Rationalizing Constrained Contingent Claims

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Abstract. Choice of contingent claims could reflect risk aversion or pessimism. Accordingly, the underlying, but hidden preferences might fit expected utility of customary von Neumann-Morgenstern form - or more generally, comply with a Choquet integral. This paper considers constrained choice and rationalizes both sorts of attitudes. Two avenues are pursued: one invokes complete orders; the other contends with partial ordering. Emphasis is on incomplete financial markets, featuring nonlinear pricing.

Keywords: nonlinear price, risk or uncertainty aversion, Choquet integral, stochastic order, incomplete preferences.

JEL Classification: C81, D01, G13.

1. Introduction

For background and motivation, recall the problem to maximize state-independent expected utility

$$U(x) := Eu(x) := \sum_{s \in \mathbb{S}} \pi_s u(x_s)$$

at linear cost $E(px)$ within prescribed budget b. The optimality conditions read: $\lambda p_s \in \partial u(x_s)$ for each state $s \in \mathbb{S}$ and some Lagrange multiplier $\lambda \geq 0$. The operator ∂ is the ordinary derivative or a generalization thereof. Most often, marginal utility ∂u has positive but decreasing values. Then λ, p_s must also be positive, and $x_s < x_{\bar{s}} \Rightarrow p_s \geq p_{\bar{s}}$; that is,

$$(x_s - x_{\bar{s}})(p_s - p_{\bar{s}}) \leq 0 \text{ for all } s, \bar{s} \in \mathbb{S}. \quad (1)$$

Since $E(px) = (\pi_s p_s) \cdot x$, one refers to p_s as the *price density* of consumption in state s. It equals the customary consumption price per unit of probability. By (1), larger consumption implies lower price density.

Peleg and Yaari [34] used (1) to characterize efficient random variables. Dybvig and Ross [13], [14] applied the same inequalities to study efficient portfolios or claims. Starting from (1) Dana [9] has developed a substantial part of competitive market theory for contingent claims.

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This paper also considers price-taking purchase of such claims, retaining the assumption of state-independent utility. Our inquiry is motivated by the fact that financial markets for contingent claims often feature incomplete dividend spaces, nonlinear pricing, ambiguous beliefs, and possibly incomplete preferences.[1]

Then, what condition could come to replace (1)? Can a suitable utility index u be recovered from observed, but more restricted choice? In what way might such choice mirror risk aversion and pessimism? Can a suitable stochastic order rationalize recorded data as constrained efficient?

All these questions are addressed below - broadly in the stated order. Related studies include [9], [13], [15], [28], [34]. The novelties here come with accounting for constraints, systematic use of nonlinear pricing, and explicit construction of Choquet expected utility.

After preliminaries on notation in Section 2 and nonlinear pricing in Section 3, the paper is planned as follows:

Section 4 considers *one* price-taking agent and follows [34] by fabricating for him a concave criterion that rationalizes a suitably modified version of inequality (1) as though it stems from *risk aversion* and constrained maximization of *vNM expected utility.*

Section 5 takes a closely related but complementary tack. It rather seeks to explain data as manifestation of *uncertainty aversion* and constrained maximization of *Choquet expected utility.*

Section 6 abandons the assumption that preferences be total (i.e.complete) and contends instead with efficient choices under the concave stochastic order. Minimal expenditure then becomes a chief object.

This paper brings diverse material, found in separated contexts, under a common umbrella. Included are concave orders [18], stochastic majorization [32], and Choquet expected utility [36].

2. Preliminaries and Notation

Considered below is mainly competitive exchange of contingent claims to *one* perfectly divisible, transferable good, say money. Chief objects and instruments are then contracts written in terms of which *state* $s \in \mathbb{S}$ will happen. The *state space* $\mathbb{S} := \{1, ..., S\}$ is finite and non-trivial: $S > 1$.

A *contingent claim* $s \in \mathbb{S} \mapsto x_s \in \mathbb{R}$ is uniquely codified as a vector $x = (x_s) \in \mathbb{R}^{\mathbb{S}}$. For typographical convenience, we often write $\mathbb{X}$ for $\mathbb{R}^{\mathbb{S}}$. Two vectors $x^*, x \in \mathbb{R}^{\mathbb{S}}$ are declared *comonotone* (*antimonotone*) iff $(x^*_s - x^*_{\bar{s}})(x_s - x_{\bar{s}}) \geq 0$ (respectively ≤ 0) for all $s, \bar{s} \in \mathbb{S}$.

Elements in the standard simplex $\Delta := \left\{\delta \in \mathbb{R}^{\mathbb{S}}_+ : \sum_s \delta_s = 1\right\}$ are referred to as probability distributions. Among those is prescribed a non-degenerate $\pi \in \Delta$, having all $\pi_s > 0$.[2] Operator E means expectation with respect to π. Any vector $x \in \mathbb{X}$

[1] Section 3 considers incompleteness and nonlinear pricing Section 5 illustrates ambiguous beliefs, subjective or distorted probabilities, and pessimism.

[2] If the setting is that of intertemporal allocation, $s \in \mathbb{S}$ could mean a *stage* affected by *discount*

can be regarded as a random variable that takes value x_s with probability π_s. The distribution π generates a probabilistic inner product $\langle x^*, x\rangle := \sum_s \pi_s x_s^* x_s = E(x^* x)$ on $\mathbb{X}$. That product will most often be used in the sequel. The ordinary inner product is denoted $x^* \cdot x = \sum_s x_s^* x_s$.

A function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ is declared *proper* if finite at least somewhere. Given a proper $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, a linear functional $x^* : \mathbb{X} \to \mathbb{R}$ - that is, an element of the dual space $\mathbb{X}^* = \mathbb{X}$ - is called a *supergradient* of f at x, and we write $x^* \in \partial f(x)$, iff $f(\chi) \leq f(x) + x^*(\chi - x)$ $\forall \chi \in \mathbb{X}$.

The *extended indicator* $\iota_X : \mathbb{X} \to \{0, +\infty\}$ of a subset $X \subseteq \mathbb{X}$ equals 0 if $x \in X$, and $+\infty$ elsewhere. At $x \in X$ such a subset has (outward) *normal cone*

$$N_X(x) := \{x^* \in \mathbb{X}^* : x^*(\chi - x) \leq 0 \quad \forall \chi \in X\} = -\partial(-\iota_X)(x).$$

Orders are important on random variables $x, y \in \mathbb{R}^{\mathbb{S}}$, the most elementary one being $x \leq y \Leftrightarrow x_s \leq y_s$ $\forall s$. We say that x *dominates* y *in concave stochastic order*, and write $x \succsim_c y$, iff

$$Eu(x) \geq Eu(y) \text{ for each concave } u : \mathbb{R} \to \mathbb{R}.$$

The concave stochastic order has manifold equivalent characterizations; see [18], [33].[3] The *concave increasing order* $\succsim_{ci}$ - commonly called *second-order stochastic dominance* - is defined by $x \succsim_{ci} y \Leftrightarrow Eu(x) \geq Eu(y)$ for each concave increasing $u : \mathbb{R} \to \mathbb{R}$. These orders satisfy

$$x \succsim_c y \Leftrightarrow [x \succsim_{ci} y \ \& \ Ex = Ey];$$

see [33]. The concave order also relates closely to the *Schur concave order* or *stochastic majorization* [32] defined here as follows: $x \succsim_{(\cdot)} y$ iff there is a permutation $s \mapsto (s)$ on $\mathbb{S}$ such that x, y are *similarly ordered:*

$$x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(S)}, \quad y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(S)}, \tag{2}$$

and

$$\pi_{(1)} x_{(1)} + \cdots + \pi_{(s)} x_{(s)} \geq \pi_{(1)} y_{(1)} + \cdots + \pi_{(s)} y_{(s)} \ \forall s < S, \text{ with } Ex = Ey. \tag{3}$$

Then, $x \succsim_{(\cdot)} y \Rightarrow x \succsim_c y$. Conversely, provided x and y are similarly ordered by some permutation, $x \succsim_c y \Rightarrow x \succsim_{(\cdot)} y$; consult [4], [32].

