LUND UNIVERSITY

Properties of a Parameterized Model Reduction Method

Sootla, Aivar

2010

Total number of authors:
1

Properties of a Parameterized Model Reduction Method

Aivar Sootla

Department of Automatic Control,

Lund University, Lund, Sweden, Box 118 SE 221 00

Email: aivar@control.lth.se

Abstract— In this contribution a recently proposed model reduction method for a class of linear time-invariant (LTI) parameterized models is investigated. The method is based on matching of the frequency response samples using the semidefinite programming methods. The main focus of this contribution is the properties of the obtained approximations. Among those properties is stability of individual LTI systems, continuity with respect to parameters, error bounds on approximation quality.

I. INTRODUCTION

Model reduction is a well-established field of research. The problem itself is non-convex and to authors best knowledge there exists no polynomial time method to obtain the optimal approximation. There does exist quite a few suboptimal and heuristic techniques to obtain one. Among them two families of methods are distinguished: singular value decomposition (SVD) based and Krylov subspace projection based methods. The methods are well-developed and an interested reader may find their description in [1], [2] and the references therein.

Model reduction as a tool is often required in modeling, where simulating large-scale models may be computationally overwhelming. Such models often arise in physical modeling, in which linear integro-differential equations (e.g. Maxwell's, heat transfer equation) are discretized using Petrov-Galerkin or finite element approach (see, [3], [4], [5]) creating LTI systems. If a low-order model with a similar behaviour was obtained, the simulation time would be considerably reduced. A common reduction technique to use is the Krylov subspace projection one (for general framework description, see [6], for application to various problems, see [7], [8], [9], [10]).

Parameterized model reduction is arguably a more important tool in modeling and design. Since the parameters of the models are subject to change or tweaking over time, it is often required to obtain a family of models that describe a particular system in various settings. An extensive research using the Krylov techniques was performed for various applications, e.g. micro-electro-mechanical systems or MEMS (see, [11]), radio-frequency (RF) inductors (see, [12]), interconnects ([13], [14], [15], [16]), general linear systems ([17], [18], [19], [20]) and nonlinear systems ([21]). A major drawback of the Krylov techniques is the lack of flexibility. Imposing extra properties on a reduced model or preserving properties of the full one is generally a hard task and requires a rigorous investigation in each case.

For example, if passivity can be preserved efficiently by extending [22] to the parameterized case, preserving stability can be done only for an extra cost.

In this contribution a different direction is taken. As a basis a method for parameterized model reduction of single-input-single-output (SISO) systems from [23] is considered. The method performs matching of the frequency response samples of the full model and its approximation. It exploits a Hankel-type relaxation to obtain a semidefinite program. The method can preserve stability, passivity of the individual LTI models or any other property that can be parameterized in a convex manner making this method a powerful tool. The computational cost for the non-parameterized model reduction mainly consists of the frequency response calculation and is equal to $O(n^3)$, where n is the order of the original model. Using [24], [25], [26] or [27] for calculating the frequency response the cost may be lowered to $O(n \log(n))$, which is valid for certain applications. For the parameterized case the cost estimates are shown in [23], [28].

The main contribution of this paper is the theoretical error bounds similar to the non-parameterized case shown in [29]. Such bounds are valid for the single-input case and were not shown in the original work. Also a multi-input-multi-output (MIMO) extension to a single-input model reduction method from [23] is presented. Some modification to the original SISO method were made in [30], [31] that may be useful depending on an application. Investigation into the applications was made in [32].

The paper is organized as follows. In Sec. II the problem formulation is described and the main ideas of solution for the non-parameterized case are shown. Sec. III presents the model reduction method for parameter-dependent systems. In Sec. IV the investigation into the properties of approximations is made and the continuity of the latter is proved. Finally, in Sec. V an example is presented, which is the modeling of a deformable telescope mirror.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider a rational transfer function $G(z, \theta)$, where for every fixed θ the function $G(\cdot, \theta)$ is an LTI model. It is also assumed that θ does not depend on time or the state-space variables of G. Basically, all the coefficients of the transfer function $G(z, \theta)$ depend on θ. The dependence of G on θ is considered to be continuous, which is a common assumption in many applications. The parameter is usually a vector $\theta = (\theta_1, \ldots, \theta_n)$, where the individual entries θ_i are bounded:

$\underline{\theta}_i \leq \theta_i \leq \overline{\theta}_i$ for all $i = 1, \ldots, n$, and $\underline{\theta}_i, \overline{\theta}_i$ are known constants. Also the infinity norm of G is uniformly bounded with respect to θ, i.e. there exists κ such that $\|G(\cdot, \theta)\|_\infty \leq \kappa$ for all θ.

For simplicity the case with a scalar θ is considered. The generalization to an arbitrary dimension n can be performed in a similar manner with the same theoretical results. Since θ is bounded it is assumed that $\theta \in [0, \pi]$. If the bounds are different, one can always map θ to this interval. It is done for a convenient notation: $\boldsymbol{\omega} = \{\omega, \theta\} \in [0, \pi]^2$. Denote $G(\boldsymbol{\omega}) = G(\omega, \theta)$ the frequency response of $G(\cdot, \theta)$ for every fixed θ. A similar notation is used for all transfer functions.

