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Karlsson, Anders

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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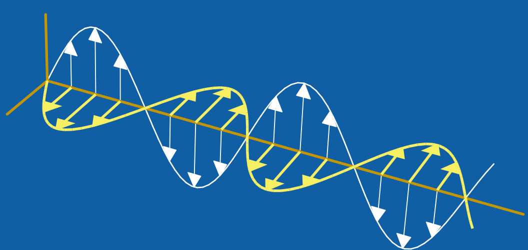
**Anders Karlsson**

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Anders Karlsson

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

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**Abstract**

The time-domain version of the optical theorem is discussed. From the theorem and causality fundamental results concerning the scattered and absorbed energies from an incident plane pulse are obtained. Some results are verified by numerical calculations.

# 1 Introduction

Consider the scattering experiment in figure 1. A plane wave with Gaussian time dependence is scattered from two identical spheres with a radius of one meter and with a distance of five meters between their centers. A question is now posed to one physicist and one non-physicist: Is the scattered energy from the two spheres equal to twice the energy that is scattered if only one sphere is present? The non-physicist says probably yes, since one plus one is two. The physicist claims that there are multiple reflections taking place and says no. Who is correct? If the spatial pulse width is long compared to the radius of the spheres, then the physicist is correct. However, for a short enough pulse the non-physicist is correct. This is one of the results that is extracted from the time-domain version of the optical theorem in this paper.

The optical theorem has been known for almost a century and has been widely used in different areas of physics. A historical review of it is found in [5]. Derivations of the frequency domain version of the optical theorem can be found in elementary textbooks in quantum mechanics, acoustics and electrodynamics, cf. [3] and [4]. The time-domain version of the optical theorem is less known. A proof of it is given in [1] for electromagnetic waves and in [2] for acoustic waves. In those two papers the time-domain version of the optical theorem is referred to as a time-domain energy theorem.

In the next section the optical theorem in the time-domain is presented for the electromagnetic case. Different implications of the theorem are discussed in section 3. A derivation of the theorem is presented in the appendix.

# 2 The optical theorem in time-domain

Assume a bounded scattering object with volume $V$. The electromagnetic properties of the object are arbitrary. The volume outside the object is a lossless homogeneous dielectric medium with permittivity $\varepsilon$ and permeability $\mu$ and is denoted $V'$. The wave speed in $V'$ is $c = 1/\sqrt{\varepsilon\mu}$. An electromagnetic plane wave with electric field

$$\boldsymbol{E}^i(\boldsymbol{r},t) = \boldsymbol{E}_0\left(t - \hat{k}_i \cdot \boldsymbol{r}/c\right) \tag{2.1}$$

impinges on the object. The incident wave then propagates in the direction $\hat{k}_i$. It is now assumed that the vector function $\boldsymbol{E}_0$ has compact support in time, i.e. $\boldsymbol{E}_0(t) = 0$ when $t < t_0$ and $t > t_1$ for some $t_0$ and $t_1$. It is possible to have $t_1 = \infty$

if $\boldsymbol{E}_0(t)$ goes to zero when $t \to \infty$. Outside the object the total field is decomposed into the incident field and the scattered field as

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}^i(\boldsymbol{r},t) + \boldsymbol{E}^s(\boldsymbol{r},t) \tag{2.2}$$

In the far zone the scattered field is a spherical wave [3]

$$\boldsymbol{E}^s(\boldsymbol{r},t) = \frac{\boldsymbol{F}(\hat{r}, t - r/c)}{r} \tag{2.3}$$

where the far-field amplitude $\boldsymbol{F}(\hat{r}, t-r/c)$ is perpendicular to $\hat{r}$. The time argument $t - r/c$ is the retarded time that tells us that the far-field is moving in the radial direction with speed $c$. Causality implies that $\boldsymbol{F}(\hat{r}, t) = 0$ for times $t < t_f \leq t_0$. If the wave speed in the scattering object is everywhere less or equal to the wave speed of the surrounding medium then $t_f = t_0$.