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factor π_s. Alternatively, in case of social planning, $s \in \mathbb{S}$ might indicate an agent assigned *welfare weight* π_s.

[3] More basically, $\succsim_c$ stems from ordering *measures* on the real line by: $m \succsim_c \bar{m} \Leftrightarrow \int u(r)m(dr) \geq \int u(r)\bar{m}(dr)$ for each concave $u : \mathbb{R} \to \mathbb{R}$. Thus $x \succsim_c y$ is shorthand for $\pi x^{-1} \succsim_c \pi y^{-1}$. In particular, $x \sim_c y$ iff x and y have the same distribution.

3. Arbitrage Free Nonlinear Pricing

Chief items here are *constrained contingent claims,* evaluated at *nonlinear price.* How do such claims arise? Why are they constrained? And for what reason is pricing nonlinear?

For the sake of background and illustration, these questions motivate a brief outline of an underlying financial market in which feasible claims are replicable (synthesized) objects, infeasible ones aren't replicable, and nonlinear prices serve to block arbitrage. Specifically, this section offers a blitz derivation of the *fundamental asset pricing theorem*, and goes thereafter on to represent the corresponding price functional. Main arguments here apply merely linear programming to financial scenario trees. Readers familiar with - or not concerned with - such issues or objects, may skip this section.

Portfolio planning along a scenario tree: Consider a (decision or scenario) *tree* with finite *node set* $\mathcal{N}$. Each node $n \in \mathcal{N}$, except *one*, has a unique immediate *ancestor* $\mathcal{A}(n) \in \mathcal{N}$. The exceptional node, called the *root*, has none. If n is the ancestor of $c \in \mathcal{N}$, the latter is declared a *child,* and we write $c \in \mathcal{C}(n)$. Nodes without children are called leafs or *scenarios.* They constitute a distinguished subset $\mathbb{S} \subseteq \mathcal{N}$.

A fixed finite ensemble J of *assets* is repeatedly listed for trade. At *node* $n \in \mathcal{N}$ the investor buys $\theta_{jn} \in \mathbb{R}$ of asset $j \in J$, and sells the amount $\theta_{j\mathcal{A}(n)}$ bought at the *ancestor node* $\mathcal{A}(n)$ (if any). Thus he acquires (outgoing) *portfolio* $\theta_n := (\theta_{jn}) \in \mathbb{R}^J$ at node n and liquidates there the (incoming) portfolio $\theta_{\mathcal{A}(n)}$ purchased at the preceding node. At price $\theta_n^* \in \mathbb{R}^J$ and no transaction costs at node n, those operations bring him there nominal *gain*

$$G_n(\theta) := \theta_n^* \cdot \left[\theta_{\mathcal{A}(n)} - \theta_n\right].$$

Naturally, let $G_{root}(\theta) := -\theta_{root}^* \cdot \theta_{root}$. The investor is apt to ask: *Is money available for free*? That simple question motivates the following

Definition: *The market allows* ***arbitrage*** *iff the inequality system:*

$$G_n(\theta) \geq 0 \text{ for all nodes } n \text{ and } \theta_s^* \cdot \theta_s \geq 0 \text{ for each scenario } s \in \mathbb{S}, \tag{4}$$

admits a solution $\theta = (\theta_n)$ with at least one strict inequality. Otherwise the market is declared ***arbitrage-free.*** □

Suppose some *numeraire* paper $b \in J$, commands a positive price $\theta_{bn}^* > 0$ at each node n. In terms of a fixed numeraire define *discount factors* $\alpha_n := \theta_{b,root}^* / \theta_{bn}^*$ that actualize (to the present) payments received at future node n. The above question about free money can now be settled:

Theorem 3.1 (The fundamental theorem of asset pricing)**.** *The market is arbitrage-free iff there exists a strictly positive probability measure μ on $\mathbb{S}$ such that the transition* *probabilities, induced by μ on $\mathcal{N}$, satisfy the martingale condition*

$$\alpha_n \theta_n^* = E_\mu \left[\alpha_c \theta_c^* | \, n \right] = \sum_{c \in \mathcal{C}(n)} \alpha_c \theta_c^* \mu(c \,|n) \quad \textit{for all } n \notin \mathbb{S}. \tag{5}$$

Proof. Fix *any* probabilities $\hat{\mu}_s > 0$ across $s \in \mathbb{S}$, and use the induced probabilities $\hat{\mu}_n$ at nonterminal nodes $n \notin \mathbb{S}$. Consider the homogeneous linear program

$$\max_\theta \quad \sum_n \alpha_n \hat{\mu}_n G_n(\theta) + \sum_{s \in \mathbb{S}} \alpha_s \hat{\mu}_s \theta_s^* \cdot \theta_s \quad \text{s.t. (4)}. \tag{6}$$

Clearly, the market is arbitrage-free iff the optimal value of (6) is 0. Associate multiplier $\alpha_n y_n \geq 0$ to inequality $G_n(\theta) \geq 0$, and $\alpha_s Y_s \geq 0$ to the terminal wealth constraint $\theta_s^* \cdot \theta_s \geq 0$ of scenario s. Maximizing the resulting Lagrangian

$$\sum_n \alpha_n (\hat{\mu}_n + y_n) G_n(\theta) + \sum_{s \in \mathbb{S}} \alpha_s (\hat{\mu}_s + Y_s) \theta_s^* \cdot \theta_n =$$

$$\sum_{n \notin \mathbb{S}} \left[\sum_{c \in \mathcal{C}(n)} \alpha_c (\hat{\mu}_c + y_c) \theta_c^* - \alpha_n (\hat{\mu}_n + y_n) \theta_n^* \right] \cdot \theta_n + \sum_{s \in \mathbb{S}} \alpha_s (Y_s - y_s) \theta_s^* \cdot \theta_s \tag{7}$$

with respect to the free variable $\theta = (\theta_n)$, we see that the dual of (6) amounts to find a vector $y \in \mathbb{R}_+^{\mathcal{N}}$ such that:

$$\alpha_n (\hat{\mu}_n + y_n) \theta_n^* = \sum_{c \in \mathcal{C}(n)} \alpha_c (\hat{\mu}_c + y_c) \theta_c^* \quad \text{for all} \quad n \notin \mathbb{S} \text{ with } y \geq 0.$$

Suppose the latter equation is indeed solvable for some $y \geq 0$. In that case, by LP duality, problem (6) has 0 as optimal value, and there are no arbitrage opportunities. Then consider the numeraire component b of the last equation to get

$$\hat{\mu}_n + y_n = \sum_{c \in \mathcal{C}(n)} (\hat{\mu}_c + y_c).$$

Therefore $\mu(c \,|n) := (\hat{\mu}_c + y_c)/(\hat{\mu}_n + y_n)$ defines strictly positive transition probabilities that satisfy (5).

