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Sihvola, Ari; Kristensson, Gerhard; Lindell, Ismo V.

1996

[Link to publication](#)

Citation for published version (APA):

Sihvola, A., Kristensson, G., & Lindell, I. V. (1996). *Non-radiating sources in time-domain transmission-line theory*. (Technical Report LUTEDX/(TEAT-7049)/1-11/(1996); Vol. TEAT-7049). [Publisher information missing].

Total number of authors:

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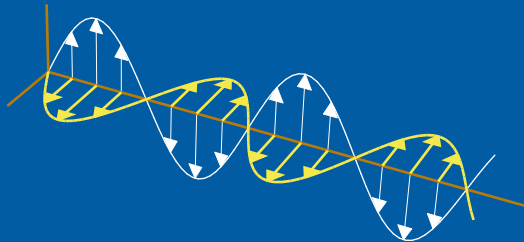
LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

Non-radiating sources in time-domain transmission-line theory

Ari Sihvola, Gerhard Kristensson and Ismo V. Lindell

Department of Electrosience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Ari Sihvola and Gerhard Kristensson
Department of Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

Ismo V. Lindell
Electromagnetics Laboratory
Faculty of Electrical Engineering
Helsinki University of Technology
Otakaari 5A
FIN-02150 Espoo
FINLAND

Abstract

The concept of non-radiating sources is introduced to transmission lines in the time-domain analysis. A method is presented to construct localized voltage and current sources which do not produce any fields outside the source domain. These sources cannot therefore be detected by measurements made outside the source region. The importance of such sources for the uniqueness of the inverse source problem is pointed out, and energy conditions for the uniqueness are discussed. The analysis can be advantageously used in the design and optimization of the EMC properties of transmission lines.

1 Introduction

Direct problems in electromagnetics have unique solutions, which means that two different fields are necessarily generated by two different sources. However, the inverse problem is not unique without additional constraints. In other words, two different sources may radiate the same electromagnetic field outside the source region. One consequence of this non-uniqueness property of the inverse source problem is that non-radiating (NR) sources exist. Non-radiating sources are such which do not generate any electric or magnetic fields outside their support.

The inverse source problem in acoustics and electromagnetics has been studied by various authors [1, 3, 4, 7]. These articles treat currents and their radiation in free space from the NR point of view, and give conditions that the source distributions have to satisfy in order not to radiate electromagnetic energy. The construction of a non-radiating source distribution starts with choice of any function that vanishes outside a finite domain. Applying the wave operator to this function gives a certain source function. Because the resulting source function is a solution of the inhomogeneous wave equation, it is an NR source because the field it corresponds to is zero outside the source domain.

One of the strong results of the non-radiating source studies is the following: a time-harmonic electric current distribution $\mathbf{J}(\mathbf{r}, \omega)$, does not radiate electromagnetic fields outside its support if

$$\mathbf{k} \times \int \mathbf{J}(\mathbf{r}, \omega) e^{i\mathbf{k} \cdot \mathbf{r}} dV = 0 \quad (1.1)$$

for $\omega = c|\mathbf{k}|$. The integral behaves well because the integration domain is the support of the current distribution which is a finite domain. In fact, equation (1.1) is a necessary and sufficient condition for a dynamic current to be non-radiating. In other words, the NR condition is that certain components vanish of the transverse part of the spatial Fourier transform of the current density; namely those components for which $|\mathbf{k}| = \omega/c$, where $c = 1/\sqrt{\mu\epsilon}$ is the radiation velocity in the medium permeating the space [3]. To give one example of a single-frequency current function satisfying this criterion, take $\mathbf{J}(\mathbf{r}) = \mathbf{u}_r J(r)$, which is a spherically symmetric vector function. This current source does not radiate in a homogeneous environment, as is well known.

