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Published in:
Control-Theory and Advanced Technology (C-TAT)

1995

Reprinted from

Control–Theory and Advanced Technology
Vol.10, No.4, Part 2, pp.993–1013, 1995

SUPERMARTINGALE ANALYSIS OF MINIMUM VARIANCE ADAPTIVE CONTROL

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PUBLISHED BY MITA PRESS
TOKYO, JAPAN
1995

CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol. 10, No. 4, Part 2, pp. 993–1013, June, 1995

Contributed Paper

SUPERMARTINGALE ANALYSIS OF MINIMUM VARIANCE ADAPTIVE CONTROL*

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Abstract. Recursive estimation in feedback operation as used in minimum variance adaptive control is considered. The purpose is to eliminate the necessity of resorting to the stochastic stability assumption which has been an unsatisfactory feature of previous work in this field, and to establish stability properties for minimum variance adaptive control based on least-squares identification. To this end, supermartingale analysis has been used, and adaptive control has been formulated as a problem of information theory. Stability results demonstrating state-error convergence are provided.

Key Words—Adaptive control, recursive estimation, self-tuning control, stability, supermartingale.

1. Introduction

Recursive parameter estimation with some subsequent feedback action has often been used in applications of signal processing, adaptive control, or artificial neural networks. The presence of feedback operation entails several theoretical problems of time series analysis concerning the interference between estimation and control. One important area is adaptive control which often is based on least-squares estimation of the adaptive control parameters. Such solutions are systematically biased in the presence of colored noise, and convergence towards correct values of the estimated control parameters is not self-evident. It was, however, stated by Åström and Wittenmark (1973) that their self-tuning controller will converge to a minimum variance regulator (if it converges at all). Ljung (1977) formulated positive real conditions for stationary parameter convergence under the assumption that the trajectories are stable and finite.

However, quasistationary analysis is not entirely satisfactory for the analysis of the time-varying adaptive phenomena. The nonstationary case was analyzed by Solo (1979), who showed convergence of pseudolinear regression to have been obtained with the "near supermartingale" approach of Neveu (1975), which presupposes the fulfillment of the necessary condition that the regressors are bounded. Landau and Silveira (1979) and Landau (1980; 1982) used the same outlines together with hyperstability analysis (Popov, 1973) to show parameter convergence of least-squares based adaptive control in the presence of noise, though

* Received by the editors September 3, 1993 and in revised form April 7, 1994.
This work was supported by grants from the Research Board of Natural Sciences (NFR) and Centre National de la Recherche Scientifique (CNRS).

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here, too, the stability condition is assumed in the proof. Becker et al. (1985) used a geometric argument to demonstrate convergence points of parameters when simple gradient methods are used.

In all these papers, stability is an assumption of the proofs in which convergence is shown. Some problems of stability and convergence have been treated by Goodwin and Sin (1984), although their approach lacks stringency in the limits of error magnitude. A stochastic Lyapunov function or a supermartingale study of the transient trajectories from an initial state value to an equilibrium point has the advantage of taking into account both the convergence aspects and the disturbance rejection aspects of stability (see Kushner, 1967).

The noise-free system and the noise-corrupted system may differ in convergence points due to the colored noise and the parameter estimation during closed-loop operation. The study of the noise-free system and the noise-corrupted system motivates an investigation of the two different convergence points and their associated control laws. Two different linear control laws are relevant as possible convergence points (or fixed points): first, the deterministic control law with pole-zero cancellation of zeros of the control object and pole assignment of other poles to the origin; second, the minimum variance control law which results in pole-zero cancellation of all zeros of the control object and its noise model. These control laws can be expressed in terms of the adaptive control parameters involved and thus, there are two convergence points:

- The convergence point θ_o for parameters associated with the purely deterministic noise-free problem corresponds to a pole assignment of the closed-loop system with pole-zero cancellation of the system zeros and with all other poles at the origin.
- The parameter convergence point θ_{MV} corresponds to minimum variance control as anticipated by Åström and Wittenmark (1973).

The purpose of this paper is to eliminate the stochastic stability assumption which has been a not entirely satisfactory feature of previous work in this field, and to establish stability properties for minimum variance adaptive control based on least-squares identification. To this end, supermartingale analysis is used. Some results on stability properties of the two convergence points are presented, and adaptive control is formulated as a problem of information theory.

2. System Description and Notations

We start with a standard system description of a control object as a discrete-time ARMAX model formulated in the backward shift operator,

$$A^*(q^{-1})y(t) = b_0 q^{-1} B^*(q^{-1})u(t) + C^*(q^{-1})w(t), \tag{1}$$

from the input u and the noise w to the output y with the following coprime polynomials in the backward shift operator q^{-1}:

$$\left.\begin{aligned} A^*(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n} \\ B^*(q^{-1}) &= 1 + b_1 q^{-1} + \cdots + b_{n-1} q^{-n+1} \\ C^*(q^{-1}) &= 1 + c_1 q^{-1} + \cdots + c_n q^{-n} \end{aligned}\right\}, \tag{2}$$

where "*" is standard notation to denote polynomials in the backward shift operator. The B^*-polynomial is assumed to have no non-minimum phase zeros, and the parameter b_0 is a gain factor. Assume that the input u and the output are available to measurement, and let φ contain the u and y as well as some reference values u_c.

$$\varphi(k) = [u(k-1) \cdots u(k-n+1) \quad y(k) \cdots y(k-n+1) \quad u_c(k) \cdots u_c(k-n)]^T . \tag{3}$$

The purpose of the direct adaptive control algorithm is to make simultaneous feedback control and the estimation of suitable regulator parameters from input-output data when the process model is unknown. The adaptive control problem is thus partitioned into a parameter estimation problem and a control problem. Let θ denote the adequate feedback parameters corresponding to φ. The direct minimum variance adaptive control algorithm then comprises the following steps:

$$\begin{cases} \hat{\theta}(k) = \hat{\theta}(k-1) + P(k)\varphi(k-1)\varepsilon(k), & \text{Parameter updating} \\ P(k) = P(k-1) - \dfrac{P(k-1)\varphi(k-1)\varphi^T(k-1)P(k-1)}{1+\varphi^T(k-1)P(k-1)\varphi(k-1)}, & \\ & P(0) = P_0 = P_0^T > 0 \\ \varepsilon(k) = y(k) - \beta u(k-1) - \hat{\theta}^T(k-1)\varphi(k-1), & \text{Prediction error} \\ u(k) = -\dfrac{1}{\beta}(\hat{\theta}^T(k)\varphi(k)), & \text{Control law} \end{cases} \tag{4}$$

where the vector of estimated parameters $\hat{\theta}$ has replaced the parameters θ of the correct desired control law, whereas β_0 is a fixed *a priori* estimate of the gain factor b_0.

$$\theta = [r_1 \cdots r_{n-1} \quad s_0 \quad s_1 \cdots s_{n-1} \quad t_0 \cdots t_n]^T . \tag{5}$$

The vectors θ and φ contain the coefficients and input-output data to express the appropriate control law as the inner product,

$$u(k) = -\frac{1}{b_0}(\theta^T \varphi(k)), \tag{6}$$

derived from the polynomial formulation,

$$R^*(q^{-1})u(t) = -S^*(q^{-1})y(t) + T^*(q^{-1})u_c(t), \tag{7}$$

where R^*, S^*, T^* are polynomials in the backward shift operator (see Fig. 1).