The reduction problem is formulated as:

$$\gamma_{min-max} = \min_{P,Q} \max_\theta \left\| G(\boldsymbol{\omega}) - P(\boldsymbol{\omega})Q(\boldsymbol{\omega})^{-1} \right\|_\infty,$$

where P and Q are polynomials in $e^{-\jmath\omega}$ with coefficients P_i, Q_i depending on θ:

$$P(\omega, \theta) = \sum_{i=0}^{k} P_i(\theta)e^{-ij\omega} \quad Q(\omega, \theta) = \sum_{i=0}^{k} Q_i(\theta)e^{-ij\omega}$$

where $\jmath$ is a complex identity. The dependence of P_i and Q_i on θ is continuous in order to achieve a continuous dependence of PQ^{-1} on θ. For every fixed θ the inverse of polynomial Q should provide a stable transfer function. Hence, Q is a minimum phase transfer function (all the poles and the zeros are inside the unit disc $\mathbb{D} = \{z \,|\, |z| \leq 1\}$, where the variable z is an analytic extension of ω to the complex plane such that $z = e^{\jmath\omega}$ if $|z| = 1$) and the feedthrough term (Q_0) has to be invertible for all values θ.

Note that the min-max problem is preferred to a max-min, which is formulated as:

$$\gamma_{max-min} = \max_\theta \min_{P,Q} \left\| G(\boldsymbol{\omega}) - P(\boldsymbol{\omega})Q(\boldsymbol{\omega})^{-1} \right\|_\infty,$$

Generally $\gamma_{max-min} \leq \gamma_{min-max}$, and in this case the equality may not be achieved since the objective function is not convex in decision variables. However, a convex formulation for max-min problem with a polynomial time algorithm has not be obtained, hence it will not be considered.

A. Sketch of the Solution for the Non-Parameterized Case.

In the non-parameterized case, the coefficients P_i and Q_i are constant matrices.

First consider the scalar case, i.e. G, P, Q are scalar valued functions. In the scalar case denote the numerator and the denominator with the small letters:

$$p = \sum_{i=0}^{k} p_i e^{-ij\omega} \quad q = \sum_{i=0}^{k} q_i e^{-ij\omega}$$

The infinity norm is reformulated as a minimization with an infinite number of constraints (for every frequency ω):

$$\min_{p,q} \gamma \quad \text{subject to} \quad \left| G(\omega) - \frac{p(\omega)}{q(\omega)} \right| < \gamma \quad \forall\omega,$$

Here $G(\omega)$ denotes the frequency response of G. Multiplication of both sides of the inequality with $|q(\omega)|^2 = q(\omega)q^\sim(\omega)$ yields:

$$\min_{p,q} \gamma \quad \text{subject to}$$

$$\left| G(\omega)|q(\omega)|^2 - p(\omega)q^\sim(\omega) \right| < \gamma|q(\omega)|^2 \quad \forall\omega$$

where operator $^\sim$ denotes an adjoint in $\mathcal{H}_\infty$ space, i.e. $q^\sim(z) = q^T(1/z)$. Using the Schur complement properties, obtain an equivalent formulation:

$$\min_{p,q} \gamma \quad \text{subject to}$$

$$\begin{pmatrix} \gamma|q(\omega)|^2 & G(\omega)|q(\omega)|^2 - p(\omega)q^\sim(\omega) \\ * & \gamma|q(\omega)|^2 \end{pmatrix} > 0 \quad \forall\omega$$

where the asterisk denotes the Hermitian transpose of the upper right element. This program is still not convex in variables p, q. Denote $a = |q|^2$ and relax the structure of the polynomial $pq^\sim$, i.e. replace $pq^\sim$ with $b = \sum_{i=-k}^{k} b_i z^{-i}$. Now the program becomes convex in the variables a, b for any given value of γ, i.e. quasiconvex:

$$\min_{\gamma>0,a,b} \gamma \quad \text{s.t.} \quad \begin{pmatrix} \gamma a(\omega) & G(\omega)a(\omega) - b(\omega) \\ * & \gamma a(\omega) \end{pmatrix} > 0 \quad \forall\omega \tag{1}$$

The denominator q_* is obtained from the spectral factorization problem $a_* = q_* q_*^\sim$ as a minimum-phase factor. If the polynomial a_* is positive then such a factor exists. Therefore the positivity of a_* guarantees the stability of q_*. The numerator is obtained with a given denominator as:

$$\min_p \|G - pq_*^{-1}\|_\infty$$

One could impose the conditions in one LMI using the KYP lemma ([33]), for example. In this case the number of constraint will depend on the order of the transfer function G. The resulting LMI will therefore provoke heavy calculations, which will be comparable to solving the Lyapunov equations. Therefore the norm constraints in (1) will be imposed only in the finite number of points. The positivity constraint $a > 0$ should be imposed for all the frequencies, since it guarantees the existence of a stable approximation.

Note also that the proposed relaxed problem has a following property:

$$(1) = \min_{a>0,b} \|G - ba^{-1}\|_\infty \geq \min_{p,q} \|G - pq^{-1}\|_H$$

Since $a > 0$ the minimization is performed both in stable and antistable transfer functions, the second inequality follows. It means that the proposed algorithm is a restricted version of the Hankel model reduction.

Assume now that G is a m-input m-output model, then $P_i, Q_i \in \mathbb{R}^{m \times m}$. Note, that if G is a m_1-input m_2-output model, where $m_1 \neq m_2$, the method is still applicable. The same scheme as in the single variable case is used. Consider the problem:

$$\min \gamma \quad \text{subject to}$$

$$(G(\omega) - P(\omega)Q^{-1}(\omega))^\sim(G(\omega) - P(\omega)Q^{-1}(\omega)) < \gamma^2 I$$

Rewrite the matrix inequality in the minimization problem as:

$$(G(\omega)A(\omega) - B(\omega))^{\sim}(G(\omega)A(\omega) - B(\omega)) < \gamma^2 A^2(\omega) \quad (2)$$

where a similar to the SISO case variables are introduced $A = QQ^{\sim}$ and B as a substitute to $PQ^{\sim}$, (i.e. relax the structure of the latter polynomial). Unlike the single variable case, here a convex LMI condition can not be derived directly from (2).