The optical theorem in the time-domain states that when the incident field has passed the scattering object the sum of the scattered and absorbed energies is obtained from

$$W_T = W_s(t) + W_a(t) = -\frac{4\pi}{\mu} \int_{t_0}^{t_1} \boldsymbol{E}_0(t') \cdot \int_{t_f}^{t'} \boldsymbol{F}(\hat{k}_i, t'') dt'' dt' \tag{2.4}$$

Here $\hat{k}_i$ is the direction of propagation for the incident plane wave, $W_s(t)$ is the energy stored in the scattered field, or equivalently, the scattered energy, and $W_a(t)$ is the energy absorbed in the object. The sum of the scattered and absorbed energies $W_T = W_s(t) + W_a(t)$ is independent of time although each of the two terms can be time-dependent. From Eq. (2.4) it is seen that:

*The sum of the scattered and absorbed energies is uniquely determined by the far-field amplitude in the forward direction during the time interval* $[t_f, t_1]$.

# 3 Implications of the theorem

In this section some fundamental results that follow from causality and the optical theorem are discussed. In the numerical examples there is always vacuum in the region outside the scattering objects and thus $c = 3 \cdot 10^8$ m/s.

## 3.1 Scattering from several objects

Let the incident field be a plane wave with compact support in time. Consider two scattering objects, as in figure 1, that give rise to a scattered field $\boldsymbol{E}^s$. If the objects are close to each other then, in general,

$$W_T \neq W_{T1} + W_{T2}, \tag{3.1}$$

where $W_{T1}$ is the sum of the scattered and absorbed energies from object 1 when object 2 is not present, and vice versa for $W_{T2}$. The inequality is due to multiple

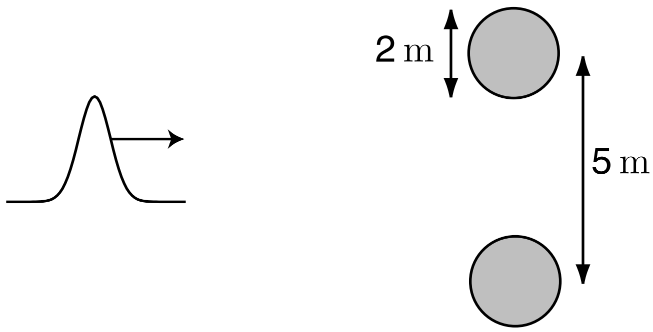


**Figure 1**: Scattering of a plane Gaussian pulse from two identical spheres.

scattering. If the pulse width is short compared to the transverse distance between the two objects then the scattering amplitude $\boldsymbol{F}(\hat{r},t)$ in the time interval $t_f < t < t_1$ is given by $\boldsymbol{F}(\hat{r},t) = \boldsymbol{F}_1(\hat{r},t) + \boldsymbol{F}_2(\hat{r},t)$, where $\boldsymbol{F}_i,\ i = 1,2$ is the scattering amplitude from object $i$ if only that object is present. The reason is that the multiply scattered waves are delayed in the forward direction at least a time $t_1 - t_f$ compared to the waves that are only scattered from one of the objects. Thus from Eq. (2.4)

$$W_T = W_{T1} + W_{T2} \tag{3.2}$$

Notice, it is not because multiple scattering effects are small that the relation holds.

Figure 1 shows two identical lossless spheres with a radius of one meter and with a distance of 5 meters between their centers. The spheres are homogeneous with wave speed $c_o/4$. An incident Gaussian pulse

$$\boldsymbol{E}^i(\boldsymbol{r},t) = \hat{x} f(t - z/c) \tag{3.3}$$

with

$$f(t) = e^{-(t-5T)^2/T^2} \tag{3.4}$$

impinges on the two spheres. The pulse width is $T = 2$ ns and the amplitude is 1 V/m. Using Mie scattering it is possible to determine the scattered energy from one sphere. It is given by $4.63 \times 10^{-11}$ Nm. The distance between the spheres is large enough for Eq. (3.2) to hold. The scattered energy from the two spheres is then $9.26 \times 10^{-11}$ Nm. A result that is hard to obtain by numerically solving the scattering problem with the two spheres.

