Risk Exchange as a Market or Production Game

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Abstract. Risk exchange is considered here as a cooperative game with transferable utility. The set-up fits markets for insurance, securities and contingent endowments. When convoluted payoff is concave at the aggregate endowment, there is a price-supported core solution. Under variance aversion the latter mirrors the two-fund separation in allocating to each agent some sure holding plus a fraction of the aggregate.

Keywords: securities, mutual insurance, market or production games, transferable utility, extremal convolution, core solutions, variance or risk aversion, two-fund separation, CAPM.

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1. Introduction

Many economic agents face risky endowments or commitments. Then, to mitigate ups and downs, it appears prudent to pool risks - often many and material in nature - and share them thereafter. For its viability the sharing had better be contingent, efficient and voluntary.

Along such lines, albeit in a purely pecuniary setting, Borch (1962) showed that reinsurance contracts may mirror a competitive equilibrium of an exchange economy.[1] By the first fundamental welfare theorem, given non-satiated consumers, any equilibrium of that sort resides in the core. Indicated thus is an *indirect* connection between risk/security markets and cooperative games. Apart from [5], [7], [28] *direct* connections have hardly been emphasized. In fact, even the most tractable instances, featuring transferable utility (TU), have received almost no attention. Yet such instances could serve a few good ends.

Accordingly, presuming TU, this paper probes beyond Pareto-optimality [1], [6], [19], [20], [36], [37] by linking risk exchange directly to cooperative contracts. One bonus comes by connecting reciprocal treaties closer to asset pricing theory [2], [14]. Another is to generate not only equilibrating prices but also slopes of the resulting

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[1] For related studies, see [11], [30], [31].

curves. On a more technical note, no fixed point arguments are needed for existence of a core solution. Instead it suffices that Lagrangian duality be attained with no *gap*. This makes for easier analysis and computation. In addition, concerns about existence of equilibrium *prices* become fully divorced from those regarding equilibrium *allocations*.

To set the stage Section 2 introduces, by way of examples, a *market game* in order to recall what is meant by a core solution. Section 3 identifies weak conditions under which such solutions can be found merely in terms of shadow prices on the aggregate endowment/risk. Section 4 elaborates on the nature and existence of shadow prices. Section 5 digresses to supplement the market perspective by regarding cooperation alternatively as a *production game*. After so much groundwork, Sections 6&7 address pricing and sharing of risk. Some results align perfectly with the two-fund separation that characterizes equilibrium in capital asset pricing models. Section 8 considers the resulting price curves and tolerances for risk. Section 9 concludes with some examples.

The paper addresses several types of readers. Included are actuaries, finance analysts or general economists interested in risk exchange, but not quite knowing how nicely Lagrangian duality produces explicit core outcomes. Also addressed are mathematicians interested in optimization, but less informed as to how extremal convolution relates to exchange markets.

2. The Game

Accommodated henceforth is a fixed, finite set I of economic agents. For background and motivation consider two different settings.

Electricity generation: Plant $i \in I$ has promised to deliver the energy amount $e_i(s)$ in state or season $s \in S$. Since one plant uses hydro-power based on short term precipitation, its production capacity is highly variable. Because another hydro-based plant merely draws melting water from under a glacier, it is practically non-operative during cold winters - but well furnished in hot summers. A third supplier owns a thermal station. By helping each other these plants may, in each state s, more easily satisfy the total commitment $e_I(s) := \sum_{i \in I} e_i(s)$. How should the overall load be allocated? And what payments would induce voluntary cooperation?

Exchange of catch quotas: Fisherman $i \in I$ is allowed to catch the amount $e_{ij}(s)$ of species $j \in J$ in state or season $s \in S$. Since his gear selects merely one specific species, he wants to exchange his allowances in other species for the one he wants. When trade is mediated by money, what exchange rates are reasonable?

In short, we think of firms that must cope with uncertain product demand or random factor supply. Firm or individual $i \in I$ owns (commitment or) *endowment* e_i. For the sake of generality - and for simple presentation - e_i is construed, until Section 6, simply as a vector in some real linear space $\mathbb{X}$.[2]

Individual i has payoff function $\pi_i : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$. The extreme value $-\infty$ reports infinite loss, or total dissatisfaction, or violation of implicit constraints. This device helps highlighting essential features and saves special mention of the *effective domain*

$$dom\pi_i := \{x_i \in \mathbb{X} : \pi_i(x_i) > -\infty\}$$

to which any feasible choice x_i must belong. Until further notice, no sort of concavity, differentiability or monotonicity is required of π_i. Also, we impose no particular functional form.[3] We presume however, that individual payoff be metered in money or some common unit of account. This feature is crucial for what follows in that utility must be transferable.[4]

Now, rather than everybody contending with his own endowment, the parties might agree upon some reallocation. In fact, the aggregate $e_I := \sum_{i\in I} e_i$ can most likely be split in ways that better suit the needs of everyone. So, we ask: *can the agents write an efficient, socially stable contract? And if so, what will be its nature?* For the argument, suppose the members of a *coalition* $C \subseteq I$ be able to cooperate among themselves. If endowments are perfectly divisible and freely transferable,[5] that coalition could foresee overall payoff

$$\pi_C(e_C) := \sup\left\{\sum_{i\in C} \pi_i(x_i) : \sum_{i\in C} x_i = \sum_{i\in C} e_i =: e_C\right\}. \tag{1}$$

Construction (1), called a *sup-convolution*, tacitly presumes that no member of C misrepresents his payoff function or endowment to own advantage. Thus, strategic communication is precluded. This assumption can be justified if the underlying data are common knowledge, or readily observed, or honestly reported. *Suppose henceforth that $\pi_I(e_I)$ is finite.*

The potential advantages of enterprise (1) are evident and twofold. First, aggregation offers the agents increased leeway and better substitution possibilities. Second, depending on the setting, it may facilitate transfers across time and contingencies. So, a key issue is whether the *grand coalition* $C = I$ can agree upon ways to share the aggregate endowment. Plainly, formation of that coalition requires that proceeds be distributed in ways not blocked by any subgroup. Reflecting on this concern, a *payoff distribution* $u = (u_i) \in \mathbb{R}^I$ is declared a *core solution* iff it entails

$$\begin{cases} \textit{Pareto efficiency:} & \sum_{i\in I} u_i = \pi_I(e_I) \quad \textit{and} \\ \textit{stability:} & \sum_{i\in C} u_i \geq \pi_C(e_C) \quad \textit{for each coalition } C \subset I. \end{cases}$$

[2]When J, S are finite sets, the above example of electricity generation gives $e_i \in \mathbb{X} := \mathbb{R}^S$, whereas the fisheries example has $e_i \in \mathbb{X} := \mathbb{R}^{J\times S}$.

[3]But clearly, objectives of ordinary or Choquet integral form are accommodated [13].

[4]At least two settings justify use of monetary payoff. In a first, i is a producer who obtains pecuniary payoff $\pi_i(x_i)$ from input bundle $x_i \in \mathbb{X}$. In another, i is a consumer who enjoys quasi-linear utility $\pi_i(x_i^a, x_i^{-a}) = x_i^a + \pi_i^a(x_i^{-a})$ from profile $x_i = (x_i^a, x_i^{-a})$, the a-th component of which refers to a common real-valued *unit of account*.

[5]Fixed factors are neither pooled nor exchanged.

Stability is easily achieved. Simply let payments be so wonderfully large that $\sum_{i\in C} u_i \geq \pi_C(e_C), \forall C \subseteq I$. Thus, the essential difficulty hides in the requirement that total payoff be efficient and not handed out excessively.

The core as solution concept, although central to cooperative game theory, does not figure prominently in the finance or insurance literature.[6] Construction (1) mimics the classical Shapley-Shubik (1969) analysis of market or production games. If all π_i are concave, the cooperative incentives become so strong and well distributed that the grand coalition can safely form. To wit, the game - and every subgame - then has non-empty core:

Proposition (Concave objectives make the game totally balanced).[7] *Suppose each π_i is concave and all values $\pi_C(e_C), C \subseteq I$, are finite. Then the TU cooperative game, featuring characteristic function $C \subseteq I \mapsto \pi_C(e_C)$ is **totally balanced**. That is, each subgame, restricted to any coalition $C \subseteq I$, has non-empty core.* □

3. Price-Generated Core Solutions

The preceding proposition is less than satisfying on two accounts. First, one would like to push beyond mere existence and seek some specific, computable core element. Second, one wonders whether less concavity would suffice. For these purposes write $\mathbf{x} = (x_i) \in \mathbb{X}^I$ for the profile $i \mapsto x_i$. Further, let $x^* : \mathbb{X} \to \mathbb{R}$ be any linear functional, and associate the standard Lagrangian

$$L_C(\mathbf{x}, x^*) := \sum_{i\in C} \pi_i(x_i) + x^*(\sum_{i\in C} e_i - \sum_{i\in C} x_i)$$

to problem (1). To simplify notation we henceforth write x^*x instead of $x^*(x)$.

Definition (Shadow prices). *Any linear $\lambda : \mathbb{X} \to \mathbb{R}$ such that $\pi_I(e_I) \geq \sup_{\mathbf{x}} L_I(\mathbf{x}, \lambda)$ will be named a **Lagrange multiplier** or **shadow price**.* □

The next section discusses existence of shadow prices. Here we note that λ *qualifies as shadow price iff $\pi_I(e_I)$ is a saddle value of L_I in that*

$$\pi_I(e_I) = \inf_{x^*} \sup_{\mathbf{x}} L_I(\mathbf{x}, x^*) = \sup_{\mathbf{x}} \inf_{x^*} L_I(\mathbf{x}, x^*).$$

In fact, these equalities - as well as $\pi_I(e_I) = \sup_{\mathbf{x}} L_I(\mathbf{x}, \lambda)$ - follow from

$$\pi_I(e_I) \geq \sup_{\mathbf{x}} L_I(\mathbf{x}, \lambda) \geq \inf_{x^*} \sup_{\mathbf{x}} L_I(\mathbf{x}, x^*) \geq \sup_{\mathbf{x}} \inf_{x^*} L_I(\mathbf{x}, x^*) \geq \pi_I(e_I).$$

To better appreciate shadow prices let the convex function

$$f^{(*)}(x^*) := \sup \{f(x) - x^*x : x \in \mathbb{X}\} \tag{2}$$

[6] Exceptions include [2], [5], [7], [27], [28].