3. State Space Model

The assumed coprimeness of A^*, B^* and C^* of Eq. (1) ensures that the input-output model of Eq. (1) also corresponds to a state-space realization of order n and also the fractional form,

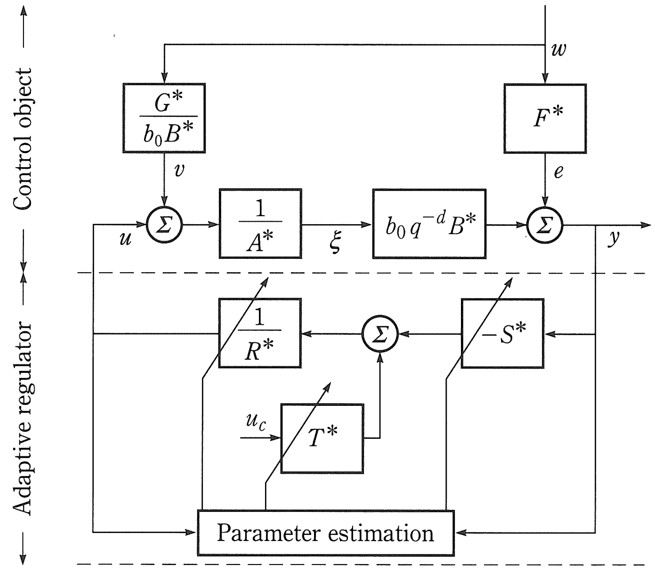

Fig. 1. Block diagram of the self-tuning regulator with a noise model according to (3)–(5) with the variable $d=1$ denoting time delay. Notice that a correctly tuned minimum variance regulator totally decouples the partial state ξ from interference of noise w.

$$A^*(q^{-1})\xi(k) = u(k) + v(k), \tag{8}$$

$$y(k) = b_0 q^{-1} B^*(q^{-1})\xi(k) + e(k), \tag{9}$$

with the partial-state variable $\xi(k)$ and the noise components

$$\left.\begin{aligned} v(k) &= \frac{G^*(q^{-1})}{b_0 B^*(q^{-1})} w(k) \\ e(k) &= F^*(q^{-1}) w(k) \end{aligned}\right\} \tag{10}$$

for polynomials F^* and G^* satisfying the polynomial Diophantine equation,

$$A^*F^* + q^{-1}G^* = C^*. \tag{11}$$

The appropriate minimum variance regulator is given by

$$\theta = \theta_{MV} \Leftrightarrow \begin{cases} R^* = b_0 B^* F^*, & \text{“Minimum variance”} \\ S^* = G^* & \\ T^* = C^* & \end{cases} \tag{12}$$

in the case of known parameters, whereas in the noise-free system it is given by

$$\theta = \theta_o \Leftrightarrow \begin{cases} R^* = b_0 B^* F_o^*, & \text{Pole assignment to the origin,} \\ S^* = G_o^*, & \\ T^* = 1, & \end{cases} \tag{13}$$

where the characteristic polynomials of the closed-loop systems are

$$\begin{aligned} &R^* A^* + S^* (b_0 q^{-1} B^*) \\ &\quad = \begin{cases} b_0 B^* (A^* F^* + q^{-1} G^*) = b_0 B^* C^*, & \text{"Minimum variance,"} \\ b_0 B^* (A^* F_o^* + q^{-1} G_o^*) = b_0 B^*, & \text{Pole assignment to the origin.} \end{cases} \end{aligned} \tag{14}$$

The following expansion based on the polynomial equation above (argument q^{-1} omitted) gives:

$$\begin{aligned} v(k) = B^*(q^{-1}) \xi(k) &= \frac{1}{b_0} (R^* A^* + S^* (b_0 q^{-1} B^*)) \xi(k) \\ &= \frac{1}{b_0} (R^* (u(k) + v(k)) + S^* (y(k) - e(k))) \\ &= \frac{1}{\beta} (\beta u(k) + \frac{\beta}{b_0} \theta_o^T \varphi(k)) + \frac{1}{b_0} (F_o G - G_o F) w(k), \end{aligned} \tag{15}$$

whereas the same expansion with respect to θ_{MV} gives

$$v(k) = B^*(q^{-1}) \xi(k) = \frac{1}{\beta C^*(q^{-1})} \left(\beta u(k) + \frac{\beta}{b_0} \theta_{MV}^T \varphi(k) \right). \tag{16}$$

The signal v may be interpreted as the error input to the compensated closed loop system or as a signal representing the error of feedback control. State vectors for the parameter estimation error can be introduced as

$$\begin{cases} \tilde{\theta}_{MV}(k) = \hat{\theta}(k) - \dfrac{\beta}{b_0} \theta_{MV}, & \text{"Minimum variance",} \\ \tilde{\theta}_o(k) = \hat{\theta}(k) - \dfrac{\beta}{b_0} \theta_o, & \text{Poles at the origin.} \end{cases} \tag{17}$$

Application of the adaptive control law $u = -\hat{\theta}^T \varphi / \beta$ described in (4) gives

$$\begin{aligned} &v(k) = B^*(q^{-1}) \xi(k) \\ &\quad = \begin{cases} \dfrac{1}{\beta} (-\tilde{\theta}_o^T \varphi(k)) + \dfrac{1}{b_0} (F_1 G - G_1 F) w, & \text{Pole assignment to the origin,} \\ \dfrac{1}{\beta C^*} (-\tilde{\theta}_{MV}^T \varphi(k)), & \text{"Minimum variance."} \end{cases} \end{aligned} \tag{18}$$

The control object was described by the fraction form of Eqs. (3)–(5), and the states of the regulator may be represented in a similar way. However, the regulator makes use of old input-output data, and it is feasible to express the control object states as well as the regulator states of φ in terms of ξ. Introduce, therefore, the state vector,

$$x(k)=[\xi(k-1) \quad \xi(k-2)\cdots\xi(k-2n+1)]^T\in\mathcal{R}^{2n-1}. \tag{19}$$

A state equation of Eqs. (18), (19) or Eqs. (8), (9) subject to control by Eq. (7) with the dynamics of x on controllable canonical form is given by

$$x(k+1)=\Phi x(k)+\Gamma v(k) \tag{20}$$

with a Φ-matrix and a Γ-vector given by

$$\Phi=\begin{bmatrix} -b_1 & \cdots & -b_n & 0 & \cdots & 0\\ 1 & 0 & \cdots & 0 & & 0\\ 0 & 1 & \ddots & \vdots & & \vdots\\ \vdots & \ddots & \ddots & 0 & & 0\\ 0 & \cdots & 0 & 1 & & 0\end{bmatrix}=\begin{bmatrix} b^T & 0\\ & 0\\ I_{(2n-2)\times(2n-2)} & \vdots\\ & 0\end{bmatrix},\quad \Gamma=\begin{bmatrix}1\\0\\ \vdots\\ 0\end{bmatrix}. \tag{21}$$

The components of the vector b are the coefficients b_i of the polynomial B^* of Eq. (2). A state space representation of the matrix $P(k)$ of Eq. (4) is needed which motivates the introduction of the vector Π defined from the components p_{ij} of the matrix $P(k)$.

$$\Pi(k)=[p_{11}(k)\cdots p_{1(2n-1)}(k) \quad p_{21}(k)\cdots p_{(2n-1)(2n-1)}(k)]^T\in\mathcal{R}^{2n^2-4n+1}. \tag{22}$$

Notice that the matrix Φ represents all poles of the closed-loop system—also those poles which cancel the zeros of the B^*-polynomial.