The question of possible electron models that would be stable in the sense that the dynamics of the charges would not lose energy through radiation puzzled physicists like Sommerfeld, Herglotz, and Ehrenfest early in the beginning of this century. The famous result by Schott [6] is that a rotating spherical shell of charge is non-radiating if the radius of the shell b satisfies the following condition:

$$b = \frac{ncT}{2}$$

where n is integer and T is the period of the charge motion. This result by Schott does not require that the rotation orbit be circular; as a matter of fact, it does not even have to be planar. Goedecke has generalized these results to more general rotating and spinning charge distributions [5]. This reference gives examples of various localized NR sources that can be asymmetric and non-spherical, and that could also include a spinning current contribution in addition to the orbital movement of the charge. The non-radiating character of these charge constellations is connected to a quantized condition for the orbital and spinning motions, which result leads to the temptation to hypothesize that all stable particles in nature would be “merely non-radiating charge-current distributions whose mechanical properties are electromagnetic in origin,” although Goedecke in [5] seems to be very careful in propagating this suggestion.

The previous results of non-radiating sources in the literature have dealt with waves in unbounded homogeneous space. In the present article, we concentrate on the problem of non-radiating sources in transmission lines. The analysis allows arbitrary time-dependence of the fields and sources. The sources in the transmission-line problem are enforced voltages and currents, which can be either distributed or lumped sources. The construction procedure of NR transmission-line sources will be presented. Also power conditions are discussed because the energy balance is obviously different for non-radiating sources from ordinary transmission-line configurations where the line is used to transmit energy from the source to the load. We might see application of the NR-source transmission-line theory in, for example, electromagnetic compatibility and interference problems. Disturbances quite often become coupled to transmission lines through unshielded parts of the line. Hence, to minimize the effect of these undesired foreign signals, one may try to design the critical contact area in such a way that the disturbance would appear as a non-radiating source, when looking from the transmission line.

2 General principle of non-radiating sources

In a very general form the concept of non-radiating sources can be introduced as follows [8, Section 6.1]. Consider a linear problem

$$Lf = g,$$

where L is a linear operator containing differentiation, g is the cause (source) and f the effect (field). In addition, linear and homogeneous boundary conditions

$$Bf = 0$$

are assumed, which make the solution unique so that the problem

$$Lf_0 = 0, \quad Bf_0 = 0$$

has only the null solution $f_0 \equiv 0$.

The simple way to construct a non-radiating source is the following. If h is a function of bounded support in space, i.e., $h = 0$ outside a certain bounded region in space, the source $g_h = Lh$ is non-radiating. This is because the corresponding field f_h satisfies

$$L(f_h - h) = 0, \quad B(f_h - h) = 0,$$

assuming $Bh = 0$, i.e., the boundary of the problem is outside the support of h . From uniqueness we now have

$$f_h - h = 0 \quad \Rightarrow \quad f_h = h,$$

which means that $f_h = 0$ outside the support of h . Thus, the source $g_h = Lh$ does not create (radiate) a field outside the support of h .

3 Application to transmission lines

In the time-domain transmission-line theory, the field f is a combination of the voltage $U(z, t)$ and current $I(z, t)$ functions, the source g is a combination of the distributed series voltage $u(z, t)$ and shunt current $i(z, t)$ functions and the linear operator L is defined by the transmission-line equations

$$Lf = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_z + \begin{pmatrix} 0 & L \\ C & 0 \end{pmatrix} \partial_t + \begin{pmatrix} 0 & R \\ G & 0 \end{pmatrix} \right] \begin{pmatrix} U \\ I \end{pmatrix} (z, t) = \begin{pmatrix} u \\ i \end{pmatrix} (z, t). \quad (3.1)$$

Here, ∂_z and ∂_t denote differentiation with respect to z and t and the line parameters L , C , R , G (inductance, capacitance, series resistance, and leakage conductance, per unit length) may be functions of the position coordinate z but not the time t . The circuit that obeys Equation (3.1) is shown in Figure 1.