[7] This result appears well known and is therefore stated without proof.

denote a *conjugate* of $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$.[8] The last section provides some examples. Conjugates are central in the following

Theorem (Shadow prices support core solutions). *Let λ be a shadow price. Then the payoff distribution that offers agent i the amount*

$$u_i(\lambda) := \pi_i^{(*)}(\lambda) + \lambda e_i \tag{3}$$

constitutes a core solution.

Proof. The argument is surprisingly short and simple. It was already given in [16] for cost sharing but is reproduced here for profit sharing - and for completeness. Note that given any linear price $x^* : \mathbb{X} \to \mathbb{R}$ and coalition $C \subseteq I$ it holds

$$\sup_{\mathbf{x}} L_C(\mathbf{x}, x^*) = \sum_{i \in C} u_i(x^*).$$

Thus, social stability obtains for arbitrary x^* because coalition C receives

$$\sum_{i \in C} u_i(x^*) = \sup_{\mathbf{x}} L_C(\mathbf{x}, x^*) \geq \inf_{x^*} \sup_{\mathbf{x}} L_C(\mathbf{x}, x^*) \geq \sup_{\mathbf{x}} \inf_{x^*} L_C(\mathbf{x}, x^*) = \pi_C(e_C).$$

The very last inequality, which holds without any qualifications, is often referred to as *weak duality.*[9] In particular, $\sum_{i \in I} u_i(\lambda) \geq \pi_I(e_I)$. The hypothesis on λ ensures the reverse inequality - commonly called *strong duality.* Thereby Pareto efficiency obtains as well: $\sum_{i \in I} u_i(\lambda) = \pi_I(e_I)$. $\square$

The above result, while adding to [8], [34], [36], can serve as spring-board for several extensions; see [16] and references therein.

For interpretation, if λ prices "input" x_i, and agent i acts as price-taker in factor markets, core solution (3) offers him profit $\pi_i^{(*)}(\lambda)$ plus payment λe_i for his endowment. As customary, a price λ should equal marginal payoffs. That feature is explored next.

[8] In terms of the *Fenchel conjugate* $f^*(x^*) := \sup_x \{x^* x - f(x)\}$, one has $f^{(*)}(x^*) = (-f)^*(-x^*)$; see [33]. Definition (2) suits here because it reflects price-taking in factor markets and the pursuit of profit. Specifically, if input $x \in \mathbb{X}$ comes at linear cost $x^* x$, and yields revenue $f(x)$, then the maximal economic rent is $f^{(*)}(x^*)$. If $\mathbb{X}$ is locally convex topological, and f is proper, upper semicontinuous, concave, then $f(x) = \inf \left\{ f^{(*)}(x^*) + x^* x : x^* \text{ continuous linear} \right\}$.

[9] Note that $\sup_{\mathbf{x}} L_C(\mathbf{x}, x^*) \geq \pi_C(e_C)$ holds for *any* functional $x^* : \mathbb{X} \to \mathbb{R}$ that satisfies $x^*(0) \geq 0$. If moreover, x^* is additive, then $\sum_{i \in C} u_i(x^*) = \sup_{\mathbf{x}} L_C(\mathbf{x}, x^*)$. Also, if for some class X^* of functionals $x^* : \mathbb{X} \to \mathbb{R}$ it holds

$$\inf \{x^* x : x^* \in X^*\} = \begin{cases} 0 & \text{for } x = 0 \\ -\infty & \text{otherwise,} \end{cases}$$

then $\sup_{\mathbf{x}} \inf_{x^* \in X^*} L_C(\mathbf{x}, x^*) = \pi_C(e_C)$.

4. THE NATURE AND EXISTENCE OF SHADOW PRICES

Our approach makes room for non-smooth functions, several goods, constrained choice - and for preferences that need not be of the expected utility format. These feature notwithstanding, we want to regard shadow prices as marginal payoffs - that is, as *derivatives*, possibly generalized. For the statement, denote by ∂ the *superdifferential* of convex analysis [33]. That is, given any proper function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, a linear mapping $x^* : \mathbb{X} \to \mathbb{R}$ is called a *supergradient* of f at x, and we write $x^* \in \partial f(x)$, iff

$$f(\hat{x}) \leq f(x) + x^*(\hat{x} - x) \quad \forall \hat{x} \in \mathbb{X}.$$

Thus, $x^* \in \partial f(x)$ iff the affine function $f(x) + x^*(\cdot - x)$ globally overestimates $f(\cdot)$ but with no discrepancy at x. What comes next is a crucial characterization of shadow prices. For brevity declare $\mathbf{x} = (x_i) \in \mathbb{X}^I$ *an optimal allocation* iff $\sum_{i \in I} [x_i, \pi_i(x_i)] = [e_I, \pi_I(e_I)]$.

Theorem (Shadow prices as supergradients).
• λ is a *shadow price iff* $\lambda \in \partial \pi_I(e_I)$. *Thus, given the payoff functions, a shadow price depends only on the aggregate endowment* e_I.
• *For any* $\lambda \in \partial \pi_I(e_I)$ *and any optimal allocation* (x_i) *we have* $\lambda \in \partial \pi_i(x_i)$ *for all* i. *Conversely, if some* λ *belongs to all* $\partial \pi_i(x_i)$ *and* $\sum_i x_i = e_I$, *then* λ *is a shadow price, and allocation* (x_i) *is optimal.*
• *Suppose some* π_i *is monotone at a point* x_i *with respect to a cone* $\mathbb{X}_i \subseteq \mathbb{X}$ *in that* $\pi(x_i + \mathbb{X}_i) \geq \pi_i(x_i) > -\infty$. *Then* $\lambda \mathbb{X}_i \geq 0$ *for each shadow price* λ.

Proof. These assertions are well known when all π_i are concave; see e.g. the nice presentation is [25]. Here, however, concavity is not presumed. So, some extra work is needed. For simplicity define the "death" penalty $\delta(\cdot)$ on $\mathbb{X}$ by $\delta(x) = +\infty$ when $x \neq 0$ and $\delta(0) = 0$. Note that this function has Fenchel conjugate $\delta^*(x^*) := \sup_x \{x^* x - \delta(x)\} \equiv 0$. Now, $\lambda \in \partial \pi_I(e_I)$

$$\Leftrightarrow \sum_{i \in I} \pi_i(x_i) - \delta(\sum_{i \in I} x_i - x) \leq \pi_I(x) \leq \pi_I(e_I) + \lambda(x - e_I) \ \forall x \in \mathbb{X}, \forall (x_i) \in \mathbb{X}^I$$

$$\Leftrightarrow \sum_{i \in I} \pi_i(x_i) + \sum_{i \in I} \lambda(e_i - x_i) + \lambda(\sum_{i \in I} x_i - x) - \delta(\sum_{i \in I} x_i - x) \leq \pi_I(e_I) \ \forall x, \forall (x_i)$$

$$\Leftrightarrow \sum_{i \in I} \{\pi_i(x_i) + \lambda(e_i - x_i)\} + \delta^*(\lambda) \leq \pi_I(e_I) \quad \forall (x_i) \in \mathbb{X}^I \quad (*)$$

$$\Leftrightarrow \sup_{\mathbf{x}} L_I(\mathbf{x}, \lambda) \leq \pi_I(e_I).$$

This proves the first bullet. For the second let $(\tilde{x}_i)$ be any optimal allocation. In the above string of equivalences (*) says

$$\lambda \in \partial \pi_I(e_I) \Leftrightarrow \sum_{i \in I} \pi_i(x_i) \leq \sum_i \{\pi_i(\tilde{x}_i) + \lambda(x_i - \tilde{x}_i)\} \ \forall (x_i) \in \mathbb{X}^I$$

$$\Leftrightarrow \pi_i(x_i) \leq \pi_i(\tilde{x}_i) + \lambda(x_i - \tilde{x}_i) \ \forall x_i \in \mathbb{X}, \forall i \Leftrightarrow \lambda \in \partial \pi_i(\tilde{x}_i) \ \forall i.$$

For the last bullet, if $\lambda \hat{x}_i < 0$ at some $\hat{x}_i \in \mathbb{X}_i$, then

$$\pi_i^{(*)}(\lambda) \geq \sup_{r>0} \{\pi_i(x_i + r\hat{x}_i) - \lambda(x_i + r\hat{x}_i)\} = +\infty, \tag{4}$$

which is impossible. □

The instance with all π_i concave stands out, making π_I concave. Then, provided some term π_i be strictly concave, the optimal x_i, if any, must be unique. Moreover, if that same π_i is differentiable at x_i, the shadow price becomes unique as well. Generally, for any shadow price λ and optimal allocation (x_i), we get $x_i \in \partial(-\pi_i^{(*)})(\lambda)$ and $e_I \in \partial(-\pi_I^{(*)})(\lambda)$.

We emphasize that concavity of π_i or π_I is not essential. What imports is rather to have global support of π_I from above at e_I by some affine function. Such support cannot come about unless every optimal allocation (x_i) entails quite similar support of π_i at x_i. Thus, no agent having strictly convex payoff π_i could be admitted here. In fact, if π_i is supported from above as just described, it could not be globally convex unless affine with slope λ. These observations beg questions as to whether and when shadow prices do exist:

Proposition (Existence of shadow prices). *Let $\mathbb{X}$ be a locally convex Hausdorff topological vector space. Denote by $\hat{\pi}_I : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ the smallest concave function that dominates π_I from above. Suppose*

$$\hat{\pi}_I(\cdot) \text{ is finite-valued, bounded below near } e_I. \tag{5}$$

Also suppose that the convoluted preference is convex at e_I, meaning that $\hat{\pi}_I(e_I) = \pi_I(e_I)$. Then there exists at least one shadow price shadow price. Moreover that price is continuous.