The full error dynamics state vector X comprising the states of the control object, controller, and the parameter estimation is now

$$X_{MV}(k)=[x^T(k) \quad \tilde{\theta}_{MV}^T(k) \quad \Pi^T(k)]^T, \qquad X_{MV}\in\mathcal{R}^{4n^2-1}, \tag{23}$$

$$X_o(k)=[x^T(k) \quad \tilde{\theta}_o^T(k) \quad \Pi^T(k)]^T, \qquad X_o\in\mathcal{R}^{4n^2-1}. \tag{24}$$

The difference between the state vectors X_{MV} and X_o is thus constant

$$\|X_{MV}(k)-X_o(k)\|=\frac{\beta}{b_0}\|\theta_{MV}-\theta_o\|, \quad \forall k. \tag{25}$$

Introduce the following assumptions.

Assumption 1. The polynomials A^*, B^* and C^* are mutually prime.

Assumption 2. The polynomial B^* has a stable inverse.

Assumption 3. The accuracy of the estimate β_0 of the gain b_0 is such that $0<b_0/\beta_0<2$.

Assumption 4. The parameter vector $\hat{\theta}(k)$ has a correct number of parameters.

For simplicity, the analysis is restricted to the following case.

Assumption 5. Reference value $u_c = 0$ and

$$\theta = [r_1 \cdots r_{n-1} \quad s_0 \quad s_1 \cdots s_{n-1}]^T .$$

Example 1. Consider the control object,

$$y_k = 0.7 y_{k-1} + u_{k-1} + w_k + 0.3 w_{k-1}$$

with ARMAX polynomials $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$ and $C^* = 1 + 0.3q^{-1}$. The feedback control law $u_k = -0.7 y_k$ (i.e., $\theta_o = 0.7$) provides pole assignment to the origin, whereas the control law $u_k = -y_k$ (i.e., $\theta_{MV} = 1.0$) provides minimum variance control. The adaptive control algorithm of Eqs. (3)–(5) with $x(0) = 1$, $\hat{\theta}(0) = 0$ and $P(0) = 1 \cdot 10^5$ was simulated (see Fig. 2). Notice that the

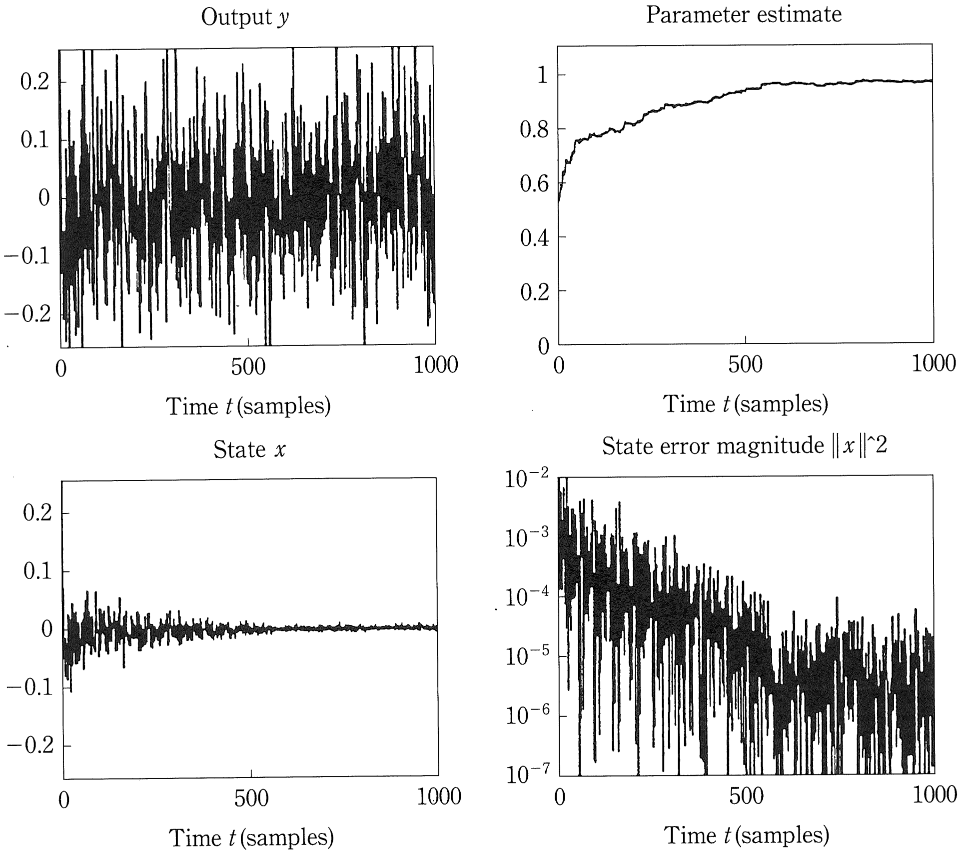

Fig. 2. Simulation of a transient of output y (*upper left*) and state x (*lower left*) vs. time in an adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$ and $C^* = 1 + 0.3q^{-1}$. Notice that the state error magnitude $\|x\|$ gradually decreases (*lower right*) and appears to be asymptotically decoupled from the noise and that the parameter estimate $\hat{\theta}$ first converges towards $\theta_o = 0.7$ and then proceeds towards $\theta_{MV} = 1.0$ (*upper right*). All graphs vs. time.

state error magnitude $||x||$ of Fig. 2 gradually decreases and appears to be asymptotically decoupled from the noise, whereas the parameter estimate $\hat{\theta}$ first converges towards $\theta_o = 0.7$ and then proceeds towards $\theta_{MV} = 1.0$.

4. Stochastic Stability Analysis

Feedback control systems may exhibit unstable behavior which results in unbounded signals under certain circumstances. A prerequisite of convergence analysis is to show that all signals remain bounded, and that motivates an investigation of the growth rate of the state vector X_o. Lyapunov analysis is suitable in this context, and a Lyapunov function candidate is introduced to represent all components of Eqs. (33), (34). A technical requirement is that Lyapunov function candidates should be continuous at the origin and grow with the magnitude of all state vector components (radial unboundedness); see LaSalle (1976) or Khalil (1992). The following result is applicable to the stability of transients in the noise-free system and ensures that the global stability properties are present.

Theorem 1. (Johansson, 1989) Suppose that $\beta = b_0$, and let the positive definite matrix Λ_o satisfy the Lyapunov equation under Assumptions 1–5

$$\Phi^T \Lambda_o \Phi - \Lambda_o = -Q - I, \quad \Lambda_o, Q \in \mathscr{R}^{2n-1 \times 2n-1}. \tag{26}$$

There are constants $\mu > 0$, $K > 0$ such that the function,

$$V_o(X_o(k)) = \frac{1}{K}\tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k) + \log(1 + \mu x^T(k)\Lambda_o x(k)) + \operatorname{tr}(P^T(k)P(k)), \tag{27}$$

decreases in each recursion at least as

$$V_o(X_o(k+1)) - V_o(X_o(k)) \leq -\mu \frac{x^T(k)Qx(k)}{1 + \mu x^T(k)\Lambda_o x(k)},$$
$$V_o(X_o(0)) = V_0. \tag{28}$$

The function V_o is a Lyapunov function for the adaptive system (1)–(21), and the system is globally stable in the sense of Lyapunov with all signals remaining bounded. The state vector converges so that $||x(k)|| \to 0$ as k increases.

Proof. (See Johansson, 1989, app. 3) The generalization to $\beta \neq b_0$ is also shown to be stable although it exhibits only a local (but large) stability region.