3.1 Construction of non-radiating voltage–current combinations

After having written down the operator L for transmission-line dynamics (Equation 3.1), the non-radiating voltage and current sources can be constructed using the principle presented in the previous section. Denote the function h by

$$h = \begin{pmatrix} U_h \\ I_h \end{pmatrix} (z, t),$$

and the non-radiating source can be expressed as

$$g_h = \begin{pmatrix} u_h \\ i_h \end{pmatrix} (z, t) = Lh = \begin{pmatrix} \partial_z & R + L\partial_t \\ G + C\partial_t & \partial_z \end{pmatrix} \begin{pmatrix} U_h \\ I_h \end{pmatrix} (z, t),$$

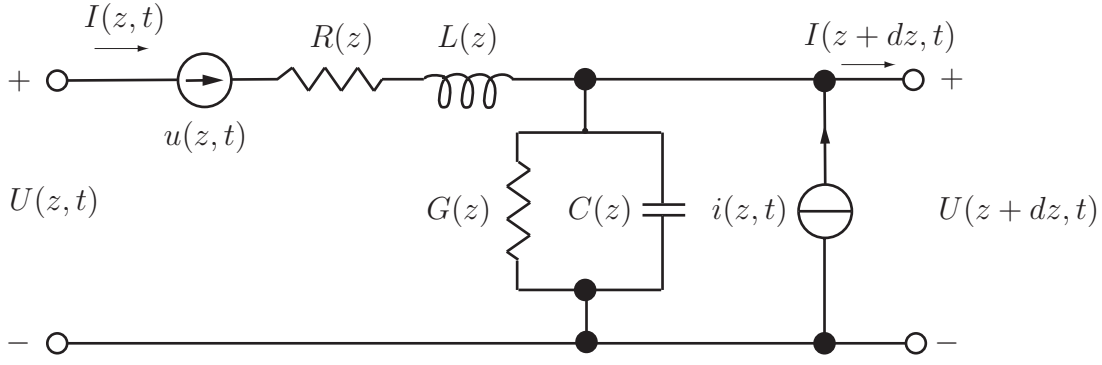


Figure 1: The transmission line with series voltage and shunt current sources $u(z, t)$ and $i(z, t)$, and the transmission-line parameters L, C, R, G . The dynamics of this circuit obeys Equation 3.1.

or, more explicitly, as

$$\begin{cases} u_h(z, t) = \partial_z U_h(z, t) + (R + L\partial_t)I_h(z, t), \\ i_h(z, t) = \partial_z I_h(z, t) + (G + C\partial_t)U_h(z, t). \end{cases} \quad (3.2)$$

Let us consider some basic examples for NR voltage–current distributions.

3.2 Simple non-radiating sources

Let us assume that the non-radiating source is limited to an interval in space, $z_1 < z < z_2$, and in time, $t_1 < t < t_2$. More specifically, let us assume that there is only one nonzero function $U_h(z, t)$ defined by:

$$U_h(z, t) = U_0 P(z_1, z, z_2) P(t_1, t, t_2), \quad I_h(z, t) = 0.$$

Here, the function $P(a, x, b)$ denotes the pulse function, which equals unity when $a < x < b$ and zero otherwise, and U_0 is a constant. Because the derivative of the pulse function is a combination of two delta functions, we have for the non-radiating source functions

$$\begin{cases} u_h(z, t) = U_0 [\delta(z - z_1) - \delta(z - z_2)] P(t_1, t, t_2), \\ i_h(z, t) = U_0 P(z_1, z, z_2) [G P(t_1, t, t_2) + C(\delta(t - t_1) - \delta(t - t_2))]. \end{cases}$$

The source is composed of two series voltage generators plus distributed shunt current generators according to Figure 2. The voltage generators are of opposite polarity, $\pm U_0$ at the points $z = z_1$ and $z = z_2$. These are turned on at $t = t_1$ and off at $t = t_2$. The current generators are distributed along the interval $[z_1, z_2]$, and flashing on with amplitude CU_0 at the moment $t = t_1$ and another time with the amplitude $-CU_0$ at the moment $t = t_2$. For a lossy transmission line, we also have a current generator distribution GU_0 .