Proof. Qualification (5) ensures that the concave function $\hat{\pi}_I(\cdot)$ is super-differentiable at e_I. That is, $\partial\hat{\pi}_I(e_I)$ is non-empty, and it can be taken to consists of only continuous linear functionals $x^* : \mathbb{X} \to \mathbb{R}$; see [15]. Now, $\hat{\pi}_I \geq \pi_I$ and $\hat{\pi}_I(e_I) = \pi_I(e_I)$ implies $\partial\hat{\pi}_I(e_I) \subseteq \partial\pi_I(e_I)$. The desired conclusion follows straightforwardly by noting that any supergradient $\lambda \in \partial\pi_I(e_I)$ is a shadow price - as pointed out in the preceding theorem. □

Thus arbitrage-free pricing obtains if an affine function supports the convoluted payoff from above at that the aggregate endowment. Assumption (5) clarifies that individual payoffs really need not be convex. Rather, it suffices that π_I has appropriate curvature with respect to e_I. Like in [35] aggregative convexity is what counts in preferences - albeit here only at e_I. This point bears on the qualitative fact that having *many* and *small* agents may mitigate adverse effects of non-convex preferences

[3], [16], [18]. As in [23] heterogeneity can also help.[10]

When will *no* shadow price exist? Plainly, as brought out in the last theorem, none is available if $\inf_{x^*} \sup_{\mathbf{x}} L_I(\mathbf{x}, x^*) > \pi_I(e_I)$. Then, the *duality gap*

$$\Delta := \inf_{x^*} \sup_{\mathbf{x}} L_I(\mathbf{x}, x^*) - \pi_I(e_I)$$

equals the smallest overall budgetary *deficit* - or the minimal overspending - that could possibly emerge by paying players according to formula (3). A positive gap might stem from some payoff function not being concave. Present many small players, each preferably having a smooth payoff functions, one may show that Δ becomes relatively small; see [3], [16], [18]. In any case, apart from existence of shadow prices, it is natural to wonder whether an optimal allocation (x_i) is available for the grand coalition $C = I$.

Proposition (Existence of optimal allocations). *Let $\mathbb{X}$ be a reflexive Banach space. Suppose the upper-level set*

$$U(r) := \left\{ \mathbf{x} = (x_i) \in \mathbb{X}^I : \sum_{i \in I} \pi_i(x_i) \geq r, \sum_{i \in I} x_i = e_I \right\}$$

is bounded and weakly closed for every real $r < \pi_I(e_I)$. Then there exists an optimal allocation. In particular, if $(x_i) \mapsto \sum_i \pi_i(x_i)$ is quasi-concave upper semi-continuous, it suffices that each set $U(r)$ be bounded.

Proof. The closed convex hull of $U(r)$ is bounded whence weakly compact for $r < \pi_I(e_I)$. Then, by reflexivity, $U(r)$ itself is weakly compact. It follows that $\cap_r \{U(r) : r < \pi_I(e_I)\}$ must be non-empty. Any element $\mathbf{x}$ in that intersection solves problem (1) for the grand coalition. When $(x_i) \mapsto \sum_{i \in I} \pi_i(x_i)$ is quasi-concave upper semi-continuous, $U(r)$ becomes closed convex whence weakly closed. $\square$

Clearly, optimal allocations do not depend on the endowment distribution.

5. PRODUCTION GAMES

This section offers a brief - and dispensable - digression, meant to emphasize three features:

- first, sharing of production and profit also fits format (1);
- second, (1) might emerge as a *reduced* model; and
- third, it is often convenient to keep original data pretty much in original, raw form.

For these purposes regard each agent $i \in I$ here as a *producer* who obtains profit $f_i(z_i)$ from plan $z_i \in Z_i$ provided $g_i(z_i) \leq e_i$. The set Z_i may lack exploitable structure, and $g_i : Z_i \to \mathbb{X}$ accounts for technological restrictions or material bounds. The linear

[10] For instance, let π_i be "concave" but defined on a disconnected domain D_i. Specifically, take $I = \{0, 1\}$, $D_0 = \{0\} \cup [1/2, 1]$, and $D_1 = [0, 1/2] \cup \{1\}$. Then π_I is concave on its domain $[0, 2]$.

space $\mathbb{X}$ is now ordered by a convex cone $\mathbb{X}_+ \subset \mathbb{X}$ in that $x \leq x' \Leftrightarrow x' - x \in \mathbb{X}_+$. Corresponding to (1) consider the planning problem

$$\pi_C(e_C) := \sup \left\{ \sum_{i \in C} f_i(z_i) : z_i \in Z_i \text{ and } \sum_{i \in C} g_i(z_i) \leq \sum_{i \in C} e_i \right\} \tag{6}$$

of coalition $C \subseteq I$. Its members share not only resources, but technologies as well. Upon setting $\pi_i(x_i) := \sup \{ f_i(z_i) : z_i \in Z_i \text{ and } g_i(z_i) \leq x_i \}$, format (1) comes up again as a reduced model. There is no need however, to synthesize the characteristic function $C \mapsto \pi_C(e_C)$. Computation could merely revolve around $\pi_I(e_I)$ - with all data kept in original form. This is seen next.

When $z_i \in Z_i$, and the linear functional $x^* : \mathbb{X} \to \mathbb{R}$ is non-negative on $\mathbb{X}_+$, let $z = (z_i)$, and associate to (6) the Lagrangian

$$\mathcal{L}_C(z, x^*) := \sum_{i \in C} \{ f_i(z_i) + x^* [e_i - g_i(z_i)] \} .$$

Write here

$$u_i(x^*) := \sup \{ f_i(z_i) - x^* g_i(z_i) : z_i \in Z_i \} + x^* e_i \tag{7}$$

and note that $\sup_z \mathcal{L}_C(z, x^*) = \sum_{i \in C} u_i(x^*)$. Arguing verbatim as for the first theorem we get

Proposition (Shadow prices support core solutions in production games). *Let λ be a shadow price in that $\pi_I(e_I) \geq \sup_z \mathcal{L}_I(z, \lambda)$. Then, paying agent i the amount* (7) *constitutes a core solution of the TU game that has* (6) *as characteristic function.* $\square$

6. ARBITRAGE-FREE, RISK-NEUTRAL PRICING

It is time now to specify a more detailed setting and seek some structure in optimal allocations. More details are available in two ways. First, the space $\mathbb{X}$ should be specified more closely; second, one might reasonable suppose some separability in the objectives across stages or states.

We begin with $\mathbb{X}$. Fix hereafter a non-empty *state space* S, equipped with a complete sigma-field $\mathcal{F}$ and a finite non-negative measure μ.[11] From here on each $x \in \mathbb{X}$ is at least a $\mathcal{F}$-measurable mapping from S into a finite-dimensional Euclidean space $\mathbb{E}$. The latter is endowed with inner product $e \cdot e'$, associated norm $|\cdot|$, and the Borel sigma-field in $\mathbb{E}$ is generated by the open sets.[12] Fix some number $p \in [1, +\infty)$ and suppose

$$\|x\| := \left(\int |x(s)|^p \, \mu(ds) \right)^{1/p} < +\infty$$

[11] When computation is a main concern, one would typically choose S finite, let σ contain all subsets of S, and have $\mu(s) > 0 \; \forall s$. Some convenience or flexibility comes with not insisting on $\mu(S) = 1$.

[12] More generally, $\mathbb{E}$ could be a separable Hilbert space.

for all $x \in \mathbb{X}$. Thus $\mathbb{X}$ is contained in the space $\mathbb{L}^p$ of all $\mathcal{F}$-measurable, p-integrable $x : S \to \mathbb{E}$. Risk or security markets are chief cases - and often incomplete. $\mathbb{X}$ may therefore be a strict, but presumably closed subset of $\mathbb{L}^p$.

Define the conjugate exponent $p^* \in [1, +\infty]$ implicitly by $\frac{1}{p} + \frac{1}{p^*} = 1$. A theorem of Riez says that any continuous linear functional x^* on $\mathbb{X}$ admits a *representation*

$$x \mapsto x^*x := \int x^*(s) \cdot x(s)\mu(ds) \tag{8}$$

for an (almost surely) unique $x^* \in \mathbb{X}^* \supseteq \mathbb{L}^{p^*}$. It is convenient to identify any such functional x^* with its Riez representation. The instance $p = 2$ stands out with $p^* = 2$ because $\mathbb{X} = \mathbb{X}^*$ becomes Hilbert with inner product (8).[13]

The present setting may naturally be construed as reflecting uncertainty about the true state $s \in S$, known ex ante only up to a probability measure μ on $\mathcal{F}$. Any $x \in \mathbb{X}$ is then a random vector $x(\cdot) \in \mathbb{E}$ and accordingly referred to as a *risk*.[14] As said, $\mathbb{X}$ should contain the already given endowments $e_i, i \in I$, and might - as a minimal requirement - even be spanned by these. Whilst insurance theory often assumes independent or weakly associated risks, no such assumption is made here.[15]

Recall that a shadow price λ is a linear functional from $\mathbb{X}$ into $\mathbb{R}$. While endogenous to the game, it helps players to evaluate various risks and securities. Clearly, unless λ blocks arbitrage it can't apply as price regime. That issue is briefly explored next.