Our goal is to obtain an LMI similar to:

$$\begin{pmatrix} \gamma A(\omega) & G(\omega)A(\omega) - B(\omega) \\ * & \gamma A(\omega) \end{pmatrix} > 0 \quad (3)$$

Note, that (3) is convex in the variables A, B, and is always equivalent to

$$(G(\omega)A(\omega) - B(\omega))^{\sim} A^{-1}(\omega)(G(\omega)A(\omega) - B(\omega)) < \gamma^2 A(\omega) \quad (4)$$

due to the Schur complement properties. The difference between (4) and (2) is the right-hand side polynomial A in (4). It can not be carried to the left-hand side in order to obtain (2) due to its matrix structure. The main idea is to replace one of the matrices A with a scalar bounded function $f(\omega)$, which is obtained by means of optimization. Also $f(\omega)I \leq A(\omega)$ will be valid for all ω's. The relaxed matrix inequalities are:

$$(G(\omega)A(\omega) - B(\omega))^{\sim}(f(\omega)\gamma)^{-1} \cdot$$
$$\cdot (G(\omega)A(\omega) - B(\omega)) < \gamma A(\omega) \quad (5)$$
$$f(\omega)I \leq A(\omega) \quad (6)$$

Now the inequality (4) follows after multiplying both sides of (5) by $\gamma f(\omega)$ and then using (6). Also using the Schur complement from (5) a convex LMI can be obtained.

Finally the relaxed problem is formulated as:

$$\min_{f,A,B} \gamma \quad \text{subject to:} \quad A \geq \mu I \quad \forall \omega \in [0, \pi] \quad (7)$$
$$\begin{pmatrix} \gamma f_i I & G(\omega_i)A(\omega_i) - B(\omega_i) \\ * & \gamma A(\omega_i) \end{pmatrix} \geq 0$$
$$\mu I \leq f_i I \leq A(\omega_i) \quad \forall i = 1, \ldots, N$$

where f_i are decision variables and μ is a positive pre-defined scalar. The reduced model PQ^{-1} is obtained in a similar manner to the SISO case.

The error bounds are guaranteed using the following result in [34]:

Theorem 2.1: Assume γ_*, A_*, B_*, P_* and Q_* are obtained from the reduction algorithm described above with the full sampling (i.e. the constraints are enforced for all the frequencies $\omega \in [0, \pi]$), then:

1) $\gamma_* \geq \sigma_{km+1}(G)$
2) $\gamma_* \leq \|G - P_* Q_*^{-1}\|_{\infty} \leq (km + 1)\gamma_*$

$\sigma_{km+1}(G)$ is the $(km + 1)$-th largest Hankel singular value of G, k is the order of matrix polynomials P, Q and m is the number of inputs of G.

Note, that for the SISO case γ_* is also a lower bound for the general reduction problem (shown in [29]). The variables of the relaxed problem are chosen as:

$$A = \sum_{i=1}^{k} A_i e^{ij\omega} + A_0 + \sum_{i=1}^{k} A_i^T e^{-ij\omega}, \quad cI \geq A_0 \geq I$$
$$B = \sum_{i=-k}^{k} B_i e^{ij\omega}$$

This form of A polynomial together with positivity of A guarantees the existence of solution to the spectral factorization problem (see, [35]). $A_0 > I$ instead of $A_0 > 0$ is used in order to normalize the calculations. One also may bound $A_0 < cI$, where c is a constant.

III. MULTIVARIABLE PARAMETERIZED QUASI-CONVEX OPTIMIZATION APPROACH.

According to the notation in the multivariable reduction framework, introduce polynomials A and B :

$$A(\boldsymbol{\omega}) = \sum_{i=-k_0}^{k_0} \sum_{l=-k_1}^{k_1} A_{i,l} \cos(l\theta) e^{-ij\omega}$$
$$B(\boldsymbol{\omega}) = \sum_{i=-k_0}^{k_0} \sum_{l=-k_1}^{k_1} B_{i,l} \cos(l\theta) e^{-ij\omega}$$

The input data to the algorithm is again a finite number of samples $G(\boldsymbol{\omega}_i)$, where $i = 1, \ldots, N$. The relaxed minimization problem is set up in the similar to the non-parameterized case:

$$\min_{f,A,B} \gamma \quad \text{subject to} \quad A \geq \mu I \quad \forall \boldsymbol{\omega} \in [0, \pi]^2, \quad (8)$$
$$\begin{pmatrix} \gamma f_i I & G(\boldsymbol{\omega}_i)A(\boldsymbol{\omega}_i) - B(\boldsymbol{\omega}_i) \\ * & \gamma A(\boldsymbol{\omega}_i) \end{pmatrix} \geq 0, \quad (9)$$
$$f_i I \leq A(\boldsymbol{\omega}_i) \quad i = 1, \ldots, N \quad (10)$$

where f_i are also decision variables and μ is a positive scalar. γ_{freq}^N will denote the optimal solution to the problem. In the case when $N = \infty$ the optimal solution will be denoted as γ_{rel}.