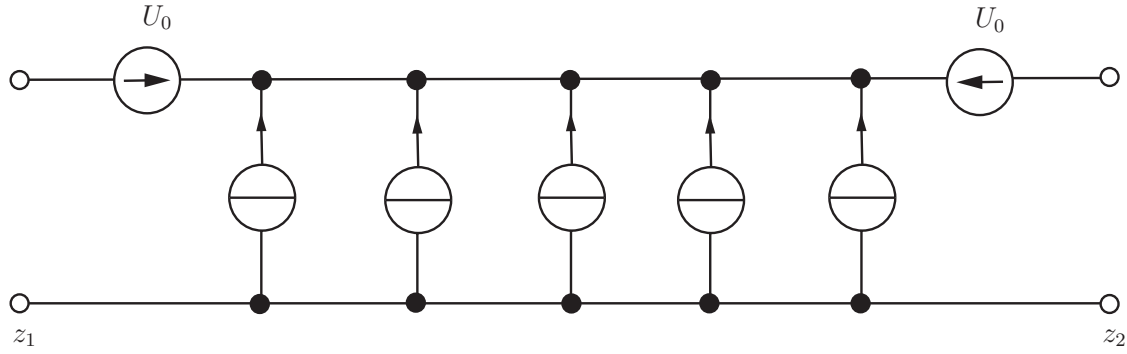


Figure 2: Non-radiating source in the interval $z_1 < z < z_2$ does not produce fields outside the interval. It cannot be detected by observing voltages and currents outside the interval. See the text for the strength of the current sources.

To understand that this combination of sources does not generate any fields propagating along the line, it is enough to note that during the time interval $[t_1, t_2]$, there exists a potential difference only within the space interval $[z_1, z_2]$. This will lead to leakage current, which is compensated by the continuous current source GU_0 . The transient that is excited at $t = t_1$ as the voltage generators are turned on is equal and opposite to the effect of the simultaneous current flash with amplitude CU_0 , and therefore no signal can be measured outside the region at any time. Similarly, this happens at the time $t = t_2$, with oppositely directed sources.

The complementary non-radiating source constellation can be constructed with the following choice of the U_h and I_h -functions:

$$U_h(z, t) = 0, \quad I_h(z, t) = I_0 P(z_1, z, z_2) P(t_1, t, t_2).$$

whence we have

$$\begin{cases} u_h(z, t) = I_0 P(z_1, z, z_2) [RP(t_1, t, t_2) + L(\delta(t - t_1) - \delta(t - t_2))], \\ i_h(z, t) = I_0 [\delta(z - z_1) - \delta(z - z_2)] P(t_1, t, t_2). \end{cases}$$

This source is illustrated in Figure 3. Two opposite shunt current sources at the points $z = z_1$ and $z = z_2$ create a circulating current which is limited within the region between these two points. For a lossy line, distributed series voltage generators RI_0 supply the voltage lost in the series resistance. And similarly to the earlier case, to extinguish the effect of the transients at $t = t_1$ and $t = t_2$, voltage flashes have to be included with amplitudes LI_0 at these moments. These voltage sources are distributed continuously along the interval.

3.3 Voltage and current sources

In the previous examples, the non-radiating sources consisted of both voltages and currents. We can also design a non-radiating source which only consists of voltage functions. In the above example, we did not use the function $I_h(z, t)$ at all, and the

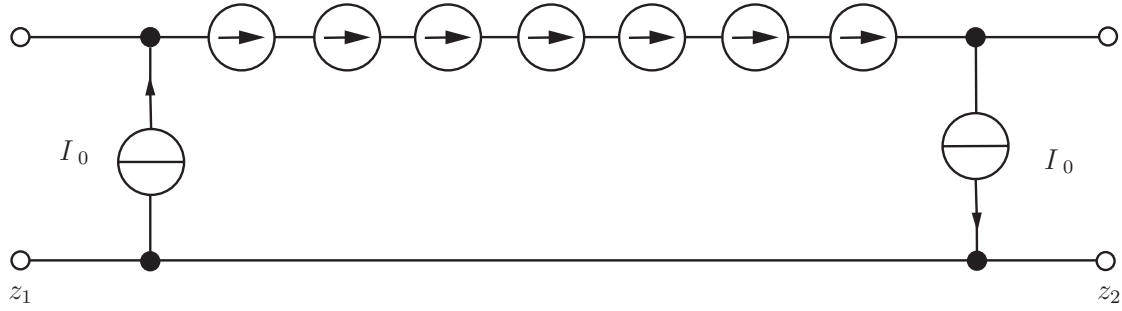


Figure 3: The non-radiating source combination complementary to that in Figure 2. See the text for the strength of the voltage sources.

