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INERTIAL NAVIGATION SYSTEMS WITH THREE SINGLE AXIS GYROS AND A PHYSICAL PENDULUM

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Karl-Johan Åström

The fundamental idea for the systems discussed in this report is the synthesis of a vertical indicating system based on a physical pendulum, proposed by Mr K-J Åström, member of the TTN Group, and Mr F Hector of the Philips Teleindustri.

This report is written in order to demonstrate how the basic idea can be adopted to the design of selfcontained position indication systems. The scope of the report is to survey the main features of the systems and the required component accuracy without too many analytical details.

The basic idea for the synthesis of the vertical indicating system is briefly described in section 1. In section 2 are shown two ways of mechanizing the systems. The single-axis systems are analysed in section 3. This analysis is the base for the discussion of the navigation error caused by component inaccuracies in section 4. Section 5 deals with some problems concerning the instrumentation of the three-axis systems and the initial alignment of the systems.

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Contents

1. INTRODUCTION

1.1 Vertical indication with a physical pendulum

1.11 We will start by analysing the motion of a physical pendulum in a gravity field. The gravity field is supposed to be directed towards a point O and the magnitude of the force depends only on the distance to that point. The pivot point of the pendulum is denoted by P. The vector OP is denoted by $\bar{r}$, the vector from the pivot point to the center of mass of the pendulum by $\bar{h}$, and the vector from the center of mass to the center of gravity by $\bar{h}'$. See figure 1.11.

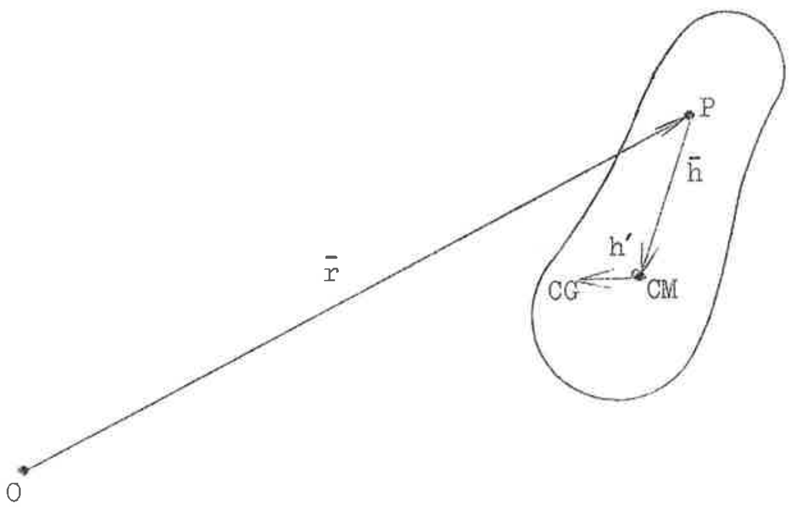

Figure 1.11

Newtons Laws of motion give

$$\begin{cases} \dot{\bar{H}}_{CM} = - \bar{h} \times \bar{F} + \bar{h}' \times \bar{G} & \quad 1.11 \\ m(\ddot{\bar{r}} + \ddot{\bar{h}}) = \bar{F} + \bar{G} & \quad 1.12 \end{cases}$$

where

$\bar{H}_{CM}$ the angular momentum of the pendulum with respect to its center of mass

$\bar{F}$ the force acting on the pendulum at the point of suspension

$\bar{G}$ the gravity force on the pendulum

m the mass of the pendulum

Equations (1.11) and (1.12) gives

$$\dot{\bar{H}}_P = (\bar{h} + \bar{h}') \times \bar{G} - m\, \bar{h} \times \ddot{\bar{r}}$$

where H_P is the angular momentum of the pendulum with respect to the pivot point

Neglecting the difference between the center of mass and the center of gravity, i.e. $|\bar{h}'| \ll |\bar{h}|$, and introducing

$$\bar{G} = m\bar{g}$$

we get

$$\dot{\bar{H}}_P = m\ \bar{h}\ x(\bar{g} - \ddot{\bar{r}}) \qquad 1.13$$

Suppose that the pivot point moves in a plane at "constant height", i.e. r=constant, and that the velocity of the pivot point is considerably less than $\sqrt{rg} \approx 8000$ m/sec.

Introduce a coordinate set 0 x y z fixed to inertial space and the notations according to figure 1.12.

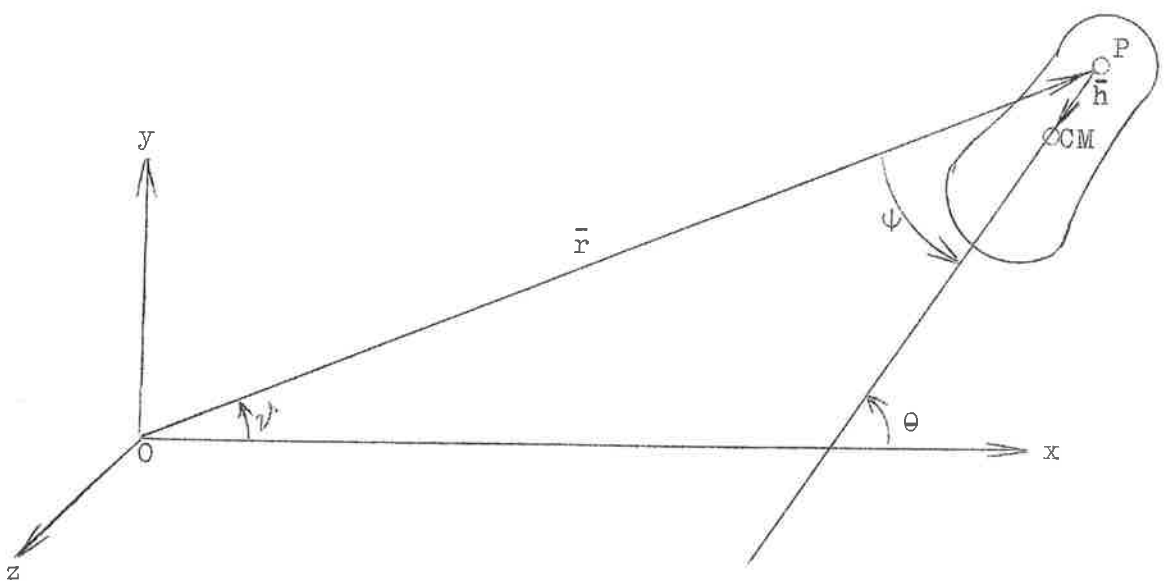

Figure 1.12

The z-component of equation (1.13) gives after linearization in ψ

$$J_P\ddot{\theta} = m\ r\ h\ \ddot{\nu} - m\ g\ h\ \psi \qquad 1.14$$

where

J_P the moment of inertia of the pendulum with respect to its pivot point.

We want the pendulum to indicate the vertical, i.e. θ should equal ν. The difference $\theta - \nu$ is called the vertical indication error.

The vertical indicating error is denoted by ψ. We have

$$\Theta = \vartheta + \psi$$

Equation (1.14) gives

$$J_p \ddot{\psi} + mgh\,\psi = (mrh - J_p)\ddot{\vartheta} \qquad 1.15$$

These equations can be illustrated by the block-diagram of figure 1.13.

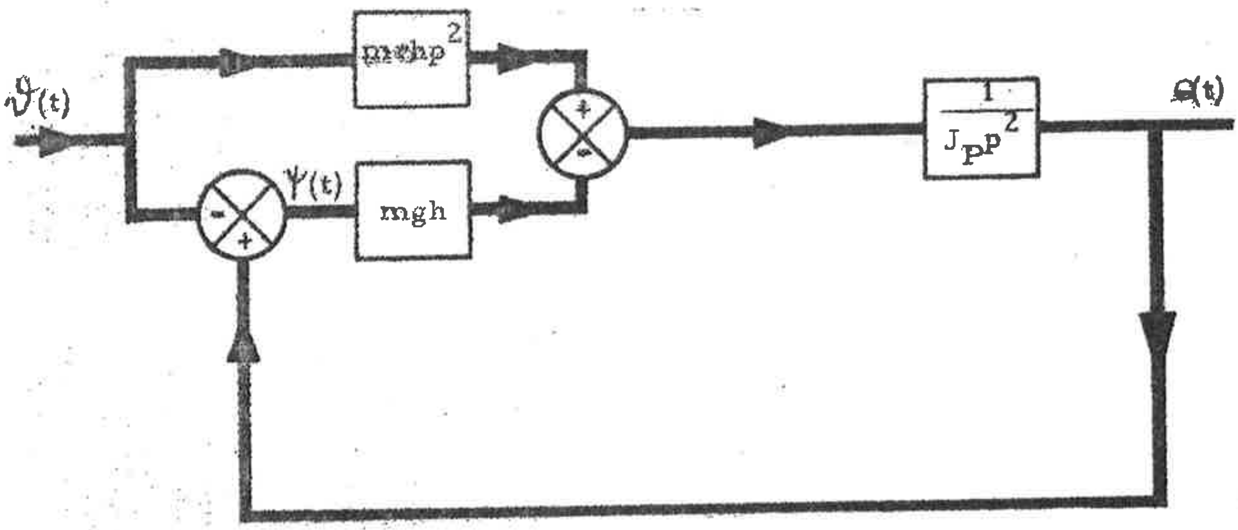

Figure 1.13

Block-diagram of a single-axis physical pendulum moving at constant height in a spherical gravity field

1.12 Suppose that the pendulum is initially indicating the vertical i.e.

$$\psi(0) = \dot{\psi}(0) = 0$$

Equation (1.15) then gives

$$\psi(t) \equiv 0 \quad \text{for all } t \text{ if}$$

$$J_P = mrh \qquad 1.16$$

This means that if the pendulum is initially indicating the vertical it will do so henceforth if the equation 1.16 is satisfied. The condition (1.16) is called the Schuler-tuning condition. See reference 1. Equation (1.16) means that the natural oscillations of the pendulum is

$$\omega_s = \sqrt{\frac{g}{r}}$$

Introducing

$$r = R = 6.4 \times 10^6 \text{ m (the radius of the earth)}$$

$$g = 9.8 \text{ m/sec}^2$$

we get

$$\omega_s = \frac{1}{800} \text{ sec}^{-1}$$

which corresponds to a period of 84 minutes.

If the pivot point no longer is restricted to move on a circle it is still indicating the vertical with no error if the gravity field is spheric symmetric, the pivot point moves at constant height and the Schuler-tuning condition is satisfied. See reference 2.

1.13 If the pivot point does not move at constant height an upper limit of the vertical indication error in case of perfect Schuler-tuning is given by

$$|\psi| < 3 \frac{\Delta r}{r} \cdot \frac{v}{\sqrt{rg}} \qquad 1.17$$

where

Δr the variation in height

v the velocity of the vehicle

g the acceleration of gravity

The relation (1.17) is derived in reference 3, section 3.

Some numerical examples are given in table 1.1 calculated for

$$r = R = 6.4 \times 10^6 \text{ m} \quad \text{(the radius of the earth)}$$

$$g = 9.8 \text{ m/sec}^2$$

<u>Table 1.1</u>

An estimate of the vertical indication error due to variations in height for a perfect Schuler-tuned system.

Δr	$3 \cdot \frac{\Delta r}{r} \frac{v}{\sqrt{rg}}$ $v = 20$ m/sec	$v = 400$ m/sec
10 m	1.2×10^{-8} rad $\approx$ 0.002"	2.4×10^{-7} rad $\approx$ 0.05"
10^2 m	1.2×10^{-7} rad $\approx$ 0.02"	2.4×10^{-6} rad $\approx$ 0.5"
10^3 m	1.2×10^{-6} rad $\approx$ 0.2"	2.4×10^{-5} rad $\approx$ 5"
10^4 m	1.2×10^{-5} rad $\approx$ 2"	2.4×10^{-4} rad $\approx$ 50"
10^5 m	1.2×10^{-4} rad $\approx$ 20"	2.4×10^{-3} rad $\approx$ 500"

As was already pointed out by Schuler, the problem of vertical indication is solved if a pendulum with the property given by eq. (1.16) is mechanized.

We will now further analyse the properties of such a pendulum.

1.14 Introducing

$$r = R = 6.4 \times 10^6 \text{ m}$$

into the Schuler-tuning condition (1.16) we get the following numerical values of the displacement h and the radius of gyration of the pendulum K.

Table 1.2

Radius of gyration K	Displacement h
10^{-2} m	0.16 Å
10^{-1} m	16 Å
1 m	0.16 μ
10 m	16 μ
10^{2} m	1.6 mm
10^{3} m	0.16 m
10^{4} m	16 m

From this table it is obvious that, for a pendulum of reasonable size, the massdisplacement h is a too small to be mechanized.

In an actual application we also have to consider various disturbing torques, e.g. suspension friction.

Introducing the disturbing torque M(t) acting on the pendulum the equation of motion of the pendulum, equation (1.15) becomes

$$J_p\ddot{\psi} + mgh\psi = (mrh-J_p)\ddot{\vartheta} + M(t) \tag{1.15a}$$

If the disturbing torque has a constant magnitude M_o we obtain a stationary vertical indicating error

$$\psi = \frac{M_o}{mgh} \tag{1.18}$$

This equation can be used to estimate the order of magnitude of the vertical indicating error obtained for disturbing torques acting on the pendulum.

Bearings today available have friction torques down to the order of 1 dyncm. Assuming a reasonable mass of the pendulum (0.5 kg) the displacement h must be of the order of 1 cm if the vertical indicating error should be less than one second of an arc ($\approx 5 \times 10^{-6}$ rad). In order to satisfy the Schuler-tuning condition the radius of gyration of the pendulum then should be of the order of 250 m. With such a pendulum the difference between the center of mass and center of gravity (h′ in figure 1.11) will not be negligible compared with the displacement h. See reference 3.

1.2 Vertical indication with a physical pendulum and an inertial reference

The analysis of section 1.1 shows that a physical pendulum can indicate the vertical if the Schuler-tuning condition is satisfied. The analysis also shows the impossibility of mechanizing the pendulum with present day technology. The main difficulty is that, with a displacement of reasonable size, the moment of inertia J_P is far too large. Yet the simple pendulum has the attractive property of indicating the vertical when the pivot point is at rest. In order to see what should be done to the simple pendulum we can argue as follows.

The only way J_P enters the equation of motion of the pendulum is as a coefficient of $\ddot{\theta}$. See equation 1.14. With a reasonable size of the system this coefficient is far to small to satisfy the Schuler-tuning condition. In order to obtain Schuler-tuning the coefficient of $\ddot{\theta}$ in the equation of motion thus must be increased. This can be achieved by applying a torque to the pendulum proportional to $\ddot{\theta}$.

Equation (1.14) is then replaced by

$$J_P \ddot{\theta} = mrh \ddot{\nu} - mgh\,\psi + M' \qquad 1.21$$

where M′ is the torque acting on the pendulum i.e.

$$M' = -A\ddot{\theta} \qquad 1.22$$

hence

$$(J_P + A)\ddot{\theta} = mrh\ddot{\nu} - mgh\,\psi \qquad 1.23$$

This equation is identical to equation (1.14) if $J_P + A$ is replaced by J_P. Similarly all equation of section 1.1 are valid if J_P is replaced by $J_P + A$, e.g. equation (1.15) corresponds to

$$(J_P + A)\ddot{\psi} + mgh\,\psi = (mrh - J_P - A)\ddot{\vartheta} \qquad 1.24$$

The Schuler-tuning condition for this system is thus

$$J_P + A = mrh$$

hence

$$A = mrh - J_P \qquad 1.25$$

This means that the torque M′ applied to the pendulum should equal

$$-\left[mrh - J_P\right]\ddot{\theta} \qquad 1.26$$

Equations (1.21) and (1,22) can be illustrated by the block-diagram of figure 1.21.