Conversely, suppose some strictly positive measure μ on $\mathbb{S}$ suits (5). In (7) let $\hat{\mu} = \mu$ and each $y_n, Y_n = 0$ to get

$$\sum_n \alpha_n \mu_n G_n(\theta) + \sum_{s \in \mathbb{S}} \alpha_s \mu_s \theta_s^* \cdot \theta_s = \sum_{n \notin \mathbb{S}} \left[\sum_{c \in \mathcal{C}(n)} \alpha_c \mu_c \theta_c^* - \alpha_n \theta_n^* \right] \cdot \theta_n = 0$$

for all θ. Thus arbitrage is impossible. $\square$

The subset $X \subseteq \mathbb{R}^{\mathbb{S}}$ could include precisely those terminal claims x obtained by self-financed market operations θ. Then formally, $X :=$

$$\left\{x \in \mathbb{R}^{\mathbb{S}} : \exists\, \theta = (\theta_n) \text{ with } G_n(\theta) \geq 0\ \forall n \notin \mathit{root} \cup \mathbb{S}, \text{ and } x_s = \theta_s^* \cdot \theta_{\mathcal{A}(s)} \text{ for } s \in \mathbb{S}\right\}.$$

Clearly, X so defined is a closed convex cone. It contains the subspace

$$\left\{x \in \mathbb{R}^{\mathbb{S}} : \exists\, \theta = (\theta_n) \text{ with } G_n(\theta) = 0\ \forall n \notin \mathit{root} \cup \mathbb{S}, \text{ and } x_s = \theta_s^* \cdot \theta_{\mathcal{A}(s)} \text{ for } s \in \mathbb{S}\right\}.$$

A set $\mathbb{P}$ of price densities emerges here. Specifically,

$$\mathbb{P} := \{p = (p_s) = (\alpha_s \mu_s / \pi_s) : \mu \in \Delta \text{ and (5) holds}\}.$$

Any $\mu \in \Delta$ that satisfies (5) is called a *martingale measure* associated to the tree and price process θ^*. Together the price densities define an evaluation functional

$$P(x) := \max\{E(px) : p \in \mathbb{P}\}. \tag{8}$$

The following result is well known, and it now derives easily:

Proposition 3.1 (On unique values, completeness, and nonlinear pricing).
• *$P(x) = p \cdot x$ for all $p \in \mathbb{P}$ when $x \in X$.*
• *The market is complete, meaning $X = \mathbb{X}$, iff P is positive and linear; that is, iff $\mathbb{P}$ consists of a unique strictly positive element.*
• *Pricing $P(\cdot)$, as defined by (8), is*
1) arbitrage free: $P(x) > 0$ *for* $x \gneqq 0$;
2) positively homogeneous: $P(rx) = rP(x)$ *for any* $r \geq 0$ *and* $x \in \mathbb{X}$;
3) subadditive: $P(x + \bar{x}) \leq P(x) + P(\bar{x})$ for any $x, \bar{x} \in \mathbb{X}$;
4) increasing: $x \leq \bar{x} \Rightarrow P(x) \leq P(\bar{x})$.
• *Conversely, any evaluation functional P that possesses these four properties is of form (8) for some compact convex set $\mathbb{P} \subset \mathbb{R}_+^{\mathbb{S}}$ of underlying price densities p, at least one of which is strictly positive.*

Proof. Only the last bullet is proven. Notice that the Fenchel conjugate

$$P^*(p) := \sup\{E(px) - P(x) : x \in \mathbb{X}\}$$

of any positively homogeneous P takes only the values 0 and $+\infty$. Hence P^* is the extended indicator of a closed convex set $\mathbb{P}$. Being continuous convex, P equals its double conjugate; that is,

$$P(x) = P^{**}(x) = \sup\{E(px) : p \in \mathbb{P}\}.$$

Further, because P is finite-valued, $\mathbb{P}$ must be bounded whence max replaces sup in the last equation, and (8) follows. Some $p^s \in \mathbb{P}$ must have its s-component $p_s^s > 0$. Otherwise the contradiction $P(\mathbf{1}_s) = 0$ would result for the unit vector

$\mathbf{1}_s = (0, .., 0, 1, 0, ..)$. To conclude, note that $p = \sum_{s \in \mathbb{S}} p^s/S$ is a non-degenerate member of $\mathbb{P}$. $\square$

Clearly, the price regime P is linear iff $\mathbb{P}$ reduces to a singleton. For any $x \in \mathbb{X}$ let

$$\partial P(x) := \{p \in \mathbb{P} : P(x) = E(px)\}.$$

In fact, $\partial P(x)$ equals the *subdifferential* of the convex function P at x. Thus, $\mathbb{P} = \partial P(0)$. One may, of course, normalize P by setting $P(\mathbf{1}) = 1$ to have $\mathbb{P} \subseteq \Delta$.

Nonlinear pricing P, with properties listed in Proposition 3.1, was motivated here by market incompleteness. Other market imperfections, that yield the same properties, include: proportional transaction costs, constraints or cost on short selling, different borrowing and lending rates; see [28] and references therein.

4. Risk Averse Preferences

The constraint set X described in the preceding section was common to all traders. It may happen of course, that somebody, for diverse reasons, faces an even smaller set X of feasible claims. A leading question is then: What relation has his choice $x \in X$ to the price density p?

To explore that question, we begin by considering the following budget-constrained consumption problem:

$$\text{maximize } U(x) \text{ subject to } x \in X \text{ and } P(x) \leq b. \tag{9}$$

Here and henceforth X *is a closed convex subset of* $\mathbb{X}$ *and* P *is an arbitrage free, positively homogeneous, subadditive, increasing price.* Also, whenever a supergradient $x^* \in \partial U(x)$ - or a normal vector to X - is chosen, its representation, as a linear functional, is tacitly understood to come in the form $\langle x^*, \cdot \rangle = E(x^* \cdot)$.

Proposition 4.1 (Optimal consumption and minimal expenditure). *Suppose the function* $U : \mathbb{X} \to \mathbb{R}$ *is concave.*

• (On constrained consumption). *If* x *solves problem* (9) *with* $P(x) = b$, *and* $X \cap \{P < b\}$ *is non-empty, there exist*

$$\lambda \geq 0, \quad p \in \partial P(x), \quad x^* \in N_X(x) \tag{10}$$

such that

$$\lambda p + x^* \in \partial U(x). \tag{11}$$

Conversely, suppose (10) & (11) *hold. Then* x *solves consumption problem* (9) *for budget* $b := P(x)$.

• (On minimal expenditure). *For specified utility level* $\bar{U}$, *suppose* x *will*

$$\textit{minimize } P(x) \quad \textit{s.t. } x \in X \cap \left\{U \geq \bar{U}\right\}. \tag{12}$$

If $X \cap \left\{U > \bar{U}\right\}$ *is nonempty, then conditions* (10) & (11) *are satisfiable. Conversely, when those conditions hold,* x *solves expenditure problem* (12) *for utility level*

$\bar{U} = U(x)$.

Proof (for the characterization of constrained consumption). x solves (9) iff

$$0 \in \partial \left\{ U - \iota_{\{P \leq b\}} - \iota_X \right\} (x). \tag{13}$$

Since $U - \iota_{\{P \leq b\}} - \iota_X$ is concave, the qualification $X \cap \{P < b\} \neq \emptyset$ ensures

$$\partial \left\{ U - \iota_{\{P \leq b\}} - \iota_X \right\} = \partial U - N_{\{P \leq b\}} - N_X; \tag{14}$$

see Theorem 6.6.7 in [31]. Here, $N_{\{P \leq b\}}(x) = \mathbb{R}_+ \partial P(x)$ because $P(x) = b$. Thus, by (13) and (14) there exist $\lambda \geq 0$, $p \in \partial P(x)$, $x^* \in N_X(x)$, $g \in \partial U(x)$ such that $\lambda p + x^* = g$, whence (11) follows.

For the converse, note that inclusion

$$\partial U - N_{\{P \leq b\}} - N_X \subseteq \partial \left\{ U - \iota_{\{P \leq b\}} - \iota_X \right\}.$$

holds with no strings attached. Hence (11) implies (13), and x must be optimal. Because $p \in \partial P(x)$, we get $P(x) = b$.

The assertions concerning minimal expenditure are proven in the same manner. □

For simplicity in notation, let

$$d := \lambda p + x^*, \tag{15}$$

with d_s now denoting the modified *price density* for consumption in state s.