The property of Eq. (28) assures global convergence of the state vector so that $||x(k)|| \to 0$ as k increases and suffices to explain the behavior of the noise-free system as well as the behavior of an adaptive system corrupted by occasional disturbances and modeled as transients from an initial state. Non-zero disturbances result in the modification,

$$V_o(X_o(k+1)) - V_o(X_o(k)) \leq -\mu \frac{x^T(k)Qx(k)}{1 + \mu x^T(k)\Lambda_o x(k)} + \mathscr{O}(||W(k)||), \tag{29}$$

where

$$W(k) = [\omega(k) \cdots \omega(k-n)]^T \tag{30}$$

is a vector of noise components affecting the state vector $X_o(k)$. The maximum possible magnitude of the disturbances $\|W(k)\|$ for the system to remain stable is obviously limited by the matrix norms of Q and Λ_o. The result of Eqs. (28), (29) is sufficient to assure the boundedness of all variables involved during disturbances of a certain maximum amplitude. During persistent disturbances, however, obviously no convergence of $\|X_o\|$ is to be expected. By adopting the notation of Markov processes (Doob, 1953), we investigate the adaptive control behavior when there are stochastic disturbances acting on inputs and outputs.

Assumption 6. Let $(\Omega, \mathscr{F}, \mathscr{P})$ be a probability space that describes the noise process $\{w(k)\}_{k=0}^{\infty}$ with the properties

$$\left.\begin{aligned} &\mathscr{E}\{w(k+1)|\mathscr{F}_k\} = 0 \\ &\mathscr{E}\{w^2(k+1)|\mathscr{F}_k\} = \sigma^2 \qquad \text{a.s.} \\ &\mathscr{E}\{w(i)w(j)|\mathscr{F}_k\} = \sigma^2\delta_{i,j} \qquad \text{a.s.,} \quad i, j > k \end{aligned}\right\}, \tag{31}$$

where $\mathscr{F}_k$ is the σ-algebra of measurements up to time k.

Assumption 7. The C^*-polynomial has a stable inverse.

The behavior is changed by the fact that the noise results in a systematic bias of the least-squares estimated parameters $\hat{\theta}$ with respect to the convergence point θ_o. The Lyapunov functions may be replaced by stochastic Lyapunov functions or by some other supermartingale analysis. Introduce the following function:

$$\begin{aligned} -\log \hat{L}(\tilde{\theta}, \Pi(k)) = &\frac{1}{2\sigma^2}\tilde{\theta}_{MV}^T(k)P^{-1}(k)\tilde{\theta}_{MV}(k) \\ &+ \frac{1}{2}\log(2\pi)^{\frac{n-1}{2}}\det(\sigma^2 P(k)), \end{aligned} \tag{32}$$

which would be the log-likelihood function of normally distributed variables $\hat{\theta}$ or $\tilde{\theta}$ attaining the Cramér-Rao lower bound of the covariance function. The mathematical expectation $\mathscr{E}\{-\log L|\mathscr{F}_k\}$ may be interpreted as the amount of information carried by $\tilde{\theta}(k)$ to be "transmitted" at the input of the control system. Similarly, the information received at the output of the control system may be quantified as the following information rate (Shannon, 1948; Gallager, 1968) about the state x contained in the noisy output signal y:

$$\begin{aligned} C(x(k)) &= b_0^2 \log\left(1 + \frac{S}{N}\right) = b_0^2 \log\left(1 + \frac{1}{\sigma^2}x^T(k)\Lambda x(k)\right) \\ &\le \frac{b_0^2}{\sigma^2}x^T(k)\Lambda x(k), \qquad \Lambda = \Lambda^T > 0, \end{aligned} \tag{33}$$

where S/N is the suitable signal-to-noise ratio, and b_0 is the gain factor of Eq. (9). The following entropy measure $\mathscr{H}$ is the aggregate of information rates at the input and the output of the adaptive system,

$$\mathscr{H}(X_{MV}(k)) = -\log \hat{L}(\tilde{\theta}(k), \Pi(k)) + C(x(k)). \tag{34}$$

This function is not quite suitable as a Lyapunov function candidate because it is not positive definite with respect to $\|\Pi(k)\| \neq 0$ (i.e., for $P(k) \neq 0$). (The boundedness of $P(k)$ is always guaranteed, however, because $\Pi^T(k)\Pi(k) = \mathrm{tr}(P^T(k)P(k))$ can be shown to be positive and to decrease as k increases.) Introduce the following function intended for supermartingale analysis of convergence:

$$V_{MV}(X_{MV}(k)) = \exp\left(-\log \hat{L}(\tilde{\theta}, \Pi(k)) + \frac{b_0^2}{\sigma^2} x^T(k)\Lambda x(k)\right) - 1 \tag{35}$$

with initial value,

$$V_{MV}(X_{MV}) \geq V_{MV}(0) = 0,$$

so that

$$\mathscr{H}(X_{MV}) \leq \log(1 + V_{MV}(X_{MV})), \tag{36}$$

and state the following theorem.

Theorem 2. Suppose that an adaptive control system is defined by Eqs. (1)–(4) with measurements and feedback at times k; $k \in N$ under Assumptions 1–7. There exists a matrix $\Lambda = \Lambda^T > 0$ so that the $\mathscr{F}_k$-measurable function $V_{MV}(X_{MV}(k))$ satisfies the conditions,

$$\left.\begin{array}{lll} \text{(i)} & \mathscr{F}_k \subset \mathscr{F}_{k+1}, & \forall k \in N \\ \text{(ii)} & \mathscr{E}\{|V_{MV}(X_{MV}(k))|\} < \infty & \\ \text{(iii)} & V_{MV}(X_{MV}(k)) \geq \mathscr{E}\{V_{MV}(X_{MV}(k+1)) | \mathscr{F}_k\}, & \forall k \in N \end{array}\right\}, \tag{37}$$

and $\{V_{MV}(X_{MV}(k)), \mathscr{F}_k\}$, $\{\mathscr{H}(X_{MV}(k)), \mathscr{F}_k\}$ of Eqs. (34), (35) are supermartingales so that the adaptive system is stable for a β restricted as

$$1 - \sqrt{1 - \Gamma^T \Lambda \Gamma} \leq \frac{\beta}{b_0} \leq 1 + \sqrt{1 - \Gamma^T \Lambda \Gamma}. \tag{38}$$

Proof. See Appendix.

5. No Stable Inverse of C^*

Consider now the case when Assumption 7 is not valid. The optimal solutions found by the adaptive control algorithm are modified in the case where C^* has no stable inverse. The minimum variance solution is given by reformulation of the noise process and spectral decomposition with reflections of zeros in the unit circle. Find a decomposition of C^* into

$$C^*(q^{-1}) = C^{*+}(q^{-1}) C^{*-}(q^{-1}) \tag{39}$$

so that C^{*-} contains nothing but the non-invertible zeros. The minimum variance adaptive control parameters converge towards the regulator,

$$\begin{cases} R^* = b_0 B^* F^*_{\text{mod}} \\ S^* = G^*_{\text{mod}} \\ T^* = C^- C^{+*} \end{cases} \tag{40}$$

with F^*_{mod} and G^*_{mod} as solutions from

$$A^*(q^{-1})F^*_{\text{mod}}(q^{-1}) + q^{-d}G^*_{\text{mod}}(q^{-1}) = C^-(q^{-1})C^{+*}(q^{-1}). \tag{41}$$

The stable closed loop pole polynomial will be

$$B^* C^{*+} C^- . \tag{42}$$

Example 2. (Non-minimum phase system) Consider the control object,

$$y_k = 0.7 y_{k-1} + u_{k-1} + w_k + 3.333 w_{k-1}$$

with ARMAX polynomials $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$ and $C^* = 1 + 3.333q^{-1}$. The feedback control law $u_k = -0.7y_k$ (i.e., $\theta_o = 0.7$) provides pole assignment to the origin, whereas the control law $u_k = -y_k$ (i.e., $\theta_{MV} = 1.0$) provides minimum variance control (Fig. 3). The adaptive control algorithm of Eqs. (3)–(5) with $x(0) = 1$, $\hat{\theta}(0) = 0$ and $P(0) = 1 \cdot 10^5$ was simulated (see Fig. 4). Note that the state error magnitude $\|x\|^2$ of Fig. 4 does not decrease over time and that the parameter estimate $\hat{\theta}$ first converges towards $\theta_o = 0.7$ and then proceeds towards $\theta_{MV} = 1.0$. In contrast to Example 1, there is not provided any asymptotical decoupling of x from the noise.