As the only result of the added control torque M′ was that J_P was replaced by J_P+A. The moment of inertia of the pendulum J_P is thus apparently increased by the action of the control torque M′. The quantity (J_P+A) is referred to as apparent moment of inertia.

The vertical indicating system thus consists of an ordinary physical pendulum provided with a device for applying a control torque proportional to the angular acceleration of the vehicle. The possibilities of designing such systems were first discussed in reference 5 and has also been proved by experimental investigations.

In order to apply a torque to the pendulum proportional to its angular acceleration with respect to inertial space a space reference is needed. But the vertical and a space-reference are sufficient to determine position. Hence there is no need for extra equipment for the position indication.

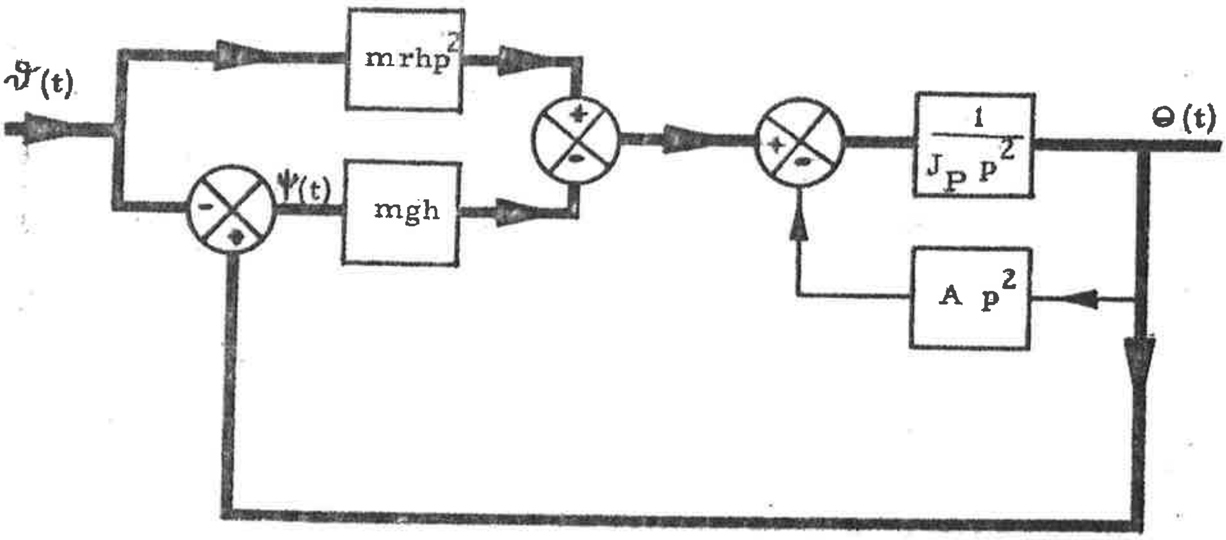

Figure 1.21

Block-diagram of a single axis physical pendulum, with internal feedback, moving at constant height in a spherical gravity field. Compare figure 1.13.

2 SOME POSSIBLE WAYS OF MECHANIZING THE NAVIGATION SYSTEM

2.1 Introduction

According to section 1.2 the vertical indicating system consists of a physical pendulum provided with some equipment which make it possible to apply a torque to the pendulum proportional to its angular acceleration with respect to inertial space. We will now describe some possible ways of mechanizing such a pendulum. We will also discuss what equipment is needed in order to complete the vertical indicating system to a navigation system.

As it is necessary to apply a torque to the pendulum the suspension must be provided with torque-motors. In order to apply the torque it is necessary to have a signal proportional to the angular acceleration of the pendulum.

This signal can e.g. be obtained in the following way

1. By measuring the angle, between a space-reference and the pendulum and differentiating the twice.

2. By providing the pendulum with rate-gyros and differentiating the output signals from the gyros. (We might as well use an integrating gyro and differentiate twice. Compare reference 5, section 4.1)

The corresponding systems are referred to as systems of type 1 and type 2 resp.

2.2 Description of a system of type 1

A system of type 1 is shown in figure 2.21.

The space-reference is a three-gyro platform system stabilized with respect to inertial space. Position indication is obtained by measuring the angles between the pendulum and the space-reference. With an arrangement according to figure 2.21 the angles directly correspond to longitude and latitude. The control signal is obtained by differentiating these angles twice. The differentiated signals are amplified and distributed to torque-motors which give a control torque proportional to the angular acceleration of the pendulum.

A system of this type can be obtained from the Draper-system shown in figure 5 of reference 1 by

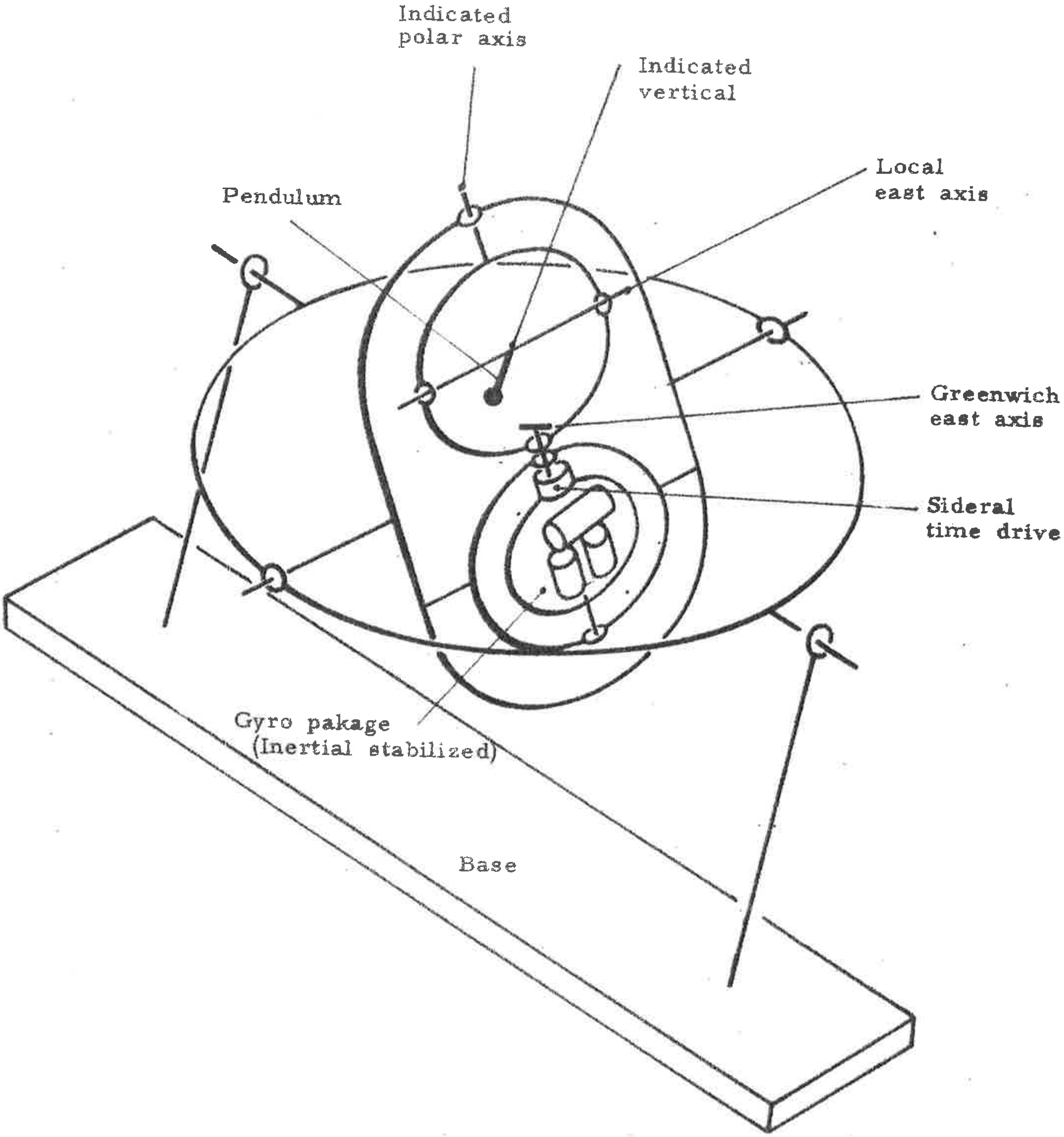

Figure 2.21 (System of type 1)

Navigation-system with an inertial stabilized three-gyro platform system and a vertical indicating system with a physical pendulum with a high apparent moment of inertia. (Compare fig. 5, ref. 1)

1. Removing the accelerometers
2. Providing the accelerometer-gimbal with an unbalance
3. Changing the internal feedback. (The essential change is that the double integration in the Draper system should be replaced by a double differentiation)

2.3 Description of a system of type 2

A system of type 2 is shown in figure 2.31.

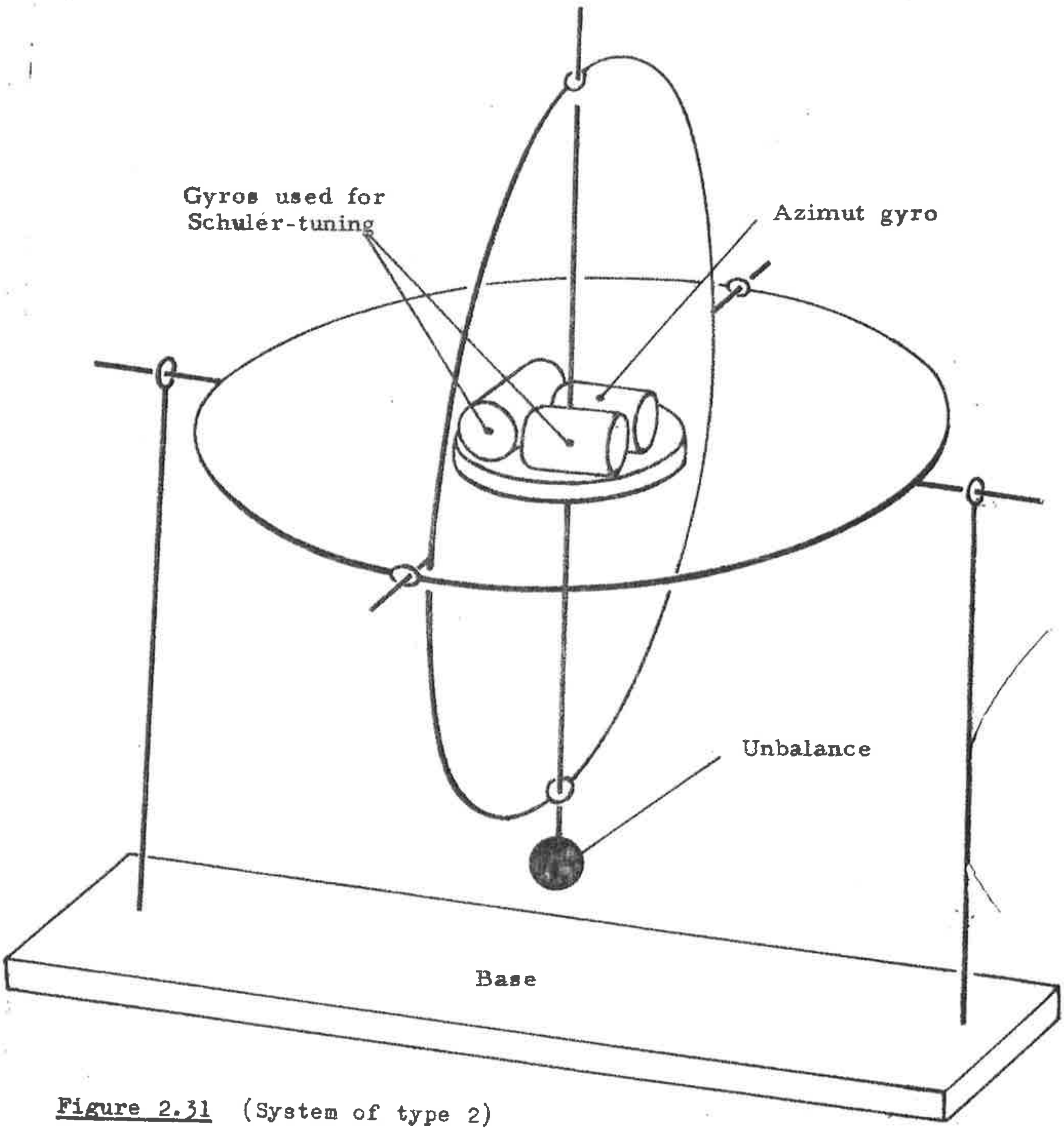

Figure 2.31 (System of type 2)
Navigation system based on a Schuler-tuned three-gyro platform system

It consists of a pendulum provided with two rate-gyros.* The control signal is obtained by differentiating the output signals of the gyros. These signals are amplified and fed to torque-motors on the gimbals, which provide the control torque proportional to the angular acceleration of the pendulum. A third gyro is required to stabilize the pendulum with respect to rotations around the vertical. Position indication is obtained by integrating the output signals of the gyros. A system of this type is analysed in reference 7.

A system of type 2 can be obtained from the Draper-system shown in figure 4 of reference 2 by

1. Removing the accelerometers
2. Providing the controlled member with an unbalance
3. Changing the internal feedback. (The essential change is that the double integration in the Draper-system should be replaced by a double differentiation).

*It is also possible to use one rate-coupled two-degree gyro.

3. A SIMPLIFIED ANALYSIS OF THE SINGLE AXIS SYSTEMS

3.1 Introduction

In order to obtain a further understanding of the systems to be able to give the component accuracy required for a given system accuracy it is necessary to analyse the systems. In order to get the order of magnitude of the accuracy an analysis of the single-axis systems is sufficient. For a more detailed discussion of the systems the reader is referred to the bibliography.

In order to further simplify the problem we make the following assumptions.

1. The distance between the center of mass and the center of gravity h' of the pendulum is considerably less than the displacement h.

2. The vehicle carrying the navigation system has a translational motion in a plane at constant height i.e.

   $r = \text{const}$

   with a velocity v whose magnitude is considerably less than $\sqrt{rg} \approx 8000$ m/sek.

3. The gravity field has a spherical symmetry.

3.2 Analysis of a single-axis system of type 1

3.21 A single-axis system of the first type is shown in figure 3.21.

The space reference is a single-axis inertial stabilized platform system. It consists of a gimbal to which a single axis gyro is mounted. Angular displacements of the gimbal are sensed by the gyro whose output signal is filtered and fed to the gimbal torquer TM_1 in such a phase that the angular displacements are counteracted. The gimbal will then essentially maintain its orientation with respect to inertial space.

The vertical indicating system is built up according to the principles described in section 1.2. It consists of a pendulum with a torque-motor TM_2. The angle between the pendulum and the space-reference is measured differentiated twice, amplified, and fed to the torque-motor TM_2 of the pendulum.