The next step in the non-parameterized case would be the spectral factorization of A. However, if there is two or more frequency variables, the factorization has much stricter conditions. For a general case such conditions are described in [36] and for the two dimensional case (one parameter in our notation) in [37]. The conditions are not convex in the chosen variables (the coefficients of A and B) and author was not able to obtain a reasonable relaxation. An approximate solution to the multivariate spectral factorization may be found in, for example, [38]. In this contribution the multivariate spectral factorization problem is avoided and the obtained approximation is a look-up table of some values of parameters.

Obtaining a stable approximation can be done in a number of ways. For a fixed θ, denote $A_* = A(\cdot, \theta)$ and $B_* =$

$B(\cdot, \theta)$. In both cases it is considered that

$$P(\boldsymbol{\omega}) = \sum_{i=0}^{k_0} P_i(\theta) e^{-i\jmath\omega}$$

- Obtain the denominator Q_* from the spectral factorization problem $A_* = Q_* Q_*^{\sim}$.
- 1) Obtain the numerator P_* as the following minimization for every θ :

$$\min_P \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty$$

  2) Consider the numerator P with P_i continuous with respect to θ. Obtain the numerator P_* as:

$$\min_P \max_\theta \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty$$

Case (1) is equivalent to solving the problem:

$$\delta_{max-min} = \max_\theta \min_P \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty \qquad (11)$$

The cheapest solution is provided by imposing the norm constraint on a finite number of frequencies ω and solving the minimization for a finite number of parameter values θ. Imposing the dependence in the coefficients P_i on parameter is not possible to authors best knowledge.

Case (2) is equivalent to:

$$\delta_{min-max} = \min_P \max_\theta \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty \qquad (12)$$

Here again the cheapest solution is provided by imposing the norm constraint for a finite number of frequencies ω and for a finite number of parameter values θ. Also one may enforce the coefficients P_i to depend on θ in a particular manner.

As the output of the algorithm two maps are obtained. In realization mappings P_θ, Q_θ can be stored as look-up tables for required values of θ or one can store polynomials A and B, and obtain the required P_θ, Q_θ when needed. Some of the properties of the obtained maps are listed in the next statements. Note, that the results are valid only if the algorithms are implemented with the full sampling (the constraints are enforced for all the frequencies $\omega \in [0, \pi]$)

Proposition 3.1: Given A and B from the reduction algorithm, then

a) B, A and BA^{-1} are continuous with respect to the variable θ.
b) The spectral factor $Q(\theta)$ of A is continuous with respect to θ.
c) Function $f(P, \theta) = \|BA^{-1} - PQ^{-1}\|_\infty$ is continuous in both P and θ.
d) $P(\theta)$ is continuous with respect to θ
   Proof:

a) Since B and A are trigonometric polynomials in variable θ the continuity of A, B is trivial. Also $A \geq \mu I$ where scalar μ is predefined, therefore A^{-1} is continuous in θ. Multiplication of two continuous maps B and A^{-1} will also give a continuous map BA^{-1}.
b) Here Theorem 6.2 can be applied. The conditions are fulfilled, since $A \geq \mu I$ and A is a trigonometric polynomial in ω.

c) Follows directly from the parameterization of P and statements a) and b).
d) See, Lemma 6.4 in Appendix.

∎

Proposition 3.2: Recall the defined above minimization programs:

$$\delta_{max-min} = \max_\theta \min_P \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty$$
$$\delta_{min-max} = \min_P \max_\theta \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty$$

Then $\delta_{min-max} = \delta_{max-min}$.
  Proof: In general $\min_x \max_y f(x,y) \geq \max_y \min_x f(x,y)$, hence $\delta_{min-max} \geq \delta_{max-min}$ To prove $\delta_{min-max} \leq \delta_{max-min}$ take a solution of

$$\delta_{max-min} = \max_\theta \min_P \|BA^{-1} - PQ^{-1}\|_\infty$$

denote it as P_*. By Proposition 3.1 every coefficient of the transfer function P_* is continuous with respect to θ. Therefore the transfer function P_* is continuous with respect to θ. The conditions $\|B_* A_*^{-1} - PQ_*^{-1}\|_\infty \leq \delta_{max-min}$ are also satisfied for every θ as a solution to max-min problem. Therefore P_* lies in the set of variables of min-max problem and $\delta_{max-min} \geq \max_\theta \|B_* A_*^{-1} - PQ_*^{-1}\|_\infty \geq \delta_{min-max}$.

∎

IV. ERROR BOUNDS AND CONTINUITY

The main result of this contribution are the theoretical properties of the approximations.

Theorem 4.1 (Error Bounds and Continuity of Solution): Consider the algorithm described in the previous section with the full sampling (the constraints are enforced for all the frequencies $\omega \in [0, \pi]$). The following statements hold:

1) The reduced model PQ^{-1} is a continuous with respect to θ map.
2) $\gamma_{max-min} \leq \gamma_{min-max}$
   $\gamma_{freq}^N \leq \gamma_{rel} \leq \gamma_{min-max}$
3) $\max_\theta \sigma_{km+1}(G) \leq \gamma_{rel} \leq \max_\theta \|G - P_* Q_*^{-1}\|_\infty \leq (km+1)\gamma_{rel}$

where $\sigma_{km+1}(G)$ is the $(km+1)$-th largest Hankel singular value of G, k is the order of matrix polynomials P, Q and m is the number of inputs of G.
  Proof:

1) Maps P, Q are continuous with respect to θ as shown in Proposition 3.1. Since $Q(\omega, \theta)$ does not have zeros, it is invertible and its inverse is also continuous. Therefore the PQ^{-1} is continuous.
2) Shown by construction.
3) $\sigma_{km+1}(G) \leq \gamma_{rel}$ is satisfied for every θ since γ_{rel} is the solution to a restricted Hankel model reduction. Therefore by taking the maximum over all θ the inequality is achieved.
   As Theorem 2.1 states $\gamma_{rel} \leq \|G - P_* Q_*^{-1}\|_\infty \leq (km+1)\gamma_{rel}$ for every parameter value θ, where P_*, Q_* are obtained using the proposed algorithm.