Corollary 4.1 (Choice and density are antimonotone). *Assume* $U(x) = \sum_{s \in \mathbb{S}} \pi_s u(x_s)$ *with* u *concave and each* π_s *positive. Then* (11) *tells that* $x_s < x_{\bar{s}} \Rightarrow d_s \geq d_{\bar{s}}$, *or equivalently,*

$$(x_s - x_{\bar{s}})(d_s - d_{\bar{s}}) \leq 0 \quad \textit{for all } s, \bar{s} \in \mathbb{S}. \ \square \tag{16}$$

Inequalities (16) generalize (1). They tell that after replacing U by $x \mapsto U(x) - E(x^* x)$, the doubly constrained choice $x \in X \cap \{P \leq b\}$ becomes optimal in the greater set $\{P \leq b\}$.

As argued before, the instance $\lambda > 0$ appears most natural in (15).[4] Then, with no loss of generality, let $\lambda = 1$ for easier interpretation of (16). Construe $\tilde{p}_s := \pi_s(p_s + x_s^*)$ as the ordinary, but modified price of an elementary Arrow-Debreu paper (maybe not marketed directly) that pays 1 unit of account in state s, nil otherwise. Thus, most is bought of elementary claims (if any) that have modified price density $d_s = p_s + x_s^*$ at minimal level. For additional interpretation, assume $d_{\bar{s}} = p_{\bar{s}} + x_{\bar{s}}^* > 0$ and restate (16) as

$$x_s < x_{\bar{s}} \Rightarrow \frac{\tilde{p}_s}{\tilde{p}_{\bar{s}}} \geq \frac{\pi_s}{\pi_{\bar{s}}}.$$

[4] When $\lambda > 0$, after replacing U by the equivalent criterion λU, one may choose $\lambda = 1$. Also, when $p = n$, the specifaction $\lambda = 1$ will do.

Regard the ordinary price ratio $\tilde{p}_s/\tilde{p}_{\bar{s}}$ as the modified *market based odds* of a bet on state s against $\bar{s}$. Compare that ratio with the *probabilistic odds* $\pi_s/\pi_{\bar{s}}$ to see that less is placed on actuarially unfair opportunities. Thus, observed behavior indicates risk aversion.

Is the derivation of (16), from expected utility to characterization of optimal choice, a one way passage? Can a criterion $U = Eu$ be recovered from (16) such that x solves (9)?[5]

For empirical reasons, the last question isn't quite well posed: At most the pair p, x is directly observable; neither λ nor x^* is reported, both being idiosyncratic. To get around this obstacle, declare a price density-quantity pair $p \in \partial P(x)$, $x \in X$ *feasible* if (15) and (16) hold for some $\lambda \geq 0$ and $x^* \in N_X(x)$. The next result states that, under such feasibility, x does indeed solve (9) for some concave criterion $U = Eu$ of von Neumann-Morgenstern form. Moreover, given λ and x^*, the integrand u can straightforwardly be constructed:

Proposition 4.2 (Rationalizing realized choice). *Suppose the price density-quantity pair p, x is feasible. Then:*
- *there exists a concave utility index $u : \mathbb{R} \to \mathbb{R}$ such that x solves consumption problem* (9) *with $U(x) = \sum_{s\in\mathbb{S}} \pi_s u(x_s)$ and budget $b = P(x)$.*
- *x also minimizes expenditures* (12) *at utility level $\bar{U} = U(x)$.*
- *u is non-smooth precisely at those points $r \in \mathbb{R}$ for which $\#\{d_s : x_s = r\} \geq 2$.*
- *If $x_s < x_{\bar{s}} \Rightarrow d_s > d_{\bar{s}}$, then u can be taken strictly concave. In that case there is no $\hat{x} \in X \cap \{P \leq P(x)\}$ such that $\hat{x} \succ_c x$.*
- *When moreover, all d_s are nonnegative, the function u must be increasing, and there is no $\hat{x} \in X \cap \{P \leq P(x)\}$ that satisfies $\hat{x} \succ_{ci} x$.*

Proof. The main argument is well known from [34] but given for completeness. The key idea is to regard all quantity-density pairs (x_s, d_s) as belonging to a decreasing curve, namely the graph of marginal utility ∂u. Let $x(\mathbb{S}) = \{x_s : s \in \mathbb{S}\}$ denote the set of observed x-values. When $r \in x(\mathbb{S})$, posit

$$\partial u(r) := conv\{d_s : x_s = r\} =: \left[u'_+(r), u'_-(r)\right].$$

When $r \notin x(\mathbb{S})$, let $\partial u(r) = D(r)$ for some continuous decreasing function D, mapping $\mathbb{R}\diagdown x(\mathbb{S})$ into $\mathbb{R}$, such that

$$\lim_{r\nearrow x_s} D(r) = u'_-(x_s) \quad \text{and} \quad \lim_{r\searrow x_s} D(r) = u'_+(x_s).$$

The correspondence $r \in \mathbb{R} \mapsto \partial u(r) \subset \mathbb{R}$ so defined is monotone decreasing with connected graph. Hence

$$u(r) := \int_{\min x_s}^{r} \partial u$$

[5] Studies on recovering utility functions from market data include [1], [12], [34].

is concave - and non-smooth precisely at those points x_s where the interval $\partial u(x_s)$ is non-degenerate. Strict concavity obtains for u when D decreases strictly.[6] Notice that $d_s \in \partial u(x_s)$ for all s. This implies (11). Proposition 4.1 tells that x solves consumption problem (9) when $U(x) = \sum_s \pi_s u(x_s)$ - and expenditure problem (12) when $\bar{U} = U(x)$.

If $\hat{x} \succ_c x$, then $Eu(\hat{x}) > Eu(x)$, hence by the optimality of x in $X \cap \{P \leq P(x)\}$, the point $\hat{x}$ can't belong to the latter set. The same argument applies to demonstrate the final assertion. □

Recall that $\lambda \geq 0$ and $p \in \mathbb{R}_+^{\mathbb{S}}$ in (15). Many economic settings have $N_X(x) \subseteq \mathbb{R}_+^{\mathbb{S}}$ - or at least $N_X(x) \cap \mathbb{R}_+^{\mathbb{S}}$ non-empty. In such cases, if all $d_s > 0$, the constructed u increases strictly, and $u(r) > 0$ iff $r > \min x_s$.

Since ∂u, as designed, is decreasing, the corresponding inverse curve $(\partial u)^{-1}$ defined by $r^* \in \partial u(r) \Leftrightarrow r \in (\partial u)^{-1}(r^*)$ also decreases. Convex analysis tells that $(\partial u)^{-1} = \partial u^*$ where

$$u^*(r^*) := \inf \{r^* r - u(r) : r \in \mathbb{R}\}$$

is a concave conjugate. Thus, while ∂u defines an *indirect demand curve*, ∂u^* gives the corresponding direct demand. These relations carry nicely over from the integrand u to objective $U = Eu$ provided one employs the inner product $E(x^* x)$. To wit,

$$U^*(x^*) := \inf \{E(x^* x) - U(x) : x \in \mathbb{X}\} = \sum_{s \in \mathbb{S}} \pi_s u^*(x_s^*),$$

and $x^* \in \partial U(x) \Leftrightarrow x \in \partial U^*(x^*)$.

As noted, the function u, just constructed, could be non-smooth, featuring jumps in its derivative. There are good economic and technical reasons for wanting u at least twice continuously differentiable. That matter is briefly considered next:

Proposition 4.3 (Smooth choice). *Suppose the density-quantity pair p, x is feasible and such that $d_s > d_{\bar{s}} \Rightarrow x_s < x_{\bar{s}}$. Then there exists a twice continuously differentiable, concave utility function $u : \mathbb{R} \to \mathbb{R}$ such x solves consumption problem* (9) *with budget $b = P(x)$ - and expenditure problem* (12) *with utility level $\bar{U} = U(x)$.*

Proof. Find a continuously differentiable and decreasing function $u' : \mathbb{R} \to \mathbb{R}$ such that $u'(x_s) = d_s$ for each $s \in \mathbb{S}$. Again, $u(r) := \int_{\min x_s}^{r} u'$ will do the job. □

In particular, if the points (x_s, d_s) are colinear, u becomes quadratic. Price density-quantity pairs then fit the CAPM.