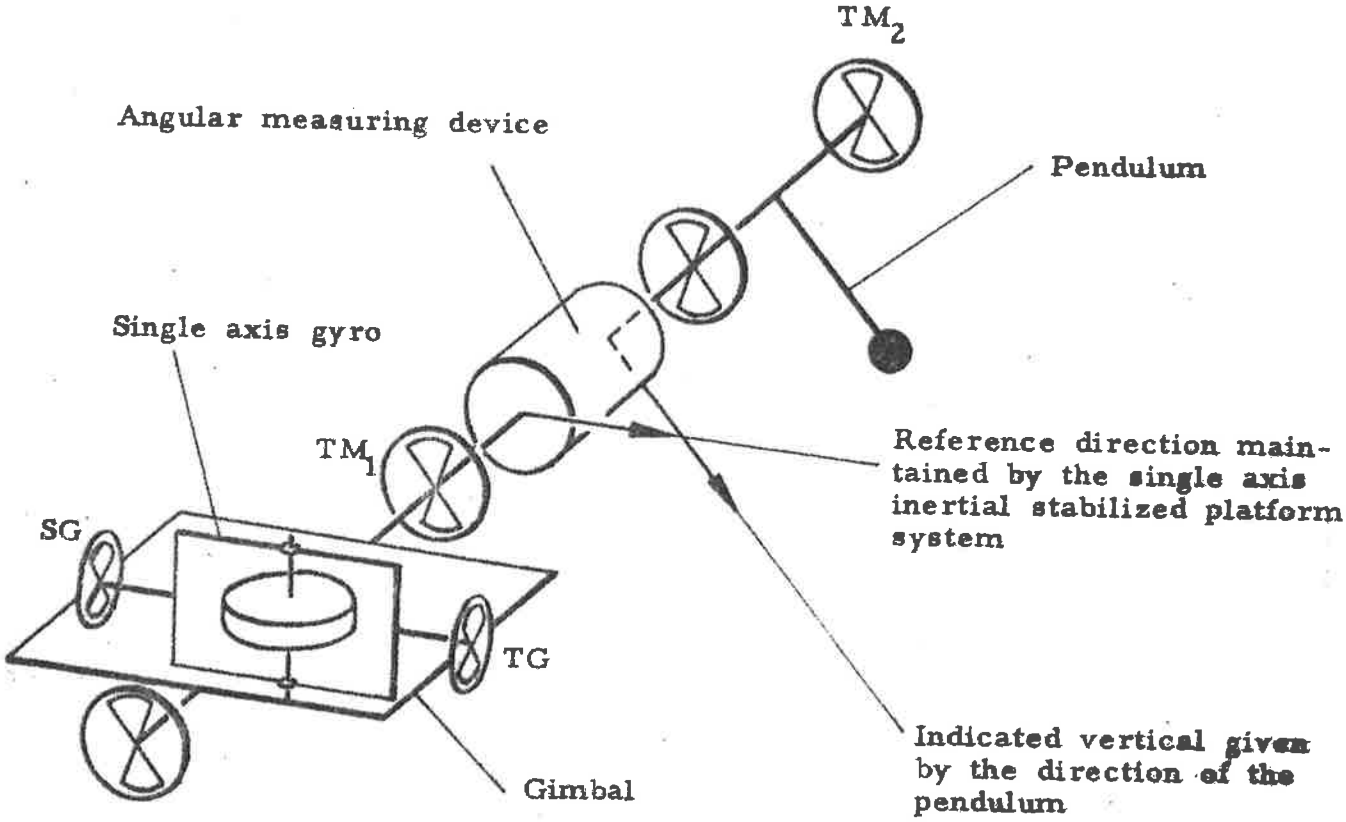

Figure 3.21

A single axis navigation system of type 1.

Introduce the notations according to figure 3.22. Also compare section 1.

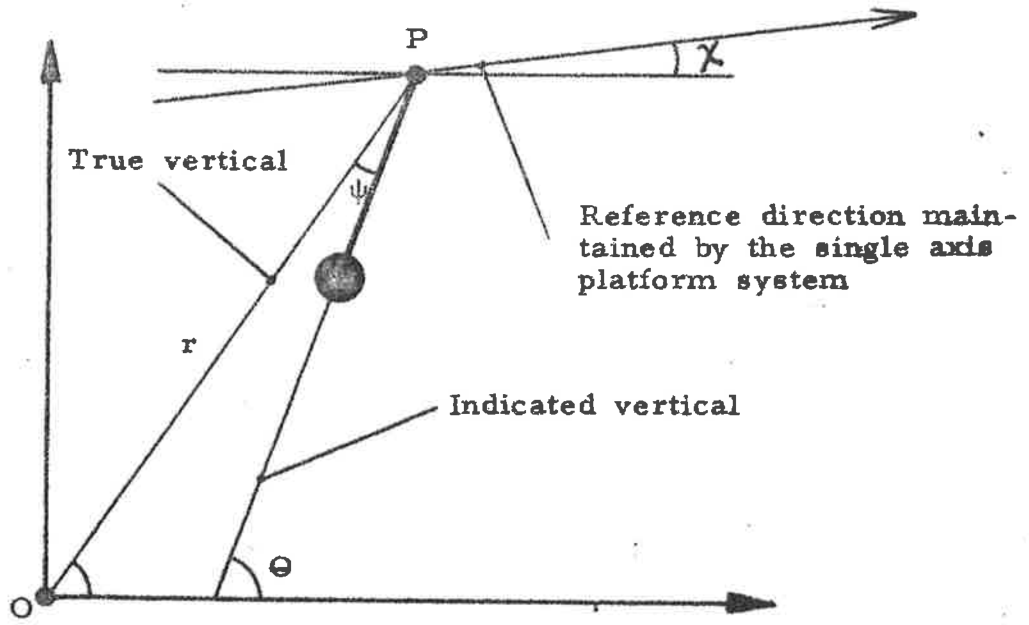

Figure 3.22

O the center of the gravity field

P the pivot point of the pendulum

ψ the vertical indicating error

the error of the single axis inertial stabilized platform system

θ the angle between the pendulum and a direction fixed to inertial space

The angle between the pendulum and the reference system is

$$\theta(t) - \chi(t)$$

The torque applied to the pendulum is thus

$$M'(p) = - A\, p^2 \left[\theta(p) - \chi(p)\right]$$

where $A\,p^2$ is the transfer function from the angle between the pendulum and the reference system to the torque applied to the pendulum.

3.22 For a Schuler-tuned system the torque M' should equal

$$-\left[mrh - J_P\right] p^2\theta \qquad \text{(See equation 1.26)}$$

where

mh the unbalance of the pendulum

J_P the moment of inertia of the pendulum with respect to the pivot point

3.23 Using the assumptions (1), (2) and (3) of section 3.1 the equation of motion of the pendulum is

$$J_P\ddot{\theta} = mrh\ddot{\vartheta} - mgh\,\psi + M'(t) + M(t)$$

where $M(t)$ is the disturbing torque acting on the pendulum. Compare equation (1.23).

Laplace-transforming and solving for the vertical indication error $\psi(t)$ we get

$$\psi(p) = \frac{mrh-A-J_p}{\left[J_P+A\; p^2\right]+ mgh} \cdot \frac{a(p)}{r}$$

$$+ \frac{1}{\left[J_P+A\; p^2\right]+ mgh} \cdot M(p) \qquad 3.23$$

$$+ \frac{Ap^2}{\left[J_P+A\; p^2\right]+ mgh} \cdot \chi(p)$$

The equations of the single-axis inertial stabilized platform system are derived in reference 6. Equation (A. 13) of reference 6 gives

$$\chi(p) = \frac{1}{bJp^2(1+Y_o(p))} M_1(p) + \frac{1}{Hp} \frac{Y_o(p)}{1+Y_o(p)} m(p) \qquad 3.24$$

where

$$Y_o(p) = \frac{\omega_o}{ab} \cdot \frac{\tau(p)+\omega_o p}{p\left[p^2+ \sigma(p)\right]} \qquad 3.25$$

and

ω_o the angular velocity of the gyroscopic element

J the moment of inertia of the gyroscopic element with respect to the spin axis

$\omega_o J=H$ the angular momentum of the gyroscopic element

aJ the moment of inertia of the float of the gyro

bJ the moment of inertia of the gimbal

$\varphi(t)$ the output signal of the gyro

$\chi(t)$ the angle between the gimbal and a direction fixed to inertial space

$M_1(t)$ the disturbing torque acting on the gimbal

$m_1(t)$ the disturbing torque acting on the float of the gyro

$J\ \tau(p)$ the transfer function from the output signal of the gyro to the torque applied by the torque-motor TM_1

$J\ \sigma(p)$ the transfer function from the output signal of the gyro to the torque applied by the torque-generator TG of the gyro. The transfer function $\sigma(p)$ includes the viscous damping of the fluid and the external feedback).

A block-diagram of the complete single-axis navigation system is shown in figure 3.23.

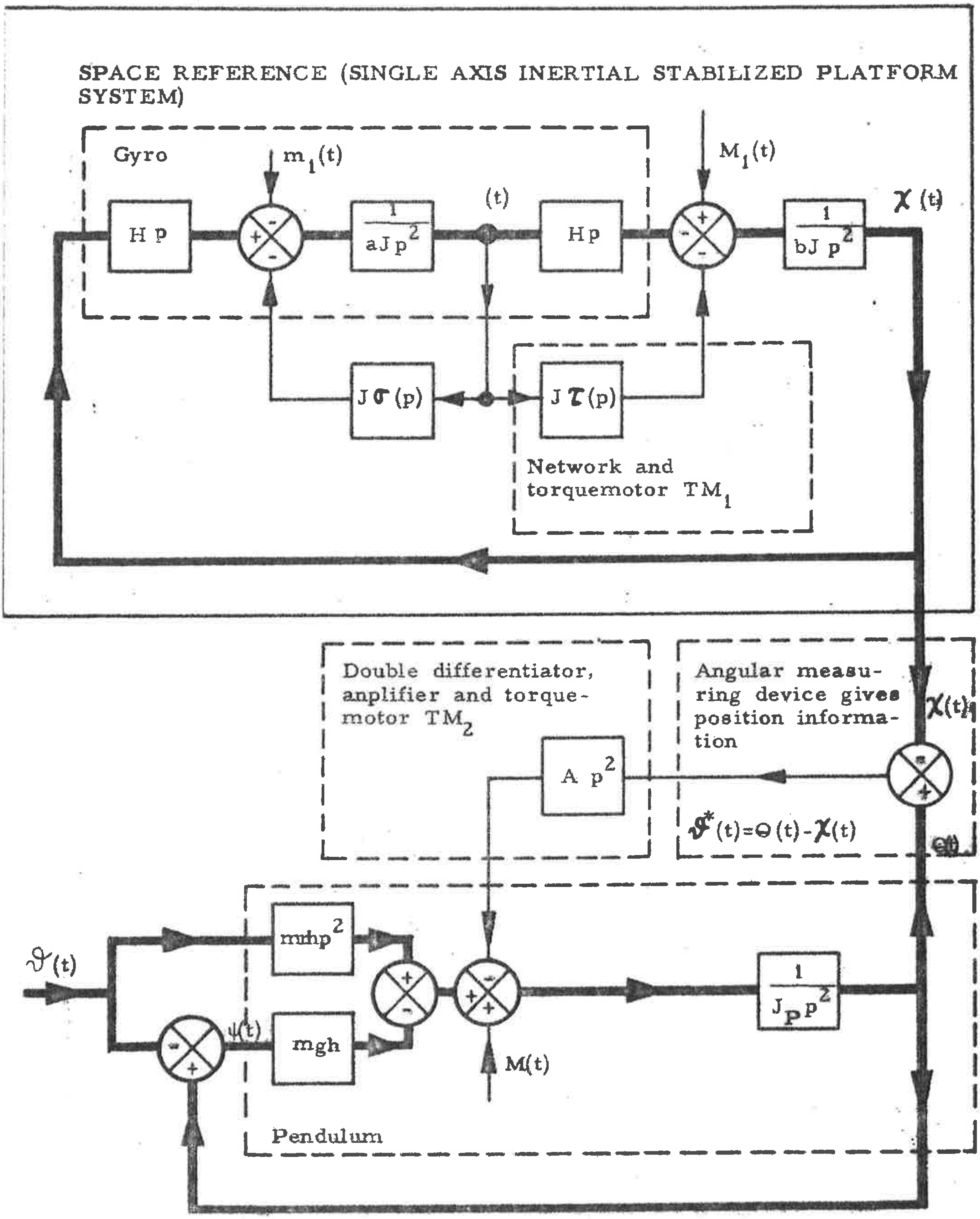

Figure 3.23

Block-diagram of a single-axis navigation system of type 1. Compare fig. 1.13

3.24 As the vehicle moves on a circle the position is given by the angle $\vartheta(t)$. The angle between the space-reference, given by the single-axis platform system, and the vertical indicated by the pendulum is

$$\vartheta^*(t) = \theta(t) - \chi(t) = \vartheta(t) + \psi(t) - \chi(t)$$

This angle is obtained from the angular measuring device and is used to estimate the angle $\vartheta(t)$.

The error is thus

$$\varepsilon(t) \equiv \vartheta^*(t) - \vartheta(t) = \psi(t) - \chi(t) \qquad 3.26$$

The angle $\varepsilon(t)$ is called the navigation error. To obtain the position error d we have to multiply the angle $\varepsilon(t)$ by r. Compare figure 3.24. (Notice that $\varepsilon = 1' \Longleftrightarrow d = 1852$ m)

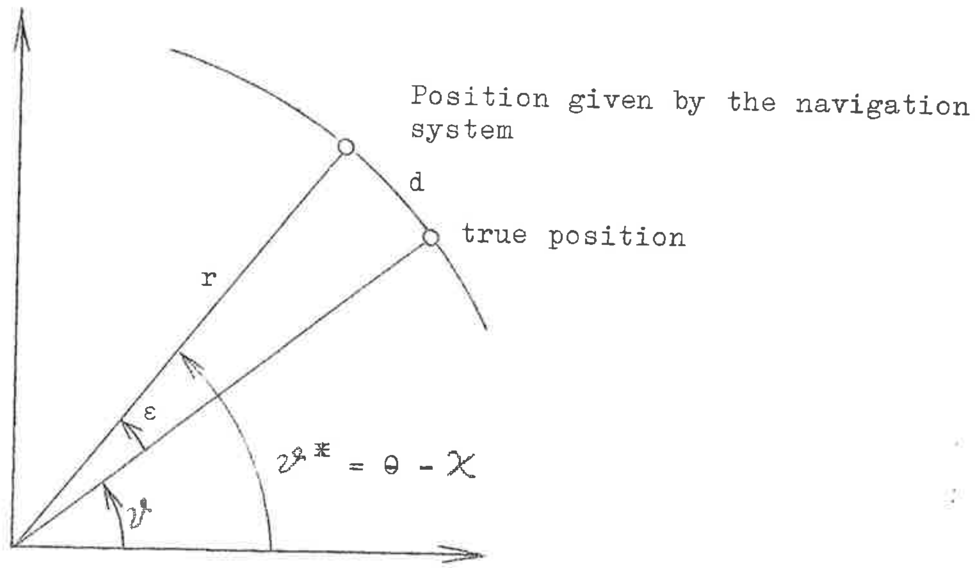

Figure 3.24

3.25 According to the equation (3.26) the navigation error $\varepsilon(t)$ is the sum of the vertical indication error $\psi(t)$ and the error of the space-reference $\chi(t)$. The equations (3.22) and (3.24) give the following expression for the navigation error

$$\varepsilon(p) = \frac{mrh-A-J_P}{[J_P+A\ p^2]+mgh}\ p^2\,\vartheta(p) + \\ + \frac{1}{[J_P+A\ p^2]+mgh}\cdot M(p) - \\ - \frac{mgh}{[J_P+A\ p^2]+mgh}\cdot \chi(p) \qquad 3.27$$

where

$$\chi(p) = \frac{1}{Jbp^2}\cdot\frac{1}{[1+Y_o(p)]}\,M_1(p) + \frac{1}{Hp}\cdot\frac{Y_o(p)}{1+Y_o(p)}\,m(p)$$

and

$$Y_o(p) = \frac{\omega_o}{ab}\cdot\frac{\omega_o p + \tau(p)}{p\,[p^2 + \sigma(p)]}$$

The first two terms are due to errors in the vertical indicating system and the last term is due to errors in the reference system. Before discussing this equation we will derive the corresponding equation for a system of type 2.