Denote $\hat{G}_-$ the antistable part of $B_* A_*^{-1}$. Since $\|G - B_* A_*^{-1}\|_\infty \leq \gamma$, by the celebrated AAK theorem:

$$\|\hat{G}_-\|_H \leq \gamma \Rightarrow \|\hat{G}_- + K\|_\infty \leq km\gamma$$

There exists a matrix $K(\theta)$ such that the infinity norm bound is also satisfied. By triangular inequality:

$$\|G - P_* Q_*^{-1}\|_\infty \leq \|G - P_* Q_*^{-1} - G_- - K\|_\infty + \\ + \|G_- - K\|_\infty \leq (km + 1)\gamma_{rel}$$

Since

$$\|G - P_* Q_*^{-1} - G_- - K\|_\infty \leq \|G - B_* A_*^{-1}\|_\infty$$

After taking the maximum over θ and using Proposition 3.2 the result follows.

∎

Note, that for the case $m = 1$ the value γ_{rel} is a lower bound for the model reduction, i.e. $\gamma_{rel} \leq \gamma_{min-max}$.

V. NUMERICAL EXAMPLE

Implementation issues and numerical complexity estimates are presented in [23], [28]. Few more numerical examples are presented in [32]

A. Deformable Mirror Modeling

The following model was studied in [39] and obtained by means of a finite element modeling approach that resulted in a system of second-order differential equations:

$$I\ddot{x} + \Lambda^2 \dot{x} + \psi\Lambda = Fu, y = F^T u$$

where a model has 420 sensors and actuators, 2000 states, and the friction coefficient ψ is chosen as a modeling parameter. The model is discretized in order to apply the described method.

The control problem in [39] is solved in a decentralized fashion. Based on the actuator and sensor topology it is assumed that a 2 input 8 output model is decoupled from the rest of the system. The objective is to obtain a family of low-order models that depend on $\psi \in \Psi = [0.01, 0.06]$. The frequency response samples are calculated for $\psi = 0.01$, 0.02, 0.03, 0.04, 0.05, 0.06 (the training grid). The order of polynomials is chosen to $k = (8, 2)$, i.e. the frequency component would have a rational dependence of order 8, and ψ would enter with order 2 in to the reduced model. The validation grid is chosen to $\psi = 0.015, 0.025, 0.035, 0.045$, 0.055. The approximation error on the training grid is 2.06% of $\max_{\psi \in \Psi}\|G\|_\infty$ and on the validation grid is 7.14%. The frequency responses of the approximations and the original models for one entry on the training and the validation grid are shown in Fig. 1.
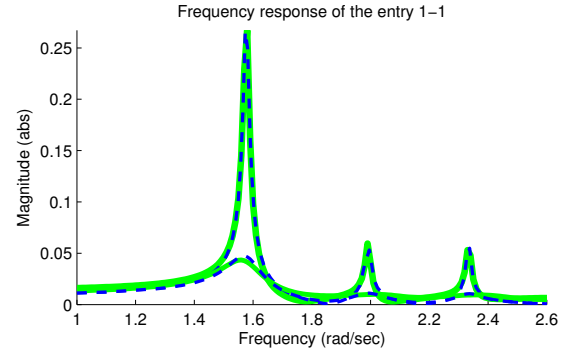
(a) Entry (1,1) for $\psi = 0.01, 0.06$ (values belong to the training grid)
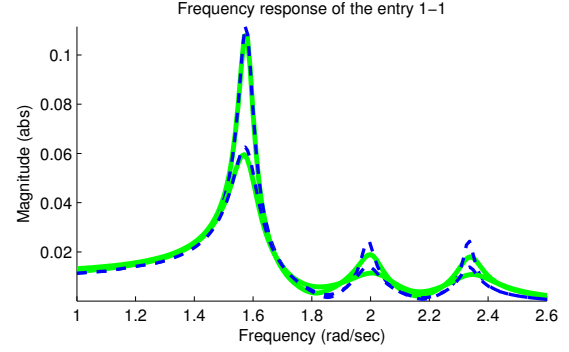
(b) Entry (1,1) for $\psi = 0.025, 0.045$ (values belong to the validation grid)

Fig. 1. Frequency responses of the reduced (green solid lines) and the full (blue dashed lines) models

VI. CONCLUSION

A multivariable extension of a known method have been presented. The properties of approximations were investigated. The continuity of the approximations with respect to the parameter is shown. Also the error bounds for the method are presented, which are valid for the scalar valued transfer function approximations, as well. A numerical example is presented in order to validate the theoretical results.

APPENDIX

The following two theorems will be used in this section:

Theorem 6.1 (Weisstrass' theorem [40]): Consider a closed proper function $f : \mathbb{R}^n \to (-\infty, \infty]$, and assume that there exist a scalar α such that the level set $\{x \mid f(x) < \alpha\}$ is nonempty and bounded. Then the set of minima of f over $\mathbb{R}^n$ is nonempty and compact.