## 3.2 Scattering from layered objects

The same arguments that were used in the first example can be used for a layered object with an inner volume $V_2$ and an enclosing layer $V_1$. The wave speed in the object is everywhere slower than the wave speed outside the object. The incident field is again assumed to have compact support in time. If the layer is thick enough, the reflections from the inner volume give no contribution to $\boldsymbol{F}(\hat{k}_i,t)$ for times $t < t_1$. From Eq. 2.4 it is then seen that the the sum of the scattered and absorbed energies is independent of the inner object.

This result is verified by numerically calculating the scattered energy from lossless layered spheres. The incident field is a Gaussian pulse given by Eqs (3.3) and (3.4) with $T = 2$ ns. Three different spheres are used, all with outer radius $b = 1$ m and with wave speeds

$$c(r) = \begin{cases} \frac{c_0}{4} & \text{when } a < r < b \\ c_0 & \text{when } r < a \end{cases} \tag{3.5}$$

The first sphere has $a = 0.7$ m, the second $a = 0.5$ m and the third is a homogeneous sphere with $a = 0$ m. The scattered field is calculated at a large number of frequencies, by Mie scattering, and then the pulse response is obtained by a Fourier transformation. The scattered energy is calculated both by the optical theorem and by integration of the power flux over a sphere and in time. The energies integrated from the power fluxes are depicted in figure 2. The horizontal line indicates the value obtained by the optical theorem. For $t > t_1 \approx 15$ ns the curves take different routes, but they are bound to approach the value obtained by the optical theorem. Notice that even after $t = 200T$ the final value has not yet been reached. In figure 3 the integral

$$W(t) = -\frac{4\pi}{\mu} \int_{t_0}^{t} \boldsymbol{E}_0(t') \cdot \int_{t_f}^{t'} \boldsymbol{F}(\hat{k}_i, t'')dt''dt' \tag{3.6}$$

is depicted for the three spheres. As expected this integral is almost the same for the three spheres. The final value is a little bit smaller for the sphere with $a = 0.7$ m than for the other two. The reason is that the time-delay caused by the layer is $\Delta t = 6(b - a)/c_0 = 6$ ns, and this is just on the limit for the region $r < a$ to contribute to the scattered energy. Notice that $W(t) = W_T$ for $t > t_1$.

## 3.3 Scattering from dispersive objects

Simple linear dispersive materials are in the time domain characterized by the electric and magnetic susceptibility kernels, $\chi_e(t)$ and $\chi_m(t)$. Consider the case where the material of the object is characterized by the constitutive relations, cf [3]

$$\begin{cases} \boldsymbol{B}(\boldsymbol{r}, t) = \mu \boldsymbol{H}(\boldsymbol{r}, t) + \mu_0 \displaystyle\int_{-\infty}^{t} \chi_m(t - t')\boldsymbol{H}(\boldsymbol{r}, t')dt' \\ \boldsymbol{D}(\boldsymbol{r}, t) = \varepsilon \boldsymbol{E}(\boldsymbol{r}, t) + \varepsilon_0 \displaystyle\int_{-\infty}^{t} \chi_e(t - t')\boldsymbol{E}(\boldsymbol{r}, t')dt' \end{cases} \tag{3.7}$$

From the optical theorem it follows that the sum of the scattered and absorbed energies only depends on $\chi_m(t)$ and $\chi_e(t)$ in the time interval $0 < t < t_1 - t_f$.