For the statement, a cone $\mathbb{X}_i(x_i) \subset \mathbb{X}$ is said to comprise the *preferable directions of agent i at $x_i \in \mathbb{X}$* if $\pi_i(x_i + \mathbb{X}_i(x_i)) \geq \pi_i(x_i) > -\infty$. As usual, a linear price $x^* : \mathbb{X} \to \mathbb{R}$ is declared *arbitrage-free* iff no agent i has a preferable direction $d_i \in \mathbb{X}_i(x_i)$ at any $x_i \in dom\pi_i$ such that $x^*d_i < 0$. Arguing as around (4) we may state forthwith:

Proposition (Shadow prices are arbitrage-free). *Given cones $\mathbb{X}_i(x_i), i \in I$, of preferable directions, each shadow price λ must satisfy*

$$\lambda(\cup_{i\in I} \cup_{x_i \in dom\pi_i} \mathbb{X}_i(x_i)) \geq 0.$$

In particular, if $dom\pi_i$ itself is a cone $\mathbb{X}_i$, and $\pi_i(x_i + \mathbb{X}_i) \geq \pi_i(x_i)$ at each $x_i \in \mathbb{X}_i$, then $\lambda(\cup_{i\in I}\mathbb{X}_i) \geq 0$. $\square$

Arbitrage is a utility-free, more primitive concept than economic equilibrium. Typically, it is described in terms of a common family of financial instruments, monotone

[13] In particular, the conjugate pair $(p, p^*) = (1, +\infty)$ is possible, $\|x\|$ then being the *essential supremum* of $s \mapsto |x(s)|$. However, unless σ is finite, the "reciprocal" pair $(p, p^*) = (+\infty, 1)$ needs special care, and is not discussed here; see [17].

[14] Risks - alias random variables - are chief objects here. Our results extend however, to other contingent items.

[15] Consequently, we shall invoke no law of large numbers or central limit theorem. In fact, our analysis is applicable for major events, say catastrophes, inflicting severe and highly correlated losses.

preferences, and *one* punctuated convex cone $\mathbb{X}_+ \diagdown \{0\} \subset \mathbb{X}$, composed of *free lunches.* A theorem of alternatives then decides whether arbitrage is possible or not. Given a shadow price that decision is straightforward:

Proposition (Shadow prices preclude free lunches). *Let $x_i \in \mathbb{X}$ iff $x_i = e_i + Wz_i$ for some "portfolio" z_i in a real vector space $\mathbb{Z}$, with $W : \mathbb{Z} \to \mathbb{X}$ linear. Suppose at least one agent i has differentiable and strictly monotone preferences:*

$$\hat{x}_i - x_i \in \mathbb{X}_+ \diagdown \{0\} \Rightarrow \pi_i(\hat{x}_i) > \pi_i(x_i).$$

Then, existence of a shadow price λ, together with an optimal x_i, ensures that $\lambda [\mathbb{X}_+ \diagdown \{0\}] > 0$, $\lambda W = 0$, and there is no $z \in \mathbb{Z}$ such that $Wz \in \mathbb{X}_+ \diagdown \{0\}$ with $\lim_{r>0} \pi_i(e_i + rWz) = +\infty$.

Proof. Let $x_i = e_i + Wz_i$ be optimal for the agent who has strictly monotone, smooth preferences. Since λ is a shadow price, the chain rule gives

$$0 = \frac{\partial}{\partial z_i} \pi_i(e_i + Wz_i) = \lambda W.$$

Further, suppose a ticket $z \in \mathbb{Z}$ is variable for a free lunch $Wz \in \mathbb{X}_+ \diagdown \{0\}$, during which agent i is never satiated: $\lim_{r>0} \pi_i(e_i + rWz) = +\infty$. This implies the contradiction

$$\pi_i^{(*)}(\lambda) \geq \sup_{r>0} \{\pi_i(e_i + Wz) - \lambda(e_i + Wz)\} = +\infty. \quad \square$$

Example: A two-stage security market. Let $W = \begin{bmatrix} -z^* \\ D \end{bmatrix}$ where the price vector $z^* = (z_j^*) \in \mathbb{R}^J$ accounts for the up-front purchase cost of various papers $j \in J$, and the $S \times J$ matrix $D = [D_j(s)]$ reports future dividends. With $\mathbb{Z} = \mathbb{R}^J$, equation $\lambda W = 0$ and $\lambda > 0$ amount to the price rule

$$z_j^* = \delta \int D_j(s) p(ds) \quad \forall j, \tag{9}$$

featuring a deflator $\delta > 0$ alongside a risk-neutral probability measure p over $\mathcal{F}$; see [29] for S finite. The nature of rule (9) is best appreciated when uncertainty resolves over several stages. We turn to such instances next. $\square$

Quite often, identification of the true state s isn't immediate. At time $t \in \{0, 1, .., T\}$ agent i can only ascertain for each event in a sigma-field $\mathcal{F}_t \subseteq \mathcal{F}$ whether it has happened or not. His decision x_{it}, made then, must therefore be $\mathcal{F}_t$-measurable. In that case $\mathbb{X} = \mathbb{X}_0 \times \cdots \times \mathbb{X}_T$ where $\mathbb{X}_t$ is a space of $\mathcal{F}_t$-measurable mappings from S into a Euclidean space $\mathbb{E}_t$. Typically, the inclusions

$$\{\emptyset, S\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T = \mathcal{F} \tag{10}$$

hold; they represent progressive acquisition of knowledge.

Example: A multi-stage security market. Suppose S is finite, and let each field $\mathcal{F}_t$ be generated by a *partition* $\mathbb{P}_t$ of S. Then $x_t : S \to \mathbb{E}_t$ is $\mathcal{F}_t$-measurable iff constant on each part $P_t \in \mathbb{P}_t$. Posit $\mathbb{P}_0 = \{S\}$ and $\mathbb{P}_T = \{\{s\} : s \in S\}$. Regard $P_t \in \mathbb{P}_t$ as a *node* $n_t \in \mathcal{N}_t$ (at height t) in a *tree*, and draw a *directed branch* from n_t to its *child node* n_{t+1} iff $n_t = P_t \subseteq P_{t+1} = n_{t+1}$. Write $n_t \in \mathcal{A}(n_{t+1})$ and $n_{t+1} \in \mathcal{C}(n_t)$ to signal that the first node is an *ancestor* and the latter a *child*. Node n_0 is named the *root*, and each terminal node - having maximal height T - is called a *leaf*; see figure below.

$$\text{root } n_0 = \begin{bmatrix} s \\ s' \\ s'' \end{bmatrix} \begin{matrix} \nearrow \\ \\ \searrow \end{matrix} \begin{matrix} \begin{bmatrix} s \\ s' \end{bmatrix} \\ [s''] \end{matrix} \begin{matrix} \nearrow \\ \rightarrow \\ \rightarrow \end{matrix} \left.\begin{matrix} [s] \\ [s'] \\ [s''] \end{matrix}\right\} \text{leafs}$$

Legend: A tree with 3 partitions/stages/states/scenarios and 6 nodes.

Denote by $z_{jn} \in \mathbb{R}$ the number of shares an investor holds in paper $j \in J$ upon leaving node n. Suppose he buys (outgoing) *portfolio* $z_n := (z_{jn}) \in \mathbb{R}^J$ at node $n \neq n_0$ and liquidates there the (incoming) portfolio $z_{\mathcal{A}(n)}$ bought at the ancestor node. Absent transaction costs, those operations bring him nominal, current *gain* $G_n(z) := z_n^* \cdot \left[z_{\mathcal{A}(n)} - z_n \right]$. (The dot denotes the standard inner product.) At the root node n_0 naturally let $G_{n_0}(z) := -z_{n_0}^* \cdot z_{n_0}$. This stylized market allows *arbitrage* iff the system

$$G_n(z) \geq 0 \textit{ for all } n \textit{ and } \; z_n^* \cdot z_n \geq 0 \; \textit{ for each leaf}, \tag{11}$$

admits a solution $z = (z_n)$ with at least one strict inequality. Suppose some paper (say a bond) $b \in J$ commands strictly positive price z_{nb}^* at each node n. In terms of that paper define discount factors $\delta_n := z_{n_0 b}^* / z_{nb}^*$. Let $\mathcal{N} := \cup_t \mathcal{N}_t$ denote the node set.

Proposition (Shadow prices and risk-neutral probabilities). *The described market, featuring many stages, is arbitrage-free iff there exists a strictly positive probability measure p across the leafs such that the transition probabilities, induced by p on the entire node set, satisfy the martingale condition*

$$\delta_n z_n^* = E_\mu \left[\delta_c z_c^* \middle| \, n \right] = \sum_{c \in \mathcal{C}(n)} \delta_c z_c^* p(c \,|n) \; \textit{ for all non-terminal } n. \tag{12}$$

Under the hypotheses of the preceding proposition any shadow price ensures absence of arbitrage.

Proof. The first part is well known but proven for completeness. Fix *any* non-degenerate probability measure $m > 0$ across the leafs, and use the induced probabilities m_n at non-terminal nodes n. Consider the homogeneous linear program

$$\max_z \; \sum_n \delta_n m_n G_n(z) + \sum_{n \in \mathcal{N}_T} \delta_n m_n z_n^* \cdot z_n \;\text{ s.t. (11)}. \tag{13}$$

Clearly, the market is arbitrage-free iff the optimal value of (13) is 0. Associate multiplier $\delta_n y_n \geq 0$ to inequality $G_n(z) \geq 0$, and $\delta_n Y_n \geq 0$ to leaf constraint $z_n^* \cdot z_n \geq 0$. Maximizing the resulting Lagrangian

$$\sum_n \delta_n(m_n + y_n)G_n(z) + \sum_{n\in\mathcal{N}_T} \delta_n(m_n + Y_n)z_n^* \cdot z_n =$$

$$\sum_{n\notin\mathcal{N}_T}\left[\sum_{c\in\mathcal{C}(n)} \delta_c(m_c + y_c)z_c^* - \delta_n(m_n + y_n)z_n^*\right] \cdot z_n + \sum_{n\in\mathcal{N}_T} \delta_n(Y_n - y_n)z_n^* \cdot z_n \qquad (14)$$

with respect to the free variable z we see that the dual of (13) amounts to solve

$$\delta_n(m_n + y_n)z_n^* = \sum_{c\in\mathcal{C}(n)} \delta_c(m_c + y_c)z_c^* \text{ for all } n \notin \mathcal{N}_T \text{ with } y \geq 0.$$

Suppose the latter system is indeed solvable. In that case, by LP duality, problem (13) has 0 as optimal value, and there are no arbitrage opportunities. Then consider component b of the last equation to get $m_n + y_n = \sum_{c\in\mathcal{C}(n)}(m_c + y_c)$. Therefore $m(c\,|n) := (m_c + y_c)/(m_n + y_n)$ defines strictly positive transition probabilities that satisfy (12).