Theorem 6.2 (Continuity of spectral factorization [41]): Let $A_i(e^{j\omega}) = I + M_i(e^{j\omega})$, $i = 1, 2$ be two positive definite Hermitian matrices defined on $-\pi \leq \omega \leq \pi$ with

$$\|M_i(e^{j\omega})\|_\infty \leq q < 1$$

Let Q_1, Q_2 be the associated normalized, minimum phase, stable spectral factors. Assume in addition that $dM_i(e^{j\omega})/de^{j\omega} \in \mathcal{L}_2^{n \times n}$ for $i = 1, 2$. Then there exists some $\widetilde{K}$ dependent only on q and $\|dM_i(e^{j\omega})/d\omega\|_2$ such that:

$$\|Q_1 - Q_2\|_\infty \leq \widetilde{K}\|M_1 - M_2\|_\infty^{1/2}$$

The next lemma is a generalization from the scalar case.

Lemma 6.3: If A, B, Q are obtained from the program (8-10) and P is obtained from $\min_P \|BA^{-1} - PQ^{-1}\|_\infty$ for given A, B, Q then for every i the coefficients P_i are uniformly bounded on θ.

Proof: First show that for a fixed θ the coefficients A_i are bounded. Consider the positivity constraint:

$$\sum_{i=1}^{k} A_i(\cos(i\omega) + \jmath\sin(i\omega)) + A_0+$$
$$+\sum_{i=1}^{k} A_i^T(\cos(i\omega) - \jmath\sin(i\omega)) > 0 \quad \forall \omega$$

Now multiply both sides with $1 + \cos(l\omega)$ for $l \leq k$, and integrate on the interval $[0, \pi]$. All the terms will cancel themselves out except:

$$\int_0^{2\pi} (A_l + A_l^T)\cos^2(l\omega)d\omega > 2\pi A_0$$

and for any l :

$$A_l + A_l^T \geq -2cI$$

Here $cI \geq A_0$ is a constraint on A. In a similar way (just by multiplying with $1 - \cos(l\omega)$ or $1 \pm \sin(l\omega)$) it can be shown that:

$$2cI \geq A_l + A_l^T \geq -2cI$$
$$2cI \geq \jmath(A_l - A_l^T) \geq -2cI$$

and hence $\overline{\sigma}(A_l) \leq 2c$. Therefore A_l is bounded for all l, and the bound does not depend on θ, hence it is uniform.

Now prove that the function A is uniformly bounded on θ.

$$\left\| \sum_{i=1}^{k}(A_i + A_i^T)\cos(i\omega) + A_0+ \right.$$
$$\left. + \jmath\sum_{i=1}^{k}(A_i - A_i^T)\sin(i\omega) \right\|_\infty \leq$$
$$\sum_{i=1}^{k} \|A_i + A_i^T\|_2 + \|A_0\|_2 + \sum_{i=1}^{k} \|A_i - A_i^T\|_2 \leq$$
$$\leq 2ck + c + 2ck = (4k+1)c$$

From the inequality $\|G - BA^{-1}\|_\infty \leq \gamma$ get a bound for every frequency ω :

$$\overline{\sigma}(BA^{-1}) \leq \gamma + \overline{\sigma}(G) \leq \gamma + \kappa$$

where $\max_\theta \|G(\cdot, \theta)\|_\infty \leq \kappa$. The bound on P polynomial is derived from the inequality $\|BA^{-1} - PQ^{-1}\|_\infty \leq \delta$:

$$\overline{\sigma}(P) \leq \underline{\sigma}(Q)(\delta + \overline{\sigma}(BA^{-1})) \leq \underline{\sigma}(Q)(\delta + \gamma + \kappa) \leq \Delta$$

Since A is uniformly bounded, then the spectral factor Q will also be uniformly bounded on θ, therefore Δ does not

depend on θ. Multiplying by $1 \pm \cos(l\omega)$ for $0 \leq l \leq k$ and integrating on the interval $[0, \pi]$:

$$\overline{\sigma}\left(\int_0^{2\pi} P(1 + \cos(l\omega))d\omega\right) \leq 2\pi\Delta$$
$$\overline{\sigma}\left(\int_0^{2\pi} P(1 - \cos(l\omega))d\omega\right) \leq 2\pi\Delta$$

Performing the integration over ω and using the orthogonality of cosines and sines gives:

$$\overline{\sigma}(P_0) \leq \Delta/2$$
$$\overline{\sigma}(P_l) \leq 2\Delta, \text{ for } l \geq 1$$

Note that Δ does not depend on θ providing a uniform bound on θ. ∎

Lemma 6.4: Denote

$$x = \text{vec}\left([P_0, \cdots, P_k]^T\right)^T$$
$$P = x \cdot \text{vec}\left([I, Iz^{-1}, \cdots, Iz^{-k}]^T\right)$$
$$f(x, \theta) = \|B(\omega, \theta)A^{-1}(\omega, \theta) - P(\omega, x)Q^{-1}(\omega, \theta)\|_\infty$$
$$\gamma(\theta) = \min_x f(x, \theta)$$
$$\mathcal{P}(\theta) = \arg\min_x f(x, \theta)$$

If A, B, Q are continuous with respect to θ and $Q(\cdot, \theta) \in \mathcal{H}_\infty$ then:

a) $\mathcal{P}(\theta)$ is a compact set for every θ. Also the level sets $L_\alpha = \{(x, \theta) \,|\, f(x, \theta) \leq \alpha\}$ are compact for a fixed α.

b) Consider a sequence $\{\theta_k\}$ with a limit $\theta_* \in \text{dom}(\gamma)$, then $\gamma(\theta_k) \to \gamma(\theta_*)$. Moreover for every sequence $\{x_k\}$, such that $x_k \in \mathcal{P}(\theta_k)$ for every k, all the limit points x_* lie in $\mathcal{P}(\theta_*)$.

c) $\gamma(\theta)$ is continuous with respect to θ.