Minimum variance control of the output in this case is not, of course, a well-posed optimal control problem because the solution gives minimal variance of

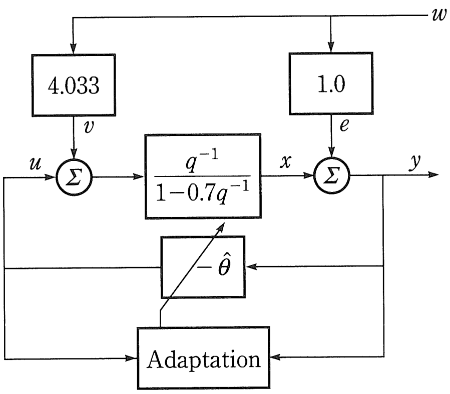

Fig. 3. An adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$, $C^* = 1 + 3.333q^{-1}$, $F^* = 1$ and $G^* = 4.0333$.

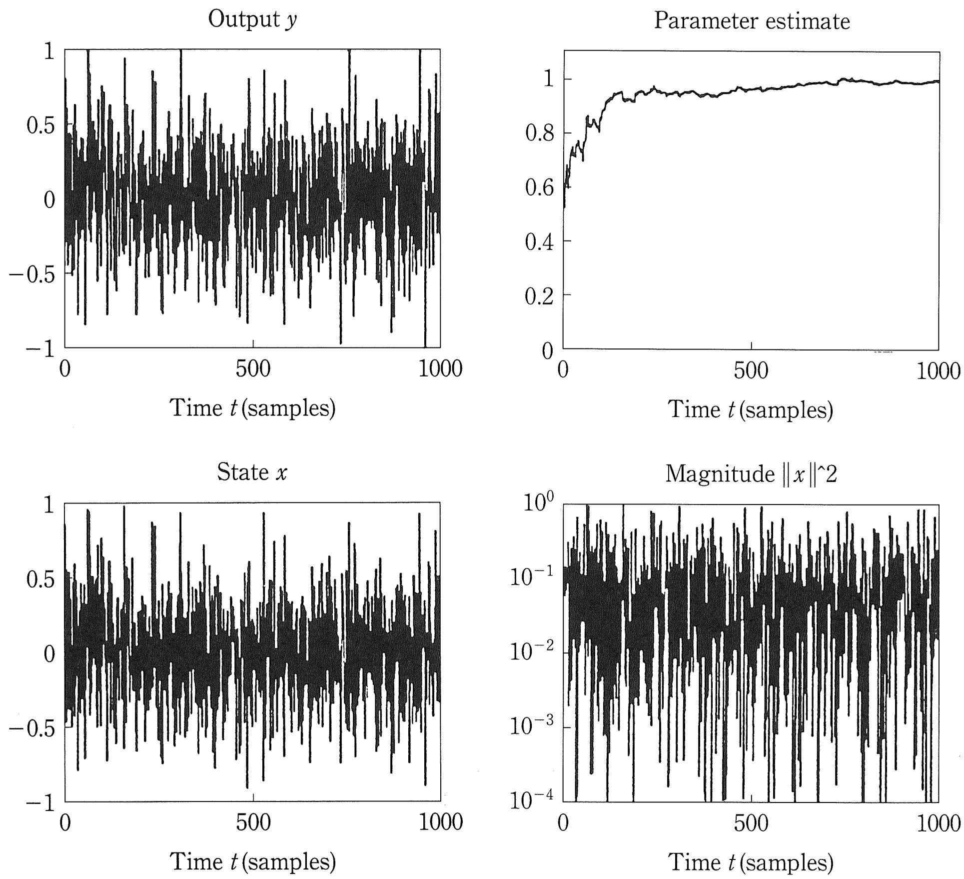

Fig. 4. A transient of output y (*upper*) and state x (*lower*) vs. time in the adaptive control system with $A^* = 1 - 0.7q^{-1}$, $B^* = q^{-1}$ and $C^* = 1 + 3.33q^{-1}$. Parameter convergence of $\hat{\theta}$ vs. time with $\theta_o = 0.7$ and $\theta_{MV} = 1.0$. Notice that $\hat{\theta}$ first converges towards θ_o and then proceeds towards θ_{MV}. Notice that x remains noisy and the magnitude $||x||^2$ does not decrease over time.

the output at the expense of variation of the state x. The stability will guarantee that $||x||$ does not grow beyond a certain limit according to Theorem 1, but x is excited by noise also for well-tuned minimum variance control. Example 2 demonstrates that it is not expected that $||x|| \to 0$. Hence, the result of state-error convergence of Theorem 2 is not achieved for non-minimum phase systems.

6. Discussion and Conclusions

The deterministic aspects of stability and the convergence towards θ_o as presented in Theorem 1 may be compared to earlier results on adaptive control presented by Johansson (1986; 1989). The results are based on the same assumptions as in Goodwin and Sin (1984, Ch. 11.3) or Egardt (1979). This work confirms earlier results of Ljung (1977) and Landau (1980; 1982) on convergence of minimum variance control. It contributes new results by eliminating the stability assumption which has been an unsatisfactory feature of previous work, and also quantifies the information properties with supermartingale analysis. The slow final convergence to the minimum variance solution is explained with an information theory argument. The originally formulated problem of least-squares estimation subject to closed-loop operation and with impact of colored noise thus results in convergence to a maximum-likelihood estimator, thus confirming the conjecture of Åström and Wittenmark (1973).

The time-variant, nonstationary behavior of adaptive control has motivated the use of stochastic Lyapunov functions or supermartingale methods. It was shown that there are two convergence points θ_o and θ_{MV}, both with cancellation of the control system zeros. A large set around θ_o is attractive for large transient trajectories, whereas θ_{MV} is locally attractive. Recovery from a large disturbance starts with initial convergence towards the deterministic solution point. It is shown in this paper that a set around θ_o is globally attractive, whereas the point θ_{MV} is only locally attractive. Stable solutions of large initial magnitudes thus start their trajectories by attraction from θ_o. Eventually, these trajectories enter the domain of attraction of θ_{MV} and converge to the minimum variance regulator. The final convergence is towards θ_{MV} and thus minimum variance control when the trajectory has reached the minimum variance solution domain of attraction. In Fig. 5 are shown phase plane trajectories and parameter transients as averaged for each point in time from fifty simulations of the adaptive control transients of Examples 1 and 2. These simulations illustrate the conclusions made about properties of the two convergence points of the adaptive control laws.

A common idealization in the literature is to consider the case of "persistent excitation" (see Anderson, 1982).

$$0 < \alpha k I_{2nx2n} \leq P^{-1}(k) \Rightarrow P(k) \leq \frac{1}{\alpha k} I_{2nx2n}, \quad \forall k \geq 1. \tag{43}$$

This permits an immediate verification of consistency of the parameter estimation, although the presence of such a condition is difficult to verify *a priori*. An assumption of persistent excitation is therefore virtually an assumption on the consistency of parameter estimation. The present analysis avoids this difficulty and shows the control error to converge even when it is not possible to show consistency.