Focus has so far been on *consumption* and on *separable* objectives $U(\pi, x) = \sum_{s \in \mathbb{S}} \pi_s u(x_s)$. The latter are *symmetric* in that

$$U(\pi, x) = U(\mathcal{P}\pi, \mathcal{P}x) \text{ for each permutation } \mathcal{P} \text{ on } \mathbb{S}. \tag{17}$$

[6] In particular, when D is piecewise linear, u becomes piecewise linear-quadratic.

More generally, one may accommodate symmetric functions $(\pi, x) \mapsto U(\pi, x)$ such that (2) and (3) imply $U(\pi, x) \geq U(\pi, y)$. Therefore, we conclude this section by briefly looking at *production* and *non-separable* revenue functions U, with special consideration of symmetric instances:

Proposition 4.4 (Production input). *Suppose a revenue function $U : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ is concave, and $X \subseteq \mathbb{X}$ is closed convex. If input bundle x maximizes profit $U - P$ over X, and U is finite at some interior point of X, then*

$$p + x^* \in \partial U(x) \ \text{ for some } p \in \partial P(x) \ \text{ and } x^* \in N_X(x). \tag{18}$$

Conversely, if (18) *holds, then* $x \in \arg\max_X [U - P]$. *If moreover, U is differentiable, depends parametrically on π, is symmetric* (17)*, and preserves stochastic majorization, then* (16) *still holds.*

Proof. The argument concerning (18) follows the proof of Proposition 4.1. Further, Proposition 14 A.6 in [32] says that inequalities

$$(x_s - x_{\bar{s}})(\partial_s U(x) - \partial_{\bar{s}} U(x)) \leq 0 \text{ for all } s, \bar{s} \in \mathbb{S}, \tag{19}$$

hold. Finally, use the inclusion $p + x^* \in \partial U(x)$ from (18) to get (16). □

Many functions U, including those of von Neumann-Morgenstern form constructed above, satisfy the assumptions in Proposition 4.4 but are differentiable only along suitable directions. The condition that corresponds to (19) is then

$$U'(x; d) \geq 0 \text{ when } x_1 \leq x_2 \leq \cdots \leq x_S \ \text{ and } d = (0, ..., 0, 1/\pi_s, -1/\pi_{s+1}, 0, ..., 0).$$

5. Uncertainty Averse Preferences

Hitherto, all arguments were coached in terms of expected utility. That paradigm - developed for *risk* by von Neumann & Morgenstern and for *uncertainty* by Savage - facilitates analysis, but brings some well known paradoxes [2], [17]. To obviate those, and to permit ambiguity in beliefs, theory has been extended by Schmeidler [36], [37] and others.[7] Notably, upon weakening the independence axiom for risk, and the sure-thing principle for uncertainty, expected utility survives - at least in form.

To accommodate such generality, recall that price densities lie on the indirect demand curve, meaning $d_s \in \partial u(x_s)$. In particular, whenever two adjacent, but distinct demands $x_s < x_{\bar{s}}$ have $\partial u(x_s) > \partial u(x_{\bar{s}})$, a quite tempting avenue is to take u strictly concave on the interval $[x_s, x_{\bar{s}}]$. But plainly, such reconstruction of preferences tends to see strict risk aversion even in cases where neutrality prevail. In short, wrong features might be attributed to the agent at hand. An example illustrates this:

[7] Early studies include Gilboa [19], Greco [21], Quiggin [35] and Yaari [39]. For a survey see [29].

Betting on the winner: Let $U(x) = \sum_{s\in\mathbb{S}} \pi_s o_s x_s$ reflect perfect risk neutrality. Construe $x_s \geq 0$ as an optimal bet on outcome s, promising odds o_s. Here $\mathbb{P}$ is the singleton $p = (1/\pi_s)$, $X = \mathbb{R}^{\mathbb{S}}_+$, and $b > 0$. So, inclusion (11) reads $\lambda + x^*_s = \pi_s o_s$ with $x^*_s \leq 0$, $x^*_s x_s = 0$ for each s, and $\lambda = \max_s \pi_s o_s$. Put differently:

$$x_s > 0 \Rightarrow o_s\pi_s \text{ is maximal in } s.$$

In this case, with π known, if all odds o_s are different, the reconstructed u incorrectly portrays the player as strictly risk averse. $\square$

Such instances motivate a complementary approach - one that assigns as little risk aversion as possible to the agent. Specifically, data allows construction of two extreme curves $\partial\bar{u}$ and $\partial\underline{u}$, both connected and piecewise constant over maximal intervals. In pictorial terms, $r \mapsto \partial\bar{u}(r)$ and $r \mapsto \partial\underline{u}(r)$ are descending staircases, all steps occurring at the observed points x_s. Let $x(\mathbb{S})$ be set of such points. At any intermediate point $r \notin x(\mathbb{S})$, belonging to the interval $\mathbb{I} = [\min x_s, \max x_s]$, the said extreme curves are defined by

$$\partial\bar{u}(r) := \min\{d_s : x_s < r\} \quad \text{and} \quad \partial\underline{u}(r) := \max\{d_s : x_s > r\}.$$

Mathematically, among all descending curves that comply with data, the upper one $\partial\bar{u}$ has maximal right-hand derivative, whereas the lower one $\partial\underline{u}$ has minimal left-hand derivative on $\mathbb{I}$. Any other descending curve ∂u, that fits data, satisfies

$$\partial\underline{u}(r) \leq \partial u(r) \leq \partial\bar{u}(r) \text{ for all } r \in \mathbb{I}.$$

Plainly, there is considerable latitude in specifying a decreasing ∂u, bracketed between the extreme staircases $\partial\underline{u}$ and $\partial\bar{u}$.[8]

Economically, the upper indirect demand curve $r \mapsto \partial\bar{u}(r)$ yields maximal consumer surplus, whereas $r \mapsto \partial\underline{u}(r)$ minimizes that entity.

Note that the functions $\bar{u}(r) := \int_{\min x_s}^{r} \partial\bar{u}$ and $\underline{u}(r) := \int_{\min x_s}^{r} \partial\underline{u}$ are piecewise linear, concave, and $\underline{u} \leq \bar{u}$ on $\mathbb{I}$.

Suppose now all $d_s > 0$ *and posit* $\tilde{p}_s := \pi_s d_s$. Clearly, $d_s \in \partial\bar{u}(x_s) = \left[\bar{u}'_+(x_s), \bar{u}'_-(x_s)\right]$ iff

$$\underline{\pi}_s := \frac{\tilde{p}_s}{\partial^-\bar{u}(x_s)} \leq \pi_s \leq \frac{\tilde{p}_s}{\partial^+\bar{u}(x_s)} =: \bar{\pi}_s.$$

These inequalities indicate some leeway for the choice of probabilities. Consider therefore the simple linear problem

$$\min \sum_{s\in\mathbb{S}} \bar{u}(x_s)\delta_s \text{ s.t. } \delta \in \Delta, \text{ and } \delta_s \in [\underline{\pi}_s, \bar{\pi}_s] \text{ for each } s. \tag{20}$$

[8] To extend utility beyond $\mathbb{I}$, one could set $\partial\bar{u}(\min x_s) := [\max d_s, +\infty)$ and $\partial\bar{u}(r) := \min_s d_s$ for $r > \max x_s$. Similarly, one could posit $\partial\underline{u}(r) = \max d_s$ for $r < \min x_s$, and $\partial\underline{u}(\max x_s) = (-\infty, \min_s d_s]$. Anyway, because derivatives of u are observed - or synthesized - merely at a few points, one can of course not hope to have u unique

