# Design of reflextionless slabs for transient electromagnetic waves

Hellberg, Rasmus; Karlsson, Anders

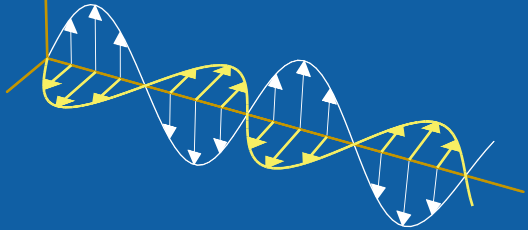
1994

CODEN:LUTEDX/(TEAT-7032)/1-21/(1994)

# Design of reflectionless slabs for transient electromagnetic waves

**Rasmus Hellberg and Anders Karlsson**

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Rasmus Hellberg

Department of Electromagnetic Theory
Royal Institute of Technology
SE-100 44 Stockholm
Sweden

Anders Karlsson

Department of Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

Editor: Gerhard Kristensson

**Abstract**

A method for modeling reflectionless conductive dispersive slabs is presented. The slabs are impedance matched to the surrounding half-spaces and are temporally dispersive with a spatially varying impedance. It is shown that the effects from the variation of the impedance can be matched by the temporal dispersive effects and the conductivity so that for a normally incident plane wave the slab does not reflect any field regardless of the shape of the incident transient field. The problem of finding reflectionless media is formulated as an inverse problem where the constitutive relation is to be determined as a function of depth given a reflection kernel which is zero. The inverse problem is solved by a time domain Green functions technique.

# 1 Introduction

The objective of this paper is to design dispersive plane-stratified slabs which are reflectionless for normally incident plane waves of all frequencies, or equivalently, reflectionless for any incident pulse. The present study is a continuation of the treatment of non-reflecting stratified non-conducting half-spaces given in [1]. The problem is formulated as the inverse problem of finding the constitutive relation as a function of depth, given that the reflection data are known to be zero up to a certain time. In general this problem has non-unique solutions. In the case of a non-dispersive non-conducting slab it is well-known that it is sufficient to keep the impedance constant throughout the slab and matched to the surrounding media in order to get a non reflecting slab. However, for a slab with temporal dispersion or a slab with conductivity it is no longer possible to obtain a reflectionless slab by keeping the impedance constant. The polarization in a dispersive medium depends on the history of the waves that have propagated through the slab. As the polarization diminishes with time, the slab sends out energy and even if the wave front experiences a constant impedance the medium will scatter energy. To model a non-reflecting conductive dispersive slab, the reflections due to dispersion and conductivity are matched by a spatial variation of the impedance.

The technique which is used in the search for reflectionless media is based upon an inverse scattering method in the time domain, referred to as the Green functions technique. The Green functions technique was originally developed for problems concerning one-dimensional direct and inverse scattering from non-dispersive media, cf [2]. It has also been applied to different types of dispersive media (see [3] and earlier references). In the Green functions technique Green operators are introduced which map the incident wave to the internal fields in the slab. The operators are represented in terms of kernels, which satisfy a system of first order hyperbolic equations. Most inverse scattering problems are ill-posed and hence scattering data that are contaminated with noise may give rise to large errors in the solution. In the present case the ill-posedness never shows up in the numerical solution, since the scattering data are prescribed and exactly known, rather than obtained from measurements. The errors in the solution of the inverse problem are then only due to discretization errors and round-off effects from the finite precision of the computer. In practice these errors can be made very small.

In the next section the constitutive relation for a dispersive medium is given in the time domain. In the third section wave splittings are introduced and the wave equation is rewritten in terms of the split fields. The fourth section contains a presentation of the Green functions technique for dispersive media. In section 5 the inverse problem of finding non-reflecting media is stated and the solution is discussed in some specific cases. The numerical section, section 6, contains three different classes of non-reflecting slabs. The dispersive models in the examples are based upon the Debye model, which is appropriate for liquids, and the Lorentz model, which is appropriate for solid materials.

# 2 Formulation of the problem

A general linear isotropic dispersive medium is described by the following constitutive relation, cf [4], [5] and [6],

$$\begin{cases} \mathbf{D}(z,t) = \varepsilon_0(\varepsilon_r(z)\mathbf{E}(z,t) + \int_{-\infty}^{t} \chi(z,t-t')\mathbf{E}(z,t')\,dt') \\ \mathbf{B}(z,t) = \mu_0\mu_r(z)\mathbf{H}(z,t), \end{cases} \tag{2.1}$$

where $\mathbf{D}(z,t)$ is the displacement field, $\mathbf{E}(z,t)$ is the electric field, $\mathbf{B}(z,t)$ is the magnetic induction and $\mathbf{H}(z,t)$ is the magnetic field. The medium thus has a non-constant relative permittivity, $\varepsilon_r(z)$, and a non-constant relative permeability, $\mu_r(z)$. The medium is furthermore dispersive, where the dispersion in the time domain is modeled by the electric susceptibility kernel, $\chi(z,t)$. A more general model for linear dispersive isotropic media can include dispersive magnetic effects and biisotropic effects, but such effects are not considered in this paper.

The stratified dispersive medium with a non-constant conductivity, $\sigma(z)$, occupies the region $0 \le z \le L$. The half-space, $z < 0$, is a non-dispersive non-conducting medium with a constant permittivity $\varepsilon_{r1}$ and permeability $\mu_{r1}$ such that $\varepsilon_{r1}/\mu_{r1} = \varepsilon_r(0)/\mu_r(0)$, i.e., the slab is impedance matched at $z = 0$. The other half-space, $z > L$, is also a non-dispersive non-conducting medium with a constant permittivity $\varepsilon_{r2}$ and permeability $\mu_{r2}$. At this stage the slab is not necessarily impedance matched to the half space $z > L$ at $z = L$.

A transient plane wave propagating in the positive $z-$direction impinges normally at $z = 0$ on the dispersive slab at time $t = 0$. The aim of the present paper is to find different $z-$dependent constitutive relations, i.e to find $\varepsilon_r(z)$, $\mu_r(z)$, $\sigma(z)$ and $\chi(z,t)$ for the slab such that the reflected field from the slab vanishes for any incident field.

# 3 The wave equation and wave splittings

In this section the wave equation for the medium described by the constitutive relation in Eq. (2.1) is presented and written as a system of first order hyperbolic equations. A wave splitting technique is introduced, in which the total electric field is split into two parts corresponding to waves traveling in the positive and negative $z$-direction. This wave splitting technique is the basis for the Green functions technique presented in the next section.

Since the incident wave is a plane wave at normal incidence it can, without loss of generality, be considered to be linearly polarized in the $x-$direction, i.e., $\mathbf{E}(z,t) = E(z,t)\hat{x}$. The corresponding magnetic field is then directed in the $y-$direction, i.e., $\mathbf{H}(z,t) = H(z,t)\hat{y}$. In the region $0 < z < L$ the constitutive relation in Eq. (2.1) and the Maxwell equations lead to

$$\begin{aligned}
\partial_z E(z,t) &= -\mu_0\mu_r(z)\partial_t H(z,t) \\
\partial_z H(z,t) &= -\varepsilon_0\left(\varepsilon_r(z)\partial_t E(z,t) + [\chi(z,\cdot) * \partial_t E(z,\cdot)](t)\right) - \sigma(z)E(z,t).
\end{aligned} \tag{3.1}$$

The conventional short hand notation for convolution

$$[f * g](t) = \int_0^t f(t-t')g(t')\,dt',$$

introduced in Eq. (3.1), is used throughout the rest of the paper. Notice that the lower integration limit in Eq. (3.1) is zero, since there is no electric field in the half space for negative times. The Maxwell equations can be written in a matrix notation as

$$\partial_z \begin{pmatrix} E \\ H \end{pmatrix} = -\begin{pmatrix} 0 & \mu_0\mu_r\partial_t \\ \sigma + \varepsilon_0\varepsilon_r\partial_t + \varepsilon_0\chi * \partial_t & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{A}\begin{pmatrix} E \\ H \end{pmatrix}. \tag{3.2}$$

This system of first order partial differential equations is equivalent to the wave equation for the electric field

$$\left(\partial_z^2 - \frac{1}{c(z)^2}\partial_t^2\right)E(z,t) - \frac{\mu_r(z)}{c_0^2}\left[\chi(z,\cdot) * \partial_t^2 E(z,\cdot)\right](t) - \mu_0\mu_r(z)\sigma(z)\partial_t E(z,t) = 0,$$

where $c_0 = (\varepsilon_0\mu_0)^{-\frac{1}{2}}$ is the velocity of light in vacuum and $c(z) = (\varepsilon_0\varepsilon_r(z)\mu_0\mu_r(z))^{-\frac{1}{2}}$ is the wavefront speed in the medium.