3.3 Analysis of a single-axis system of type 2

3.31 A single-axis system of the second type is shown in figure 3.31.

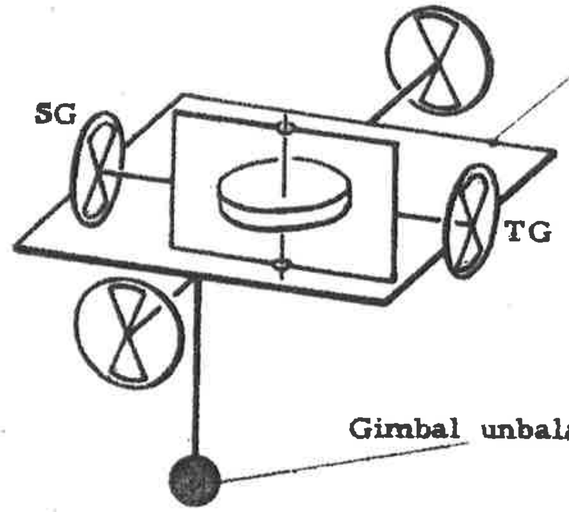

Figure 3.31

A single-axis system of type 2.

A single-axis gyro is mounted to the pendulum. The gyro could e.g. be rate-coupled*. This is assumed for the present. The output signal of the gyro $\varphi(t)$ is thus proportional to the angular velocity of the pendulum. The signal $\varphi(t)$ is differentiated, amplified and fed to the torque-motor TM. A torque proportional to the angular acceleration of the pendulum is thus applied to the pendulum.

The torque applied to the pendulum is

$$M'(p) = -Hp \cdot \varphi(p) - J\,\tau(p)\,\varphi(p) \qquad 3.31$$

* Other schemes of coupling the gyro are discussed later in section 3.33 and in section 4 of reference 5

where

H	the angular momentum of the gyroscopic element
$J\,\tau(p)$	the transfer function from the output signal of the gyro to the torque applied by the torque motor TM

The first term in the above expression is due to the reaction torque of the gyro and the second term is the torque applied by the torque-motor TM.

The output signal of the gyro $\varphi(t)$ is related to the angular velocity of the pendulum $\dot{\theta}(t)$ by the equation

$$\varphi(p) = \frac{Hp}{aJ\left[p^2 + \sigma(p)\right]}\,\theta(p) = \frac{\omega_o p}{a\left[p^2 + \sigma(p)\right]}\,\theta(p) \qquad 3.32$$

where

J	the moment of inertia of the gyroscopic element
aJ	the moment of inertia of the float of the gyro
ω_o	the angular velocity of the gyroscopic element
$H=J\omega_o$	the angular momentum of the gyroscopic element
$J\sigma(p)$	the transfer function from the output signal of the gyro to torque acting on the float of the gyro (the viscous damping as well as external feedback is included in $\sigma(p)$. An integrating gyro has $\sigma(p) = \alpha p$. A rate gyro has $\sigma(p) = \alpha p + \beta$)

Equation (3.32) tells how the changes in the angle $\theta(t)$ are sensed by the gyro and is referred to as the <u>signal equation</u>.

Eliminating $\varphi(p)$ between the equations (3.31) and (3.32) we get the torque applied to the pendulum

$$M'(p) = -A\,p^2\,\theta(p) \qquad 3.33$$

where

$$A = A(p) = \frac{J\omega_o\left[\omega_o p + \tau(p)\right]}{ap\left[p^2 + \sigma(p)\right]} \qquad 3.33'$$

3.22 For a Schuler-tuned pendulum the torque M' should equal

$$-\left[mrh - J_P\right] p^2\Theta$$

The Schuler-tuning condition for this system is thus

$$J\omega_o\left[\omega_o p + \tau(p)\right] = ap\left[mrh - J_P\right]\cdot\left[p^2 + \sigma(p)\right] \qquad 3.34$$

3.33 Both of the transfer functions $\sigma(p)$ and $\tau(p)$ are to the designers disposal. The Schuler-tuning condition, equation (3.34), can thus be satisfied in many ways. Some examples are given below

(i) $$\tau(p) = 0, \quad \sigma(p) = \frac{J\,\omega_o^{\,2}}{a\left[mrh - J_P\right]} - p^2$$

(ii) $$\tau(p) = \frac{ap\left[mrh - J_P\right]\left[p^2 + \alpha p\right]}{J\,\omega_o} - \omega_o p$$

$$\sigma(p) = \alpha p$$

(iii) $$\tau(p) = \frac{ap\left[mrh - J_P\right]\left[p^2 + \alpha p + \mathcal{H}\right]}{J\,\omega_o} - \omega_o p$$

$$\sigma(p) = \alpha p + \mathcal{H}$$

Compare reference 5, section 4.1.

The first scheme means that no gimbal torquers are used. The desired control-torque is obtained from the reaction torque of the gyro. The tuning condition means that $\sigma(p)$ should equal a constant for low frequencies at least. This is obtained simply by spring-restraining the float of the gyro. The spring-coefficient should be

$$k = \frac{H^2}{aJ\left[mrh - J_P\right]} \approx \frac{H^2}{a\,J\,mrh} \qquad 3.35$$

This scheme of mechanizing the Schuler-tuning condition is used in the Anschütz Gyro Compass, but is not suited for high precision equipment. The main reason is that the gyro is used both as sensing and actuating device, which means difficulties. This is further discussed in reference 9.

Scheme (ii) means that the gyro is integrating and that $\tau(p)$ essentially includes two differentiations. This method is not very attractive since the output signal of the gyro is proportional to the distance travelled, which means difficulties in keeping the output signal of the gyro within reasonable limits. The output signal of the gyro should thus be reset to zero. One way of doing this is by spring-restraining the float of the gyro to zero. The gyro then works as a rate gyro. Scheme (iii) is a system of this type.

A system of type (iii) is discussed in reference 5. The reader is refered to this report for a discussion of the choice of the coefficient $\mathcal{H}$, ref. 5, section 4.1.

3.34 Introduce the notations according to figure 3.32.

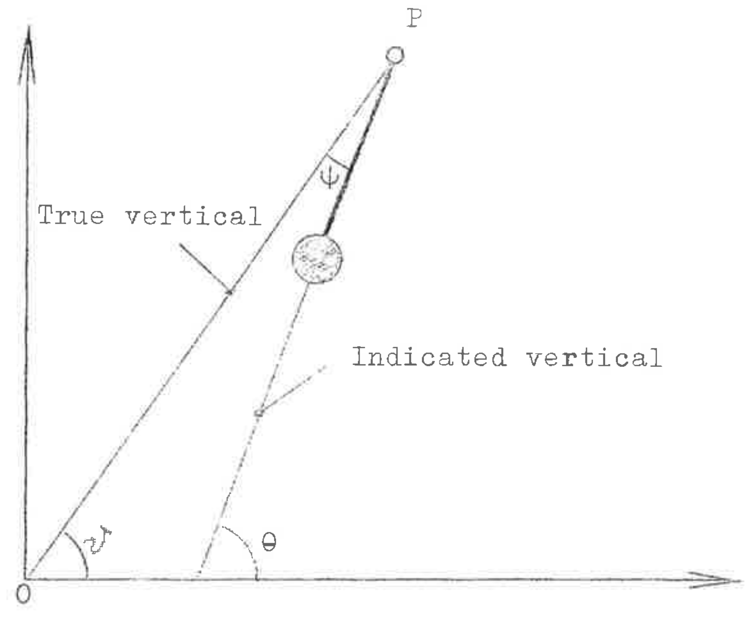

Figure 3.32

O	the center of the gravity field
P	the pivot point of the pendulum
ψ	the vertical indicating error

Using the assumptions (1), (2) and (3) of section 3.1 the equation of motion of the pendulum is

$$J_P \ddot{\Theta} = mrh \ddot{\vartheta} - mgh \psi + M'(t) \qquad 3.36$$

where

$M'(t)$ is the torque applied to the pendulum by the torque-motor and the gyro.

Laplace-transforming and solving for the vertical indication error we get,

$$\psi(p) = \frac{mrh - A(p) - J_P}{\left[J_P + A(p)\right] p^2 + mgh} \frac{a(p)}{r}$$

where

$a(p)$ is the acceleration of the vehicle i.e.

$$a(p) = r\, p^2 \vartheta(p)$$

Introducing the disturbing torque $m(t)$ acting on the float of the gyro the signal equation (3.32) becomes

$$\varphi(p) = \frac{Hp}{Ja\left[p^2 + \sigma(p)\right]} \Theta(p) - \frac{m(p)}{aJ\left[p^2 + \sigma(p)\right]} \qquad 3.32$$

Also introducing the disturbing torque acting on the pendulum $M(t)$ we get

$$\psi(p) = \frac{mrh - A(p) - J_P}{\left[J_P + A(p)\right] p^2 + mgh} \frac{a(p)}{r}$$

$$+ \frac{1}{\left[J_P + A(p)\right] p^2 + mgh} M(p)$$

$$+ \frac{A(p)\, p^2}{\left[J_P + A(p)\right] p^2 + mgh} \cdot \frac{1}{Hp} \cdot m(p) \qquad 3.37$$

where

$$A(p) = \frac{J\, \omega_o\left[\omega_o p + \tau(p)\right]}{ap\left[p^2 + \sigma(p)\right]}$$

A block-diagram of the system is shown in figure 3.33.

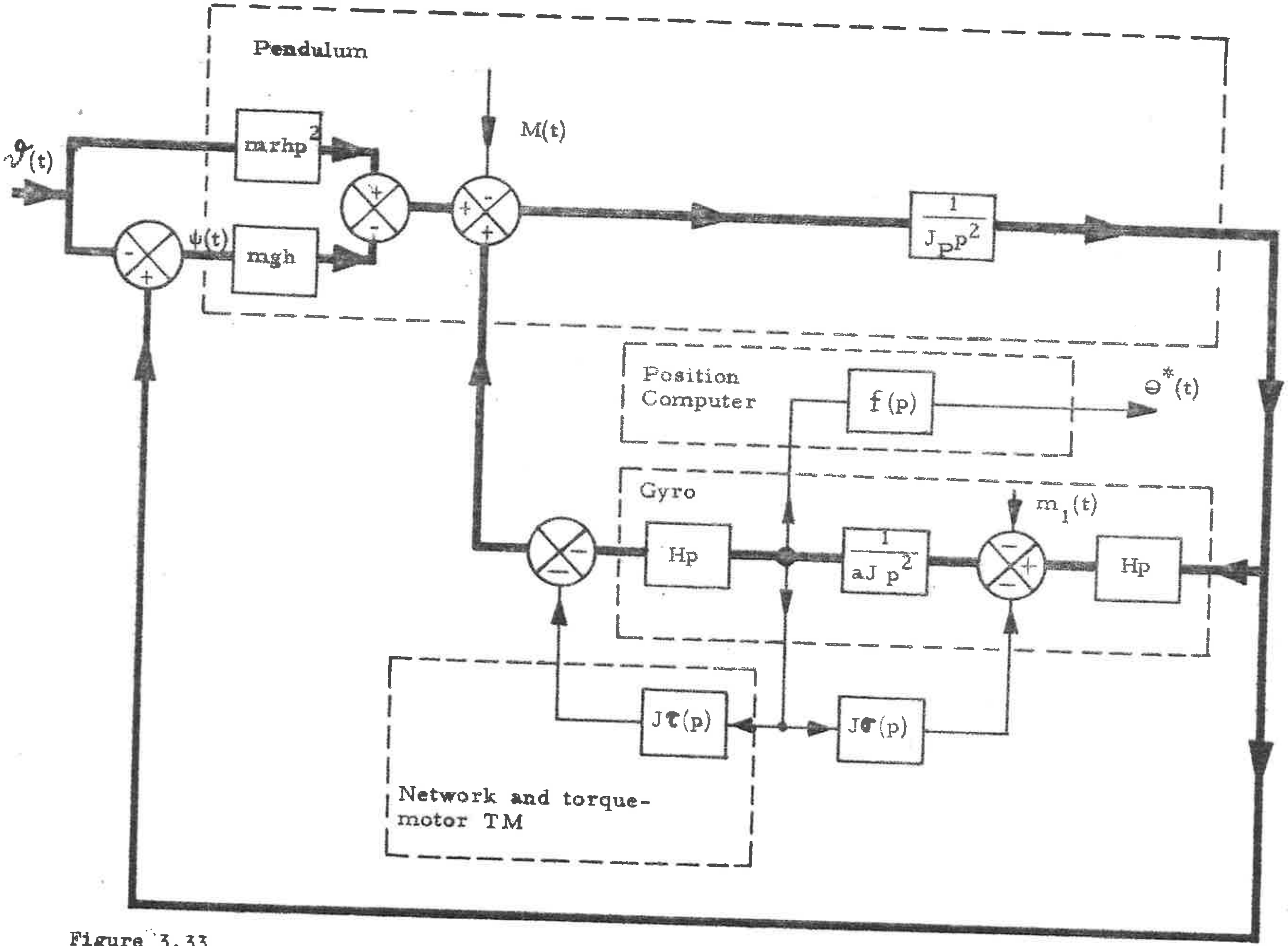

Figure 3.33

Block-diagram of a single-axis navigation system of type 2. Compare figure 1.13.

3.35 As the vehicle moves on a circle position indication consists of determining the angle $\vartheta(t)$. See figure 3.34.