The original distribution π is of course feasible for (20). Moreover, if $\sum_{s\in\mathbb{S}} \underline{\pi}_s = 1$ or $\sum_{s\in\mathbb{S}} \bar{\pi}_s = 1$, no other distribution is feasible. So, only the instance $\sum_{s\in\mathbb{S}} \underline{\pi}_s < 1$ & $\sum_{s\in\mathbb{S}} \bar{\pi}_s > 1$ merits further consideration. With no loss of generality, assume $x_1 \leq x_2 \leq \cdots \leq x_S$ to have $\bar{u}(x_1) \leq \bar{u}(x_2) \leq \cdots \leq \bar{u}(x_S)$. Let $\bar{s}$ be that largest element in $\mathbb{S}$ for which some optimal solution $\delta \in \Delta$ of (20) has $\delta_s = \bar{\pi}_s$. Define an additive capacity v on proper subsets of $\mathbb{S}$ by $v(s) = \bar{\pi}_s$ for $s \leq \bar{s}$, $v(s) = \underline{\pi}_s$ for $s > \bar{s}$, and posit $v(\mathbb{S}) = 1$. This specification fits a pessimistic view in that maximal probability is assigned to "worst states" where the price density is large. In terms of

$$Core(v) := \{\delta \in \Delta : \delta \geq v\}$$

the optimal value of (20) equals the Choquet integral

$$\int \bar{u}(x)dv = \min\left\{\sum_{s\in\mathbb{S}} \bar{u}(x_s)\delta_s : \delta \in Core(v)\right\}. \tag{21}$$

With Choquet expected utility, consumption problem (9) assumes the more general form:

$$\text{maximize } U(x) = \int u(x)dv \text{ subject to } x \in X \text{ and } P(x) \leq b. \tag{22}$$

The preceding elaborations prove a result that corresponds to Proposition 4.2:

Proposition 5.1 (Rationalizing choice by Choquet utility). *Suppose the pair p, x is feasible with strictly positive density $d = \lambda p + x^*$. Then there exists a piecewise linear, strictly increasing, concave function $\bar{u} : \mathbb{R} \to \mathbb{R}$ and a convex capacity v such that x solves* (22) *with utility index $\bar{u}$ and budget $b = P(x)$.* $\square$

There is of course an entirely similar result that invokes the lower function $\underline{u}$.

It is fitting to conclude this section with a brief consideration of the Choquet integral $\int \cdot dv$ as it applies to consumer problem (22) with a finite sample space $\mathbb{S}$. That integral is defined in terms of a monotone set function $v : 2^{\mathbb{S}} \to [0,1]$, called a *capacity*, that satisfies $v(\emptyset) = 0$, $v(\mathbb{S}) = 1$, and $A \subset B \Rightarrow v(A) \leq v(B)$.

$\int \cdot dv$ operates on mappings $\mathbf{u} : \mathbb{S} \to \mathbb{R}$ as follows: Order the values of $\mathbf{u}$ increasingly: $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(S)}$ - and posit

$$\int \mathbf{u}dv := \sum_{s\in\mathbb{S}} u_s\delta_s \tag{23}$$

where $\delta_{(s)} = v\left\{\mathbf{u} \geq u_{(s)}\right\} - v\left\{\mathbf{u} \geq u_{(s)+1}\right\}$ and $\left\{\mathbf{u} \geq u_{(S)+1}\right\} = \emptyset$. The integral so defined isn't additive unless v is so. However, because values of the integrand $\mathbf{u}$ were ordered, $\int \cdot dv$ becomes additive across comonotone mappings $\mathbf{u}, \hat{\mathbf{u}} : \mathbb{S} \to \mathbb{R}$.

A capacity v is called *convex* if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B) \text{ for all } A, B \subseteq \mathbb{S}.$$

In that case, $Core(v) := \{\delta \in \Delta : \delta \geq v\}$ is non-empty, and $\int \mathbf{u} dv = \min \{E_\delta \mathbf{u} : \delta \in \Delta\}$ as in (21).

A weak order[9] $\succsim$ on $\mathbb{U} := \mathbb{R}^{\mathbb{S}}$ is said to display *uncertainty aversion* if

$$[\mathbf{u} \sim \tilde{\mathbf{u}} \ \& \ (\tilde{\mathbf{u}} \text{ and } \hat{\mathbf{u}} \text{ are comonotone})] \Rightarrow \mathbf{u} + \hat{\mathbf{u}} \succsim \tilde{\mathbf{u}} + \hat{\mathbf{u}}.$$

Intuitively, uncertainty aversion tells that while $\hat{\mathbf{u}}$ can't hedge $\tilde{\mathbf{u}}$, it may still provide some hedging of $\mathbf{u}$, thus making $\mathbf{u} + \hat{\mathbf{u}}$ preferable to $\tilde{\mathbf{u}} + \hat{\mathbf{u}}$. Provided a weak order be continuous and monotone, under uncertainty aversion it has a nice representation:

Theorem 5.1 (Choquet integral representation of a uncertainty averse weak order [6], [7], [21], [36]).
Let the weak order $\succsim$ on $\mathbb{U}$ be continuous and monotone. Then the following two statements are equivalent:
- *$\succsim$ displays uncertainty aversion;*
- *there exists a convex capacity v such that $\mathbf{u} \succsim \hat{\mathbf{u}} \Leftrightarrow \int \mathbf{u} dv \geq \int \hat{\mathbf{u}} dv$.* □

Basic for the agent considered here are contingent claims $x \in \mathbb{X}$ and his utility function $u : \mathbb{R} \to \mathbb{R}$. If his weak preference order $\succsim$ on contingent claims is such that $x \succsim \hat{x} \Leftrightarrow \int \mathbf{u} dv \geq \int \hat{\mathbf{u}} dv$ where $\mathbf{u} := [u(x_s)]$ and $\hat{\mathbf{u}} := [u(\hat{x}_s)]$, then his criterion $U : \mathbb{X} \to \mathbb{R}$ assumes the Choquet expected utility form $U(x) = \int u(x) dv$. Particularly interesting are instances with concave integrand u and convex capacity v. Such criteria emerge under surprisingly weak assumptions:

Theorem 5.2 (Quasi-concave Choquet expected utility [8]). *Suppose $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, differentiable on $\mathbb{R}_{++}$ and strictly increasing. Then the following statements are equivalent:*
- *The functional $U(\cdot) = \int u(\cdot) dv$ is quasi-concave.*
- *The utility index u is concave and the capacity v is convex.* □

Arguing as for Proposition 4.1, we get a corresponding result:

Proposition 5.2 (Optimal consumption). *Suppose $u : \mathbb{R} \to \mathbb{R}$ is concave and v is convex.*
- *If x solves consumer problem* (22), *and $X \cap \{P < b\}$ is non-empty, with $P(x) = b$, there exist*

$$\pi \in Core(v), \quad \lambda \geq 0, \quad p \in \partial P(x), \quad x^* \in N_X(x) \tag{24}$$

such that (11) *holds with*

$$[u(x_s)] \cdot (\delta - \pi) \geq 0 \text{ for all } \delta \in Core(v). \tag{25}$$

- *Conversely, if* (11), (24) and (25) *hold, then x solves consumer problem* (22) *for budget $b := P(x)$.* □

[9] A *weak order* is a reflexive, total and transitive binary relation.

6. Demand, Expenditure and Efficient Choice

Aumann (1962), and others, argued that completeness isn't really a prerequisite for rational choice. He extended the von Neumann-Morgenstern utility theory to allow incomparable alternatives. His generalization preserves the expected utility format, but dispenses with (affine) uniqueness of the representing function. As representative comes instead a *class* of functions.