A wave splitting can now be introduced according to the principal part, $\partial_z^2 - c(z)^{-2}\partial_t^2$, of the wave equation. The following change of independent variables is introduced:

$$E^{\pm}(z,t) = \frac{1}{2}\left\{E(z,t) \pm Z(z)H(z,t)\right\},$$

where $Z(z)$ is the wave impedance

$$Z(z) = \sqrt{\frac{\mu_0\mu_r(z)}{\varepsilon_0\varepsilon_r(z)}}.$$

In a matrix form the splitting reads

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & Z(z) \\ 1 & -Z(z) \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{P}\begin{pmatrix} E \\ H \end{pmatrix}. \tag{3.3}$$

Thus the split fields $E^+$ and $E^-$ satisfy the following relations

$$\begin{aligned} E^+ + E^- &= E \\ E^+ - E^- &= Z(z)H. \end{aligned}$$

It is also seen that in a non-dispersive and non-conducting medium the matrix $\mathbf{P}$ diagonalizes the matrix $\mathbf{A}$ in Eq. (3.2). In a non-dispersive region with constant permittivity, the split fields $E^+$ and $E^-$ are left (negative $z$ direction) and right (positive $z$ direction) moving waves, respectively. There are other wave splittings that can be applied to the Maxwells equations. A natural choice is a splitting that includes the conductivity of the medium. This splitting diagonalizes the matrix $\mathbf{A}$ for a non-dispersive conducting medium. For non-dispersive conducting media the matrix $\mathbf{A}$ reads:

$$\partial_z \begin{pmatrix} E \\ H \end{pmatrix} = -\begin{pmatrix} 0 & \mu_0\mu_r\partial_t \\ \sigma + \varepsilon_0\varepsilon_r\partial_t & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{A} \begin{pmatrix} E \\ H \end{pmatrix}. \tag{3.4}$$

The diagonalization of Eq. (3.4) gives the $\mathbf{P}$ matrix, (cf Appendix A).

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \mathcal{K}^{-1} \\ 1 & -\mathcal{K}^{-1} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{P} \begin{pmatrix} E \\ H \end{pmatrix}, \tag{3.5}$$

where

$$\mathcal{K}^{-1}E(z,t) = Z(z)\left(E(z,t) + \frac{\sigma}{2\varepsilon_o\varepsilon_r}[(I_1(\frac{\sigma}{2\varepsilon_o\varepsilon_r}\cdot) - I_0(\frac{\sigma}{2\varepsilon_o\varepsilon_r}\cdot))\exp(-\frac{\sigma}{2\varepsilon_o\varepsilon_r}\cdot) * E(z,\cdot)](t)\right),$$

and where $I_0(t)$ and $I_1(t)$ are the modified Bessel functions. Note that for a non-conducting media this wave splitting is identical to the simple wave splitting given in Eq. (3.3). Either of the two wave splittings presented above can be be utilized for the solution of the Maxwell equations for dispersive media. The Maxwell equations are then written in terms of $E^\pm$ as a system of first order hyperbolic equations

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = ((\partial_z\mathbf{P})\mathbf{P}^{-1} + \mathbf{PAP}^{-1}) \begin{pmatrix} E^+ \\ E^- \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}. \tag{3.6}$$

The operators $\alpha, \beta, \gamma$ and $\delta$ are dependent on the splitting that is used. It turns out that the simple wave splitting given in Eq.(3.3) gives shorter expressions, both for the operators in Eq. (3.6) as well as for the equations that are based upon Eq. (3.6), than the splitting given by Eq. (3.5). For this reason the analysis in the rest of the paper is based upon the splitting defined in Eq. (3.3).

The elements $\alpha$, $\beta$, $\gamma$ and $\delta$ corresponding to the splitting in Eq. (3.3) read

$$\begin{cases} \alpha = \dfrac{1}{2}(\partial_z \ln Z(z) - \sigma(z)Z(z)) - \dfrac{1}{c(z)}\partial_t - \dfrac{\varepsilon_o Z(z)}{2}\chi * \partial_t \\ \\ \beta = -\dfrac{1}{2}(\partial_z \ln Z(z) + \sigma(z)Z(z)) - \dfrac{\varepsilon_o Z(z)}{2}\chi * \partial_t \\ \\ \gamma = -\dfrac{1}{2}(\partial_z \ln Z(z) - \sigma(z)Z(z)) + \dfrac{\varepsilon_o Z(z)}{2}\chi * \partial_t \\ \\ \delta = \dfrac{1}{2}(\partial_z \ln Z(z) + \sigma(z)Z(z)) + \dfrac{1}{c(z)}\partial_t + \dfrac{\varepsilon_o Z(z)}{2}\chi * \partial_t. \end{cases}$$

# 4 The Green operators

In this section, the Green operators for a slab with impedance matched front-wall and back-wall are introduced. In Appendix B these operators are related to the reflection operator for a slab with an impedance mismatched back-wall.

From arguments based upon invariance under time translation and causality, the following representation of the internal split fields are seen to hold for a slab impedance matched to its surrounding, cf [2] and [3],

$$\begin{aligned} E^+(z, t + \tau(z)) &= \mathcal{G}^+ E^+(0, t) = a(z)E^+(0, t) + [G^+(z, \cdot) * E^+(0, \cdot)](t) \\ E^-(z, t + \tau(z)) &= \mathcal{G}^- E^+(0, t) = [G^-(z, \cdot) * E^+(0, \cdot)](t). \end{aligned} \tag{4.1}$$

Here $\tau(z) = \int_0^z 1/c(z')\, dz'$ is the travel time for the wavefront from the front-wall to $z$, $a(z)$ is the attenuation of the wave-front at $z$ and $E^+(0, t)$ is the incident field at $z = 0$. In these representations wavefront time is used, i.e., $t = 0$ at a point $z$ when the wave front reaches that point. The operators $\mathcal{G}^\pm$ are referred to as the Green operators and the kernels $G^\pm(z, t)$ as the Green kernels. The representations in Eq. (4.1) imply the following boundary values of the Green kernels

$$\begin{cases} G^+(0, t) = 0 \\ G^-(0, t) = R(t) \\ G^+(L, t) = T(t) \\ G^-(L, t) = 0, \end{cases}$$

where $R(t)$ is the reflection kernel and $T(t)$ is the transmission kernel of the slab. The representation in Eq. (4.1) implies that $G^-(0, t) \equiv 0$ for a slab which is non-reflecting for any incident field $E^+(0, t)$.

A set of first order hyperbolic equations can be obtained for the Green kernels, cf [2] and [7]. Differentiation with respect to $z$ of the representations in Eq. (4.1) gives

$$\begin{aligned} &\begin{pmatrix} \partial_z E^+ \\ \partial_z E^- \end{pmatrix}(z, t + \tau(z)) + \partial_z \tau(z) \begin{pmatrix} \partial_t E^+ \\ \partial_t E^- \end{pmatrix}(z, t + \tau(z)) = \\ &\qquad = \begin{pmatrix} \partial_z a(z) \\ 0 \end{pmatrix} E^+(0, t) + \left[ \begin{pmatrix} \partial_z G^+(z, \cdot) \\ \partial_z G^-(z, \cdot) \end{pmatrix} * E^+(0, \cdot) \right](t). \end{aligned} \tag{4.2}$$

The $z-$derivatives of $E^{\pm}$ are eliminated by utilizing the dynamics in Eq. (3.6). Furthermore, $E^{\pm}(z,t+\tau)$ and $\partial_t E^{\pm}(z,t+\tau)$ are expressed in terms of $E^{+}(0,t)$ using Eq. (4.1). The only field appearing in the resulting equations is $E^{+}(0,t)$ and since this is an arbitrary incident field, the equations and initial condition for the Green kernels follow

$$\begin{aligned}
\partial_z G^{+}(z,t) &= \frac{1}{2}(\partial_z \ln Z(z)(G^{+}(z,t)-G^{-}(z,t)) \\
&-\frac{1}{2}\sigma(z)Z(z)(G^{+}(z,t)+G^{-}(z,t)) - \\
&-\frac{\varepsilon_o Z(z)}{2}\left(a(z)\partial_t\chi(z,t)+\partial_t\left[\chi(z,\cdot)*(G^{+}(z,\cdot)+G^{-}(z,\cdot))\right](t)\right) \quad (4.3)\\
\partial_z G^{-}(z,t) &-\frac{2}{c(z)}\partial_t G^{-}(z,t) = \frac{1}{2}\left(\partial_z \ln Z(z)\right)\left(G^{+}(z,t)-G^{-}(z,t)\right) \\
&+\frac{1}{2}\sigma(z)Z(z)(G^{+}(z,t)+G^{-}(z,t)) + \\
&+\frac{\varepsilon_o Z(z)}{2}\left(a(z)\partial_t\chi(z,t)+\partial_t\left[\chi(z,\cdot)*(G^{+}(z,\cdot)+G^{-}(z,\cdot))\right](t)\right) \quad (4.4)
\end{aligned}$$

$$G^{-}(z,0) = \frac{1}{4}a(z)c(z)\left(\partial_z \ln Z(z) - Z(z)\left(\sigma(z)+\varepsilon_0\chi(z,0)\right)\right). \tag{4.5}$$