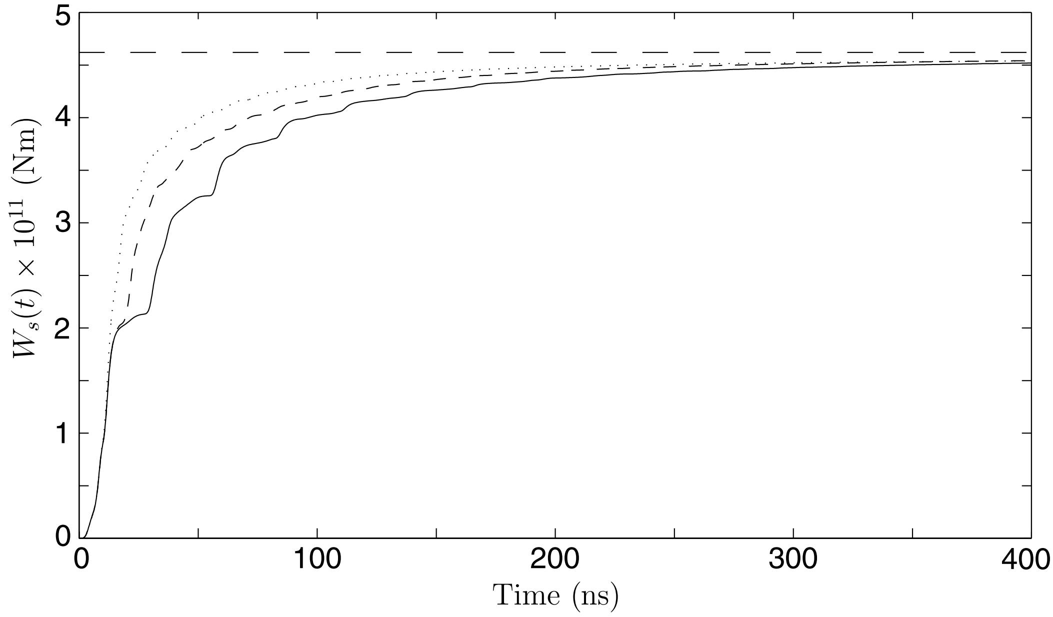


**Figure 2**: The scattered energy as a function of time for three different spheres with wave velocities as in Eq. (3.5). The incident pulse is given by Eqs (3.3) and (3.4). The solid line is for a sphere with $b = 1$ m and $a = 0$ m, the dashed line for a sphere with $b = 1$ m and $a = 0.5$ m, and the dotted line for a sphere with $b = 1$ m and $a = 0.7$ m.

## 3.4 Scattering of discontinuous pulses

Consider the incident pulse

$$\boldsymbol{E}^i(\boldsymbol{r}, t) = \boldsymbol{E}_0\left(t - \hat{k}_i \cdot \boldsymbol{r}/c\right) \tag{3.8}$$

Let $\boldsymbol{E}_0(t)$ be a smooth function and let the wave speed of the scattering volume $V$ be smaller than the speed in $V'$. For this pulse one can calculate the integral

$$-\frac{4\pi}{\mu}\int_{t_0}^{t} \boldsymbol{E}_0(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}(\hat{k}_i, t'')dt''dt' \tag{3.9}$$

for any $t$. Then consider an incident pulse as in Eq. (3.8) but with time-dependence

$$\boldsymbol{E}_1(t) = \begin{cases} \boldsymbol{E}_o(t) & t_0 < t \leq t_m \leq t_1 \\ \boldsymbol{0} & t > t_m \end{cases} \tag{3.10}$$

i.e. a discontinuous function. The sum of the scattered and absorbed energies for this pulse is

$$W_{T1} = W_{a1}(t) + W_{s1}(t) = -\frac{4\pi}{\mu}\int_{t_0}^{t_m} \boldsymbol{E}_1(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}_1(\hat{k}_i, t'')dt''dt' \tag{3.11}$$