In our setting, the suitable classes identify themselves straightforwardly. To wit, since concave, concave & increasing, and Schur concave orders occupy center stage, the classes consist of corresponding functions. The resulting partial orders expressly capture that variability around the mean is undesirable. Of course, upon assuming so little, less sharp conclusions obtain on asset pricing and portfolio choice. Quite a few results still stay intact though; see [9], [10], [13], [15], [28].

In such settings, it's natural to analyze behavior merely by means of partial orders. Along that line, this section considers an agent about whom we only know that he waists no money on inefficient choices. That is, having made his choice, no preferred prospect should still be available at lower cost.

Hypothesis (Efficient choice). *When consumption profile $x \in X$ has been chosen, none of the sets*

$$\{P \leq P(x)\} \cap \{\succ_{ci} x\} \quad \textit{and} \quad \{P < P(x)\} \cap \{\succsim_{ci} x\}$$

should intersect X. More generally, for any feasible x the agent has selected, it should hold:

$$0 \in bd\,[X \cap \{P < P(x)\} - X \cap \{\succ_{ci} x\}]\,.$$

When improvement is impossible, the budget must be binding and expenditure minimal:

Proposition 6.1 (Minimal expenditure).

• *Under the above hypothesis there exists an underlying price $p \in \partial P(x)$ such that the expenditure problem:*

$$\textit{minimize } P(\chi) \textit{ s.t. } \chi \succsim_{ci} x \ \textit{ and } \chi \in X, \tag{26}$$

has optimal value $p \cdot x$, and x as optimal solution.

• *Suppose that $X \cap int\,\{\succsim_{ci} x\}$ is nonempty. Then, for any optimal solution χ to* (26) *there are vectors $p \in \partial P(\chi)$, $x^* \in N_X(\chi)$ and $g \in -N_\chi\,\{\succsim_{ci} x\}$ such that*

$$p + x^* = g \tag{27}$$

• *In particular, let $p \in \partial P(x)$, $x^* \in N_X(x)$ be such that x solves* (27)*. Then inequality* (16) *holds.*

Proof. Both sets $\{P < P(x)\}$ and $X \cap \{\succ_{ci} x\}$ are convex. The hypothesis ensures that some hyperplane $\{p \cdot \; = p \cdot x\}$ passes through x and separates the said sets; that is:

$$p \cdot \{P < P(x)\} \leq p \cdot x \leq p \cdot X \cap \{\succ_{ci} x\}.$$

Then

$$p \cdot \{P \leq P(x)\} \leq p \cdot x \leq p \cdot X \cap \{\succsim_{ci} x\}$$

In the last string of inequalities the left-hand one tells that x solves the problem $\max p \cdot \chi$ s.t. $P(\chi) \leq P(x)$. Then $p \in \lambda \partial P(x)$ for some $\lambda > 0$. Replace p by p/λ to have $p \in \partial P(x)$ and $P(x) = p \cdot x$. Now, because $p \in \mathbb{P}$, the right-hand inequality in the last string says that

$$P(x) = p \cdot x \leq p \cdot \chi \leq P(\chi) \text{ for all } \chi \in X \cap \{\succsim_{ci} x\}.$$

From this (26) follows. □

Sometimes X consists of precisely those random variables that are measurable with respect to a field $\mathcal{F} \subseteq 2^{\mathbb{S}}$. If all underlying prices $p \in \mathbb{P}$ are $\mathcal{F}$-measurable as well, then problem (26) can be relaxed:

Proposition 6.2 (Relaxed minimization of expenditure). *Suppose $x \in X$ iff $E[x|\mathcal{F}] = x$ for a specified field $\mathcal{F} \subseteq 2^{\mathbb{S}}$. If $\mathbb{P} \subset X$, then for any $x \in X$, the optimal value and solution set of* (26) *equals that of the relaxed problem*

$$\textit{minimize } P(\chi) \textit{ s.t. } \chi \succsim_{ci} x.$$

Proof. Take any $\chi \succsim_{ci} x$ to have $E[\chi|\mathcal{F}] \succsim_{ci} x$, and $E[\chi|\mathcal{F}] \in X$. Further, for arbitrary $p \in \mathbb{P}$ it holds $E(p\chi) = E(pE[\chi|\mathcal{F}])$. □

Economic theory and practice deals with reallocation of income and production factors across various stages, states or agents. Notably, finance and insurance are the fields - and the institutions - that specialize in analyzing or providing such reallocation. Major forces stem there from agents who prefer averaged or smoothened claims over more dispersed ones. Clearly, when possible, any such agent would appreciate to have a more concentrated claim at lower cost. The next result isolates a simple, two-state setting in which that sort of double improvement is indeed achievable:

Preposition 6.3 (On majorizing, mean-preserving change of cost). *Given here is a two-point, nondegenerate probability distribution $\delta_1, \delta_2 > 0$, $\delta_1 + \delta_2 = 1$, with expectation E_δ. Let $\succsim \in \{\succsim_{ci}, \succsim_c, \succsim_{(\cdot)}\}$.*

• (On majorizing, mean preserving, cheaper consumption). *For prescribed prices $0 \leq p_1 \leq p_2$, consider consumption choices $c_1 < c_2$. Then there is another consumption profile $c' \succ c$ with the the same expectation which costs no more. That* *is,*

$$\begin{cases} c_1 < c'_1 & \leq & c'_2 < c_2, \\ E_\delta c' & = & E_\delta c, \quad \textit{and} \\ E_\delta(pc') & \leq & E_\delta(pc). \end{cases}$$

For any strictly concave $u : \mathbb{R} \to \mathbb{R}$ *it holds* $E_\delta u(c') > E_\delta u(c)$*. Also,* $E_\delta(pc') < E_\delta(pc)$ *unless* $\delta_1 = \delta_2 = 1/2$ *and* $p_1 = p_2$*.*

• (On majorizing, mean preserving, costlier pricing). *For prescribed consumption choices* $c_1 \geq c_2 \geq 0$*, consider prices* $p_1 > p_2$*. Then there is another price profile* $p' \succ p$ *with the the same expectation which yields greater expenditure. That is,*

$$\begin{cases} p_1 > p'_1 & \geq & p'_2 > p_2, \\ E_\delta p' & = & E_\delta p, \quad \textit{and} \\ E_\delta(p'c) & \geq & E_\delta(pc). \end{cases}$$

For any strictly concave $u : \mathbb{R} \to \mathbb{R}$ *it holds* $E_\delta u(p') > E_\delta u(p)$*. Also,* $E_\delta(p'c) > E_\delta(pc)$ *unless* $\delta_1 = \delta_2 = 1/2$ *and* $c_1 = c_2$*.*

Proof. Let $c'_1 = w_1 c_1 + (1 - w_1)c_2$ and $c'_2 = (1 - w_2)c_1 + w_2 c_2$ with weights $w_1, w_2 \in \,]0, 1[$. Clearly, $c_1 < c'_1, c'_2 < c_2$. Notice that $c'_1 \leq c'_2$ iff $w_1 + w_2 \geq 1$. To have

$$\begin{aligned} E_\delta c' &= \delta_1 [w_1 c_1 + (1 - w_1)c_2] + \delta_2 [(1 - w_2)c_1 + w_2 c_2] \\ &= [\delta_1 w_1 + \delta_2(1 - w_2)] c_1 + [\delta_1(1 - w_1) + \delta_2 w_2] c_2 \\ &= \delta_1 c_1 + \delta_2 c_2 = E_\delta c, \end{aligned}$$

it must hold: $\delta_1 w_1 + \delta_2(1 - w_2) = \delta_1$ and $\delta_1(1 - w_1) + \delta_2 w_2 = \delta_2$. (Each of the last two equations follows from the other). There are two cases, to be discussed separately:

$\underline{\delta_1 < \delta_2}$. Then, for chosen $w_1 \in \,]0, 1[$, take $w_2 = 1 - \frac{\delta_1}{\delta_2}(1 - w_1)$. With this specification, since $w_1 + w_2 = w_1(1 + \frac{\delta_1}{\delta_2}) + 1 - \frac{\delta_1}{\delta_2}$ increases in w_1, the smallest w_1 for which $w_1 + w_2 \geq 1$, is $w_1 \geq \underline{w}_1 := \delta_1/(\delta_1 + \delta_2)$.