Also the equation for the attenuation $a(z)$ follows from Eq. (4.2)

$$\partial_z a(z) = \frac{a(z)}{2}\left(\partial_z \ln Z(z) - Z(z)\left(\sigma(z)+\varepsilon_0\chi(z,0)\right)\right).$$

Since $a(0)=1$ it is seen that

$$a(z) = \sqrt{\frac{Z(z)}{Z(0)}}\exp\left\{-\frac{1}{2}\int_0^z Z(z')(\sigma(z')+\varepsilon_0\chi(z',0))\,dz'\right\}.$$

The kernel $G^{+}(z,t)$ is continuous for $t>0$, whereas $G^{-}(z,t)$ in general has a discontinuity along the characteristic $\frac{dz}{dt}=-\frac{2}{c(z)}$ of Eq. (4.4), i.e., along the line $t=2(\tau(L)-\tau(z))$. This discontinuity is induced at $z=L$ since the initial condition (4.5) gives that $G^{-}(L^{-},0)\not\equiv 0$ and the representation in Eq. (4.1) implies that $G^{-}(L^{+},t)\equiv 0$. The discontinuity of the Green kernel at $z=L$, $t=0$, $[G^{-}(L,0)] = G^{-}(L,0^{+}) - G^{-}(L,0^{-}) = -G^{-}(L,0^{-})$, is then propagated along the characteristic line $t=2(\tau(L)-\tau(z))$. The discontinuity in the reflection kernel $R(t)=G^{-}(0,t)$ occurs at one roundtrip, $t=2\tau(L)$, i.e, when the wave has traveled forward and backward once in the medium. From propagation of singularity arguments and from Eq. (4.5) it follows that

$$\begin{aligned}
[R(2\tau(L))] =\ & [G^{-}(0,2\tau(L))] = [G^{-}(L,0)]b(L) = \\
& -\frac{a(L)c(L)}{4}\left(\frac{Z'(L)}{Z(L)} - Z(L)(\sigma(L)+\varepsilon_0\chi(L,0))\right)b(L), \quad (4.6)
\end{aligned}$$

where

$$b(L) = \sqrt{\frac{Z(0)}{Z(L)}} \exp\left\{-\frac{1}{2}\int_0^L Z(z')(\sigma(z') + \varepsilon_0 \chi(z',0))\,dz'\right\},$$

is the attenuation of the discontinuity from $z = L$ to $z = 0$.

The equation for the reflection kernel of an impedance mismatched slab $R_b(t)$, is obtained by relating the reflection operator, $\mathcal{R}_b$, to the Green operators for the matched slab. The details of the derivation are found in Appendix B. The representation of the operator $\mathcal{R}_b$ can be shown from arguments based upon invariance under time translation and causality to be, cf [8]

$$\mathcal{R}_b E^+(0,t) = r_1 \frac{Z(0)}{Z(L)} a(L)^2 E^+(0, t - 2\tau(L)) + [R_b(\cdot) * E^+(0,\cdot)](t - 2\tau(L)). \quad (4.7)$$

The first term represents the hard reflected wave that arrives at $z = 0$ after one roundtrip, $t = 2\tau(L)$, and $r_1$ is the reflection coefficient for the boundary at $z = L$. The second term is the contribution from the scattering inside the slab. In appendix B it is shown that the reflection kernel $R_b(t)$ in general is discontinuous at $t = 2\tau(L)$, at one roundtrip, and at $t = 4\tau(L)$, at two roundtrips. The first term is always present when there is a mismatch and it can not be compensated by the second term for an arbitrary incident field $E^+(0,t)$. Thus it is not possible to obtain a non-reflecting slab with an impedance mismatch at $z = L$. Instead one can try to minimize the reflected field, in some sence. This problem can be studied by optimization techniques but is not addressed in this paper. In the rest of the paper only slabs that are impedance matched to the surrounding half-spaces are considered.

# 5 Non-reflecting slabs

In order to find non-reflecting slabs that are impedance matched at $z = L$, Eqs. (4.3) and (4.4) are to be solved for $\varepsilon_r(z)$, $\mu_r(z)$, $\sigma(z)$ and $\chi(z,t)$ using the initial condition in Eq. (4.5) and the boundary conditions

$$\begin{cases} G^+(0,t) = 0 \\ G^-(0,t) = 0 \\ G^-(L,t) = 0. \end{cases} \quad (5.1)$$

This problem has non-unique solutions, since one function of both $z$ and $t$ and two functions of $z$ are to be determined from one function $G^-(0,t)$. This non-uniqueness is necessary in order to find non-reflecting slabs with realistic material parameters.

The dispersive half-space was discussed in [1]. In the first part of this section the results in that paper are reviewed and the generalization to a conductive dispersive half-space is commented upon. It is important to notice that for times less than one roundtrip there is no difference in the reflected field for a finite slab and a half-space.

In the case of a dispersive slab the inverse problem is non-unique. In order to formulate an inverse problem which has the potential of being uniquely solvable, the functions to be determined have to be reduced to a single function of one argument. There are five natural choices of the function to be determined, namely, the impedance $Z(z)$, the conductivity $\sigma(z)$, the phase velocity $c(z)$, the susceptibility kernel $\chi(z,t)$ with a given time dependence, and the susceptibility kernel with a given $z-$dependence. The last choice is not a good one since $\chi(z,t)$ has to satisfy certain conditions in order for the medium to satisfy energy conservation, cf [6]. Moreover, even a $\chi(z,t)$ that satisfies these conditions might be unphysical, cf [6]. The first four choices are assumed to be applicable and are further analyzed in the sections below. Of course one may use the permittivity, $\varepsilon_r(z)$, and the permeability, $\mu_r(z)$, as independent functions instead of the impedance, $Z(z)$, and phase velocity, $c(z)$.

It is hard to find sufficient conditions for a solution to exist for the inverse problem of finding a reflectionless slab. However, necessary conditions are easier to find and below two such conditions are presented. The first condition says that the impedance has to be an increasing function of $z$ at $z=0$ if the susceptibility kernel is non-zero at $z=0$. Thus

$$\partial_z \ln Z(z)|_{z=0} \geq 0. \tag{5.2}$$

The condition follows from energy conservation, cf [1] and [6]. For a non-magnetic dispersive slab, $\mu_r = 1$, the condition (5.2) implies that $\varepsilon_r(z)$ has to be a decreasing function of $z$ at $z=0$. Hence a non-magnetic dispersive slab can not be reflectionless for a transient wave incident from vacuum, since a non-reflecting non-magnetic slab has to have $\varepsilon_r > 1$ for $z > 0$. A magnetic dispersive slab with a constant permittivity $\varepsilon_r = 1$, can be reflectionless for a wave incident from vacuum. The condition (5.2) only demands that $\mu_r(z)$ is an increasing function of $z$ at $z=0$.

The other necessary condition is that the initial condition of the Green kernel $G^-$ has to be a $C^\infty$ function, i.e., infinitely differentiable with respect to the $z-$coordinate. This condition results from the initial condition, Eq. (4.5), and propagation of singularity arguments, which say that a discontinuity in the $p$:th $z$-derivative of $G^-(z,0)$ propagates along the characteristic of Eq. (4.4) and give rise to a discontinuity in the $p$ :th $t$-derivative of $G^-(0,t)$. Thus for a non-reflecting medium $G^-(z,0)$ has to be a $C^\infty$ function on the interval $0 < z < L$, since $G^-(0,t) \equiv 0$ is a $C^\infty$ function of time. This condition is satisfied if $\varepsilon_r(z)$, $\mu_r(z)$ and $\chi(z,0)$ are $C^\infty$ functions of $z$.