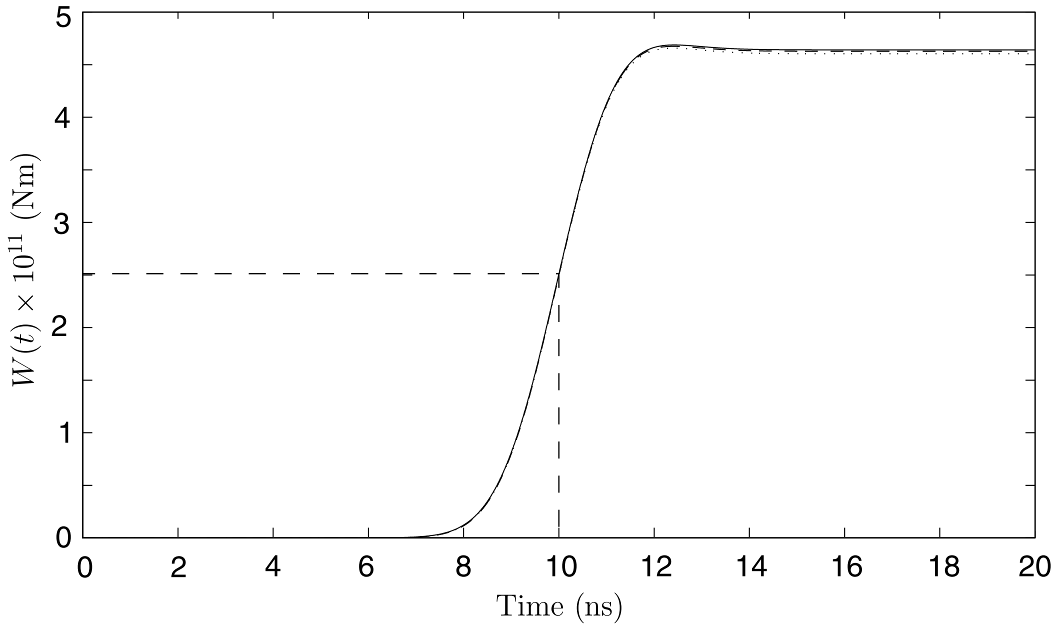


**Figure 3**: The energy $W(t)$ given by Eq. (3.6) for three different spheres with wave velocities as in Eq. (3.5). The incident pulse is given by Eqs (3.3) and (3.4). The solid line is for a sphere with $b = 1$ m and $a = 0$ m, the dashed line for a sphere with $b = 1$ m and $a = 0.5$ m, and the dotted line for a sphere with $b = 1$ m and $a = 0.7$ m. Notice that $W(t) = W_T$ for $t > t_1$ where $t_1 \approx 16$ ns. The value of $W_T$ for the pulse in the example in subsection 3.4 is indicated.

Causality implies that for $t < t_m$

$$\boldsymbol{F}_1(\hat{k}_i, t) = \boldsymbol{F}(\hat{k}_i, t) \tag{3.12}$$

and hence

$$W_{T1} = -\frac{4\pi}{\mu} \int_{t_0}^{t_m} \boldsymbol{E}_0(t') \cdot \int_{t_0}^{t'} \boldsymbol{F}(\hat{k}_i, t'') dt'' dt' \tag{3.13}$$

Thus the sum of the scattered and absorbed energies for the discontinuous pulse $\boldsymbol{E}_1(t)$ can be obtained from the far field of the continuous pulse $\boldsymbol{E}_0(t)$. Numerically this is of great help since it is much harder to calculate the far field from a discontinuous pulse than for a continuous. If the wave speed in the scattering volume $V$ is larger than in $V'$ then Eq. (3.12) does not hold and it is not possible to obtain $W_{T1}$ from Eq. (3.13).

As an example consider a discontinuous pulse given by Eq. (3.3) and with

$$f(t) = \begin{cases} e^{-(t-5T)^2/T^2} & t < t_m \\ 0 & t \geq t_m \end{cases} \tag{3.14}$$

where $t_m = 5T$ and $T = 2$ ns. The pulse is scattered from one of the spheres described in figure 3. Then $W_T = W(5T) = 2.5 \times 10^{-11}$ Nm, as seen from figure 3.