Conversely, suppose some strictly positive measure m on $\mathcal{N}_T$ suits (12). In (14) let $m = p$ and each $y_n, Y_n = 0$ to get

$$\sum_n \delta_n G_n(z)p_n + \sum_{n\in\mathcal{N}_T} \delta_n z_n^* \cdot z_n p_n = \sum_{n\notin\mathcal{N}_T}\left[\sum_{c\in\mathcal{C}(n)} \delta_c p_c z_c^* - \delta_n p_n z_n^*\right] \cdot z_n = 0$$

for all z. Thus arbitrage is impossible.

For the final assertion, let $\mathbb{X} = \mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{\mathcal{N}_T}$ with the customary non-negative orthant $\mathbb{X}_+$. Posit $\mathbb{Z} := \mathbb{R}^{J\times\mathcal{N}}$, and define the linear operator $W : \mathbb{Z} \to \mathbb{X}$ by

$$Wz = \left[[G_n(z)]_{n\in\mathcal{N}}, [z_n^* z_n]_{n\in\mathcal{N}_T}\right].$$

Absence of arbitrage means that no $z \in \mathbb{Z}$ yields $Wz \in \mathbb{X}_+\diagdown 0$. By the preceding proposition there exists a positive λ such that $\lambda W = 0$. Choose $y_n \geq 0$ to have $\lambda_n = \delta_n(\mu_n + y_n)$ for $n \in \mathcal{N}$ and posit $Y_n = y_n$ at leaf n. Consequently, (14) becomes feasible. $\square$

Example: Two-stage risk-neutral pricing. If available up front, how much is the risk-free asset worth that offers guaranteed future dividend 1? As seen next λ complies with the well known risk-neutral, arbitrage-free evaluation:

Suppose there are merely two stages with $\{\emptyset, S\} = \mathcal{F}_0 \subset \mathcal{F}_1 = \mathcal{F}$ and only one commodity ($\mathbb{E} = \mathbb{R}$). Given a shadow price $\lambda \gneqq 0$ a.s., suppose the system

$$b := (b_0, b_1) = (-\delta, 1) \quad \textit{and} \ \lambda b = 0,$$

is solvable for some riskless bond $b \in \mathbb{X}$ together with a unique discount factor $\delta > 0$. Then, $\delta = \int_{\mathbb{S}} \lambda(s)\mu(ds)/\lambda(0)$, and the measure

$$p(A) := \int_A \lambda(s)\mu(ds) / \int_S \lambda(s)\mu(ds)$$

defines a risk-neutral probability p over $\mathcal{F}$ that satisfies $-x(0) = \delta \int_S x(s)p(ds)$ *for each* $x \in \mathbb{X}$ *such that* $\lambda x = 0$.

7. Risk Sharing

We stress that states can sometimes be seen not as "*events*" but alternatively as "*stages*" or decision epochs.[16] The measure μ then discounts the future. More generally, the description of any specific state refers to the circumstances that defines its appearance. This more broad perspective justifies speaking of any $x \in \mathbb{X}$ as a *contingent commodity bundle* in $\mathbb{E}$.

In either set-up sharing, as captured by (1), takes the form of a *contract*, specifying agent i's part $x_i(s)$ of $e_I(s)$, and his payment, in state s. A natural question is whether and when the concerned parties think the writing of such contracts worth their while. Instead of committing to a promise or policy right away, why not just wait and see?

Clearly, what explains and justifies the existence of insurance institutions is the temporal resolution of uncertainty - and the time windows that affect some decisions. Intuitively, if the restriction $x \in \mathbb{X}$ does not preclude that $x(s)$ be fully adapted to the realized state s ex post - and moreover, agents agree on probabilities - then contracts seem superfluous. This exceptional setting is briefly explored next.

Following [33] declare $\mathbb{X}$ *decomposable* iff for each $x \in \mathbb{X}$ the modified mapping

$$\mathbf{1}_B\beta + \mathbf{1}_{S \setminus B}x := \begin{cases} \beta & \text{if } s \in B \\ x & \text{otherwise} \end{cases}$$

belongs to $\mathbb{X}$ whenever the bounded $\beta : S \to \mathbb{E}$ is measurable, and $B \in \mathcal{F}$. Further, call an integrand $\Pi : S \times \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ *normal* if the point-to-set correspondence $s \mapsto \{(e, r) \in \mathbb{E} \times \mathbb{R} : \Pi(s, e) \geq r\}$ is measurable [33].

Decomposability is demanding. For instance, when S is finite, and $\mathcal{F}$ contains all singletons, a decomposable $\mathbb{X}$ must generate a *complete market space.* That is, seen as space of marketable assets, a decomposable $\mathbb{X}$ contains all elementary Arrow-Debreu securities. Also, if $\mathcal{F}_t \subsetneq \mathcal{F}$ for some $t < T$ in (10), choose a bounded $\mathcal{F}$-measurable $\beta_t : S \to \mathbb{E}_t$ which is *not* $\mathcal{F}_t$-measurable. Posit $\beta_\tau \equiv 0$ for $\tau \neq t$, and $B = S$ to have $\mathbf{1}_B\beta + \mathbf{1}_{S \setminus B}x = \beta \notin \mathbb{X}$.

In short, decomposability doesn't fit settings where information unfolds gradually. Despite their lack of realism, the extreme properties of decomposable instances shed some light on insurance:

[16] Examples include exchange of time-dependent property rights, say fish quotas or pollution permits.

Proposition (Sharing ex post, on the spot). *Suppose $\mathbb{X}$ is decomposable. For each $i \in I$, let*

$$\pi_i(x_i) = \int \Pi_i(s, x_i(s))\mu(ds), \tag{15}$$

featuring a normal integrand $\Pi_i : S \times \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$ and a common measure μ. If λ is a shadow price for the overall game, then almost surely so is $\lambda(s)$ for the ex post, contingent game that emerges in state s, with characteristic function

$$C \mapsto \Pi_C(s, e_C(s)) := \sup \left\{ \sum_{i \in C} \Pi_i(s, x_i(s)) : \sum_{i \in C} x_i(s) = e_C(s) \right\}.$$

Invoking the contingent conjugate $\Pi_i^{(*)}(s, \cdot)$, that ex post game admits a payoff distribution

$$i \mapsto U_i(s, \lambda(s)) := \Pi_i^{(*)}(s, \lambda(s)) + \lambda(s) \cdot e_i(s)$$

which belongs to its core. Further,

$$i \mapsto u_i(\lambda) = \int U_i(s, \lambda(s))\mu(ds) = \int \left\{ \Pi_i^{(*)}(s, \lambda(s)) + \lambda(s) \cdot e_i(s) \right\} \mu(ds).$$

belongs to the ex ante core.

Proof. Since objectives are separable across states, for any $\boldsymbol{\chi} = (\chi_i) \in \mathbb{E}^I$ and $\chi^* \in \mathbb{E}$ coalition C has ex post Lagrangian

$$\mathbf{L}_C(s, \boldsymbol{\chi}, \chi^*) := \sum_{i \in C} \Pi_i(s, \chi_i) + \chi^* \cdot \left[\sum_{i \in C} e_i(s) - \sum_{i \in C} \chi_i \right]$$

in state s. Ex ante it holds $L_C(x, x^*) = \int \mathbf{L}_C(s, x(s), x^*(s))\mu(ds)$. Now, by decomposition,

$$\pi_I(e_I) = \sup_x L_I(x, \lambda) \iff \Pi_I(s, e_I(s)) = \sup_{x(s)} \mathbf{L}_I(s, x(s), \lambda(s)) \text{ almost surely;}$$

see [33]; Theorem 11.40. Thus λ is a shadow price iff almost surely $\Pi_I(s, e_I(s))$ equals $\sup_{x(s)} \mathbf{L}_I(s, x(s), \lambda(s))$. Hence it equals the saddle value of $\mathbf{L}_I(s, \cdot, \cdot)$. From here on the argument goes as before. $\square$

Granted a decomposable space $\mathbb{X}$ and normal format (15), sharing may almost surely be done ex post. If moreover, the integrands are state-independent of the form $\Pi_i : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$, the state-$s$ shadow price $\lambda(s)$ depends only on the realized aggregate $e_I(s)$.

As said, the preceding proposition should not lure one into thinking that ex ante contracts are superfluous. Casual observation indicates the opposite. So, decomposability is a rare property. Most often some component of x_i must be decided before uncertainty resolves - and stays non-maleable ex post. A simple example is fire insurance: The premium paid up front cannot be altered after the event.[17]

It may happen of course, that there is only *one* stage. Such a setting allows us to consider instances where players perceive uncertainty diversely. Often probability assessments differ across agents - and typically much on exceptionally important states, occurring with very low frequencies. Nonetheless, there are prospects for risk sharing - implemented by contracts signed ex ante. Arguing as in the preceding proposition we get:

Corollary (Diverse probability assessments). *For each $i \in I$, suppose $\pi_i(x_i) = \int \Pi_i(s, x_i(s))\mu_i(ds)$ where Π_i is a normal integrand, μ_i is absolutely continuous with respect to μ, and $\mathbb{X}$ is decomposable. Let $\varphi_i = \frac{d}{d\mu}\mu_i$ be the corresponding density. Then, λ is shadow price iff almost surely*

$$\sup_{(\chi_i)} \left\{ \sum_{i \in I} \Pi_i(s, \chi_i)\varphi_i(s) : \sum_{i \in I} \chi_i = e_I(s) \right\} \geq \sum_{i \in I} U_i(s, \lambda(s)) \tag{16}$$

where

$$U_i(s, \lambda(s)) := \varphi_i(s)\Pi_i^{(*)}(s, \frac{\lambda(s)}{\varphi_i(s)}) + \lambda(s) \cdot e_i(s) \text{and } u_i(\lambda) = \int U_i(s, \lambda(s))\mu(ds).$$

In case allocation (x_i) is optimal, $\lambda(\cdot)$ is a shadow price iff, for each i,

$$\lambda(s) \in \frac{\partial}{\partial \chi_i}\Pi_i(s, x_i(s))\varphi_i(s) \ \text{ a.s.} \quad \Box \tag{17}$$

(17) dates back to [8], [36]. The Corollary shows that players who hold different (but absolutely continuous) beliefs cannot implement the overall contract ex post unless their realized payoffs be scaled by respective densities. If someone believes a particular state more likely, its realization should benefit him ex post. The rarity or non-practicality of decomposable spaces, indicates that one hardly have a realistic theory of syndicates unless members commit themselves up front.