There are properties of the supermartingales used with obvious links to features of information theory. Obviously, the stability of the adaptive system is closely related to its ability to extract adequate information from input-output data. An adaptive system may generate appropriate input-output data of this kind during the transient of an initial state. However, a system controlled by a minimum variance regulator exhibits an output that contains only a noise sequence without any information about the input-output properties between u and y. The signal-to-noise ratio thus decreases significantly and identifiability is gradually impaired as the adaptive system converges towards minimum variance control, a property that is reflected in the behavior of the entropy function $\mathscr{H}$.

Unfortunately, the use of the term "information" is somewhat ambiguous in the context of adaptive control. For example, the first term of $\mathscr{H}$ of Eq. (32) is a least-squares based criterion with a matrix $P^{-1}(k)$ that grows with time. It is quite standard to argue that P^{-1} contains the accumulated (Fisher-) "information" collected since initial time $k=0$ (see Goodwin and Payne, 1977), so that the covariance of $\tilde{\theta}(k)$ may be estimated via the Cramér-Rao lower bound,

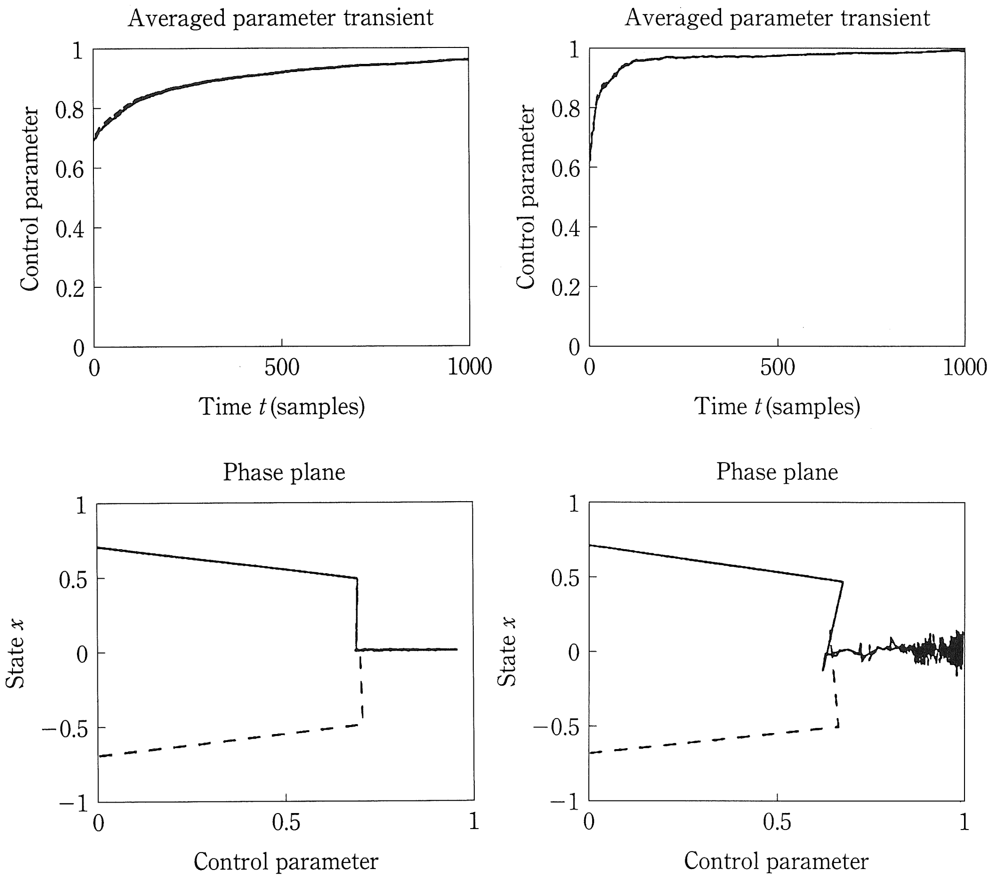

Fig. 5. Averaged control parameter transients ($\hat{\theta}$ vs. time) and phase plane trajectories (state x vs. control parameter $\hat{\theta}$) for 50 realizations of the system of Example 1 (*left*) and that of Example 2 (*right*) without stable inverse of the polynomial C^*.

$$\mathscr{E}\{\tilde{\theta}(k)\tilde{\theta}^T(k)\} \geq \sigma^2 P(k). \tag{44}$$

The uncertainty or entropy (see Shannon (1948) or Gallager (1968)) represented by $\mathscr{H}$ may be increased by noise and decreased by a non-zero $\|x\|$. The stability may be interpreted as follows. Any information in the output signal y about the state error x results in a decrease in the parameter uncertainty so that the entropy of the adaptive system represented by $\mathscr{H}$ decreases at the recursion of each measurement; cf. Eq. (32). Furthermore, the control input error v depends on $\tilde{\theta}$ and results in a new state error x, which information is transmitted to the system output. The information content in the input and output gradually decrease as the magnitudes of state errors x and parameter errors decrease. Because the signal-to-noise ratio decreases as the parameter error $\tilde{\theta}_{MV}$ approaches zero, no information is obtained at the solution point. This information theory interpretation of the supermartingales also explains the slow final convergence towards the minimum variance solution θ_{MV}.

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Appendix

The proof consists of three steps, each corresponding to a proposition of the theorem. The first step follows from the definition of the σ-algebra of measurements at times $k=1,2,3,\cdots$. The state vectors $X_o(k)$ and $X_{MV}(k)$ are measurable with respect to $\mathscr{F}_k$.

The second step is to show the boundedness of all variables involved and which follow from the properties of the Lyapunov function. In particular, both X_o and X_{MV} and thus V_{MV} remain bounded. The third step is to ascertain that the mathematical expectation of $V_{MV}(X_{MV}(k))$ decreases in each step.

Consider the adaptive control algorithm

$$\hat{\theta}(k+1)=\hat{\theta}(k)+P(k+1)\varphi(k)\varepsilon(k+1), \tag{4i}$$

$$P(k+1)=P(k)-\frac{P(k)\varphi(k)\varphi^T(k)P(k)}{1+\varphi^T(k)P(k)\varphi(k)}, \tag{4ii}$$

$$\varepsilon(k+1)=y(k+1)-\beta u(k)-\hat{\theta}^T(k)\varphi(k), \tag{4iii}$$

$$u(k)=-\frac{1}{\beta}\hat{\theta}(k)\varphi(k), \tag{4iv}$$

for which the prediction error one step ahead is determined by

$$\begin{aligned}
&\varepsilon(k+1)\\
&\quad=y(k+1)-\beta u(k)-\hat{\theta}^T(k)\varphi(k)\\
&\quad=y(k+1)=b_0B^*(q^{-1})\xi(k)+w(k+1)\\
&\quad=\frac{b_0}{\beta C^*(q^{-1})}(-\hat{\theta}^T_{MV}(k)\varphi(k))+w(k+1)=b_0\nu(k)+w(k+1).
\end{aligned} \tag{A.1}$$

The first equality of Eq. (A.1) is determined by means of Eq. (4iii) and the second equality according to Eq. (4iv).