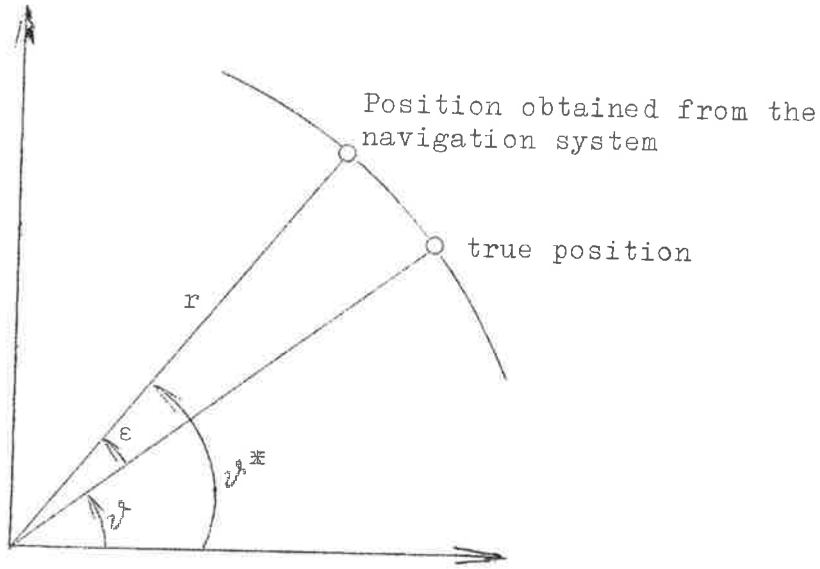

Figure 3.34

The angle $\Theta(t)$ is an estimate of $\vartheta(t)$. The output signal of the gyro $\varphi(t)$ is related to $\Theta(t)$ by the equation 3.32. An estimate $\vartheta^*(t)$ of $\vartheta(t)$ can thus be obtained by feeding the output signal through a linear network i.e.

$$\vartheta^*(p) = f(p) \cdot \varphi(p)$$

where f(p) is the transfer function of the network. Equation (3.32) gives

$$f(p) = \frac{a[p^2 + \sigma(p)]}{\omega_o \, p}$$

Compare the way position information is obtained in systems of type 1. Introduce the navigation error defined by

$$\varepsilon(t) = \vartheta^*(t) - \vartheta(t)$$

we get from equations (3.32)

$$\varepsilon(p) = \psi(p) + \frac{1}{Hp} \cdot m(p)$$

3.36 Introduce the vertical indication error $\psi(t)$ from the equation (3.37) we get

$$\varepsilon(p) = \frac{mrh - A(p) - J_P}{\left[J_P + A(p)\right] p^2 + mgh} \frac{a(p)}{r}$$

$$+ \frac{1}{\left[J_P + A(p)\right] p^2 + mgh} \cdot M(p) \qquad 3.38$$

$$+ \frac{J_P\, p^2 + mgh}{\left[J_P + A(p)\right] p^2 + mgh} \cdot \frac{1}{Hp} \cdot m(p)$$

where A(p) is given by the equation (3.33′).

3.4 Limitations of the simplified single-axis approach

In an actual navigation problem, navigation take place over a rotating earth whose gravity field is elliptic rather than spheric. Besides the vehicle can not be supposed to move at constant height. This is a situation rather different from the idealized problem dealt with in the previous sections. A questions now arises. To what extend is the simplified analysis valid. To answer this completely it is necessary to solve the complete problem. As this calls for a complex analytical work quite outside the scope of this report we just mention a few results obtained, concerning the simplifications.

1. The distance h′ between the center of mass and the center of gravity of the pendulum is for a pendulum whose inertia ellipsoid has circular symmetric

$$h' \approx 3\,\frac{J-J'}{mr} \cdot \cos\psi$$

where

J, J′	the moments of inertia of the pendulum with respect to the pivot point
m	the mass of the pendulum
r	the distance from the pivot point to the center of the earth
ψ	the angle between the pendulum and the vertical

See reference 3.

2. As the vehicle is no longer restricted to move at constant "height" even a perfect Schuler-tuned system moving in a spherical gravity field would have an indication error. The order of magnitude of the error can be estimated by equation (1.17)

$$|\psi| < 3\,\frac{\Delta r}{r} \cdot \frac{v}{\sqrt{rg}}$$

See section 1.13.

Using external information of height this error can largely be eliminated. Similarly it is possible to compensate for deviations from the spherical gravity field by introducing a loop which changes the tuning according to the position of the vehicle.
Compare section 5.

3. In the analyses of the previous sections no damping is introduced in the systems. This means that errors obtained will never fade out. As damping also means misstuning of the system the introduction of damping is a compromize between solution time and forced error. This problem is the same for all Schuler-tuned system. It is extensive dealt with in reference 2.

4. The single-axis approach means that the interactions of the different loops are neglected.

To summarize the single-axis simplified single-axis approach is not satisfactory for a detailed analysis of the system but is sufficient to show the main features of the system and to give an estimate of the order of magnitude of the required accuracy of some of the components.

4. A PRELIMINARY ESTIMATE OF THE NAVIGATION ERROR CAUSED BY CERTAIN COMPONENT INACCURACIES

4.1 Introduction

We will now use the single axis models developed in section 3 in order to obtain estimates of the navigation error caused by certain component inaccuracies. As we have no special application in mind we cannot expect detailed information about the nature of the various disturbances. Hence there is no need for a complex model.

This section is by no means complete. It is merely some examples of the error obtained for some special deterministic disturbances and preliminary estimates to get the order of magnitude of the required component accuracy.

As the simplified equations are linear we can calculate the error from different sources and add to get the total navigation error. The navigation errors for systems of type 1 and type 2 are given by the equations (3.27) and (3.38).

4.2 Errors occuring in both types of systems

We will first discuss errors which are the same for systems of both types.

4.21 Nonperfect tuning means that an error is obtained in the vertical indicating system when the vehicle accelerates. The deviations from perfect tuning may be intentionally introduced in order to obtain damping in the system or may be due to component inaccuracy. The analytical relationships are the same for all Schuler-tuned systems. For a single-axis system where the carrying vehicle moves at constant height, i.e. r=const, with a velocity considerable less than $\sqrt{rg} \approx 8000$ m/sec we have

$$\varepsilon_1(p) = \frac{mrh - A - J_P}{\left[J_P+A\right]p^2 + mgh} \cdot \frac{a(p)}{r} \qquad 4.21$$

where

$\varepsilon_1(t)$	the navigation error
$a(t)$	the acceleration of the vehicle
mh	the unbalance of the pendulum
J_p	the moment of inertia of the pendulum with respect to the pivot point
g	the acceleration of gravity
r	the distance between the pivot point of the pendulum and the center of the gravity field
A	transfer function from the angular acceleration of the pendulum to torque applied to the pendulum

If the system is perfectly tuned, i.e.

$$A = mrh - J_p \quad \text{(Compare section 1.2, equation 1.25)}$$

the error $\varepsilon_1(t)$ is zero independent of the acceleration of the vehicle. As was previously mentioned the transfer function A is mechanized with physical components which means that the Schuler-tuning condition cannot be accurately satisfied. Errors of the same type are obtained if damping is introduced in the system. See reference 2 and 4. We will give some examples of the error obtained.

Example 1

Suppose that

$$A = (mrh - J_p)(1 - \delta)$$

where δ is an overall figure of the component accuracy. Equation 4.21 gives

$$\varepsilon_1(p) = \frac{\delta(mrh - J_p)}{\left[mrh - \delta(mrh - J_p)\right]p^2 + mgh} \cdot a'(p)$$

Supposing

$$\delta \ll 1$$
$$J_p \ll mrh$$

we get

$$\varepsilon_1(p) = \frac{\delta}{p^2 + \omega_s^2} \cdot \frac{a(p)}{r}$$

Assume that the vehicle accelerates to the velocity v_o in a time considerably shorter than $\frac{1}{\omega_s} \approx 800$ sec we get

$$\varepsilon_1(p) = \frac{\delta v_o}{\omega_s r} \cdot \frac{\omega_s}{p^2+\omega_s^2}$$

hence

$$\varepsilon_1(t) = \frac{\delta v_o \omega_s}{g} \sin \omega_s t$$

A maximum error of

$$\varepsilon_{10} = \frac{\delta v_o \omega_s}{g} = \frac{\delta v_o}{\sqrt{Rg}} \qquad 4.22$$

will thus be obtained for

$$t = \frac{\pi}{2\omega_s} \approx 1250 \text{ sec}$$

Some numerical values of the maximum navigation error are given in table 4.21. This table is calculated for

$r = 6.4 \times 10^6$ m (the radius of the earth)

$g = 10$ m/sec^2

δ	ε_{10}: $v_o = 20$ m/sec	ε_{10}: $v_o = 400$ m/sec
10^{-2}	2.5×10^{-5} rad $\approx 5''$	5×10^{-4} rad $\approx 100''$
10^{-3}	2.5×10^{-6} rad $\approx 0.5''$	5×10^{-5} rad $\approx 10''$
10^{-4}	2.5×10^{-7} rad $\approx 0.05''$	5×10^{-6} rad $\approx 1''$
10^{-5}	2.5×10^{-8} rad $\approx 0.005''$	5×10^{-7} rad $\approx 0.1''$

Table 4.21

Maximum navigation error ε_{10} due to misstuning, for a system with the average component accuracy δ, when the carrying vehicle accelerates to the velocity v_o in a time considerably shorter than $\frac{1}{\omega_s} \approx 800$ sec.

The synthesis of the vertical indicating system essentially was to apply a torque to the pendulum proportional to the angular acceleration of the pendulum. The synthesis involved two differentiations. Because of the noise generated when differentiating high frequency signals it is highly desirable to cut the high frequency part of the signal. By doing so the Schuler-tuning condition is only satisfied at low frequencies. The following example shows the order of magnitude of the error and how this depends on the cut off frequency.

Example 2

Suppose that

$$A = \frac{mrh - J_P}{1 + pT}$$

This means that the Schuler-tuning condition is exactly satisfied at zero-frequency. Equation 4.21 gives

$$\varepsilon_1(p) = \frac{(mrh - J_P)Tp}{J_P T p^3 + mrh\ p^2 + mgh\ Tp + mgh} \cdot \frac{a(p)}{r}$$

Introducing

$$\gamma = \frac{J_P}{mrh}$$

$$\omega_s^2 = \frac{g}{r}$$

and assuming

$$\gamma \ll 1$$

$$\gamma\ \omega_s^2 T^2 \ll 1$$

we get

$$\varepsilon_1(p) = \frac{Tp}{(1+\gamma Tp)(p^2+\omega_s^2 Tp+\omega_s^2)} \cdot \frac{a(p)}{r}$$

Assuming that the vehicle accelerates to the velocity v_o in a time considerably shorter than $\frac{1}{\omega_s} \approx 800$ sec we get the error

$$\varepsilon_1(p) = \frac{v_o}{r} \cdot \frac{pT}{(1+\gamma Tp)(p^2+\omega_s^2 Tp+\omega_s^2)}$$

Further assuming

$$\omega_s T \ll 1$$

we get

$$\varepsilon_1(t) \approx \frac{v_o T}{r} \cdot \left[e^{-\frac{\omega_s^2 T}{2} t} \cos \omega_s t - e^{-\frac{t}{\gamma T}} \right] \qquad t \geqslant T$$

Notice that damping is introduced in the system in this way. For a further discussion of this question we refer to reference 2 and 4.

The maximum value of the navigation error is

$$\varepsilon_{10} \approx \frac{v_o T}{r} = \frac{v_o \, \omega_s T}{\sqrt{R \cdot g}} \qquad 4.23$$

Compare equation (4.22). A timeconstant T thus gives the same error as an overall component accuracy of $\delta = \omega_s T$. Introduce the cut-off frequency $\omega_o = \frac{1}{T}$.

Some numerical values are given below

$\omega_o = \frac{1}{T}$	$\delta = \omega_s T$
0.125 rad/sec ≈ 0.02 p/sec	10^{-2}
1.25 rad/sec ≈ 0.2 p/sec	10^{-3}
12.5 rad/sec ≈ 2 p/sec	10^{-4}
125 rad/sec ≈ 20 p/sec	10^{-5}

The corresponding values of the navigation error can then be obtained from table 4.21.

Notice the order of magnitude of the cut-off frequency required for a given navigation error. It is thus no need for a high frequency differentiation, which decreases the noise problem.

This example just shows the effect of having a frequency-depending tuning. In practice the transfer-function A can depend on frequency in a much more complicated way than was the case in this example. Compare reference 4 and example 2 of reference 5, section 3.

4.22 We will now analyse the navigation error due to disturbing torque acting on the pendulum. The analytical relationship is

$$\varepsilon_2(p) = \frac{1}{(J_p+A)p^2 + mgh} \cdot M(p) \qquad 4.24$$

where

$\varepsilon_2(t)$	the navigation error due to disturbing torque acting on the pendulum
$M(t)$	the disturbing torque acting on the pendulum
J_p	the moment of inertia of the pendulum with respect to the pivot point
mh	the unbalance of the pendulum
g	the acceleration of gravity

If a torque with constant magnitude M_o is applied to the pendulum according to equation 4.22 the steady state error is

$$\varepsilon_{20} = \frac{M_o}{mgh} \qquad 4.25$$

Hence the greater the unbalance mh the less is the amplitude of the error. The unbalance mh is limited by the torque-capacity of the torque-motor acting on the pendulum.

Introduce

T	the torque-capacity of the torque-motor
a_{max}	the maximum acceleration of the vehicle

then

$$T = mh \cdot a_{max} \qquad 4.26$$

Some numerical examples of the equations (4.25) and (4.26) are given in table 4.22

Table 4.22

mh [m]	T [Nm]		M_o [dyncm]	
	$a_{max}=5\ m/s^2$	$a_{max}=50\ m/s^2$	$\varepsilon_{20}=1''$	$\varepsilon_{20}=100''$
10^{-4}	0.0005	0.005	0.05	5
10^{-3}	0.005	0.05	0.5	50
10^{-2}	0.05	0.5	5	500
10^{-1}	0.5	5	50	5000
10^{0}	5	50	500	50000

From this table we can estimate the order of magnitude of the unbalance and the quality of the suspension.

Torque-motors of reasonable size today available, have a torque capacity of the order of 1 Nm. A system for a vehicle with the maximum acceleration 5 m/sec^2 e.g. should have an unbalance of 0.2 kgm. If "second of an arc accuracy" is wanted the suspension shall have a constant friction torque less than 50 dyncm.

4.3 Errors specific to systems of type 1

4.31 There are two more terms in equation (3.27) for the navigation error of the system of type 1. The first of these terms is the error due to unbalance-torque acting on the float of the gyro.

$$\varepsilon_3(p) = \frac{1}{Hp} \cdot \frac{Y_o(p)}{1+Y_o(p)} \cdot \frac{mgh}{\left[Jp + A\right] p^2 + mgh}\, m(p)$$

where H is the angular momentum of the gyrorotor. The properties of the transfer function $Y_o(p)$ depends on the inertial stabilized platform system. The properties of $Y_o(p)$ are extensively discussed in the appendix of reference 6. One possible $Y_o(p)$ is

$$Y_o(p) = \frac{p^2 + 2\,\zeta\,\beta\,p + \beta^2}{p}$$

where β has the order of magnitude 100 rad/sec and $\zeta \approx 0.7$.

For disturbing torques m(t) with very low frequencies, we get

$$\varepsilon_3(t) \approx \frac{1}{H} \int_0^t m(\tau)\, d\tau$$

This means that a constant disturbing torque m_o gives a navigation error increasing linearly with time

$$\varepsilon_3(t) = \frac{1}{H} m_o t$$

If it is assumed that the disturbing torque is a stationary random process with zero average, we get the following asymptotic expression for the R.M.S. of the navigation error

$$\sqrt{E\left\{\varepsilon_3(t)^2\right\}} = Q \cdot \sqrt{t}\ ^{*}$$

where Q is the quality figure of the gyro

$$Q^2 = \frac{1}{H^2} \int_{-\infty}^{\infty} R_{mm}(\tau) d\tau$$

and $R_{mm}(\tau)$ is the autocorrelation function of the disturbing torque m(t). Compare reference 6, section 9.