## 5.1 Non-dispersive slabs

Consider first the simple case of a non-dispersive and non-conducting slab, i.e., when $\chi(z,t) \equiv 0$. All media which have constant impedance, $Z(z) = const.$, for $-\infty < z < \infty$ are then reflectionless. This well-known fact follows immediately from Eqs. (4.3) and (4.4), since the source term in these equations as well as the initial condition, Eq. (4.5), vanish.

For a non-dispersive conducting slab, it is no longer possible to keep the wave impedance constant to obtain a non-reflecting slab. From the initial condition in

Eq. (4.5) it is seen that $G^-(z, 0) = 0$ if the conductivity $\sigma(z)$ is chosen such that

$$\sigma(z) = \frac{\partial_z \ln Z(z)}{Z(z)} \tag{5.3}$$

From Eq. (4.3) and (4.4) it is seen that the choice in Eq. (5.3) also gives $G^-(z, t) \equiv 0$ and $G^+(z, t) \equiv 0$, since the initial values and the source term are zero. The integration of condition (5.3) gives an equation for $Z(z)$ :

$$Z(z) = \frac{Z(0)}{1 - Z(0) \int_0^z \sigma(z') dz'} \tag{5.4}$$

From Eq. (5.4) it is seen that $Z(z)$ must be an increasing or constant function of $z$ since $0 \leq \sigma(z)$. Furthermore, the condition (5.3) implies a constant damping factor, $a(z) = 1$, as seen from Eq.(4). Thus, for a non-dispersive slab, the condition (5.3) gives a distortionless and non-reflecting slab. The result can also be obtained directly from the wave equation for the split fields, Eq. (3.6). It is seen that a sufficient condition for the slab to be reflectionless is that $E^-(z, t)$ is zero everywhere. This is the case if the matrix element $\gamma$ in Eq. (3.6) is zero, which is satisfied by the condition in Eq. (5.3). It is thus seen that the conditions for having a distortionless non-reflecting non-dispersive slab are very simple to obtain. The result is probably known among those who are working with non-reflecting media, even though the authors have not found the condition stated explicitly in any paper.

## 5.2 Dispersive slabs

In the case of a dispersive slab, the susceptibility kernel introduces a source term in the equations for the Green kernels, as seen from Eqs. (4.3) and (4.4). It is then not possible to obtain a reflectionless slab by using a homogeneous initial condition for the Green kernels. A necessary condition for a reflectionless slab is that the kernel $G^-(z, t)$ has no discontinuity that propagates along any characteristic line , $\frac{dz}{dt} = -\frac{2}{c(z)}$, of Eq. (4.4). In particular there is a discontinuity if $G^-(L^-, 0^-) \neq 0$. From the initial condition in Eq. (4.5) it is seen that $G^-(L^-, 0^-) = 0$ only if

$$\partial_z \ln Z(L^-) - Z(L^-) \left(\sigma(L^-) + \varepsilon_0 \chi(L^-, 0)\right) = 0 \tag{5.5}$$

Two different inverse problems are considered in order to find a non-reflecting slab. In both cases the susceptibility kernel is independent of $z$. In the first problem the susceptibility kernel is assumed to be known and the impedance or the conductivity are to be determined as functions of $z$, given a reflection kernel that is zero. In the other problem the conductivity and the impedance are given as functions of $z$ and the susceptibility kernel is to be determined as a function of $t$.

In the case of a given susceptibility kernel the numerical algorithm indicates that it is impossible to satisfy the necessary condition (5.5). In order to get a reflection kernel which is very close to zero the slab is partitioned into two halves $0 \leq z < L/2$ and $L/2 \leq z \leq L$. The first part is dispersive while the second part is non-dispersive,

$\chi(t) \equiv 0$. In that case the discontinuities can be made very small and the slab almost reflectionless. However, a slab for which the reflection kernel is exactly zero for all times seems impossible to realize.

In the second problem the susceptibility kernel is to be determined. It is possible in this case to satisfy the necessary condition (5.5) and to construct a susceptibility kernel for which the slab is reflectionless for all times. In the numerical examinations made so far all of these susceptibility kernels are discontinuous as functions of time at one roundtrip. A discontinuous susceptibility kernel is not physical since it corresponds to an active material. For this reason only numerical examples for times less than one roundtip are presented in the next section.

# 6 Numerical examples

In this section three different classes of non-reflecting slabs are exemplified: the non-dispersive conducting slab, the dispersive slab with a prescribed susceptibility kernel and the dispersive slab with a prescribed impedance and conductivity.

An example of a non-dispersive non-reflecting slab is presented in figure 1. The conductivity is chosen to increase linearly from $\sigma = 0$ to $\sigma = 1 \cdot 10^3\,(\Omega\mathrm{m})^{-1}$ and $\varepsilon_r(z) = const = 1$. Both $G^+(z,t)$ and $G^-(z,t)$ are zero for this slab since the condition (5.5) is satisfied and thus the slab also is distortionless. The corresponding impedance is easily obtained from the condition in Eq. (5.3). Of course infinitely many other slabs that are reflectionless can be obtained in the same way. As mentioned before the impedance has to be an increasing function of $z$. Thus if there is vacuum in the half-space $z < 0$ the material in the slab has to be magnetic, $\mu_r(z) > 1$. Notice also that the conductivity increases from 0 to $10^3\,(\Omega\mathrm{m})^{-1}$) and that the slab thickness then is only one millimeter. As a rule of thumb the slab thickness measured in meters is of the order of one over the conductivity measured in $(\Omega\mathrm{m})^{-1}$.

In the case of a dispersive slab with prescribed susceptibility kernel the slab can only be reflectionless for times less than one roundtrip. In order to obtain very small reflections for times later than one roundtrip the slab is divided into two pieces $0 \le z < L/2$ and $L/2 \le z \le L$, were the first half is dispersive and the second is non-dispersive. This means that the susceptibility kernel has a discontinuity at $z = L/2$ which has to be matched by a discontinuity in the derivative of the wave impedance or by a discontinuity in the conductivity, cf Eq. (4.5). The reason why $L/2 < z < L$ is non-dispersive is to make $G^-(L^-, 0)$ sufficiently small and approximately satisfy condition (5.5). A given susceptibility kernel for the medium is assumed and the permittivity, permeability or conductivity as functions of $z$ are to be determined. The equations for the Green kernels were discretized by the trapezoidal rule, cf [2]. The convergence of the numerical algorithm is then quadratic. The thickness of the slab is in all examples chosen to be 20 cm.

The first example of a dispersive slab with prescribed susceptibility kernel is a

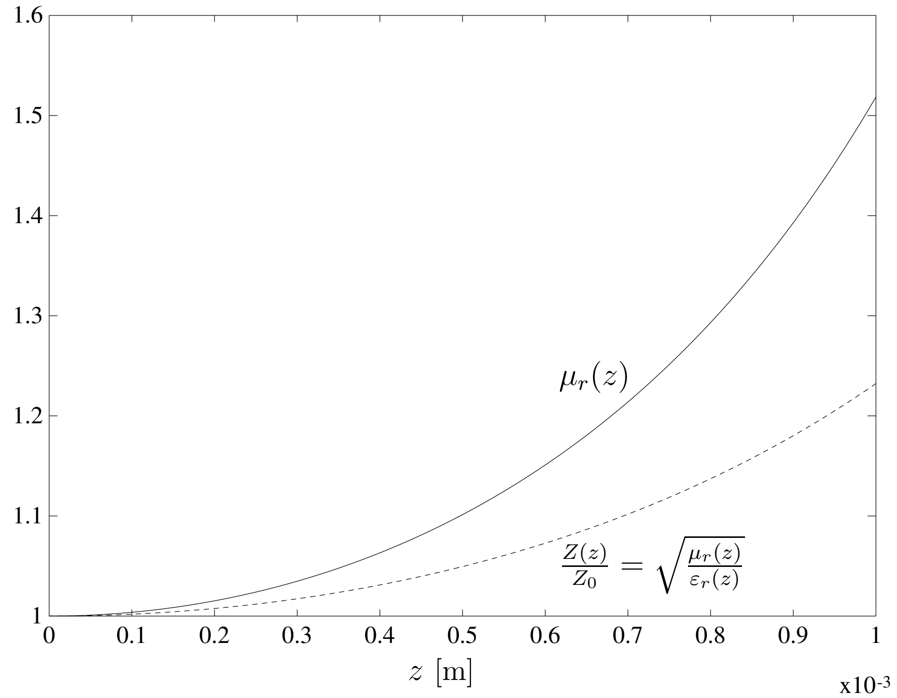



**Figure 1**: The relative permeability (solid line) and the wave impedance, scaled by the impedance of vacuum, (dotted line) as a function of depth for a non-dispersive media with linearly increasing conductivity from $\sigma(0) = 0$ to $\sigma(L) = 1 \cdot 10^3 (\Omega\mathrm{m})^{-1}$ and $\varepsilon_r(z) = const = 1$.