Now consider the following positive, radially growing function:

$$v_\theta(\tilde{\theta}(k))=\frac{1}{2}\tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k). \tag{A.2}$$

The development of v_θ one step ahead is determined as

$$\begin{aligned}
\Delta v_\theta&=v_\theta(\tilde{\theta}(k+1))-v_\theta(\tilde{\theta}(k))\\
&=\frac{1}{2}\tilde{\theta}^T(k+1)P^{-1}(k)\tilde{\theta}(k+1)-\frac{1}{2}\tilde{\theta}^T(k)P^{-1}(k)\tilde{\theta}(k)\\
&=\frac{1}{2}(\tilde{\theta}^T(k)\varphi(k))^2+\tilde{\theta}^T(k)\varphi(k)\varepsilon(k)+\frac{1}{2}\frac{\varphi^T(k)P(k)\varphi(k)}{1+\varphi^T(k)P(k)\varphi(k)}\varepsilon^2(k+1)\\
&=\frac{1}{2}(\tilde{\theta}^T(k)\varphi(k)+\varepsilon(k+1))^2-\frac{1}{2}\frac{1}{1+\varphi^T(k)P(k)\varphi(k)}\varepsilon^2(k+1).
\end{aligned} \tag{A.3}$$

Let Δ_θ and Δ_ε designate

$$\Delta_\theta = \frac{1}{2}(\tilde{\theta}^T(k)\varphi(k) + \varepsilon(k+1))^2,$$

$$\Delta_\varepsilon = \frac{1}{2}\frac{1}{1+\varphi^T(k)P(k)\varphi(k)}\varepsilon^2(k+1).$$

The positive term of Eq. (A.3) for $\tilde{\theta} = \tilde{\theta}_{MV}$ (i.e., Δ_θ) may be developed by means of Eq. (18),

$$\begin{aligned}\Delta_\theta &= \frac{1}{2}(\tilde{\theta}_{MV}^T(k)\varphi(k) + \varepsilon(k+1))^2\\ &= \frac{1}{2}\left(-\beta\begin{bmatrix}\left(1-\dfrac{b_0}{\beta}\right) & c_1 \cdots c_n\end{bmatrix}\begin{bmatrix}\nu(k)\\ \vdots\\ \nu(k-n)\end{bmatrix} + w(k+1)\right)^2\\ &= \frac{1}{2}\left(\gamma\begin{bmatrix}\nu(k)\\ \vdots\\ \nu(k-n)\end{bmatrix} + w(k+1)\right)^2,\end{aligned} \tag{A.4}$$

where the c_i :s are the coefficients of the C^*-polynomial of Eq. (2) and with γ and υ denoting

$$\left.\begin{aligned}\gamma &= -\beta\begin{bmatrix}\left(1-\dfrac{b_0}{\beta}\right) & c_1 \cdots c_n\end{bmatrix}^T \in \mathscr{R}^{n+1}\\ \upsilon &= \begin{bmatrix}\nu(k)\\ \vdots\\ \nu(k-n)\end{bmatrix} = U\begin{bmatrix}\nu(k)\\ x(k)\end{bmatrix}\\ U &= \begin{bmatrix}1 & 0 & \cdots & 0 & 0 & \cdots & 0\\ 1 & b_1 & \cdots & b_{n-1} & 0 & \cdots & 0\\ 0 & 1 & b_1 & \cdots & b_{n-1} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & & \ddots & \vdots\\ 0 & \cdots & 0 & 1 & b_1 & \cdots & b_{n-1}\end{bmatrix} \in \mathscr{R}^{(n+1)\times 2n}\end{aligned}\right\} \tag{A.5}$$

The mathematical expectation with respect to $\mathscr{F}_k$ of expressions of Eqs. (A.2)–(A.4) is

$$\begin{aligned}\mathscr{E}\{\varepsilon^2(k+1)|\mathscr{F}_k\} &= \mathscr{E}\{(b_0\nu(k) + w(k+1))^2|\mathscr{F}_k\}\\ &= b_0^2\nu^2(k) + \sigma^2 \quad \text{a.s.}\end{aligned}$$

and

$$\begin{aligned}\mathscr{E}\{\Delta_\theta\} &= \mathscr{E}\left\{\frac{1}{2}(\tilde{\theta}_{MV}^T(k)\varphi(k) + \varepsilon(k+1))^2|\mathscr{F}_k\right\}\\ &= \frac{1}{2}(\gamma^T\upsilon)^2 + \sigma^2 \quad \text{a.s.}\end{aligned}$$

Summarizing, one finds that

$$\begin{aligned}
&\mathscr{E}\{\Delta v_\theta(\tilde{\theta}(k+1))|\mathscr{F}_k\} \\
&\quad = \mathscr{E}\left\{\frac{1}{2}(\tilde{\theta}_{MV}^T(k)\varphi(k)+\varepsilon(k+1))^2|\mathscr{F}_k\right\} \\
&\qquad - \mathscr{E}\left\{\frac{1}{2}\frac{\varepsilon^2(k+1)}{1+\varphi^T(k)P(k)\varphi(k)}\middle|\mathscr{F}_k\right\} \\
&\quad = \frac{1}{2}((\gamma^T v(k))^2+\sigma^2)-\frac{1}{2}\frac{b_0^2 v^2(k)+\sigma^2}{1+\varphi^T(k)P(k)\varphi(k)} \quad \text{a.s.}
\end{aligned} \tag{A.6}$$

The mathematical expectation of the second term of Eq. (32) is

$$\begin{aligned}
&\mathscr{E}\{\Delta v_P(k+1)|\mathscr{F}_k\} \\
&\quad = \mathscr{E}\left\{\frac{1}{2}\log(2\pi)^{\frac{n-1}{2}}\det(\sigma^2 P(k+1))-\frac{1}{2}\log(2\pi)^{\frac{n-1}{2}}\det(\sigma^2 P(k))|\mathscr{F}_k\right\} \\
&\quad = \mathscr{E}\left\{\frac{1}{2}\log\det P(k+1)P^{-1}(k)|\mathscr{F}_k\right\} \\
&\quad = \frac{1}{2}\mathscr{E}\{-\log(1+\varphi^T(k)P(k)\varphi(k))|\mathscr{F}_k\} \\
&\quad = -\frac{1}{2}\log(1+\varphi^T(k)P(k)\varphi(k)) \\
&\quad = \frac{1}{2}\log\frac{1}{1+\varphi^T(k)P(k)\varphi(k)} \\
&\quad = \frac{1}{2}\log\left(1-\frac{\varphi^T(k)P(k)\varphi(k)}{1+\varphi^T(k)P(k)\varphi(k)}\right) \\
&\quad \le -\frac{1}{2}\frac{\varphi^T(k)P(k)\varphi(k)}{1+\varphi^T(k)P(k)\varphi(k)},
\end{aligned} \tag{A.7}$$

where the inequality is motivated by a standard inequality stating that the convex function $\log_e(1+x)<x$ for all $x>-1$.

The growth of the function $\hat{L}$ of Eq. (32) one step ahead is therefore determined from Eqs. (A.6), (A.7)

$$\begin{aligned}
\Delta_L &= \mathscr{E}\{-\hat{L}(\hat{\theta}(k+1),\Pi(k+1))|\mathscr{F}_k\}-(-\hat{L}(\hat{\theta}(k),\Pi(k))) \\
&= \frac{1}{\sigma^2}\mathscr{E}\{\Delta v_\theta(k+1)|\mathscr{F}_k\}+\mathscr{E}\{\Delta v_P(k+1)|\mathscr{F}_k\} \\
&\le \frac{1}{2\sigma^2}(\gamma^T v(k))^2-\frac{1}{2\sigma^2}\frac{b_0^2 v^2(k)}{1+\varphi^T(k)P(k)\varphi(k)} \quad \text{a.s.} \\
&= \frac{1}{2\sigma^2}\left([\nu(k) \quad x^T(k)]U^T\gamma\gamma^T U\begin{bmatrix}\nu(k)\\ x(k)\end{bmatrix}-\frac{b_0^2 v^2(k)}{1+\varphi^T(k)P(k)\varphi(k)}\right).
\end{aligned} \tag{A.8}$$