Proof: The proof and the statement itself are based on ideas from [40, p. 158].

a) Note that L_α is bounded according to Lemma 6.3. The fact that L_α is closed follows from the continuity of f with respect to x and θ (shown in Proposition 3.1) and the definition of L_α. Then using the Weierstrass theorem (Theorem 6.1) yields that $\mathcal{P}(\theta)$ are compact for every θ. Note that the objective function satisfies the conditions of the theorem.

b) Consider a sequence $\{\theta_k\} \to \theta_*$ and a sequence $x_k \in \mathcal{P}(\theta_k)$. Consider a scalar α such that $\gamma(\theta_*) < \alpha$. Then for a sufficiently big k : $f(x_k, u_k) \leq \alpha$. Hence the sequence $\{x_k\}$ is bounded and there exist a limit point x_*. The point:

$$\{x_*, \theta_*\} \in L_\alpha = \{(x, \theta) \,|\, f(x, \theta) \leq \alpha\}$$

Since $\{x_k, \theta_k\} \in L_\alpha$ and L_α is compact. Since it is true for an arbitrary number α such that $\gamma(\theta_*) < \alpha$, the inequality is valid:

$$f(x_*, \theta_*) \leq \alpha \Rightarrow f(x_*, \theta_*) \leq \gamma(\theta_*)$$

And finally by definition of $\gamma(\theta_*)$ the statement is proved:

$$x_* \in \mathcal{P}(\theta_*)$$

c) For any θ_k a convergent sequence $x_k \in \mathcal{P}(\theta_k)$ may be chosen (as in the previous statement). Then for sufficiently big k there exist $\varepsilon > 0$ such that:

$$|f(x_k, \theta_k) - f(x_*, \theta_*)| < \varepsilon$$

Due to the continuity of f with respect to both variables x and θ. Since $\gamma(\theta_k) = f(x_k, \theta_k)$, and $\gamma(\theta_*) = f(x_*, \theta_*)$ the continuity of γ is proved.

■

REFERENCES

[1] G. Obinata and B. Anderson, *Model Reduction for Control System Design*. London,UK:Springer-Verlag, 2001.

[2] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems (Advances in Design and Control) (Advances in Design and Control)*. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 2005.

[3] A. Ruehli, "Equivalent circuit models for three-dimensional multiconductor systems," *Microwave Theory and Techniques, IEEE Transactions on*, vol. 22, no. 3, pp. 216–221, Mar 1974.

[4] N. Ottosen and H. Petersson, *Introduction to the Finite Element Method*. Prentice Hall, 1992.

[5] P. Monk, *Finite element methods for Maxwell's equations*, ser. Numerical mathematics and scientific computation, Oxford science publications. Great Clarendon Street, Oxford OX2 6DP: Oxford University Press, 2003.

[6] R. W. Freund, "Model reduction methods based on Krylov subspaces," *Acta Numerica*, vol. 12, p. 2003, 2003.

[7] A. Antoulas, "An overview of model reduction methods and a new result," in *Proceedings of 48th Conference on Decision and Control*, Shanghai, PRC, 2009.

[8] L. M. Silveira, M. Kamon, I. Elfadel, L. Miguel, S. Mattan, K. I. Elfadel, and J. White, "A coordinate-transformed arnoldi algorithm for generating guaranteed stable reduced-order models of arbitrary rlc circuits." 1996.

[9] R. W. Freund, "Sprim: structure-preserving reduced-order interconnect macromodeling," in *ICCAD '04: Proceedings of the 2004 IEEE/ACM International conference on Computer-aided design*. Washington, DC, USA: IEEE Computer Society, 2004, pp. 80–87.

[10] Y. Liu, L. Pileggi, and A. Strojwas, "Model order-reduction of rc(l) interconnect including variational analysis," *Design Automation Conference, 1999. Proceedings. 36th*, pp. 201–206, 1999.

[11] E. B. Rudnyi, C. Moosmann, A. Greiner, T. Bechtold, and J. G. Korvink, "Parameter preserving model reduction for mems system-level simulation and design," in *Vienna University of Technology*. ISBN, 2006, pp. 3–901 608.

[12] L. Daniel and J. White, "Automatic generation of geometrically parameterized reduced order models for integrated spiral RF-inductors," in *International Workshop on Behavioral Modeling and Simulation*. IEEE, January 2003, pp. 18–23.

[13] L. Daniel, C. S. Ong, S. C. Low, K. H. Lee, and J. White, "Geometrically parameterized interconnect performance models for interconnect synthesis," in *ISPD '02: Proceedings of the 2002 international symposium on Physical design*. New York, NY, USA: ACM, 2002, pp. 202–207.

[14] L. Daniel, O. Siong, L. Chay, K. Lee, and J. White, "Multiparameter moment matching model reduction approach for generating geometrically parameterized interconnect performance models," *Trans. on Computer Aided Design of Integrated Circuits*, vol. 23, no. 5, pp. 678–93, May 2004.

[15] X. Li, P. Li, and L. T. Pileggi, "Parameterized interconnect order reduction with explicit-and-implicit multi-parameter moment matching for inter/intra-die variations," in *ICCAD '05: Proceedings of the 2005 IEEE/ACM International conference on Computer-aided design*. Washington, DC, USA: IEEE Computer Society, 2005, pp. 806–812.