result was a combination of voltage and current sources. With a suitable choice for $I_h(z, t)$, the current source $i_h(z, t)$ can be required to vanish. With this requirement, we find the following condition from Equation (3.2)

$$I_h(z, t) = -(G + C\partial_t) \int_{z_1}^z U_h(z', t) dz'$$

for the current function. Denoting the integral of the voltage by $M(z, t)$:

$$M(z, t) = \int_{z_1}^z U_h(z', t) dz'$$

we have from (3.2)

$$\begin{cases} u_h(z, t) = \partial_{zz} M(z, t) - (R + L\partial_t)(G + C\partial_t)M(z, t) \\ i_h(z, t) = 0 \end{cases}$$

which is an expression for a non-radiating purely-voltage-type source. The function $M(z, t)$ is an arbitrary function with compact support in space: $M(z, t) \equiv 0$ for $z < z_1$ and $z > z_2$. This is sufficient to guarantee that the I_h and U_h functions, too, vanish outside the support.

Similarly, we can write an expression for an NR source of purely-current-type, by requiring $u_h(z, t) \equiv 0$ from Equation (3.2), and using the condition for the current function. The result is

$$\begin{cases} u_h(z, t) = 0 \\ i_h(z, t) = \partial_{zz} N(z, t) - (R + L\partial_t)(G + C\partial_t)N(z, t) \end{cases}$$

where again, $N(z, t)$ is an arbitrary function with bounded support in space.

As an example of a non-radiating voltage-only source, choose the following function:

$$M(z, t) = \sin^2(kz)P(0, z, \pi/k) \sin(\omega t). \quad (3.3)$$

This is a monochromatic function, vibrating with the angular frequency ω . In time it is therefore not bounded but spatially it is restricted by the pulse function within a finite interval. The z -dependence has been intentionally chosen in a manner such that the function and its first derivative are continuous, with the goal of finding a “soft” source function. The source function contains second-order derivatives of the test function in this case.

The source function is

$$u_h(z, t) = [(2k^2 \cos 2kz - RG \sin^2 kz + \omega^2 LC \sin^2 kz) \sin \omega t - (RC + LG)\omega \sin^2 kz \cos \omega t] P(0, z, \pi/k)$$

and it is illustrated in Figure 4. The transmission-line parameters in this example are chosen to be those of a 75Ω coaxial cable with 1 cm diameter and dielectric insulator of relative permittivity $\epsilon_r = 2.5$ and loss tangent $\tan \delta = 0.001$. The conductor is assumed to be copper with conductivity $\sigma = 5.7 \cdot 10^7$ S/m. Microwave engineering formulas [2, Section 9.3] give the following transmission-line parameters for this cable at 1 GHz frequency:

$$C \approx 7.9\epsilon_0, \quad L \approx 0.32\mu_0, \quad G \approx 4.4 \cdot 10^{-4} \text{ S/m}, \quad R \approx 2.2 \Omega/\text{m}$$

where $\epsilon_0 \approx 8.854 \cdot 10^{-12}$ As/Vm and $\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am are the free-space permittivity and permeability, respectively. The width $L = \pi/k$ of the source function is chosen to be 30 cm. Note that this length does not need to have any connection to the frequency of the wave nor to its free-space wavelength. L is only the width of the support of the non-radiating source function and it is determined by the manner the source is excited.

In the numerical calculation of Figure 4, the pulse function is approximated by a combination of hyperbolic tangent functions:

$$P(x_1, x, x_2) \approx \frac{1}{2} \{ \tanh[a(x - x_1)] - \tanh[a(x - x_2)] \}$$

where the parameter a adjusts the steepness of the step. In the calculations the value $a = 200$ was used.