In general, shadow prices depend on all underlying data. Also, by belonging to $\mathbb{X}$, any shadow price is a mapping $s \in S \longmapsto \lambda(s) \in \mathbb{E}$. This raises the question whether s affects $\lambda(s)$ merely via $e_I(s)$? If so, $\lambda(\cdot)$ should be measurable with respect to the sigma-field $\mathcal{F}(e_I)$ generated by e_I.[18] In that case, for simplicity, declare λ *adapted.* At this juncture the implicit function theorem immediately yields:

Proposition (Dependence on the aggregate endowment). *Let $(\hat{x}_i)$ be an optimal allocation for some aggregate endowment $\hat{e}_I$. Suppose each π_i is twice continuously*

[17] Along the same line, if knowledge is asymmetric, players may, for the sake of verifiability and ex post implementation, have to contend with contracts that differ in measurability; see [24].

[18] $\sigma(e_I)$ is the smallest sigma-field with respect to which e_I is measurable.

differentiable near $\hat{x}_i$ with $\pi_i''(\hat{x}_i)$ non-singular. Then, in some neighborhood of $\hat{e}_I$, the system

$$\pi_i'(x_i) = \lambda \ \textit{ for all } \ i \in I \ \textit{ and } \ \sum_{i \in I} x_i = e_I$$

admits continuous solutions $e_I \mapsto x_i(e_I) \in \mathbb{X}, i \in I$, and $e_I \mapsto \lambda(e_I) \in \mathbb{X}$. In particular, if $\pi_i(x_i) = \int \Pi_i(x_i(s))\mu(ds)$ with Π_i twice continuously differentiable near $\hat{x}_i(s)$ and $\Pi_i''(\hat{x}_i(s))$ non-singular, then, in some neighborhood of $\hat{e}_I(s)$, the system

$$\Pi_i'(x_i) = \lambda \ \textit{ for all } \ i \in I \ \textit{ and } \ \sum_{i \in I} x_i = e_I(s)$$

admits continuous solutions $e_I(s) \mapsto x_i(e_I(s)) \in \mathbb{E}, i \in I$, and $e_I(s) \mapsto \lambda(e_I(s)) \in \mathbb{E}$ becomes adapted. $\Box$

Since individual payoffs need not be concave, parts of the analysis proceeds without assuming risk aversion. To illustrate, we emphasize next one advantage and consequence of having adapted shadow prices. When μ is a probability measure, write $\mathcal{E}$ for the expectation operator.

Proposition. (Mean-preserving shifts are undesirable). *Let μ here be a probability measure. Suppose shadow price λ is adapted and allocation (x_i) is optimal.*
- *If $\Delta e_I \in \mathbb{X}$, satisfying $\mathcal{E}\left[\Delta e_I | \, e_I\right] = 0$, is added to e_I, then $\pi_I(e_I + \Delta e_I) \leq \pi_I(e_I)$.*
- *Similarly, if $\Delta x_i \in \mathbb{X}$ with $\mathcal{E}\left[\Delta x_i | \, e_I\right] = 0$ be added to x_i, then $\pi_i(x_i + \Delta x_i) \leq \pi_i(x_i)$.*

Proof. The subgradient inequality yields $\pi_I(e_I + \Delta e_I) \leq \pi_I(e_I) + \lambda \Delta e_I$. However, since λ depends merely on e_I,

$$\lambda \Delta e_I = \mathcal{E}(\lambda \cdot \Delta e_I) = \mathcal{E}(\mathcal{E}\left[\lambda \cdot \Delta e_I \,| e_I\right]) = \mathcal{E}(\lambda \cdot \mathcal{E}\left[\Delta e_I \,| e_I\right]) = 0.$$

The second assertion is proven in the same manner. $\Box$

The last two bullets required no risk aversion, only the availability of an adapted shadow price. Also note that, so far, no properties were required of π_i. We still want to avoid separability, be it over time or events.[19] On that account, with $\mathbb{X}$ Hilbert, it turns out that a generalized form of variance aversion is expedient.

[19] The finance/insurance literature mostly considers additive, concave, state-independent, smooth payoff functions of the customary von Neumann-Morgenstern sort. That optic - apart from smoothness - appears reasonable for low-consequence, conventional risks such as minor damage on cars or theft of bicycles. It need not, however, fit major events like severe illness or catastrophes. Admittedly, the use of expected payoffs is best justified under repeated realizations, these allowing probabilities to be estimated from observed data. Nothing precludes though, that mutual insurance company be set up to protect its members against rare events the "statistics" of which merely reflect expert judgements.

Lemma (Generalized variance aversion). *Consider any inner product xx' on a Hilbert space $\mathbb{X}$ with associated norm $\|\cdot\|$. Suppose a function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ and a subset $X^* \subset \mathbb{X}$ are such that*

$$x^*\tilde{x} = x^*x \ \forall x^* \in X^* \text{and } \ \|\tilde{x}\| < \|x\| \ \textit{ implies } \ f(x + r(\tilde{x} - x)) > f(x) \textit{ for some real } r.$$

Then, any solution x to (2) *belongs to the closed linear subspace of $\mathbb{X}$ spanned by λ and X^*.*

Proof. Let $\tilde{x}$ denote the orthogonal projection of x onto the said subspace. Thus $x^*\tilde{x} = x^*x$ for all $x^* \in X^*$. Suppose $\tilde{x} \neq x$. Then, because $\tilde{x}(x - \tilde{x}) = 0$ and $\|x - \tilde{x}\|^2 > 0$,

$$\|x\|^2 = \|\tilde{x} + x - \tilde{x}\|^2 = \|\tilde{x}\|^2 + 2\tilde{x}(x - \tilde{x}) + \|x - \tilde{x}\|^2 > \|\tilde{x}\|^2 ;$$

that is, $\|\tilde{x}\| < \|x\|$ and thereby $f(\hat{x}) > f(x)$ with $\hat{x} := x + r(\tilde{x} - x)$ for some real r. However, because $\lambda\hat{x} = \lambda x$, it holds

$$f(\hat{x}) - \lambda\hat{x} > f(x) - \langle \lambda, x \rangle ,$$

an inequality which contradicts the maximality of x in (2). The upshot is that $\tilde{x} = x$, and the conclusion follows. □

Proposition (Variance aversion). *Let the commodity space $\mathbb{E}$ be $\mathbb{R}^G$ for a finite set G of economic goods. Suppose each vector $\mathbf{1}_g \in \mathbb{R}^G, g \in G$, having 1 in component g and 0 elsewhere, belongs to $\mathbb{X} \subseteq \mathbb{L}^2$. Also suppose a function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ is such that*

$$\mathcal{E}\tilde{x} = \mathcal{E}x \ \textit{ and } \ var(\tilde{x}) < var(x) \textit{ implies } \ f(x + r(\tilde{x} - x)) > f(x) \textit{ for some real } r.$$

Then, any solution x to (2) *belongs to the linear subspace of $\mathbb{X}$ spanned by λ and $\{\mathbf{1}_g : g \in G\}$.*

Proof. Use inner product (8). Further, letting $X^* := \{\mathbf{1}_g : g \in G\}$ one sees that $\mathcal{E}\tilde{x} = \mathcal{E}x \Leftrightarrow x^*\tilde{x} = x^*x \ \forall x^* \in X^*$. Clearly, when $\mathcal{E}\tilde{x} = \mathcal{E}x$, it holds $var(\tilde{x}) < var(x) \Leftrightarrow \|\tilde{x}\| < \|x\|$. Now invoke the preceding lemma to conclude. □

As above, declare $x \in \mathbb{X}$ *adapted* if measurable with respect to the sigma-field $\mathcal{F}(e_I)$ generated by e_I.

Proposition (Variance aversion and two-fund separation). *Suppose there exists a shadow price and that e_I is not constant. Let the commodity space $\mathbb{E}$ be $\mathbb{R}^G$ for a finite set G of economic goods. Suppose each $\mathbf{1}_g \in \mathbb{X} \subseteq \mathbb{L}^2$ and that every payoff function $\pi_i : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ satisfies*

$$\mathcal{E}\tilde{x} = \mathcal{E}x \quad \textit{and } var(\tilde{x}) < var(x) \textit{ implies } \ \pi_i(x + r(\tilde{x} - x)) > \pi_i(x) \textit{ for some real } r. \tag{18}$$

Then, any optimal allocation (x_i) *to game* (1) *is adapted and of the form*

$$x_i = \mathbf{r}_i + \varepsilon_i e_I$$

with unique non-random vectors $\mathbf{r}_i \in \mathbb{R}^G$ *and coefficients* ε_i *that satisfy* $\sum_{i\in I} \mathbf{r}_i = 0$ *and* $\sum_{i\in I} \varepsilon_i = 1$.

Proof. Fix any shadow price λ. From the preceding proposition $x_i \in \mathbb{V} := span\left\{\mathbb{R}^G, \lambda\right\}$ hence $e_I = \sum_{i\in I} x_i \in \mathbb{V}$. Because e_I isn't constant, $\mathbb{V} = span\left\{\mathbb{R}^G, e_I\right\}$, and the vectors $\mathbf{1}_g, g \in G$, e_I form a basis of $\mathbb{V}$. The conclusion is now immediate. $\square$

Proposition (Risk aversion and contingent two-fund separation). *Suppose* $\mathbb{X} \subseteq \mathbb{L}^2$. *Let a core allocation* (x_i) *be supported by a shadow price and suppose agent* i *has payoff of the form* $\pi_i(x_i) = \mathcal{E}\Pi_i(x_i)$ *with concave integrands* $\Pi_i : \mathbb{E} \to \mathbb{R} \cup \{-\infty\}$. *Then we may assume* x_i *adapted and there exist adapted* $\mathbf{r}_i \in \mathbb{X}$ *and* $\varepsilon_i \in \mathbb{R}$ *such that*

$$x_i = \mathbf{r}_i + \varepsilon_i e_I, \;\; \mathbf{r}_i(s) \cdot e_I(s) = 0 \text{ a.s.}$$

If all agents are of the described sort, $\sum_{i\in I} \mathbf{r}_i = 0$, *and* $\sum_{i\in I} \varepsilon_i = 1$.

