i] \models_{\nu'} \psi$, where ν' is a x-variant of ν in $[i]$ such that $\nu'(x) = (i{:}\,p)^{\mathcal{M},\nu}_{[i]} = \overline{i{:}p}$ (note that $\overline{i{:}p} \in \mathcal{D}([i])$). But then $\mathcal{M}, [i] \models_\nu (\exists x)\psi$ follows. Assume now that $@_i\neg(\exists x)\psi \in S$. If $\mathcal{M}, [i] \models_\nu (\exists x)\psi$, then there is an x-variant ν' of ν in $[i]$ such that $\mathcal{M}, [i] \models_{\nu'} \psi$. But by the definition of $\mathcal{D}([i])$ this implies that there is a parameter p such that $\nu'(x) = \overline{i{:}p} = (i{:}\,p)^{\mathcal{M},\nu}_{[i]}$. It thus follows that $\mathcal{M}, [i] \models_\nu \psi[i{:}\,p/x]$. But on the other hand using lemma 4 on $(\neg\exists)$ it also follows that $@_i\neg\psi[i{:}\,p/x] \in S$ and further by the induction hypothesis that $\mathcal{M}, [i] \not\models_\nu \psi[i{:}p/x]$. This is a contradiction and thus $\mathcal{M}, [i] \not\models_\nu (\exists x)\psi$ must be the case.

In the case φ is on the form $@_j\psi$, it first follows that

$$@_i@_j\psi \in S \implies @_j\psi \in S \implies \mathcal{M}, [j] \models_\nu \psi \implies \mathcal{M}, [i] \models_\nu @_j\psi,$$

using lemma 4 on **(@)** and the induction hypothesis. If $@_i\neg@_j\psi \in S$ it follows from lemma 4 and $(\neg\mathbf{@})$ that $@_j\neg\psi \in S$, which further by the induction hypothesis implies that $\mathcal{M}, [j] \not\models_\nu \psi$. But then

$$\mathcal{M}, [i] \not\models_\nu @_j\psi.$$

Finally assume that φ is $\downarrow v.\psi$. Then $@_i\downarrow v.\psi \in S$ implies that $@_i\psi[i/v] \in S$ by lemma 4 and $(\downarrow)$, which further by the induction hypothesis implies that $\mathcal{M}, [i] \models_\nu \psi[i/v]$. But from this follows that $\mathcal{M}, [i] \models_{\nu'} \psi$, where ν' is a v-variant of ν such that $\nu'(v) = (i)^{\mathcal{M},\nu}_{[i]} = \ell(i) = [i]$. Finally this implies that $\mathcal{M}, [i] \models_\nu \downarrow v.\psi$. Now assume that $@_i\neg\downarrow v.\psi \in S$. Then from lemma 4 and $(\neg\downarrow)$ it follows that $@_i\neg\psi[i/v] \in S$ and further by induction that $\mathcal{M}, [i] \not\models_\nu \psi[i/v]$. As before it follows that $\mathcal{M}, [i] \not\models_{\nu'} \psi$, whenever ν' is a v-variant of ν such that $\nu'(v) = (i)^{\mathcal{M},\nu}_{[i]} = [i]$, and thus that $\mathcal{M}, [i] \not\models_\nu \downarrow v.\psi$. $\square$

Before ending this section a remark is in its place. If one is interested in a first-order hybrid language just containing nominals and satisfaction operators (on both formulas and terms) and not the down-arrow binder, one can simply remove the rules $(\downarrow)$ and $(\neg\downarrow)$ from the tableau system without destroying the completeness proof. Thus getting a sound and complete tableau system for the weaker language. In the next section we will see that if one is interested in a more expressive language than **FHL** a tableau system for such a language is easily obtainable.

4. Adding the universal modality

Even though the language **FHL** is very expressive, there are things it cannot express. However it is easy to extend the language even further by adding the universal modality E to **FHL**. Let **FHLU** be the language obtained by adding the unary operator E to the language of **FHL**. Terms are as before and the definition of formulas is extended with the clause that if φ is a formula then $E\varphi$ is also a formula. The semantics for $E\varphi$ is given by

$$\mathcal{M}, w \models_\nu E\varphi \quad \textit{iff} \quad \text{there is a } w' \in W \text{ s.t. } \mathcal{M}, w' \models_\nu \varphi\ .$$

The dual operator A to E is defined by

$$A\varphi \stackrel{df}{=} \neg E\neg\varphi.$$

A tableau system for **FHLU** is obtained by adding the rules of figure 2 to the tableau system of **FHL**. The soundness of the new tableau system is trivial. For the completeness proof we first add the notion of a set of @-formulas being *E-complete* if;

$$@_iE\varphi \in S \quad \implies \quad @_j\varphi \in S, \text{ for some nominal } j.$$

To lemma 3 we add the further conclusion that S' can assumed to be E-complete. In the proof of the lemma we then need to add a new clause in the construction of S_{n+1}, namely that:

$$S_{n+1} = S_n \cup \{@_{i_n}\varphi_n, @_j\psi\}, \qquad \text{if } \varphi_n \text{ is of the form } E\psi,\ j \text{ is a new nominal not occurring in } S_n \text{ or } @_{i_n}\varphi_n, \text{ and the set } S_n \cup \{@_{i_n}\varphi_n\} \text{ is consistent.}$$

E rules:

$$\frac{@_i E\varphi}{@_j \varphi}\,(E)^1 \qquad \frac{@_i \neg E\varphi}{@_j \neg\varphi}\,(\neg E)^2$$

[1] The nominal j is new to the branch. [2] Where j is any nominal.

FIGURE 2. The extra tableau rules for **FHLU**.

The rest of the proof of lemma 3 goes through as before. In the proof of the truth lemma we also need to add the case where φ is $E\psi$: Assume that φ is on the form $E\psi$. Then if $@_i E\psi \in S$, $@_j\psi \in S$ for some nominal j, by the E-completeness of S. But then by the induction hypothesis $\mathcal{M}, [j] \models_\nu \psi$ and it follows that $\mathcal{M}, [i] \models_\nu E\psi$. Assume now that $@_i\neg E\psi \in S$. Now if $\mathcal{M}, [i] \models_\nu E\psi$ then there is a nominal j such that $\mathcal{M}, [j] \models_\nu \psi$. On the other hand it follows from lemma 4 and $(\neg E)$ that $@_j\neg\psi \in S$. But by the induction hypothesis this implies that $\mathcal{M}, [j] \not\models_\nu \psi$, which is a contradiction and thus $\mathcal{M}, [i] \not\models_\nu E\psi$ must be the case. The rest of the completeness proof goes through as before. Thus a sound and complete tableau system for **FHLU** has been presented.

5. Concluding remarks and further perspectives

This paper contains a fully internalized tableau system for a first-order hybrid logic that is both sound and complete. The language presented (**FHL**) contains nominals, a down-arrow binder as well as satisfaction operators on both formulas and terms. The notions of terms are as general as in classical first-order logic. Furthermore the tableau system it made to deal with varying domains. It turns out that tableau systems for **FHL** behave nicely, even when dealing with varying domain semantics. In [9] this requires an amount of meta notions, such as prefixes, parameters associated with prefixes, and grounding of terms. This is completely internalised in the tableau system for **FHL**. The only thing that might not look that nice for this tableau system is all the term rules needed. However it might be possible to find simpler rules.

Moreover as in [2] and [6] it would be interesting to see how automatic completeness proofs looks in the case of the presented tableau system. Besides automatic completeness results for different frame conditions given by pure formulas, first-order hybrid logic also allows for automatic completeness results for different domain conditions as discussed in for instance [2].

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