$\underline{\delta_1 \geq \delta_2}$. Then, for chosen $w_2 \in \,]0, 1[$, take $w_1 = 1 - \frac{\delta_2}{\delta_1}(1 - w_2)$. Now, since $w_1 + w_2 = w_2(1 + \frac{\delta_2}{\delta_1}) + 1 - \frac{\delta_2}{\delta_1}$ increases in w_2, the smallest w_2 for which $w_1 + w_2 \geq 1$, is $w_2 \geq \underline{w}_2 := \delta_2/(\delta_1 + \delta_2)$.

In either situation, $c_1 < c'_1 \leq c'_2 < c_2$ and $E_\delta c = E_\delta c'$. Because

$$\begin{aligned} E_\delta [p(c' - c)] &= \delta_1 p_1 [w_1 c_1 + (1 - w_1)c_2 - c_1] + \delta_2 p_2 [(1 - w_2)c_1 + w_2 c_2 - c_2] \\ &= [\delta_1 p_1(1 - w_1) - \delta_2 p_2(1 - w_2)] (c_2 - c_1), \end{aligned}$$

we have $E_\delta(pc') \leq E_\delta(pc) \Leftrightarrow \delta_1 p_1(1 - w_1) \leq \delta_2 p_2(1 - w_2)$. Clearly, if $p_2 = 0$, then $p_1 = 0$, and the last inequality becomes vacuous. So, assume $p_2 > 0$ to arrive at the equivalent inequality that characterizes cost reduction:

$$\delta_1 \frac{p_1}{p_2}(1 - w_1) \leq \delta_2(1 - w_2).$$

Clearly, $w_1 = \underline{w}_1 = \delta_1/(\delta_1+\delta_2)$ and $w_2 = \underline{w}_2 = \delta_2/(\delta_1+\delta_2)$ is *one* solution, giving $c'_1 = c'_2$.

Other solutions also exist: when $\delta_1 < \delta_2$, take any $w_1 \in \left[\frac{\delta_1}{\delta_1+\delta_2}, 1\right)$ and $w_2 = 1 - \frac{\delta_1}{\delta_2}(1-w_1)$ to obtain genuine cost reduction. Similarly, when $\delta_1 > \delta_2$, take any $w_2 \in \left[\frac{\delta_2}{\delta_1+\delta_2}, 1\right)$ and $w_1 = 1 - \frac{\delta_2}{\delta_1}(1-w_2)$ to have strictly lower cost.

Finally, the instance $\delta_1 = \delta_2 = 1/2$ yields $w_1 = w_2 = 1/2$, $c'_1 = c'_2 = (c_1+c_2)/2$, and $E_\delta(pc') < E_\delta(pc) \iff p_1 < p_2$.

The second bullet is proven in entirely the same manner. □

The preceding result clarifies that some choices appear unlikely. To make this precise we need a

Definition (On averaged consumption). *The feasible set X allows averaged consumption in states $s, \bar{s} \in \mathbb{S}$ if whenever some feasible x has $x_s < x_{\bar{s}}$, then any x' is also feasible that differs from x only in components $s, \bar{s}$, and has*

$$\begin{array}{rcl} x_s < x'_s & \leq & x'_{\bar{s}} < x_{\bar{s}} \text{ and} \\ \pi_s x_s + \pi_{\bar{s}} x_{\bar{s}} & = & \pi_s x'_s + \pi_{\bar{s}} x'_{\bar{s}}. \; \square \end{array}$$

For any order $\succsim \in \{\succsim_{ci}, \succsim_c, \succsim_{(\cdot)}\}$, price functional P of form (8), and budget $b \in \mathbb{R}$, define the *constrained, order-efficient demand set*

$$\mathcal{D}_{\succsim}(P,b) := \left\{ \begin{array}{l} c \in X \cap \{P \leq b\} : \\ \text{neither } \{\succ c\} \cap \{P \leq b\} \text{ nor } \{\succsim c\} \cap \{P < b\} \text{ intersect } X \end{array} \right\}.$$

Clearly, $\mathcal{D}_{\succsim_c}(P,b) \subseteq \mathcal{D}_{\succsim_{ci}}(P,b)$.

Proposition 6.4 (Order efficient, strictly antimonotone demand). *Let $\succsim \in \{\succsim_{ci}, \succsim_c\}$ and suppose the feasible set X allows averaged consumption in states $s, \bar{s}$.*

- *Then, if $c \in \mathcal{D}_{\succsim}(P,b)$ has $c_s < c_{\bar{s}}$ and $P(c) = E(pc)$, it must hold that $p_s > p_{\bar{s}}$.*
- *If X is a linear subspace which contains $\mathbf{1}$, and $P(\beta\mathbf{1} + c) = \beta P(\mathbf{1}) + P(c)$ for any $\beta > 0$, then each $c \in \mathcal{D}_{\succsim_{ci}}(P,b)$ makes full use of the budget: $P(c) = b$.*

Proof. Let $c_s < c_{\bar{s}}$ be two components of $c \in \mathcal{D}_{\succsim}(p,b)$. Without loss of generality, let $s = 1$, $\bar{s} = 2$. Posit $\delta_1 := \pi_1/(\pi_1+\pi_2)$ and $\delta_2 := \pi_2/(\pi_1+\pi_2)$. Suppose $p_1 \leq p_2$. With reference to Proposition 6.3 let

$$c'_s := \left\{ \begin{array}{ll} c'_s & \text{if } s \in \{1,2\} \\ c_s & \text{otherwise.} \end{array} \right.$$

Then $E(pc') \leq E(pc)$ and $Ec' = Ec$, but $c' \succ c$ for each order $\succsim \in \{\succsim_{ci}, \succsim_c\}$. This contradicts the presumed efficiency of c. Thus we must have $p_1 > p_2$. This takes care of the first bullet.

For the second, suppose $P(c) < b$. Let $\beta := \{b - P(c)\} / P(\mathbf{1})$ and posit $c' := \beta\mathbf{1} + c$ to have $P(c') = b$ and $c' \succ_{ci} c$. $\square$

We note that there often is a way to construct antimonotone density-quantity pairs:

Proposition 6.5 (Antimonotone density-quantity). *Let X be a linear subspace of $\mathbb{R}^{\mathbb{S}}$ such that $\mathbf{1} \in X$. Also suppose some underlying price $p \in \mathbb{P} \cap X$ has distinct components. Then the problem*

$$\max \sum_{s \neq \bar{s}} \ln \left[(p_s + x_s^{\perp} - p_{\bar{s}} + x_{\bar{s}}^{\perp})(x_{\bar{s}} - x_s) \right] \text{ s.t. } p \in \mathbb{P},\ E(px) \leq b,\ (x, x^{\perp}) \in X \times X^{\perp}$$

is feasible. Any feasible solution pair $(x, x^{\perp})$ has quantity x and density $d = p + x^{\perp}$ antimonotone.

Proof. For feasibility take any $p \in \mathbb{P} \cap X$ which has distinct components. Let

$$x = \frac{b + E(pp)}{E(p\mathbf{1})}\mathbf{1} - p$$

in X, and posit $x^{\perp} = 0$. The pair $(x, x^{\perp})$ is feasible. The rest is straightforward. $\square$

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