Next consider a lossless dielectric scattering object. If the pulse in Eq. (3.8) is slowly varying then the scattered energy from the discontinuous pulse in Eq. (3.10) can be obtained by a quasi-static calculation. Since the electric field changes slowly for $t < t_m$ the electromagnetic energy stored in the object is approximately given by, cf. [3],

$$W_e = \frac{1}{2}\boldsymbol{p} \cdot \boldsymbol{E}^i(\boldsymbol{0}, t), \qquad t < t_m \tag{3.15}$$

Here $\boldsymbol{p}$ is the induced dipole moment of the object. This dipole moment can be obtained by solving the boundary value problem of a dielectric object in a homogeneous electric field. At $t = t_m$ the incident field is shut off and the stored energy radiates and transforms into scattered energy. Thus the scattered energy from a slowly varying incident pulse with a discontinuity at $t = t_m$ is given by

$$W_s = \frac{1}{2}\boldsymbol{p} \cdot \boldsymbol{E}^i(\boldsymbol{0}, t_m) \tag{3.16}$$

This is exemplified by calculating the scattered energy from a homogeneous dielectric sphere with relative permittivity $\varepsilon_r = 2$. The incident Gaussian pulse is given by Eq. (3.14) and has $T = 50$ ns. The dipole moment of the sphere is given by, cf [3]

$$\boldsymbol{p} = 4\pi\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\boldsymbol{E}^i(\boldsymbol{0}, t)$$

The scattered energy is given by $W_{T1} = W(t_m)$ where $W(t_m)$ is given by Eq. (3.13). In figure 4 the curve of $W(t_m)$ obtained from the optical theorem, Eq. (3.13), is compared to the corresponding curve obtained from Eq. (3.16). It is seen that for such a broad pulse the simple formula (3.16) gives accurate results for the discontinuous pulse. For the entire pulse, $t_m = \infty$, Eq. (3.16) gives $W_T = 0$, whereas the optical theorem gives the correct value $W_T = 5.17 \cdot 10^{-15}$ Nm for the scattered energy.

# Appendix A Derivation of the theorem

Most of the techniques that are used to derive the optical theorem in the frequency domain can be generalized to the time-domain. In [1] a derivation of the time-domain version of the optical theorem is given in detail for the electromagnetic case. The proof is based upon a surface integral representation of the scattered field. From the proof it is clearly seen that the theorem holds regardless of the electromagnetic properties of the scattering object. In this appendix a simpler, but less general and rigorous, proof is given. It utilizes that the sum of the electric and magnetic energies in a volume $V_0$ is given by

$$W_e(t) + W_m(t) = \frac{1}{2}\int_{V_0} \varepsilon|\boldsymbol{E}(\boldsymbol{r}, t)|^2 + \mu|\boldsymbol{H}(\boldsymbol{r}, t)|^2 dv \tag{A.1}$$

Assume that time is large enough so that the incident field has passed the object and is in the far zone. The absorbed energy in the volume of the scattering object, $V$,