Proof. Denote by λ a shadow price that supports core allocation (x_i). Introduce the conditional expectation $\tilde{x}_i := \mathcal{E}\left[x_i \,| e_I\right]$ to have $\tilde{x}_i$ adapted. Since λ is adapted, we get $\lambda x_i = \lambda \tilde{x}_i$. Finally, Jensen's inequality yields $\pi_i(x_i) \le \pi_i(\tilde{x}_i)$. This takes care of the first assertion. Further, on any atom in $\mathcal{F}(e_I)$ project x_i orthogonally onto $\mathbb{R}e_I$ to get a unique component $\varepsilon_i e_I$ along that line, and let $\mathbf{r}_i$ be the residual. Since $\sum_i x_i = e_I$, the conclusion follows. $\square$

8. PRICE CURVES AND RISK TOLERANCE

The correspondence $e_I \mapsto \lambda(e_I)$ from aggregate endowment to shadow price may naturally be seen as a *price curve*. As such its should "slope downwards":

Proposition (The law of demand). *Shadow prices comply with the* ***law of demand*** *in that*

$$(\lambda - \hat{\lambda})(e - \hat{e}) \le 0 \tag{19}$$

whenever $\lambda \in \partial\pi_I(e)$ *and* $\hat{\lambda} \in \partial\pi_I(\hat{e})$.

Proof. $\lambda \in \partial\pi_I(e)$ implies $\pi_I(\hat{e}) \le \pi_I(e) + \lambda(\hat{e} - e)$. Similarly, $\hat{\lambda} \in \partial\pi_I(e)$ implies $\pi_I(e) \le \pi_I(\hat{e}) + \hat{\lambda}(e - \hat{e})$. Addition of the last two inequalities gives (19). $\square$

As customary, given a price curve, its slope is of chief importance.

Proposition (The slope of the price curve). *Let* (x_i) *be an optimal allocation supported by a shadow price* λ. *For each* i, *suppose* π_i *is concave with a second Fréchet* *derivative near x_i which is continuous and non-singular at x_i.Then $\pi_I^{(*)}$ is twice Fréchet differentiable at λ with*

$$\pi_I^{(*)\prime\prime}(\lambda) = -\sum_{i \in I} \pi_i''(x_i)^{-1}. \tag{20}$$

In addition, if for each i, π_i'' is continuous near x_i, then $(\pi_I^{(*)})''$ is continuous near λ. If moreover, $(\pi_I^*)''$ is *non-singular at λ, the market curve has slope*

$$\lambda'(e_I) = \pi_I{}''(\sum_{i \in I} x_i) = \left[\sum_{i \in I} \pi_i''(x_i)^{-1}\right]^{-1}. \tag{21}$$

Under these conditions individual demand $x_i = x_i(e_I)$ is differentiable and

$$x_i' = \pi_i''(x_i)^{-1} \pi_I{}''(x_I) \text{ where } x_I := \sum_{i \in I} x_i \tag{22}$$

Proof. We use the following result on inversion [12]: *If $x^* = f'(x)$ with f concave, twice Fréchet differentiable near x and $f''(x)$ non-singular, it holds*

$$f^{(*)\prime\prime}(x^*) = -f''(x)^{-1}$$

with $f^{(*)\prime\prime}$ continuous near x^* when f'' is continuous near x. Here, since $\pi_I^{(*)} = \sum_{i \in I} \pi_i^{(*)}$, we get $\pi_I^{(*)\prime\prime} = \sum_{i \in I} \pi_i^{(*)\prime\prime}$ whenever the last sum is well defined. Thus (20) follows. Invoking the above inversion result once again, $\pi_I^{(*)\prime\prime}(\lambda) = -\pi_I''(e_I)^{-1}$ and (21) obtains. Finally, (22) is a direct consequence of differentiating $\pi_i'(x_i) = \pi_I'(e_I)$. $\square$

For a function $\pi : \mathbb{R} \to \mathbb{R}$ Pratt [32] described *risk aversion* as twice the premium per unit of infinitesimal variance. Provided π be sufficiently smooth at x, with $\pi'(x) \neq 0$, the said premium $A_\pi(x) := -\pi''(x)/\pi'(x)$ is called the *absolute risk aversion* at x. The reciprocal entity $T_\pi(x) := -\pi''(x)^{-1}\pi'(x)$, called *risk tolerance,* is thus half the tolerable variance per unit of compensating premium [36]. The latter entity is often more amenable to handle. Here however, payoff functions may be defined on higher-dimensional spaces. Accordingly, a multi-dimensional version of risk tolerance.

When there is only one good ($\mathbb{E} = \mathbb{R}$), differential equation (22) amounts to

$$x_i' = T_i(x_i)/T_I(x_I) \tag{23}$$

where $T_i := T_{\pi_i}$ and $T_I := T_{\pi_I}$. Solutions to (23) have been studied in [9], [10], [26], [37].

Definition (Risk tolerance). *For any function f, mapping* $\mathbb{E}$ or $\mathbb{X}$ into $\mathbb{R} \cup \{-\infty\}$

that has a non-singular second Fréchet derivative at x, define its ***risk tolerance*** *at x as*

$$T_f(x) := -\left[f''(x)\right]^{-1} f'(x).$$

Corollary (Aggregate and individual risk tolerances). *Under the conditions of the preceding proposition,*

$$T_{\pi_I}(\sum_{i\in I} x_i) = \sum_{i\in I} T_{\pi_i}(x_i).$$

Similarly, if for each i, $\pi_i(x_i) = \int \Pi_i(x_i(s))\mu(ds)$ with state-independent integrand $\Pi_i : \mathbb{E} \to \mathbb{R}\cup\{-\infty\}$ and a measure $\mu > 0$,

$$T_{\Pi_I}(\sum_{i\in I} \chi_i) = \sum_{i\in I} T_{\Pi_i}(\chi_i).$$

Proof. From (21) follows $\left[\pi_I''(e_I)\right]^{-1} = \sum_{i\in I} \pi_i''(x_i)^{-1}$. Apply the left hand operator on the antigradient $a := -\pi_I'(e_I)$ to get $T_{\pi_I}(e_I)$. Apply the right operator on the same object $a = -\pi_i'(x_i)$ to conclude. □

9. EXAMPLES

This section provides a set of examples. Since conjugate functions are central, we first sample a few of them, emphasizing for each function f its *effective domain* $f^{-1}(\mathbb{R})$, denoted *dom*f. The second part of the section displays some games.

Conjugate functions: Note that if $f(x) = \varphi(c_0(x - x^0)) + lx + c_1$, with $\varphi : \mathbb{X} \to \mathbb{R}\cup\{-\infty\}$, a linear $l : \mathbb{X} \to \mathbb{R}$, a fixed vector $x^0 \in \mathbb{X}$, and real constants $c_0 \neq 0, c_1$, then

$$f^{(*)}(x^*) = \varphi^{(*)}(c_0^{-1}(x^* - l)) + (l - x^*)x^0 + c_1.$$

Thus, one may easily account for the effect of scaling, adding affine functions, or translating the space.

Examples of uni-variate conjugate functions $f : \mathbb{R} \to \mathbb{R}\cup\{-\infty\}$: For any number $p > 0$ define its conjugate number p^* by $\frac{1}{p} + \frac{1}{p^*} = 1$.

function $f(x)$	*dom*f	*conjugate function* $f^{(*)}(x^*)$	*dom*$f^{(*)}$
$-\lvert x\rvert^p/p, p > 1$	$\mathbb{R}$	$\lvert x^*\rvert^{p^*}/p^*$	$\mathbb{R}$
$-\lvert x\rvert^p/p, p > 1$	$\mathbb{R}_+$	$\lvert\max\{0, -x^*\}\rvert^{p^*}/p^*$	$\mathbb{R}$
$\lvert x\rvert^p/p, 0 < p < 1$	$\mathbb{R}_+$	$-(x^*)^{p^*}/p^*$	$\mathbb{R}_{++}$
$-\sqrt{1 + x^2}$	$\mathbb{R}$	$-\sqrt{1 - (x^*)^2}$	$[-1, 1]$
$\log x$	$\mathbb{R}_{++}$	$-1 - \log x^*$	$\mathbb{R}_{++}$
$-\exp(-x)$	$\mathbb{R}$	$\begin{cases} x^*\log x^* - x^* & \text{when } x^* > 0 \\ 0 & \text{when } x^* = 0 \end{cases}$	$\mathbb{R}_+$

Associated prices and choices:

$payoff\ \pi$	$\lambda = \pi'(x) =$	$x = -\pi^{(*)'}(\lambda) =$
$-\lvert x\rvert^p/p, p>1, x\in\mathbb{R}$	$-\lvert x\rvert^{p/p^*}$	$-\lvert\lambda\rvert^{p^*/p}$
$-\lvert x\rvert^p/p, p>1, x\geq 0$	$-\lvert x\rvert^{p/p^*}$	$\begin{cases} \lambda^{p^*/p} \text{ when } \lambda<0 \\ 0 \text{ otherwise} \end{cases}$
$\lvert x\rvert^p/p, 0<p<1, x\geq 0$	x^{p/p^*}	$-\lambda^{p^*/p}, \lambda>0.$
$-\sqrt{1+x^2}$	$-x/\sqrt{1+x^2}$	$\lambda/\sqrt{1-\lambda^2}$
$\log x, \quad x>0$	$1/x$	$1/\lambda$
$-\exp(-x),\ x\in\mathbb{R}$	e^{-x}	$-\log\lambda$

Piecewise linear concave functions: Any proper, upper semicontinuous, concave function $f:\mathbb{X}\to\mathbb{R}\cup\{-\infty\}$ equals the pointwise infimum of a family of affine functions. In computation - or for practical purposes - important instances have the said family finite. So, consider a finite set J of linear functionals $x_j^*:\mathbb{X}\to\mathbb{R}$, and constants $r_j\in\mathbb{R}$, and let

$$f(x) := \min\left\{x_j^* x + r_j : j\in J\right\}. \tag{24}$$

Proposition (The conjugate of a piecewise linear concave function). *Suppose the real-valued function f is piece-wise linear on a reflexive Banach space $\mathbb{X}$ - and given by formula* (24). *Then*

$$f^{(*)}(x^*) = \inf\left\{\sum_{j\in J} r_j^* r_j : r_j^*\geq 0, \quad \sum_{j\in J} r_j^* = 1, \quad \sum_{j\in J} r_j^* x_j^* = x^*\right\},$$

with the understanding that $\inf\emptyset = +\infty$. *Thus,* $f^{(*)}(x^*) = +\infty$ *iff* $x^*\notin conv\left\{x_j^* : j\in J\right\}$. *When $\mathbb{X}$ is finite-dimensional, it suffices to have at most* $\dim\mathbb{X}+1$ *coefficients* $r_j^*>0$.