The time required for the RMS of the navigation error to reach the asymptotic values depends on the properties of the correlation function $R_{mm}(t)$ and the transfer function

$$\frac{Y_o(p)}{1+Y_o(p)} \cdot \frac{mgh}{\left[Jp+A\right]p^2 + mgh}$$

After a time of the order of magnitude of the step-function response time** for the above transfer function we have

$$Q^2 \approx \frac{1}{H} \int_{-t}^{t} R_{mm}(\tau) d\tau$$

The character of the expected value of the error in case of a constant unbalance and a stationary disturbing torque acting on the float of the gyro is shown in figure 4.31.

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* $E\{x\}$ denotes the ensemble average of x.

** As the characteristic frequency and the damping coefficient for the transfer function is very low the step-function response time is very high, of the order of several hours.

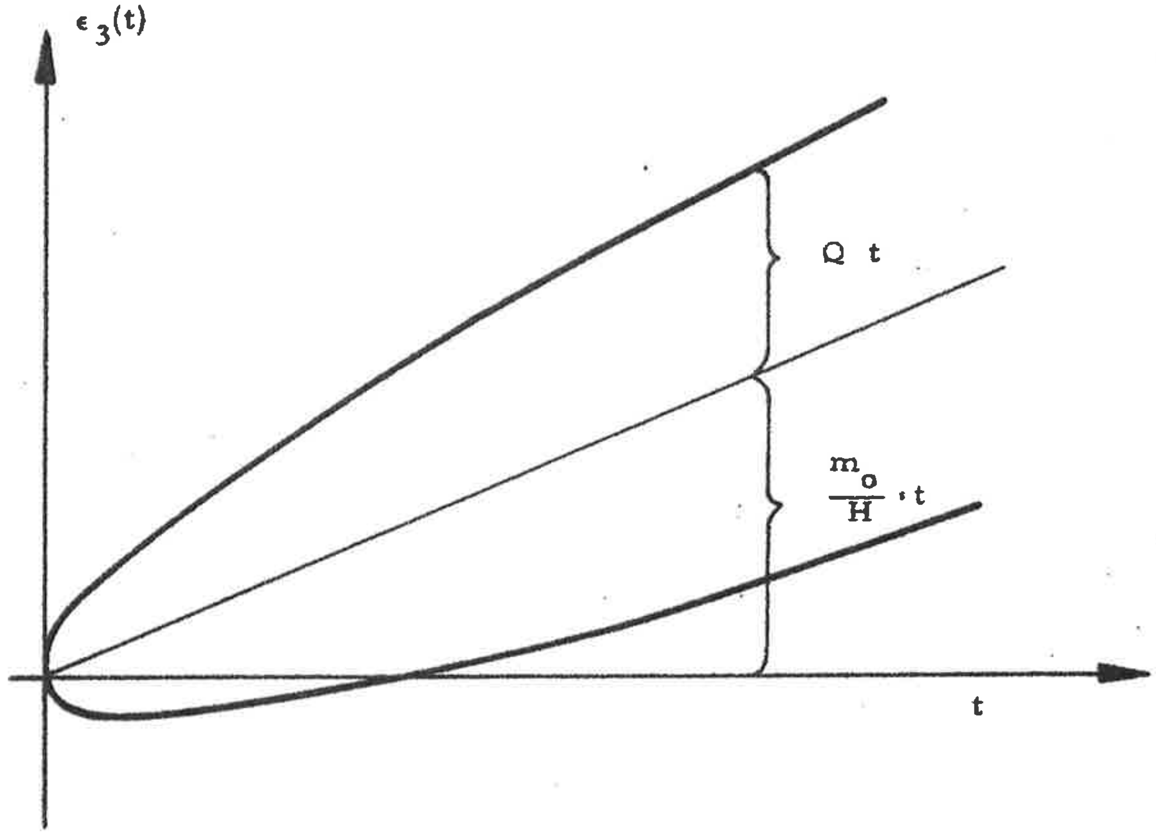

Figure 4.31

Expected navigation error in case of a constant unbalance and a stationary disturbing torque acting on the float of the gyro

4.32 The last term in the equation (3.27) is the navigation error $\varepsilon_4(t)$ caused by disturbing torques acting on the gimbal of the single-axis inertial stabilized platform system. Equation (3.27) gives

$$\varepsilon_4(p) = \frac{1}{bJp^2} \cdot \frac{1}{\left[1+Y_o(p)\right]} M_1(p)$$

where

$\varepsilon_4(t)$	the navigation error caused by disturbing torque acting on the gimbal of the reference system
$M_1(t)$	the disturbing torque acting on the gimbal of the reference system
bJ	the moment of inertia of the gimbal of the reference system including all moving parts attached to it

The platform is usually designed in such a way that this error is negligible. This problem is discussed in the appendix of reference 6.

4.33 In systems of type 1 position information is obtained simply by measuring the angle between the pendulum and the space-reference mechanized by the single-axis inertial stabilized gyroplatform. In order to Schuler-tune the pendulum, the angle was differentiated twice, amplified and fed to torquemotors on the pendulum. If the angular measuring device measures the angle with an error the signal, differentiated twice, can deviate considerably from its desired value. We will give an example of the navigation error obtained for this reason.

<u>Example</u>

Assume that the system is perfectly Schuler-tuned, and that the disturbing torques are neglected but that the angular measuring device has an error $\delta(t)$. Equation (3.23) then gives

$$\frac{d^2\psi}{dt^2} + \omega_s^2\psi = -\frac{d^2\delta}{dt^2}$$

Assuming

$$\psi(0) = \dot{\psi}(0) = \delta(0) = \dot{\delta}(0) = 0$$

we get

$$\psi(t) = -\frac{1}{\omega_s}\int_0^t \ddot{\delta}(\tau)\sin\omega_s(t-\tau)d\tau$$

hence

$$\psi(t) = -\delta(t) + \omega_s\int_0^t \delta(\tau)\sin\omega_s(t-\tau)d\tau$$

Assume

$$t < \frac{1}{\omega_s}$$

then

$$|\delta(t)| < \delta_o$$

then

$$|\psi(t)| < \delta_o\left[1 + \frac{1}{2}(\omega_s t)^2\right] = \delta_o$$

$$t >> \frac{1}{\omega_s}$$

$$|\psi(t)| < \delta_o\left[1 + \frac{\omega_s t}{\pi}\right] \approx \delta_o\frac{\omega_s t}{\pi}$$

This is the worst case andit requires that $\delta(\tau)$ oscillates between the values $-\delta_o$ and δ_o in phase with $\sin\omega_s(t-\tau)$.

4.4 Errors specific to systems of type 2

4.41 Disturbing torques acting on the float of the gyro give the error

$$\varepsilon_3(p) = \frac{1}{Hp}\cdot\frac{J_p p^2 + mgh}{\left[J_p + A(p)\right]p^2 + mgh}\cdot m(p)$$

Hence the transfer function only differs from the transfer function discussed in section 4.31 at high frequencies. As the low frequency region is of highest importance we obtain the same character of the error as was the case in section 4.31.

5. SOME ASPECTS ON THE INSTRUMENTATION OF THREE-AXIS NAVIGATION SYSTEMS

5.1 Introduction

In the preceding sections we have analysed the single-axis systems in order to obtain the order of magnitude of the required component accuracy. We will now briefly discuss some aspects on the instrumentation of the complete systems. We will assume that the vehicle is moving over a rotating earth. In order to slightly simplify the algebraic work we will assume that the earth is spherical. The position of the vehicle is given in polar coordinates:

r the distance from the center of the earth to the vehicle

ℓ the longitude

L the latitude

Compare figure 5.11.

In order to further simplify the problem we will assume that the pendulum is aligned with the vertical i.e. the pendulum is coincident with the z_E-axis in figure 5.11. Assume further that another axis, fixed to the pendulum, is pointing to the north. We will then analyse the torques which should be applied to the pendulum in order to maintain this orientation of the pendulum.

The torques acting on the pendulum are

$$\bar{M} - m \cdot \bar{h} \times \bar{a}$$

where

$\bar{M}$ the torque applied to the pendulum by the torque-motors

$\bar{a}$ the acceleration of the center of mass of the pendulum with respect to inertial space

Calculating the components of the acceleration of the center of gravity with respect to inertial space we get

$$\begin{aligned} a_x &= r\ddot{L} + 2\dot{r}\dot{L} + r(\omega_{IE}+\dot{\ell})^2 \sin L \cos L \\ a_y &= r\ddot{\ell} \cdot \cos L + 2(\omega_{IE}+\dot{\ell})(\dot{r} \cos L - r\dot{L} \sin L) \qquad 5.11 \\ a_z &= -\ddot{r} + r\dot{L}^2 + r(\omega_{IE}+\dot{\ell})^2 \cos^2 L \end{aligned}$$

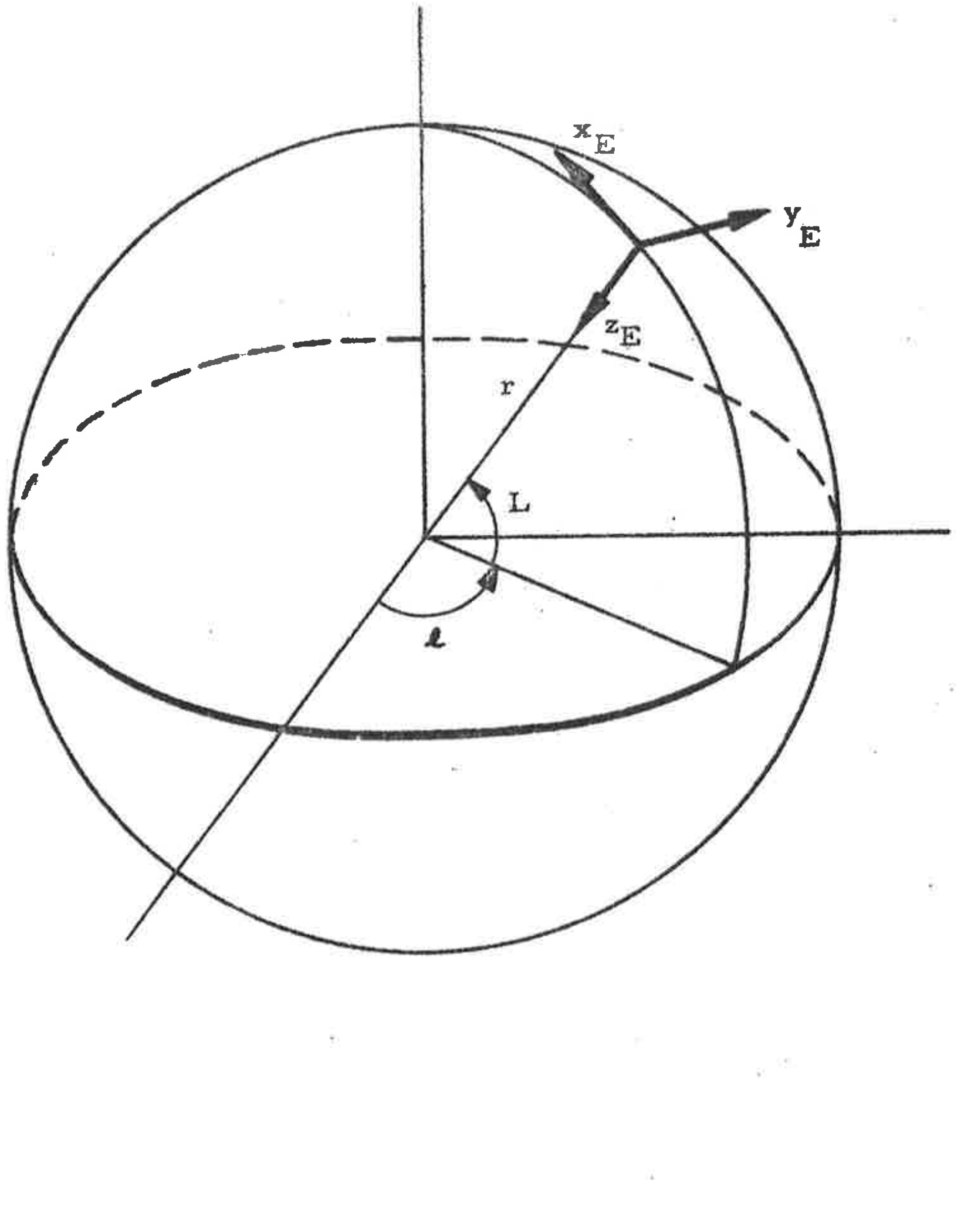

Figure 5.11

where

ω_{IE} is the angular velocity of the earth.

The angular velocity of the coordinate system, the geographical coordinates, is $\bar{\omega}_I$

$$\begin{aligned}
\omega_{Ix} &= (\omega_{IE} + \dot{\ell}) \cos L \\
\omega_{Iy} &= - \dot{L} \\
\omega_{Iz} &= - (\omega_{IE} + \dot{\ell}) \sin L
\end{aligned} \qquad 5.12$$

5.2 An approach to the instrumentation of a system of type 1

We will now discuss some instrumentational problems of a system of type 1. Applying Newtons second law of motion to the pendulum we get

$$\bar{M} - m\bar{h} \times \bar{a} = \frac{d}{dt} \bar{H}_p$$

where

$\bar{H}_p$ the angular momentum of the pendulum

The time derivative of the angular momentum of the pendulum contains terms of the type

$$J \ddot{\Theta} \quad \text{and} \quad J \dot{\Theta}^2$$

Introduce

$$J = 10^{-2} \text{ kgm}^2$$

$$\ddot{\Theta}_{max} = \frac{a_{max}}{R}$$

$$\dot{\Theta}_{max} = \frac{v_{max}}{R}$$

and assume

$$a_{max} = 64 \text{ m/sec}^2$$

$$v_{max} = 640 \text{ m/sec}$$

we get

$$J\ \ddot{\theta} \approx 10^{-7}\ \text{Nm} = 1\ \text{dyncm}$$

$$J\ \dot{\theta}^2 \approx 10^{-10}\ \text{Nm} = 10^{-3}\ \text{dyncm}$$

At present we will neglect these terms. Doing this we get

$$\bar{M} = m\ \bar{h}\ x\ \bar{a}$$

Evaluating the cross-product we get the following expression for the components of the torque

$$M_x = mh\left[r\ddot{\ell}\cos L + 2(\omega_{IE}+\dot{\ell})(\dot{r}\cos L - r\dot{L}\sin L)\right]$$

$$M_y = -mh\left[r\ddot{L} + 2\,\dot{r}\dot{L} + r(\omega_{IE}+\dot{\ell})^2 \cdot \sin L\cos L\right] \qquad 5.21$$

The terms $mhr\ddot{\ell}\cos L$ and $mhr\ddot{L}$ give rise to the high apparent moment of inertia of the pendulum. In the single axis case the corresponding term was $mrh\ddot{\theta}$, compare section 1.2.

The other terms in the equation 5.21 are compensation torques for the corioli and centrifugalaccelerations.