[16] P. Li, F. Liu, X. Li, L. T. Pileggi, and S. R. Nassif, "Modeling interconnect variability using efficient parametric model order reduction," in *In Design, Automation and Test Conference in Europe*, 2005, pp. 958–963.

[17] O. Farle, V. Hill, P. Ingelstrom, and R. Dyczij-Edlinger, "Multiparameter polynomial order reduction of linear finite element models," *Mathematical and Computer Modelling of Dynamical Systems*, 2008.

[18] U. Baur, "Multipoint padé approximation for parametric model order reduction," 2006. [Online]. Available: https://www-user.tu-chemnitz.de/ benner/talks/BBC-MOR/baur.pdf

[19] L. Feng and P. Benner, "A robust algorithm for parametric model order reduction," 2007.

[20] Y.-T. Li, Z. Bai, Y. Su, and X. Zeng, "Parameterized model order reduction via a two-directional arnoldi process," in *ICCAD '07: Proceedings of the 2007 IEEE/ACM international conference on Computer-aided design*. Piscataway, NJ, USA: IEEE Press, 2007, pp. 868–873.

[21] B. Bond and L. Daniel, "Parameterized model order reduction of nonlinear dynamical systems," in *ICCAD '05: Proceedings of the 2005 IEEE/ACM International conference on Computer-aided design*. Washington, DC, USA: IEEE Computer Society, 2005, pp. 487–494.

[22] A. Odabasioglu, M. Celik, and L. Pileggi, "Prima: passive reduced-order interconnect macromodeling algorithm," *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, vol. 17, no. 8, pp. 645–654, Aug 1998.

[23] K. C. Sou, A. Megretski, and L. Daniel, "A quasi-convex optimization approach to parameterized model order reduction," *Computer-Aided Design of Integrated Circuits and Systems, IEEE Transactions on*, vol. 27, no. 3, pp. 456–469, March 2008.

[24] T. Moselhy, X. Hu, and L. Daniel, "pFFT in FastMaxwell: a fast impedance extraction solver for 3D conductor structures over substrate," in *DATE '07: Proceedings of the conference on Design, automation and test in Europe*. San Jose, CA, USA: EDA Consortium, 2007, pp. 1194–1199.

[25] Z. Zhu, B. Song, and J. White, "Algorithms in FastImp: a fast and wideband impedance extraction program for complicated 3-D geometries," in *DAC '03: Proceedings of the 40th annual Design Automation Conference*. New York, NY, USA: ACM, 2003, pp. 712–717.

[26] S. Kapur and D. E. Long, "Ies3: Efficient electrostatic and electromagnetic simulation," *IEEE Comput. Sci. Eng.*, vol. 5, no. 4, pp. 60–67, 1998.

[27] M. Kamon, N. Marques, and J. White, "FastPep: a fast parasitic extraction program for complex three-dimensional geometries," Washington, DC, USA, pp. 456–460, 1997.

[28] A. Sootla, "Model reduction using semidefinite programming," Department of Automatic Control, Lund University, Sweden, Licentiate Thesis LUTFD2/TFRT--3247--SE, Nov. 2009. [Online]. Available: http://www.control.lth.se/publications/

[29] A. Megretski, "H-infinity model reduction with guaranteed suboptimality bound," *American Control Conference, 2006*, pp. 448–453, June 2006.

[30] A. Sootla and K. C. Sou, "Frequency domain model reduction method for parameter-dependent systems," in *Proceedings of the American Control Conference*, July 2010.

[31] A. Sootla, "Hankel-type model reduction based on frequency response matching," *Swedish Control Meeting*, June 2010. [Online]. Available: http://www.control.lth.se/publications/

[32] A. Sootla and A. Rantzer, "Parameter dependent model reduction framework with applications," *Swedish Control Meeting*, June 2010. [Online]. Available: http://www.control.lth.se/publications/

[33] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Syst. Control Lett.*, vol. 28, no. 1, pp. 7–10, 1996.

[34] A. Sootla, A. Rantzer, and G. Kotsalis, "Multivariable optimization-based model reduction," *Automatic Control, IEEE Transactions on*, vol. 54, no. 10, pp. 2477–2480, Oct. 2009.

[35] A. H. Sayed and T. Kailath, "A survey of spectral factorization methods," *Numerical linear algebra with applications*, vol. 8, no. 6-7, pp. 467–496, 2001.

[36] M. A. Dritschel and H. J. Woerdeman, "Outer factorizations in one and several variables," *Transactions of the American Mathematical Society*, vol. 357, no. 11, 2005.

[37] J. S. Geronimo and M.-J. Lai, "Factorization of multivariate positive laurent polynomials," *J. Approx. Theory*, vol. 139, no. 1, pp. 327–345, 2006.

[38] M. Ekstrom and J. Woods, "Two-dimensional spectral factorization with applications in recursive digital filtering," *Acoustics, Speech and Signal Processing, IEEE Transactions on*, vol. 24, no. 2, pp. 115–128, Apr 1976.

[39] P. Giselsson, "Modeling and control of a 1.45 m deformable mirror," Department of Automatic Control, Lund University, Sweden, Master's Thesis ISRN LUTFD2/TFRT--5775--SE, Oct. 2006. [Online]. Available: http://www.control.lth.se/publications/

[40] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar, *Convex Analysis and Optimization*. Athena Scientific, 2003.

[41] B. D. Anderson, "Continuity of the spectral factorization operation," *Mat. Aplic. Comp.*, vol. 4, no. 2, pp. 139–156, 1985.