Debye medium. The susceptibility kernel is then given by

$$\chi(t) = \alpha \exp(-\frac{t}{\tau}).$$

This model is relevant for polar liquids, cf [5]. The parameters $\tau$ and $\alpha$ in the examples are independent of $z$ and their values are $\alpha = 5 \cdot 10^8\,\mathrm{s}^{-1}$ and $\tau = 2 \cdot 10^{-10}\,\mathrm{s}$. The conductivity is in this case chosen to be constant, $\sigma = 5 \cdot 10^{-4} (\Omega\mathrm{m})^{-1}$, throughout the slab. The wave impedance is in both cases matched to vacuum at $z = 0$. Two different cases of design of the slab are considered and the results are presented in Figure 2. In the first case the permittivity $\varepsilon_r(z)$ is determined and the permittivity $\mu_r(z)$ is kept at the constant value $\mu_r(z) = 2$. Since the derivative of the impedance has to be positive at $z = 0$ it follows that the derivative of the permittivity has to be negative at $z = 0$. In the second case the permeability is constructed when the permittivity is kept at the constant value $\mu(z) = 2$. In this case the derivative of the permeability has to be positive at $z = 0$. Both of these profiles give a slab which is reflectionless for times less than one round trip and almost reflectionless for times after one round trip.

To see that the constructed permittivity profile in figure 2 really results in a low reflecting slab, the constitutive relation for the constructed material was used

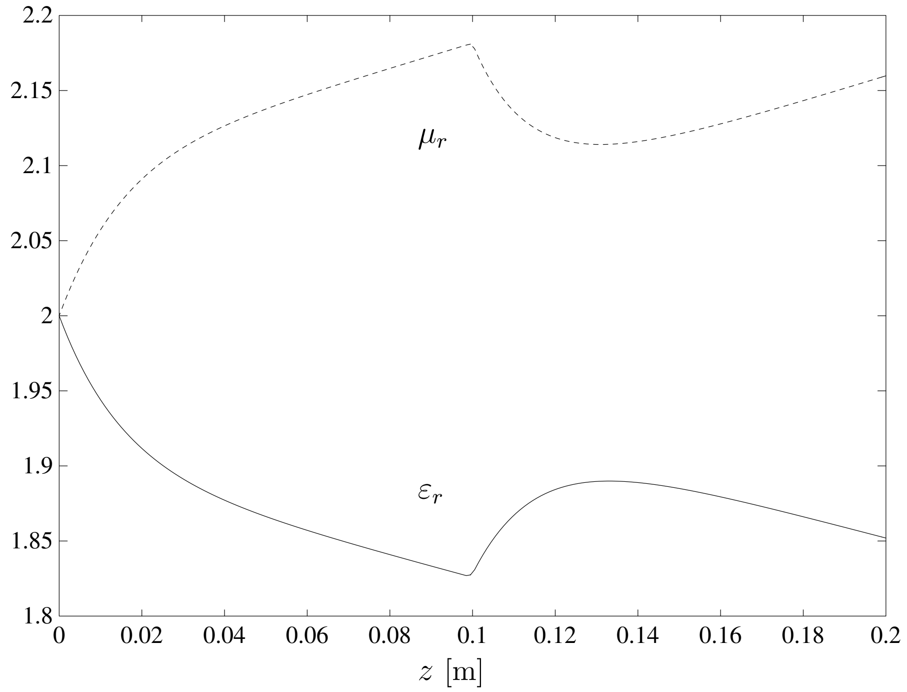



**Figure 2**: The relative permittivity (solid line) and the relative permeability (dotted line) as a function of depth for a reflectionless Debye medium with $\alpha = 5 \cdot 10^8 \mathrm{s}^{-1}$ and $\tau = 2 \cdot 10^{-10}\mathrm{s}$.

in the solution of the direct scattering problem. The resulting reflection kernel was compared to the reflection kernel for a non-dispersive medium with the permittivity profile in figure 2. As seen from figure 3 the reflection kernel for the low-reflecting slab is negligible. Note that the reflection kernel $R(t)$ is only identically zero up to one roundtrip since there is a small discontinuity in $G^-(z,0)$ at $z = L$ that is propagated along the characteristic line $t = 2(\tau(L) - \tau(z))$ cf Eq. (4.6).

In the next example for the Debye medium, the permittivity and permeability are prescribed and the conductivity is determined. In this case, $\varepsilon_r(z)$ and $\mu_r(z)$, cannot be constant functions since the condition (5.2) demands that $Z(z)$ has to be an increasing function of $z$ at $z = 0$. The permeability is therefore linearly increasing from $\mu_r(0) = 1$ to $\mu_r(L/2) = 2$ and then kept at the constant value $\mu_r(z) = 2$ for $L/2 \le z \le L$. The permittivity is kept at the constant value $\varepsilon_r(z) = 1$ in the entire slab. The constructed conductivity profile is shown in Figure 4.

Finally an example of reflectionless Lorentz medium is presented. The susceptibility kernel of the Lorentz medium

$$\chi(t) = \omega_p^2 \frac{\sin \nu t}{\nu} e^{-\gamma t}. \tag{6.1}$$

is prescribed, cf [4]. The model is relevant for most materials in the frequency region above the microwave region. It is a model for the dispersion caused by resonances of

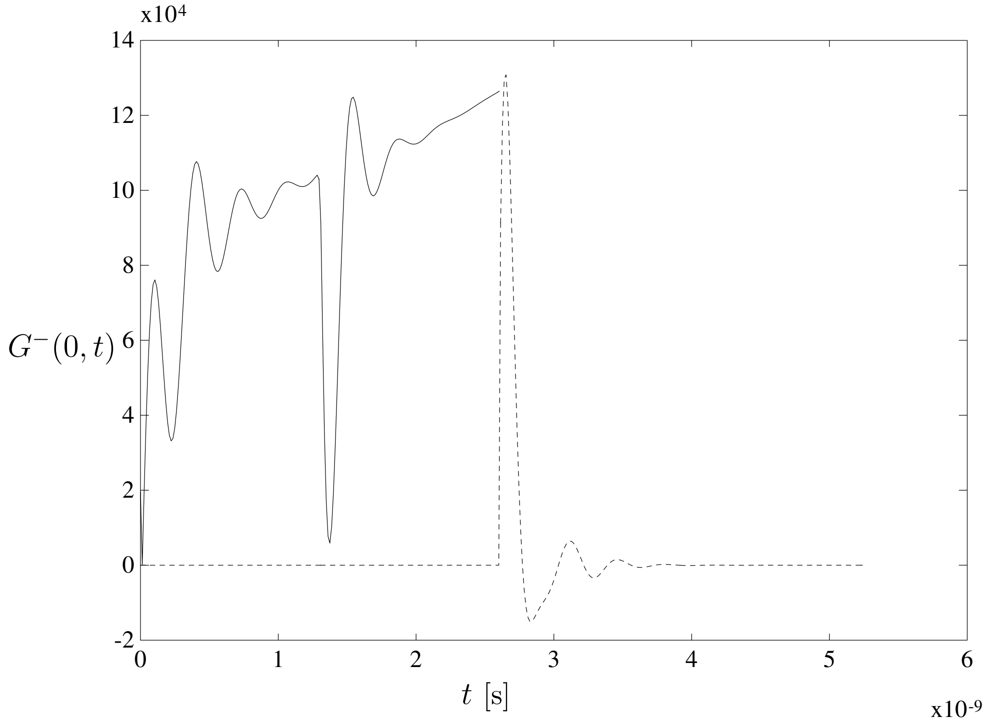



**Figure 3**: The reflection kernel $R(t) = G^-(0,t)$ for the permittivity profile given in figure 2. The solid line represents $R(t)$ scaled by a factor of $10^{-3}$ for one roundtrip of a non-dispersive slab and the dotted line the corresponding non-reflecting medium for two roundtrips.

atoms and molecules. There are however media where the Lorentz model is relevant at lower frequencies, e.g. gyrotropic media. In the expression for the susceptibility kernel $\omega_p$ is the plasma frequency and the frequency $\nu$ is related to the resonance frequency $\omega_0$ and the damping constant $\gamma$ as $\nu = \sqrt{\omega_0^2 - \gamma^2}$. The parameters in the example are independent of $z$ and their values are $\omega_p = 5 \cdot 10^9\,\mathrm{s}^{-1}$, $\omega_0 = 2 \cdot 10^{10}\,\mathrm{s}^{-1}$ and $\gamma = 1 \cdot 10^{10}\,\mathrm{s}^{-1}$. Figure 5 is the Lorentz model counterpart of figure 2, i.e., the profiles of $\varepsilon_r(z)$ and $\mu_r(z)$ resulting in a non-reflecting slab are shown. The conductivity is in this case constant, $\sigma = 5 \cdot 10^{-4}(\Omega\mathrm{m})^{-1}$ throughout the slab. The oscillating profiles of the constructed permittivity and permeability, cf. figure 5, is expected, since the variation of the constructed parameter is to compensate the oscillatory behavior of the susceptibility kernel.