Hence if the pendulum should indicate the vertical the torques M_x and M_y should be applied to the pendulum. Besides the pendulum should be rotated around an axis parallel to the polaraxis of the earth with the velocity

$$\omega_{Ip} = \omega_{IE} + \dot{\ell} \qquad 5.22$$

A functional diagram of the system is shown in figure 5.21.

In order to apply the torques M_x and M_y and the angular velocity ω_{Ip} the control signals to the torquemotors and to the polar-axis drive must be computed.

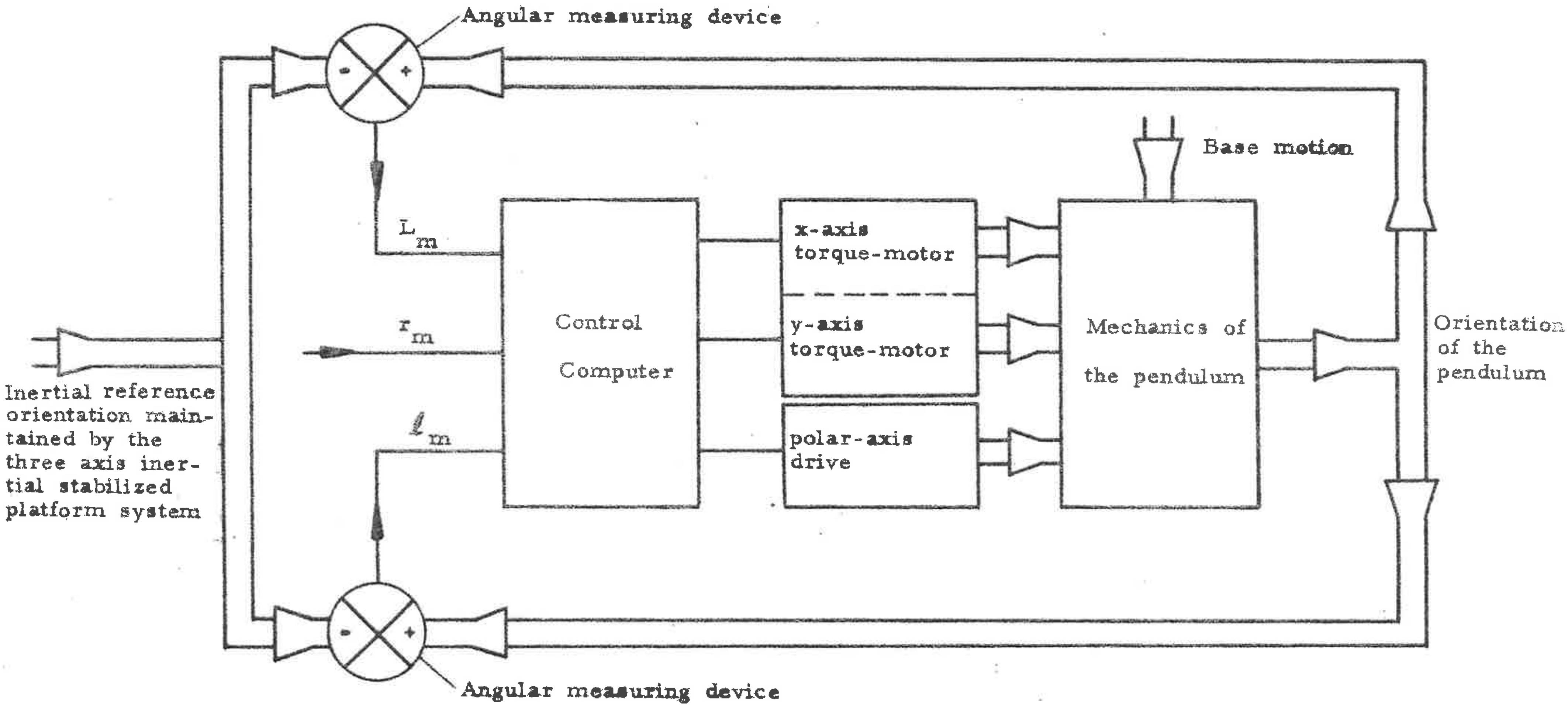

Figure 5.21

Functional diagram of a three-axis navigation system of type 1.

In the system are available the following signals

ℓ_m* the longitude, L_m the latitude — these are obtained by measuring the angle between the pendulum and the inertial stabilized platform system

r_m the height is obtained from the vertical accelerometer and/or barometric and radio devices

$\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$ — the output signals of the gyros

From the equations (5.21) and (5.22) it is obvious that the desired control signals can be computed from the signals ℓ_m, L_m and r_m. Figure (5.21) shows the block-diagram of the system. One way of calculating the control signals is shown in figure (5.22).

* The subscript m stands for "measured". In the discussed case the pendulum is aligned to the geographic coordinates, and the inertial reference has no errors hence

$$\ell_m = \ell, \quad L_m = L$$

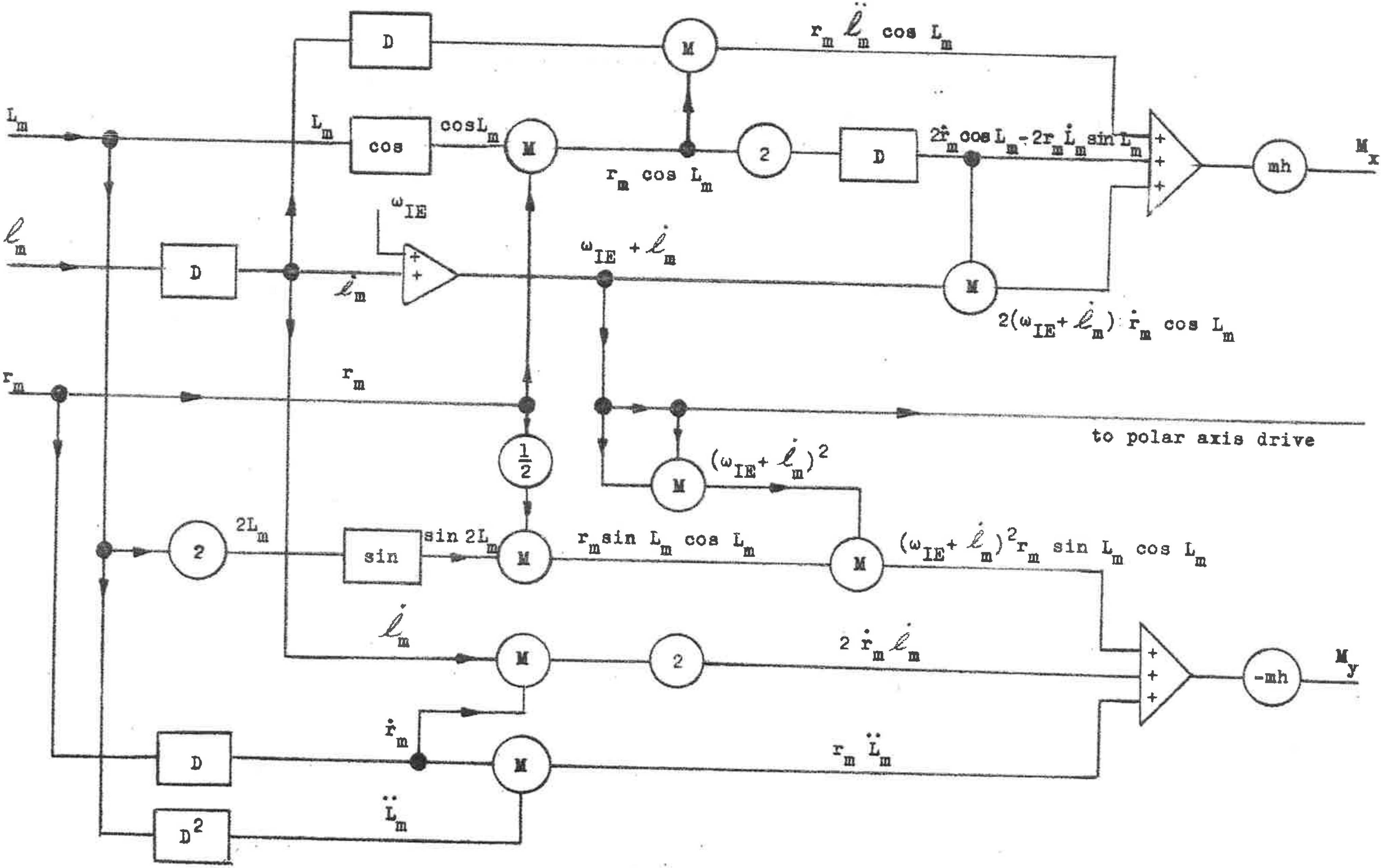

Figure 5.22

The control computer for the system of figure 5.21.

5.3 An approach to the instrumentation of a system of type 2

In a system of type 2 the gyros are attached to the pendulum. We assume that the gyros have the x,y- and z-axes as their input axes. It is further assumed that the gyros are rate-coupled. Neglecting the time-derivative of the angular momentum of the pendulum including the gyros, and assuming that the pendulum is aligned to the geographic coordinates Newtons second law of motion gives

$$\bar{M} = m\, \bar{h} \times \bar{a}$$

where

$\bar{M}$ the torque which should be applied to the pendulum if it should maintain the desired orientation

$\bar{a}$ the acceleration of the vehicle

$\bar{h}$ the vector from the pivot point to the center of mass of the pendulum

The components of the torque $\bar{M}(t)$ is given by the equation (5.11). If the pendulum should be aligned to the geographic coordinates the torques M_x and M_y should be applied to the pendulum. Besides the pendulum should be rotated around the z-axis with the angular velocity

$$\omega_{Iz} = -(\omega_{IE} + \dot{\ell})\sin L \qquad 5.31$$

This angular rotation can be obtained by arranging the z-axis loop as an integrating drive and feeding the signal-generator of the z-gyro with a signal proportional to ω_{Iz}. A block-diagram of the system is shown in figure 5.31.

In order to apply the torques M_x and M_y and the signal to the integrating drive the control signals for these devices must be computed.

In the system are available, the output signals of the gyros and a signal proportional to the height r of the vehicle. Assuming the gyros to work as perfect rate-gyros with sensitivities 1, the output signals of the gyros are equal to the angular velocity of the pendulum i.e.

$$\varphi^{(x)} = W_{Ix}$$

$$\varphi^{(y)} = W_{Iy}$$

$$\varphi^{(z)} = W_{Iz}$$

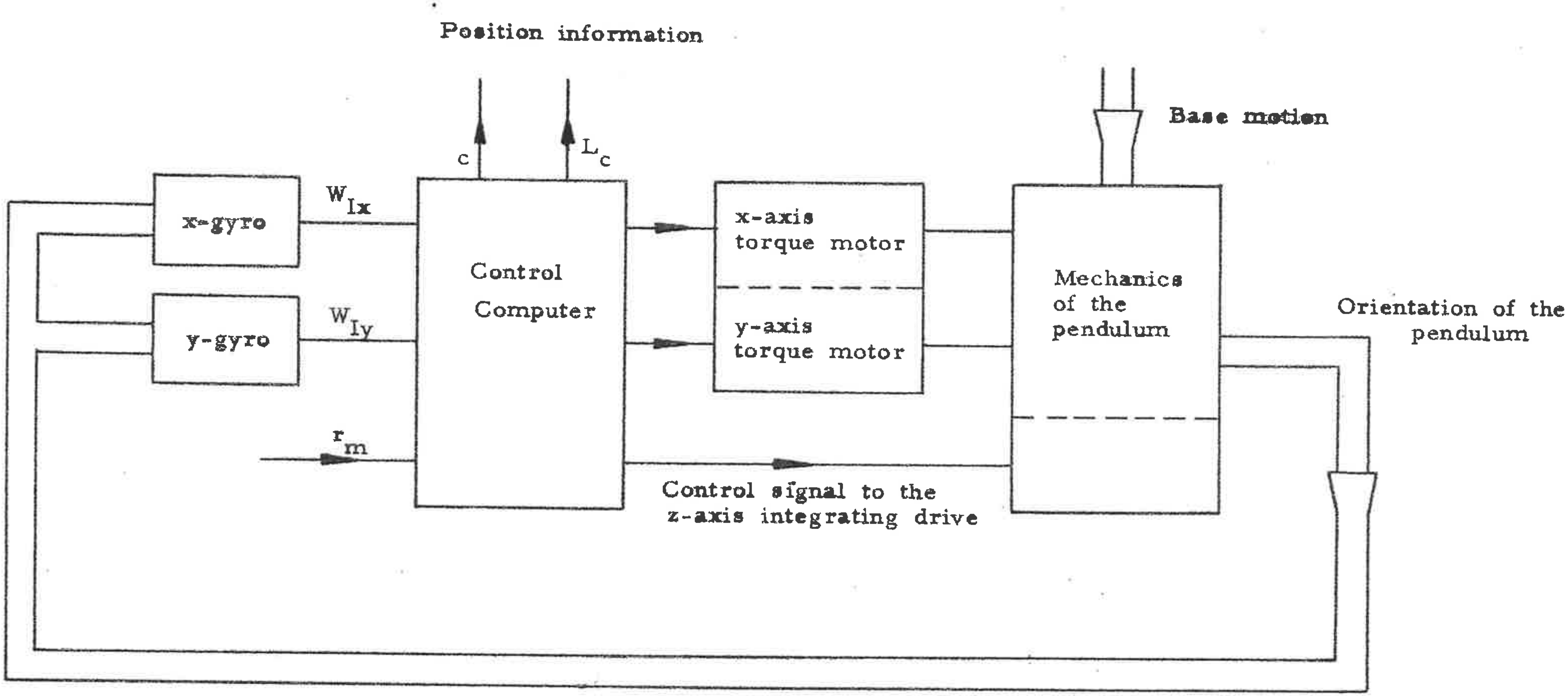

Figure 5.31

Functional diagram of a three-axis system of type 2. The z-axis loop is coupled as an integrating drive.

The angular velocity of the pendulum is denoted by $\bar{W}_I$. In the discussed case the pendulum was assumed to be aligned to geographical coordinates, which means

$$\bar{W}_I = \bar{\omega}_I$$

Eliminating ℓ and L between the equations 5.12 and 5.21 we get

$$\left\{\begin{aligned} M_x &= mh\left[r\dot{\omega}_{Iy} + 2\ r\omega_{Iy} - r\omega_{Ix}^2\ tg\ L\right] \\ M_y &= mh\left[r\dot{\omega}_{Ix} + 2\ \dot{r}\omega_{Ix} + r\omega_{Ix}\omega_{Iy}\ tg\ L\right] \\ \omega_{Iz} &= -\ \omega_{Ix}\ tg\ L \\ L &= \int^{t} -\ \omega_{Iy}\ dt \end{aligned}\right. \qquad 5.32$$

The desired control signals can thus be computed from the output signals of the gyro. The computation is illustrated in figure 5.32.