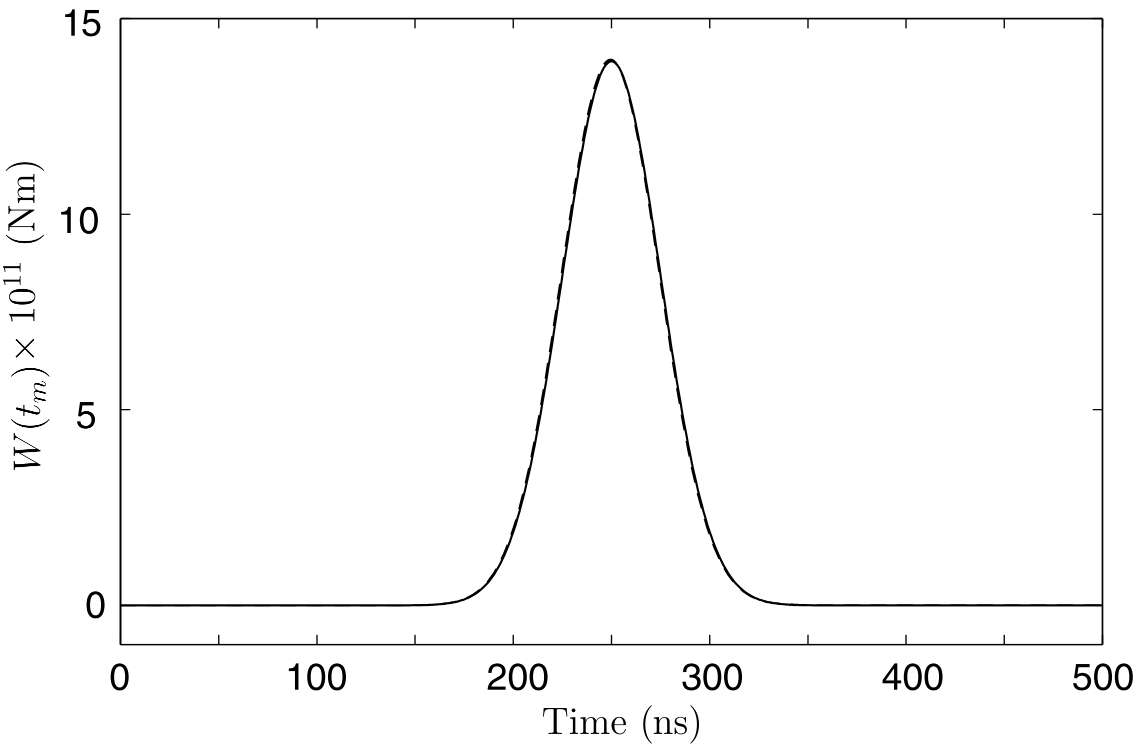


**Figure 4**: The scattered energy $W(t_m)$ given by Eq. (3.13) (solid line) and by Eq. (3.16) (dashed line) for the discontinuous Gauss pulse in Eq. (3.14) with $T = 50$ ns. The pulse is scattered from a homogeneous sphere with relative permittivity $\varepsilon_r = 2$ and with a radius of one meter.

is the difference between the energy of the incident field, $W_i$, and the electromagnetic energy in the region outside the object, i.e. $V'$, at time $t$, thus

$$\begin{aligned} W_a(t) &= W_i - \frac{1}{2}\int_{V'} \varepsilon|\boldsymbol{E}^i(\boldsymbol{r},t) + \boldsymbol{E}^s(\boldsymbol{r},t)|^2 + \mu|\boldsymbol{H}^i(\boldsymbol{r},t) + \boldsymbol{H}^s(\boldsymbol{r},t)|^2 dv \\ &= -W_s(t) - \int_{V'} \varepsilon \boldsymbol{E}^i(\boldsymbol{r},t)\cdot\boldsymbol{E}^s(\boldsymbol{r},t) + \mu\boldsymbol{H}^i(\boldsymbol{r},t)\cdot\boldsymbol{H}^s(\boldsymbol{r},t) dv \end{aligned} \tag{A.2}$$

where

$$W_s(t) = \frac{1}{2}\int_{V'} \varepsilon \boldsymbol{E}^s(\boldsymbol{r},t)\cdot\boldsymbol{E}^s(\boldsymbol{r},t) + \mu\boldsymbol{H}^s(\boldsymbol{r},t)\cdot\boldsymbol{H}^s(\boldsymbol{r},t) dv \tag{A.3}$$

is the energy stored in the scattered field, or equivalently, the scattered energy. The energy of the incident wave, $W_i$, is independent of time. Thus

$$W_s(t) + W_a(t) = -\int_{V'} \varepsilon \boldsymbol{E}^i(\boldsymbol{r},t)\cdot\boldsymbol{E}^s(\boldsymbol{r},t) + \mu\boldsymbol{H}^i(\boldsymbol{r},t)\cdot\boldsymbol{H}^s(\boldsymbol{r},t) dv \tag{A.4}$$