For a Lorentz media, it is not possible to obtain a reflectionless slab by only varying the conductivity as seen from Eq. (4.5) since $\chi(z,0) = 0$. The varying sign of $G^-(z,0)$ cannot be compensated by the conductivity $\sigma$ which has to be positive.

In figure 6 the initial values, $G^-(z,0)$, of the Green kernels corresponding to the examples depicted in figures 2 and 4 are shown. One can see that $G^-(z,0)$ decreases in the non-dispersive half giving a very small value of $G^-(L^-,0)$ so that the reflection kernel has a almost negligible discontinuity at one roundtrip.

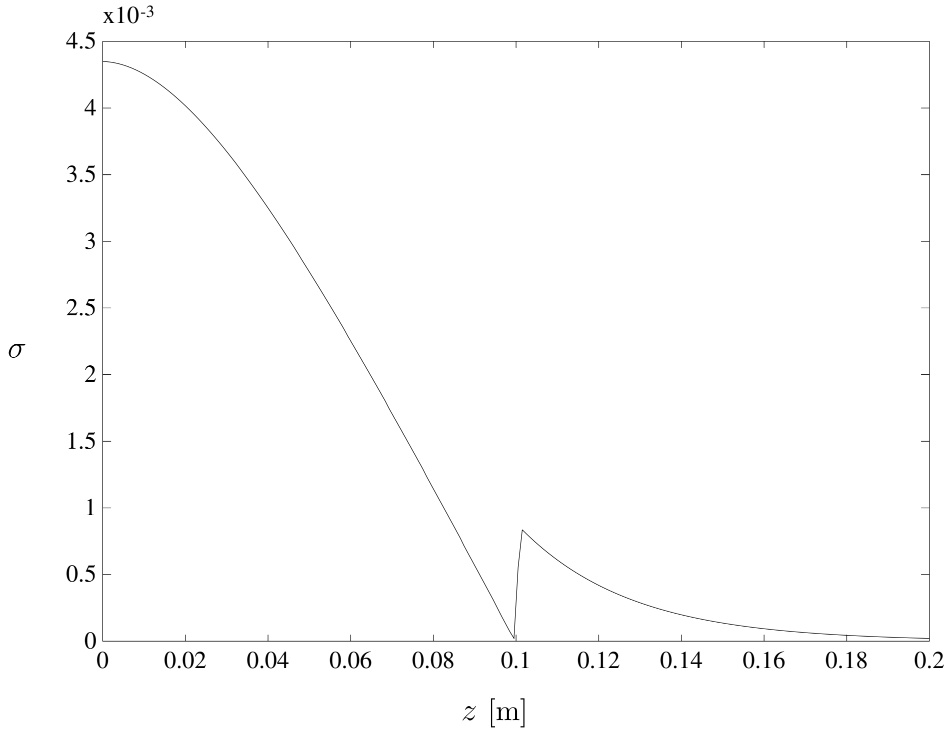



**Figure 4**: The conductivity as a function of depth for a reflectionless Debye medium with $\alpha = 5 \cdot 10^8 \mathrm{s}^{-1}$ and $\tau = 2 \cdot 10^{-10}$s.

The last example, the bottom graph in figure 7, shows the construction of the time behavior of the susceptibility kernel for a non-reflecting slab with a permittivity profile given in the upper graph in figure 7. The conductivity and permeability are both constant, $\mu_r = 2$ and $\sigma = 1 \cdot 10^{-4}$. The constructed susceptibility kernel is probably not physical, but the example shows that it is possible to reconstruct the time behavior of the susceptibility kernel for a prescribed or measured reflection kernel.

# 7 Conclusions

The present paper gives an example of media-design using inverse scattering methods. The prescribed reflection properties of the slab is used to construct one parameter in the constitutive relation. The attractive feature is that clean data can be used in the solution of the inverse problem which then is well posed and well conditioned. There are a number of similar problems that can be looked upon with the present technique. A natural extension of the project is to prescribe both reflection and transmission properties and to construct two parameters in the constitutive relation. An important problem is to find low reflecting slabs on a conducting half-space. By utilizing the technique described in this paper it should be possible to design such slabs.

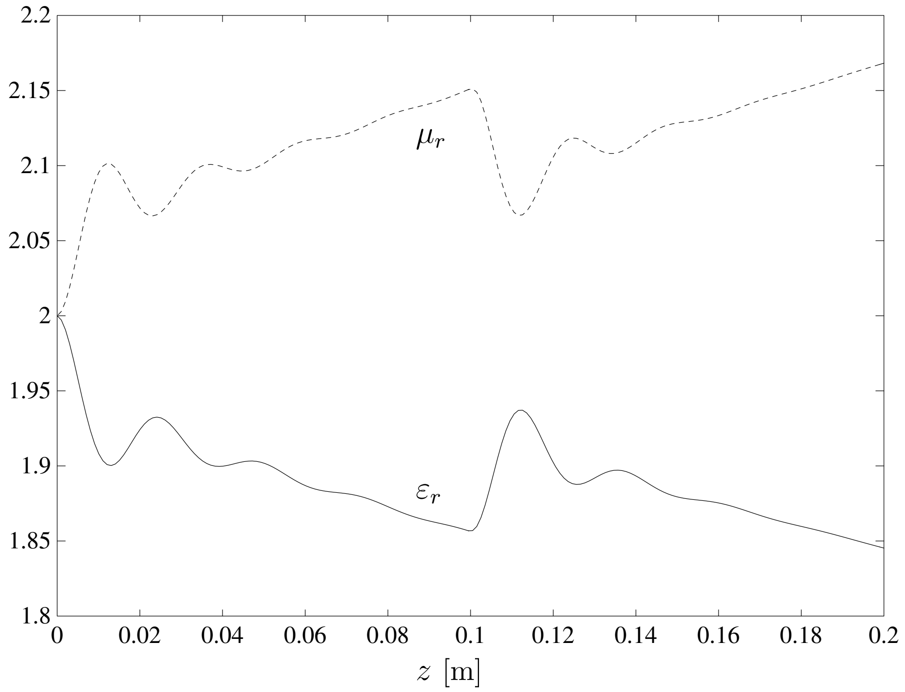



**Figure 5**: The relative permittivity (solid line) and the relative permeability (dotted line) as a function of depth for a reflectionless Lorentz medium with $\omega_p = 5 \cdot 10^9\,\mathrm{s}^{-1}$, $\omega_0 = 2 \cdot 10^{10}\,\mathrm{s}^{-1}$ and $\gamma = 1 \cdot 10^{10}\,\mathrm{s}^{-1}$.

The reflection data do not have to be prescribed, the basic theory and the algorithms can be applied also to measured data. In that case the inverse problem is ill-posed since the the reflection kernel has to be obtained by deconvolution.

## Acknowledgment

The partial support of the Swedish Research Council for Engineering Sciences is gratefully acknowledged.