5.4 Some remarks concerning the approximation used. The equations of motion in case of missalignment between the instrumented and the geographical coordinates

The analysis of the previous sections shows how the systems can be instrumented. The calculations were based on the assumption that the instrumented coordinates were aligned to the geographical coordinates and that the earth was spherical. If we instead assume an elliptic earth the torques M_x and M_y are given by

$$M_x = mh\left[r(1-2e\ \cos^2 L)\dot{\omega}_{Ey} + 2\dot{r}(1-2e\ \cos\ 2\ L)\omega_{Iy} + \right.$$
$$\left. + \ r\ \omega_{Ix}^2\ tg\ L + \ddot{r}\ e\ \sin\ 2\ L + \sigma(e)\right]$$

$$M_y = mh\left[r(1+2e\ \sin^2 L)\dot{\omega}_{Ix} + 2\dot{r}(1+2e\ \sin^2\ L)\omega_{Ix} + \right.$$
$$\left. + \ r\ \omega_{Ix}\omega_{Iy}\ tg\ L + \sigma(e)\right]$$

where e is the ellipticity of the earth. See reference 2.

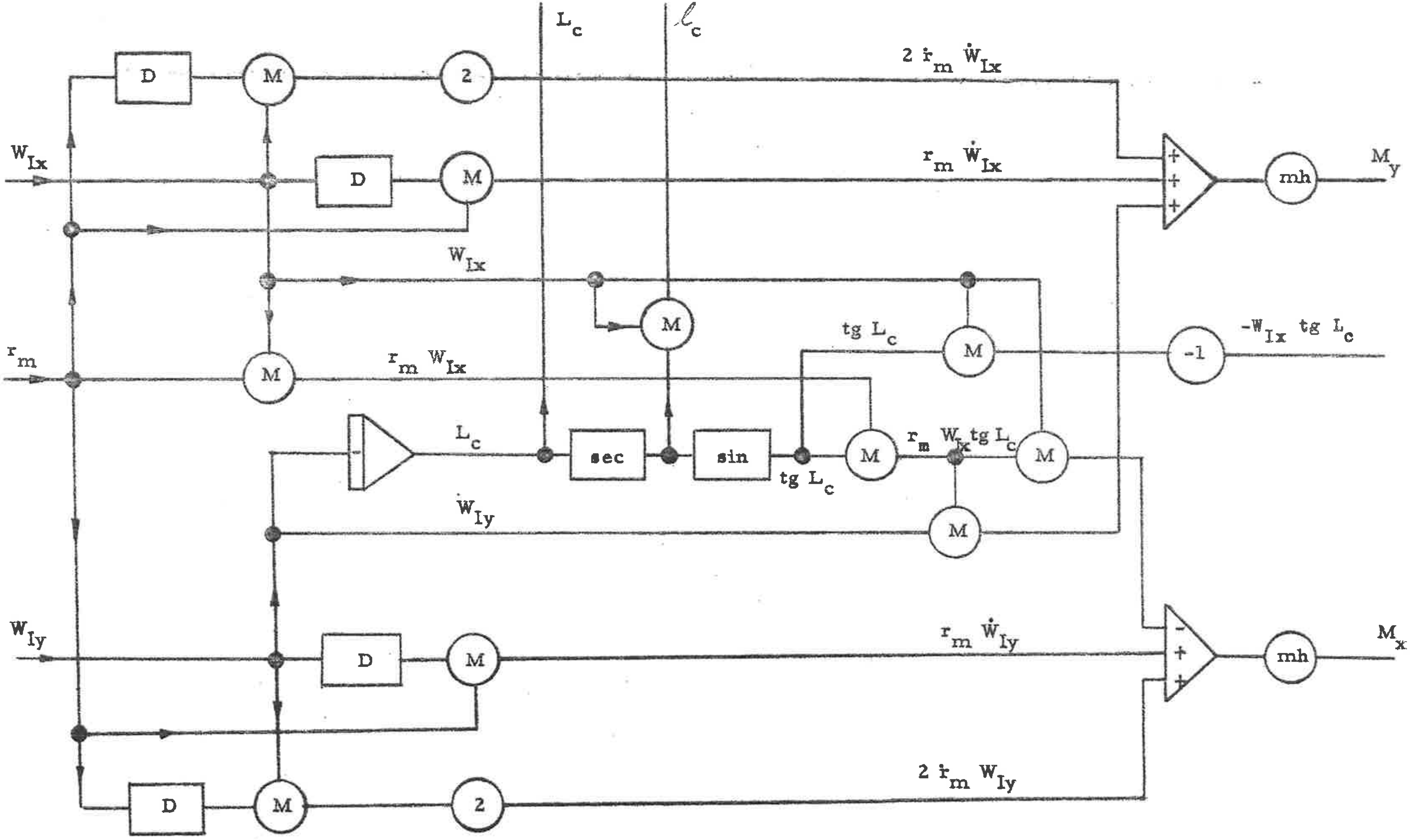

Figure 5.32

The control computer for the system shown in figure 5.31. D=d/dt differential operator

In this case the computation changes somewhat but there is no essential difference to the case with the spherical earth we have been discussing.

When we derived the equations of motion in section 5.1 we assumed that the coordinates indicated by the pendulum were aligned to the geographical coordinates. We will now briefly discuss what happens if this is not the case. Assume that the coordinates are missaligned and let the successive Euler angles for the transformation from the instrumented coordinates to the geographical coordinates be c_x, c_y and c_z.

If we assume that the angles c_x, c_y and c_z are small the transformations commutate. The transformation matrix from indicated to geographic coordinates is then

$$\begin{pmatrix} 1 & c_z & -c_y \\ -c_z & 1 & c_x \\ c_y & -c_x & 1 \end{pmatrix}$$

The angular velocity of the indicated coordinate system with respect to inertial space is thus

$$\bar{W}_I = \begin{pmatrix} 1 & -c_z & c_y \\ c_z & 1 & -c_x \\ -c_y & c_x & 1 \end{pmatrix} \bar{\omega}_I - \begin{pmatrix} \dot{c}_x \\ \dot{c}_y \\ \dot{c}_z \end{pmatrix}$$

hence

$$\left\{ \begin{aligned} W_{Ix} &= \omega_{Ix} - c_z\omega_{Iy} + c_y\omega_{Iz} - \dot{c}_x \\ W_{Iy} &= c_z\omega_{Ix} + \omega_{Iy} - c_x\omega_{Iz} - \dot{c}_y \\ W_{Iz} &= -c_y\omega_{Ix} + c_x\omega_{Iy} + \omega_{Iz} - \dot{c}_z \end{aligned} \right. \qquad 5.41$$

Neglecting the time-derivative of the angular momentum of the pendulum, including the gyros, the x- and y-components of the equation of motion of the pendulum becomes

$$\begin{cases} -gc_x+r(\dot{W}_{Ix}-\dot{\omega}_{Ix})+2\dot{r}(W_{Ix}-\omega_{Ix})+r(W_{Ix}W_{Iy}tgL_c-\omega_{Ix}\omega_{Iy}tgL)=0 \\ -gc_y+r(\dot{W}_{Iy}-\dot{\omega}_{Iy})+2\dot{r}(W_{Iy}-\omega_{Iy})+r(W_{Ix}^2tgL_c-\omega_{Ix}^2tgL)=0 \end{cases} \qquad 5.42$$

Assuming the integrating drive for the z-axis to be perfect the angular velocity of the pendulum about the z-axis is

$$W_{Iz} = - W_{Ix} \text{ tg } L_c \qquad 5.43$$

Given the motion of the vehicle and the initial conditions the missalignment of the coordinate system can be computed from the equations (5.41), (5.42) and (5.43).

5.5 The initial alignment of the system

In the systems we have been discussing the coordinates indicated by the pendulum should be aligned to the geographic coordinates north-east and down. Aligning the system to the vertical is easily obtained as the pendulum has an unbalance. The aligning in azimuth is a little more difficult to obtain. One scheme for this used in some MIT constructions is to convert the vertical indicating system to a gyrocompass. A very elegant way of doing this is given in reference 2, where it is shown that the north finding property can be obtained by introducing a velocity coupling between the north - and azimuth channels of the vertical indicating system. We will now show that the same method can be used in the system discussed in this report. We will show this for the system of the second type discussed in section 5.3.

The coupling between the z-axis loop and the y-axis loop is obtained by feeding a signal proportional to the angular velocity of the pendulum about the y-axis to the integrating drive on the z-axis. We thus take the output signal from the y-gyro and feed it through a sensitivity S_z to the torque-generator of the z-gyro.

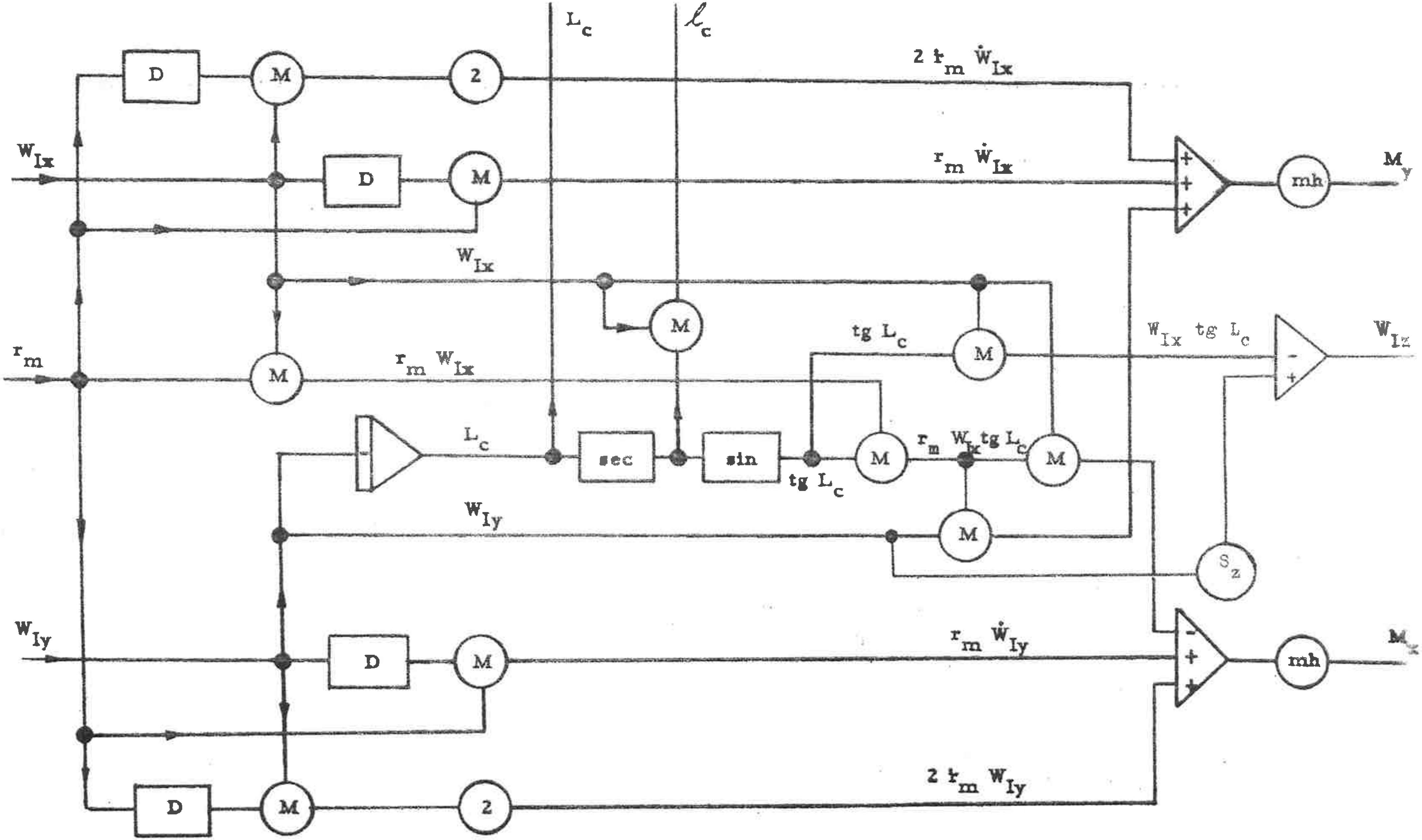

Figure 5.51

The control computer for the system shown in figure 5.31, with gyrocompassing for initial alignment.

Neglecting the errors in the integrating drive we get

$$W_{Iz} = - W_{Ix} tgL_c + S_z W_{Iy} \qquad 5.51$$

In order to do this we have to change the computer according to figure 5.51. Besides we introduce damping in the system. This is obtained e.g. by substituting the pure differentiation in the y-axis loop by a filter DF(D). See figure 5.51.

Example

We will now show that the system really has a north-finding property. In order to simplify the algebraic work we make the following assumptions. Compare reference 2.

1. The vehicle is restricted to move in the meridian plane and at constant height, i.e.

$$\omega_{Ix} = \omega_{IE} \cos L = \omega_{IEH}$$

$$\omega_{Iy} = - \dot{L}$$

$$\omega_{Iz} = - \omega_{IE} \sin L = - \omega_{IEV}$$

$$r = \text{const}$$

2. The pendulum does not tilt about the north-south line, i.e.

$$c_x = 0$$

With these assumptions the angular velocity components is as follows

$$W_{Iy} = c_z \, \omega_{IEH} - \dot{L} - \dot{c}_y$$
$$\qquad 5.52$$
$$W_{Iz} = - c_y \, \omega_{IEH} - \omega_{IEV} - \dot{c}_z$$

Using the assumptions the y-component of the equation of motion of the pendulum becomes

$$- \frac{g}{r} c_y + \left[F(D) \, \dot{W}_{Iy} + \ddot{L} \right] = 0 \qquad 5.53$$

In a Schuler-tuned system we have

$$F(D) = 1$$

Due to the assumption 2 the angular velocity of the pendulum about the z-axis is

$$W_{Iz} = - W_{Ix} \, tg \, L_c + S_z \, W_{Iy} \tag{5.54}$$

Introducing the angular velocity components according to the equation (5.52) we get

$$\begin{pmatrix} D^2F(D) + \frac{g}{R} & - DF(D)\omega_{IEH} \\ -S_z D + \omega_{IEH} & D + S_z \omega_{IEH} \end{pmatrix} \begin{pmatrix} c_y \\ c_z \end{pmatrix} = \begin{pmatrix} \left[1-F(D)\right] D^2 L \\ S_z DL \end{pmatrix}$$

From this equation it is obvious that the system is north-seeking. The stationary error is obviously zero.

This simple example shows the north-seeking property of the system. In practice the situation is more complicated as we have to consider arbitrary motions of the vehicle and the fact that $c_x \neq 0$. An analysis of this case is far beyond the scope of this report.

For systems of type 1 the north-seeking properties are obtained analogously by feeding a signal proportional to the y-component of the angular velocity of the pendulum to the polar-axis drive.

6. CONCLUSIONS

The characteristic feature of the systemsdiscussed in this report is the method of synthesizing the vertical indicating system by using an ordinary physical pendulum whose apparent moment of inertia is made very high by electromechanical aids. The navigation systems synthesized in this way require no accelerometers. This is perhaps a little missleading to say as the pendulum of cause can be regarded as an accelerometer. Anyway there seems to be a possibility of reducing the number of expensive components by using the described method of synthesizing the navigation system. Another feature of the system is that there are differentiators in the feedback loop instead of the integrators used in the MIT systems. The drawbacks of the proposed system are the high demands for linearity in the gimbal torquemotors and the requirements on the low-friction suspension of the pendulum.

There are many questions left out of this discussion, such as the resolution of the gyros, non-linear methods for damping the system etc. Another question of great interest is the possibility of using a computer controlled system which offer many interesting possibilities.

7. REFERENCES

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