In the last integral the domain of integration is reduced to the intersection of the supports of $\boldsymbol{E}^i$ and $\boldsymbol{E}^s$, as depicted in figure 5. For a plane wave $\boldsymbol{H}^i = \sqrt{\frac{\varepsilon}{\mu}}\hat{k}_i \times \boldsymbol{E}^i$

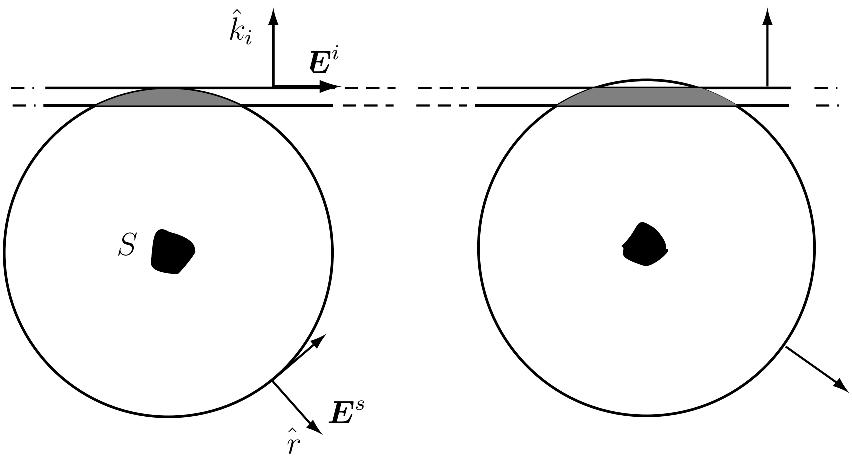


**Figure 5**: The scattering object, the incident wave and the scattered wave. The figure to the right shows a case when the wave speed inside the scattering object is larger than for the surrounding medium.

and since the intersection is in the far zone $\boldsymbol{H}^s = \sqrt{\frac{\varepsilon}{\mu}}\hat{r} \times \boldsymbol{E}^s$. Introduce cylindrical coordinates $\rho, \phi, z$ where $\hat{z} = \hat{k}_i$ and use the farfield expression (2.3) for $\boldsymbol{E}^s$. Since $\hat{r} = \hat{k}_i = \hat{z}$ in the region of intersection, the integral is reduced to

$$\begin{aligned} W_s(t) + W_a(t) &= -4\pi\varepsilon \int\limits_{c(t-t_1)}^{c(t-t_0)} \boldsymbol{E}_0(t - z/c) \cdot \int\limits_0^\infty \frac{\rho}{\sqrt{\rho^2 + z^2}} \boldsymbol{F}(\hat{k}_i, t - \sqrt{\rho^2 + z^2}/c) d\rho dz \\ &= -4\pi\varepsilon c \int\limits_{c(t-t_1)}^{c(t-t_0)} \boldsymbol{E}_0(t - z/c) \cdot \int\limits_{-\infty}^{t-z/c} \boldsymbol{F}(\hat{k}_i, t'') dt'' dz \end{aligned} \tag{A.5}$$

Now $\boldsymbol{F}(t'')$ is zero for $t'' < t_f$ and then the substitution $t' = t - z/c$ transforms the integral to

$$W_s(t) + W_a(t) = -\frac{4\pi}{\mu} \int_{t_0}^{t_1} \boldsymbol{E}_0(t') \cdot \int_{t_f}^{t'} \boldsymbol{F}(\hat{k}_i, t'') dt'' dt', \tag{A.6}$$

which is the optical theorem in the time-domain. Notice that the sum $W_s(t) + W_a(t)$ is independent of time although each of the two terms can be time-dependent.

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