# References

[1] Hellberg R, Karlsson A and Thärning P 1992 Non-reflecting dispersive media *Smart Mater. Struct.* **1** 341-346

[2] Krueger R J and Ochs R L, 1989 A Green's Function Approach to the Determination of internal Fields *Wave Motion* **11** 525-543

[3] Kristensson G 1989 Direct and inverse scattering problems in dispersive media – Green's functions and invariant imbedding techniques, in Kleinman R,

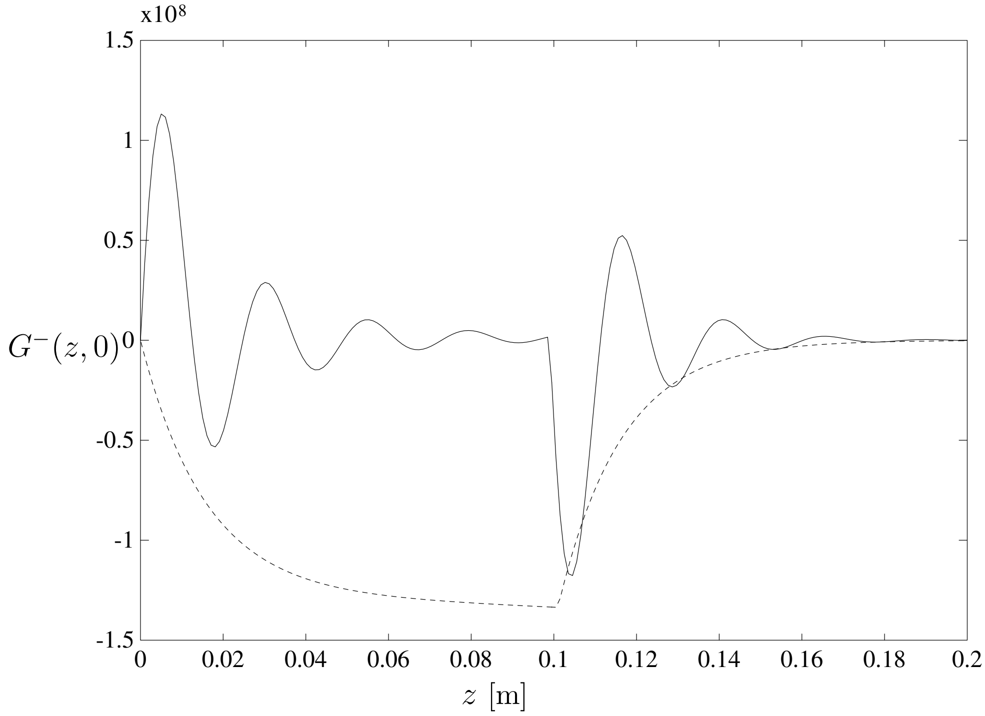



**Figure 6**: The function $G^-(z,0)$, scaled by a factor of 10, as a function of depth for the Debye medium described in figure 2 (dotted line), and $G^-(z,0)$ for the Lorentz medium described in figure 5 (solid line).

Kress R and Martensen E, editors *Dirtect and Inverse Boundary Value Problems, Methoden und Verfahren der Matematischen Physik, Band 37, pp 105-109, Mathematisches Forschungsinstitut Oberwolfach, FRG*

[4] Jackson J D 1975 *Classical Electrodynamics*, (New York: Wiley)

[5] Bohren C F and Huffman D R 1983 *Absorption and Scattering of Light by Small Particles* (New York: Wiley)

[6] Karlsson A and Kristensson G 1992 Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain *J. Electro. Waves Applic.* **6** 537-551

[7] Karlsson A, Otterheim H and Stewart R 1993 Electromagnetic fields in an inhomogeneous plasma from obliquely incident transient plane waves *Radio Science* **28** 365-378

[8] Fuks P, Karlsson A and Larson G 1993 Direct and inverse scattering from dispersive media *Inverse Problems* (in press)

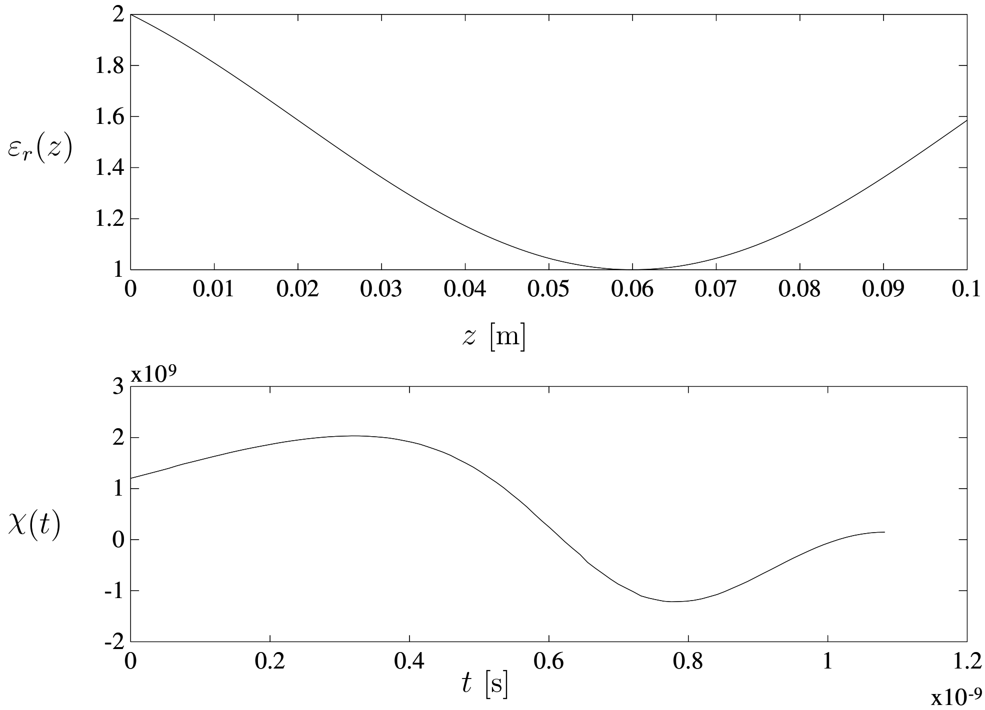



**Figure 7**: The reconstructed susceptibility kernel (bottom) for the first roundtrip and the prescribed permittivity profile (top). The permeability and conductivity are constant, $\mu_r = 2$, $\sigma = 1 \cdot 10^{-4}$.

# Appendix A Appendix

In this appendix the splitting matrix, $\mathbf{P}$, is derived for a dissipative wave equation. The matrix $\mathbf{A}$ for a non-dispersive and conducting medium reads

$$\partial_z \begin{pmatrix} E \\ H \end{pmatrix} = -\begin{pmatrix} 0 & \mu_0\mu_r\partial_t \\ \sigma + \varepsilon_0\varepsilon_r\partial_t & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{A} \begin{pmatrix} E \\ H \end{pmatrix}. \tag{A.1}$$

The goal is to diagonalize the $\mathbf{A}$ matrix by a similarity transform, i.e, to determine the matrix $\mathbf{P}$ such that the matrix $\mathbf{PAP}^{-1}$ is diagonal.

$$(\mathbf{PAP}^{-1}) \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}. \tag{A.2}$$

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $\mathbf{A}$. Introduce the eigenvectors $\begin{pmatrix} 1 \\ \mathcal{K} \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -\mathcal{K} \end{pmatrix}$ for the eigenvalues $\lambda_1$ and $\lambda_2$, where $\mathcal{K}$ is an operator. This means that the wave splitting matrix $\mathbf{P}$ reads

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \mathcal{K}^{-1} \\ 1 & -\mathcal{K}^{-1} \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = \mathbf{P} \begin{pmatrix} E \\ H \end{pmatrix}. \tag{A.3}$$

The inverse matrix $\mathbf{P}^{-1}$ is

$$\begin{pmatrix} E \\ H \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \mathcal{K} & -\mathcal{K} \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \mathbf{P}^{-1} \begin{pmatrix} E^+ \\ E^- \end{pmatrix}. \tag{A.4}$$

Introducing Eqs. (A.3) and (A.4) into Eq. (A.2) gives the eigenvector equations

$$\mathbf{A} \begin{pmatrix} 1 \\ \mathcal{K} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ \mathcal{K} \end{pmatrix}, \mathbf{A} \begin{pmatrix} 1 \\ -\mathcal{K} \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ -\mathcal{K} \end{pmatrix} \tag{A.5}$$

By eliminating $\lambda_1$ and $\lambda_2$ in Eq.(A.5), a relation for the operator $\mathcal{K}$ is obtained

$$(\sigma + \varepsilon_0\varepsilon_r\partial_t) = \mu_0\mu_r\partial_t\mathcal{K}\mathcal{K} \tag{A.6}$$

The Laplace transform of Eq. (A.6) reads

$$\mathcal{L}\left\{\mathcal{K}^{-1}(t)\right\} = \hat{\mathcal{K}}^{-1}(s) = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} s \left( \frac{1}{\frac{\sigma}{\varepsilon_0\varepsilon_r} s + s^2} \right)^{\frac{1}{2}} \tag{A.7}$$

By an inverse Laplace transform the final expression is obtained

$$\mathcal{L}^{-1}\left\{\hat{\mathcal{K}}^{-1}(s)\right\} = \mathcal{K}^{-1}(t) = \\ Z(z)(\delta(t) + \frac{\sigma}{2\varepsilon_o\varepsilon_r}(I_1(\frac{\sigma}{2\varepsilon_o\varepsilon_r}t) - I_0(\frac{\sigma}{2\varepsilon_o\varepsilon_r}t)) \exp(-\frac{\sigma}{2\varepsilon_o\varepsilon_r}t) \tag{A.8}$$

where $I_0(t)$ and $I_1(t)$ are the modified Bessel functions of zeroth and first order. The operator $\mathcal{K}^{-1}$ operates by convolution.