Proof. Recall that for any finite set $\left\{\rho_j : j\in J\right\}\subset\mathbb{R}$ it holds

$$\min\left\{\rho_j : j\in J\right\} = \min\left\{\sum_{j\in J} r_j^*\rho_j : r_j^*\geq 0, \sum_{j\in J} r_j^* = 1\right\}.$$

Consequently,

$$\begin{aligned}
f^{(*)}(x^*) &= \sup_{\rho\geq 0}\sup_{\|x\|\leq\rho}\left\{\min_{j\in J}(x_j^*x+r_j)-x^*x\right\}\\
&= \sup_{\rho\geq 0}\sup_{\|x\|\leq\rho}\left\{\min\left\{(\sum_{j\in J}r_j^*x_j^*-x^*)x+\sum_{j\in J}r_j^*r_j : r_j^*\geq 0,\quad \sum_{j\in J}r_j^*=1\right\}\right\}\\
&= \sup_{\rho\geq 0}\min_{r^*}\sup_{\|x\|\leq\rho}\left\{(\sum_{j\in J}r_j^*x_j^*-x^*)x+\sum_{j\in J}r_j^*r_j : r_j^*\geq 0,\quad \sum_{j\in J}r_j^*=1\right\}\\
&= \sup_{\rho\geq 0}\min_{r^*}\left\{\left\|\sum_{j\in J}r_j^*x_j^*-x^*\right\|\rho+\sum_{j\in J}r_j^*r_j : r_j^*\geq 0,\quad \sum_{j\in J}r_j^*=1\right\}\\
&= \inf\left\{\sum_{j\in J}r_j^*r_j : r_j^*\geq 0,\quad \sum_{j\in J}r_j^*=1,\ \sum_{j\in J}r_j^*x_j^*=x^*\right\}. \quad \square
\end{aligned}$$

For example, when $J=\{1,2\}$ and $f(x):=\min\{x_1^*x+r_1, x_2^*x+r_2\}$, we get

$$f^{(*)}(x^*)=\inf\{r_1^*r_1+r_2^*r_2 : r_1^*,r_2^*\geq 0,\quad r_1^*+r_2^*=1,\quad r_1^*x_1^*+r_2^*x_2^*=x^*\}.$$

The particular instance $\mathbb{X}=\mathbb{R}$ and $f(x)=-|x|=\min\{-x,+x\}$ gives $f^{(*)}(x^*)=0$ if $x^*\in[-1,+1]$ and $f^{(*)}(x^*)=+\infty$ otherwise.

Linear-quadratic functions: Let $\mathbb{X}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and posit $f(x)=-\langle x,Ax\rangle/2$ with A symmetric and positive semidefinite. If A is non-singular surjective, then

$$f^{(*)}(x^*)=\frac{1}{2}\left\langle x^*,A^{-1}x^*\right\rangle.$$

More generally, suppose the range of A is closed. Then

$$f^{(*)}(x^*)=\begin{cases}\frac{1}{2}\langle x^*,x\rangle & \text{when } x^*\in rangeA \text{ and } x\in A^{-1}x^*\\ +\infty & \text{otherwise.}\end{cases}$$

Extending to the linear-quadratic case $f(x)=-\langle x,Ax\rangle/2+\langle a,x\rangle+\alpha$, with $rangeA$ closed, we obtain

$$f^{(*)}(x^*)=\begin{cases}\frac{1}{2}\langle x^*-a,x\rangle+\alpha & \text{when } x\in A^{-1}(x^*-a)\\ +\infty & \text{when } x^*-a\notin rangeA.\end{cases}$$

Letting $A^\dagger$ denote the pseudo-inverse of A we get $f^{(*)}(x^*)=\frac{1}{2}\left\langle x^*-a,A^\dagger(x^*-a)\right\rangle+\alpha$ when $x^*-a\in RangeA$, and $f^{(*)}(x^*)=+\infty$ otherwise; see [4] or [33].

A multi-stage, stochastic, production game: Agent $i \in I$ must make a $\mathcal{F}_t$-measurable decision $z_{it} \in Z_{it}$ at time $t = 0, ..., T$. The production plan $z_i = (z_{it})$ gives him payoff $f_i(z_i)$ subject to

$$g_i(z_i) := \begin{cases} g_{i0}(z_{i0}) \\ g_{i1}(z_{i0}, z_{i1}) \\ \vdots \\ g_{iT}(z_{i0}, ..., z_{iT}) \end{cases} \leq \begin{cases} e_{i0} \\ e_{i1} \\ \vdots \\ e_{iT} \end{cases}$$

Here $g_{it}(z_{i0}, ..., z_{it}) \leq e_{it}$ is shorthand for inequality $g_{it}(s, z_{i0}(s), ..., z_{it}(s)) \leq e_{it}(s) \in \mathbb{E}_t$ holding almost surely, with $g_{it}(s, z_{i0}(s), ..., z_{it}(s))$ presumed $\mathcal{F}_t$-measurable.

In condensed form, i faces the problem to maximize $f_i(z_i)$ s.t. $g_i(z_i) \leq e_i$. Thus game format (6) emerges again. Note that a shadow price λ assumes the form $(\lambda_0, ..., \lambda_T)$, its time-$t$ component λ_t being a $\mathcal{F}_t$-measurable function $s \in S \mapsto \lambda_t(s)$ with values in the non-negative cone $(\mathbb{E}_t)_+$.

Linear, stochastic production games: Specializing on the stochastic production game just outlined, let μ be a probability measure and

$$f_i(z_i) := z_i^* z_i := \mathcal{E}(z_i^* \cdot z_i) = \mathcal{E} \sum_{t=0}^{T} z_{it}^* \cdot z_{it},$$

z_{it} belonging to the non-negative cone Z_{it} in some Euclidean space $\mathbb{Z}_{it}$. The random evaluation vector $z_{it}^* \in \mathbb{Z}_{it}$ is $\mathcal{F}_t$-measurable. Posit

$$g_{it}(z_{i0}, ..., z_{it}) := \sum_{\tau=0}^{t} A_{i\tau t} z_{i\tau}$$

for $\mathcal{F}_\tau$-measurable matrices $A_{i\tau t}$ of appropriate size. Then $g_i(z_i) \leq e_i$ iff $A_i z_i \leq e_i$ where the block matrix

$$A_i := \begin{bmatrix} A_{i00} & 0 & 0 & ... \\ A_{i01} & A_{i11} & 0 & ... \\ A_{i02} & A_{i12} & A_{i22} & 0 \\ ... & ... & ... & ... \end{bmatrix}$$

has transpose A_i^*. Now (6) amounts to

$$\pi_C(e_C) := \max \left\{ \sum_{i \in C} z_i^* z_i : \sum_{i \in C} A_i z_i \leq e_C \text{ with } \mathcal{F}_t\text{-measurable } z_{it} \geq 0 \right\}. \tag{25}$$

λ is a shadow price iff it solves the grand dual problem:

$$\max \ \lambda e_I \ \text{ s.t. } A_i^* \lambda \geq z_i^* \text{ for each } i \text{ and } \lambda \geq 0.$$

In the corresponding core solution agent i receives payment $u_i = \lambda e_i$ only for his endowment.

Linear-quadratic market games: Posit $\pi_i(x_i) = -\frac{1}{2}\langle x_i, A_i x_i\rangle + \langle a_i, x_i\rangle$ with $a_i \in \mathbb{X}$, and a symmetric, positive definite matrix A_i that defines a linear auto-transformation on $\mathbb{X}$. Thus, with π_i strictly concave, agent i is strictly risk averse. Choose $C = I$ and take supremum in (1) to have $x_i = A_i^{-1}(a_i - \lambda)$. So, summing across the agents,

$$\lambda = \left[\sum_{i\in I} A_i^{-1}\right]^{-1} \sum_{i\in I} \left\{A_i^{-1}a_i - e_i\right\}.$$

Consequently, *two-fund separation* and *linear sharing* obtain in that

$$\begin{aligned} & x_i = \mathbf{a}_i + \mathbf{b}_i && \text{where} \\ & \mathbf{a}_i := A_i^{-1}\left\{a_i - \left[\textstyle\sum_j A_j^{-1}\right]^{-1} \textstyle\sum_j A_j^{-1} a_j\right\} && \text{and} \\ & \mathbf{b}_i := A_i^{-1}\left[\textstyle\sum_{i\in I} A_i^{-1}\right]^{-1} e_I \end{aligned}$$

with $\sum_{i\in I}\mathbf{a}_i = 0, \sum_{i\in I}\mathbf{b}_i = e_I$. If each a_i is constant, then that $\mathbf{a}_i$ is risk-free whereas $\mathbf{b}_i$ equals a share of the aggregate risk; the "larger" A_i the smaller $\mathbf{b}_i$. □

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