Consider now the state equation (20). The growth of x is determined by

$$\begin{aligned}\Delta_x &= \mathscr{E}\{x^T(k+1)\Lambda x(k+1)|\mathscr{F}_k\} - x^T(k)\Lambda x(k)\\ &= [v(k) \quad x^T(k)]\begin{bmatrix}\Gamma^T\Lambda\Gamma & \Gamma^T\Lambda\Phi\\ \Phi^T\Lambda\Gamma & \Phi^T\Lambda\Phi-\Lambda\end{bmatrix}\begin{bmatrix}v(k)\\ x(k)\end{bmatrix}.\end{aligned} \tag{A.9}$$

We summarize the mathematical expectation, cf. Eq. (35),

$$\begin{aligned}\Delta_V &= \mathscr{E}\{\log(1+V_{MV}(X_{MV}(k+1)))|\mathscr{F}_k\} - \log(1+V_{MV}(X_{MV}(k)))\\ &= \mathscr{E}\left\{-\hat{L}(\tilde{\theta}(k+1), \Pi(k)) + \frac{b_0^2}{2\sigma^2}x^T(k+1)\Lambda x(k+1)|\mathscr{F}_k\right\}\\ &\quad -\left(-\hat{L}(\tilde{\theta}(k+1), \Pi(k)) + \frac{b_0^2}{2\sigma^2}x^T(k)\Lambda x(k)\right)\\ &\le \frac{1}{2\sigma^2}\left([v(k) \quad x^T(k)]\left(U^T\gamma\gamma^T U + b_0^2\begin{bmatrix}\Gamma^T\Lambda\Gamma & \Gamma^T\Lambda\Phi\\ \Phi^T\Lambda\Gamma & \Phi^T\Lambda\Phi-\Lambda\end{bmatrix}\right)\begin{bmatrix}v(k)\\ x(k)\end{bmatrix}\right.\\ &\quad \left. -\frac{b_0^2 v^2(k)}{1+\varphi^T(k)P(k)\varphi(k)}\right).\end{aligned} \tag{A.10}$$

According to the Kalman-Szegö-Popov lemma (see Hitz and Anderson (1969), or Popov (1973)), there are matrices K, L, c and δ such that

$$b_0^2\begin{bmatrix}\Gamma^T\Lambda\Gamma & \Gamma^T\Lambda\Phi\\ \Phi^T\Lambda\Gamma & \Phi^T\Lambda\Phi-\Lambda\end{bmatrix} = -\begin{bmatrix}K^T\\ L\end{bmatrix}[K \quad L^T] + \begin{bmatrix}2\delta & c\\ c^T & 0\end{bmatrix}, \tag{A.11}$$

if and only if the transfer function Eq. (A.12) is strictly positive real (SPR).

$$G(z) = c^T(zI-\Phi)^{-1}\Gamma + \delta > 0, \quad z \in \{z \in \mathscr{C} : |z| = 1\}. \tag{A.12}$$

Introduce a decomposition (cf. Eq. (A.5))

$$U^T\gamma = \begin{bmatrix}\kappa\\ \lambda\end{bmatrix}. \tag{A.13}$$

The following choice of the matrices:

$$\left.\begin{aligned}LL^T &= Q + \lambda\lambda^T \qquad \text{for } Q = Q^T > 0\\ K &= L^{-1}\lambda\kappa\end{aligned}\right\} \tag{A.14}$$

corresponds to the solution

$$\begin{cases} c = 0,\\ 0 < \delta = \dfrac{1}{2}(K^TK + \Gamma^T\Lambda\Gamma),\\ \Phi^T\Lambda\Phi - \Lambda = -\dfrac{1}{b_0^2}(Q + \lambda^T\lambda),\end{cases} \tag{A.15}$$

so that

$$\begin{bmatrix} \kappa^T \\ \lambda \end{bmatrix} [\kappa \quad \lambda^T] + b_0^2 \begin{bmatrix} \Gamma^T \Lambda \Gamma & \Gamma^T \Lambda \Phi \\ \Phi^T \Lambda \Gamma & \Phi^T \Lambda \Phi - \Lambda \end{bmatrix}$$
$$= -\begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} + \begin{bmatrix} \kappa^2 + b_0^2 \Gamma^T \Lambda \Gamma & 0 \\ 0 & 0 \end{bmatrix} \tag{A.16}$$

with a positive definite solution Λ. The solution presented also satisfies the strictly positive real condition Eq. (A.12) because $H(z) = \delta > 0$. From Eqs. (A.10) and (A.16), it now follows that

$$\Delta_V < \frac{1}{2\sigma^2} \Bigg(- x^T(k) Q x(k)$$
$$+ \left(\kappa^2 + b_0^2 \Gamma^T \Lambda \Gamma - \frac{b_0^2}{1 + \varphi^T(k) P(k) \varphi(k)} \right) v^2(k) \Bigg). \tag{A.17}$$

According to Eqs. (A.5) and (A.13), it follows that $\kappa^2 = (\beta - b_0)^2$, and we determine the values for which $\Delta_V < 0$. This condition is satisfied if

$$(\beta - b_0)^2 + b_0^2 \Gamma^T \Lambda \Gamma - \frac{b_0^2}{1 + \varphi^T(k) P(k) \varphi(k)} < 0 \Rightarrow \Delta_V < 0. \tag{A.18}$$

A necessary condition of Eq. (A.18) is obviously that β of Eq. (4) is restricted so that

$$1 - \sqrt{1 - \Gamma^T \Lambda \Gamma} < \frac{\beta}{b_0} < 1 + \sqrt{1 - \Gamma^T \Lambda \Gamma}. \tag{A.19}$$

The Jensen inequality (see Chung, 1974) states that for any convex function $f(x)$, it holds that $f(E(x)) \leq E(f(x))$. Applying the Jensen inequality for the convex function $f(x) = \log(1 + x)$, $x \geq 0$ gives

$$\log(1 + \mathscr{E}\{V_{MV}(X_{MV}(k+1)) | \mathscr{F}_k\}) \leq \mathscr{E}\{\log(1 + V_{MV}(X_{MV}(k+1))) | \mathscr{F}_k\}. \tag{A.20}$$

Finally, let Δ_X denote $\mathscr{E}\{V_{MV}(X_{MV}(k+1)) | \mathscr{F}_k\} - V_{MV}(X_{MV}(k))$ so that

$$\begin{aligned} \Delta_X &= \mathscr{E}\{V_{MV}(X_{MV}(k+1)) | \mathscr{F}_k\} - V_{MV}(X_{MV}(k)) \\ &\leq \exp(\mathscr{E}\{\log(1 + V_{MV}(k+1)) | \mathscr{F}_k\}) - (1 + V_{MV}(X_{MV}(k))) \\ &\leq -(1 + V_{MV}(X_{MV}(k))(1 - \exp(\Delta_k))) < 0, \quad \|x(k)\| \neq 0 \end{aligned} \tag{A.21}$$

as $\Delta_V < 0$, cf. Eq. (A.18) and

$$\mathscr{E}\{V_{MV}(X_{MV}(k+1)) | \mathscr{F}_k\} - V_{MV}(X_{MV}(k)) < 0, \quad \|x(k)\| \neq 0, \tag{A.22}$$

which establishes the supermartingale property of $\{V_{MV}(X_{MV}(k)), \mathscr{F}_k\}$ and $\{\mathscr{H}(X_{MV}(k)), \mathscr{F}_k\}$ as stated in Theorem and Eq. (37).

This finishes the proof.

Rolf Johansson received the Master-of-Science degree in Technical Physics in 1977, the Bachelor-of-Medicine degree in 1983, the doctorate in control theory in 1983, was appointed *Docent* in 1985, and received the Doctor-of-Medicine (M.D.) degree in 1986, all from the Lund University, Lund, Scandinavia.

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