# Appendix B Appendix

In this appendix the Volterra equation for the reflection kernel for the slab with an impedance mismatch at the back-wall is derived. The slab is impedance matched to the half-space $z < 0$ but there is an impedance mismatch at $z = L$. Let $Z(L)$ be the impedance in the slab at the boundary, i.e., at $z = L^-$, and let $Z_1$ be the impedance in the half-space $z > L$. The boundary conditions at $z = 0$ and $z = L$ are that both the electric and magnetic fields are continuous. In terms of the split fields the boundary conditions are reduced to the following conditions

$$E^-(L^-,t) = r_1 E^+(L^-,t) \tag{B.1}$$
$$E^t(L^+,t) = t_1 E^+(L^-,t). \tag{B.2}$$

The reflection and transmission coefficients are defined as

$$r_1 = \frac{Z_1 - Z(L)}{Z(L) + Z_1}$$
$$t_1 = \frac{2Z_1}{Z(L) + Z_1}.$$

The following operators are introduced

$$E^r(0,t) \quad = \mathcal{R}_b E^+(0,t) \tag{B.3}$$
$$E^t(L,t) \quad = \mathcal{T}_b E^+(0,t) \tag{B.4}$$
$$E^r(0,t) \quad = \mathcal{G}^- E^+(0,t) + \mathcal{F}^- E^-(L^-,t) \tag{B.5}$$
$$E^+(L^-,t) = \mathcal{G}^+ E^+(0,t) + \mathcal{F}^+ E^-(L^-,t) \tag{B.6}$$

The interpretation of these operators are:

$\mathcal{R}_b =$ The reflection operator for the entire slab.
$\mathcal{T}_b =$ The transmission operator for the entire slab.
$\mathcal{G}^\pm =$ The Green operators for incidence from the left for the impedance matched slab.
$\mathcal{F}^\pm =$ The Green operators for incidence from the right for the impedance matched slab.

From reciprocity it follows that

$$\mathcal{F}^- = \frac{Z(0)}{Z(L)} \mathcal{G}^+ \tag{B.7}$$

It is convenient to introduce the inverse of the operator $\mathcal{G}^+$ as

$$\mathcal{W}\mathcal{G}^+ = \mathcal{I} = \text{the identity operator.} \tag{B.8}$$

This means that

$$E^{+}(0,t) = \mathcal{W}\left(E^{+}(L^{-},t) - \mathcal{F}^{+}E^{-}(L^{-},t)\right). \tag{B.9}$$

Now combine Eqs. B.3 and B.5 to get

$$\mathcal{R}_b E^{+}(0,t) = \mathcal{G}^{-}E^{+}(0,t) + \mathcal{F}^{-}E^{-}(L^{-},t). \tag{B.10}$$

In this relation $E^{+}(0,t)$ can be expressed in terms of $E^{\pm}(L^{-},t)$ from Eqs. (B.9) and (B.5) and thus

$$\begin{aligned} \mathcal{R}_b\mathcal{W}\left(E^{+}(L^{-},t) - \mathcal{F}^{+}E^{-}(L^{-},t)\right) &= \\ = \mathcal{G}^{-}\mathcal{W}\left(E^{+}(L^{-},t) - \mathcal{F}^{+}E^{-}(L^{-},t)\right) &+ \mathcal{F}^{-}E^{-}(L^{-},t). \end{aligned} \tag{B.11}$$

By using the boundary condition in Eq. (B.1) the following operator identity is obtained from Eq. (B.11)

$$\mathcal{R}_b(\mathcal{W} - r_1\mathcal{F}^{+}\mathcal{W}) = \mathcal{G}^{-}(\mathcal{W} - r_1\mathcal{F}^{+}\mathcal{W}) + r_1\mathcal{F}^{-}.$$

By applying the operator $\mathcal{G}^{+}$ and using the identity in Eq. (B.8), the final relation between $\mathcal{R}_b$ and $\mathcal{G}^{\pm}$ follows

$$\mathcal{R}_b - r_1\mathcal{R}_b\mathcal{F}^{+} = \mathcal{G}^{-} - r_1(\mathcal{G}^{-}\mathcal{F}^{+} - \mathcal{G}^{+}\mathcal{F}^{-}). \tag{B.12}$$

The representation of the operator $\mathcal{F}^{+}$ is analogous to the representation of the operator $\mathcal{G}^{-}$ in Eq. (4.1)

$$\mathcal{F}^{+}E^{-}(L^{-},t) = [R^{-}(\cdot) * E^{-}(L^{-},\cdot)](t) \tag{B.13}$$

The representation of the operator $\mathcal{R}_b$ can be shown from arguments based upon invariance under time translation and causality to be

$$\mathcal{R}_b E^{+}(0^{+},t) = r_1\left(\frac{Z(0)}{Z(L)}a(L)\right)^2 E^{+}(0^{+},t-2\tau(L)) + [R_b(\cdot) * E^{+}(0^{+},\cdot)](t-2\tau(L)), \tag{B.14}$$

When Eqs. (4.1), (B.7), (B.13), and (B.14) are inserted into Eq. (B.12) the Volterra equation for the reflection kernel $R_b(t)$ is obtained

$$\begin{aligned} &R_b\,(t) - G^{-}(0,t) + r_1\left([G^{-}(0,\cdot) * R^{-}(\cdot)](t) - [R_b(\cdot) * R^{-}(\cdot)](t)\right) \\ &-\frac{Z(0)}{Z(L)}\left(r_1^2(a(L))^2 R^{-}(t-2\tau(L)) + 2r_1 a(L)T(t-2\tau(L)) + r_1[T(\cdot) * T(\cdot)](t-2\tau(L))\right) = 0. \end{aligned} \tag{B.15}$$

The reflection kernel $R_b(t)$ is in general discontinuous at $t = 2\tau(L)$, after one roundtrip, and at $t = 4\tau(L)$, after two roundtrips. The values of the discontinuities are found from Eq. (B.15) and the values of the discontinuity and initial values of $G^{-}(0,t)$, $R^{-}(t)$, and $T(t)$. From Eq. (B.15) it follows that

$$[R_b(2\tau(L))] = [G^{-}(0,2\tau(L))] + \frac{Z(0)}{Z(L)}\left(r_1^2(a(L))^2 R^{-}(0) + 2r_1 a(L)T(0)\right) \tag{B.16}$$

The value of the discontinuity $[G^-(0,\tau)]$ is given in Eq.(4.6). The initial value of $R^-(t)$ is obtained from the initial value of $G^-(0,t)$, given by Eq. (4.5), by changing $z=0$ for $z=L$ and $Z'(0)$ for $-Z'(L)$. Thus

$$R^-(0) = -\frac{c(L)}{4}\left(\frac{Z'(L)}{Z(L)} + Z(L)(\sigma(L) + \varepsilon_0\chi(L,0))\right)$$

The initial value of the transmission kernel is found by integrating Eq. (4.3) for $t=0$ from $z=0$ to $z=L$, thus

$$T(0) = a(L)\int_0^L \left\{ \frac{c(z)}{8}\left( Z(z)^2(\sigma(z)+\varepsilon_0\chi(z,0))^2 - \left(\frac{Z'(z)}{Z(z)}\right)^2\right) - \frac{\varepsilon_0 Z(z)}{2}\chi_t(z,0)\right\} dz$$

The discontinuity at $t = 4\tau(L)$ also follows from Eq. (B.15)

$$[R_b(4\tau(L))] = \frac{Z(0)}{Z(L)} r_1^2 (a(L))^2 [R^-(2\tau(L))]$$

The discontinuity $[R^-(2\tau(L))]$ is obtained from Eq. (4.6) by changing $z=0$ for $z=L$ and $Z'(0)$ for $-Z'(L)$.

$$[R^-(2\tau(L))] = -\left(\frac{Z(0)}{Z(L)}\right)^2 \frac{(a(L))^2 c(0)}{4}\left(\frac{Z'(0)}{Z(0)} - Z(0)(\sigma(0) + \varepsilon_0\chi(0,0))\right)$$