4 Power conditions

4.1 Lossless transmission line

It was pointed out earlier that the existence of non-radiating sources is tantamount to the non-uniqueness of the inverse source problem. To ensure uniqueness for this inverse source problem, certain conditions for the source must be imposed. In fact, the existence of a non-radiating source implies that no energy is given by the source to propagate along the transmission line. Thus, if part of the source gives energy to the line, another part must absorb the energy, or all supplied energy shall have to be dissipated into losses in the source region. Therefore, one may suggest that non-radiating sources that are at the same time non-absorbing cannot be excited in a

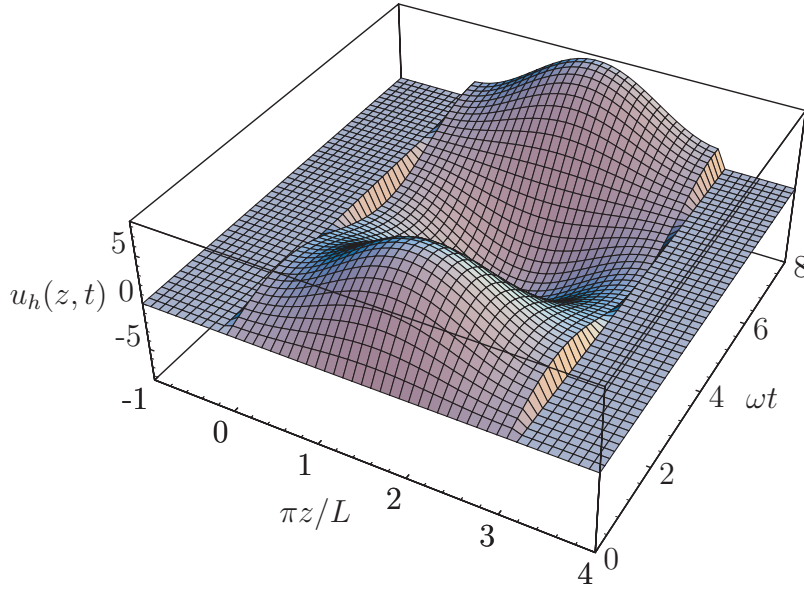


Figure 4: The non-radiating voltage source resulting with the choice of the test function (3.3), for the 75 Ω coaxial cable (see text for parameters).

lossless transmission line. In other words, enforcing the condition that no absorption of energy by the source is allowed, uniqueness for the inverse source problem can be certified in a lossless line with $R = 0$ and $G = 0$.

The power absorbed per unit length of the line can be expressed as

$$\begin{aligned}\partial_z P(z, t) &= -\partial_z(UI) = -I\partial_z U - U\partial_z I \\ &= RI^2 + GU^2 + \partial_t(LI^2/2) + \partial_t(CU^2/2) - uI - iU.\end{aligned}$$

The last two terms denote the absorption due to the sources. If such an absorption is not allowed, these terms should not give a positive number, i.e., the condition

$$Q(z, t) = u(z, t)I(z, t) + i(z, t)U(z, t) \geq 0$$

should be valid for all points z and all times t .

Let us consider this condition for the non-radiating source (Equation 3.2):

$$\begin{aligned}Q_h(z, t) &= \begin{pmatrix} U_h \\ I_h \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_h \\ i_h \end{pmatrix} \\ &= \begin{pmatrix} U_h \\ I_h \end{pmatrix}^T \begin{pmatrix} G + C\partial_t & \partial_z \\ \partial_z & R + L\partial_t \end{pmatrix} \begin{pmatrix} U_h \\ I_h \end{pmatrix} \\ &= \partial_z(U_h I_h) + \frac{1}{2}\partial_t(CU_h^2 + LI_h^2) + GU_h^2 + RI_h^2 \\ &\geq 0.\end{aligned}\tag{4.1}$$

For the previous example with $I_h(z, t) = 0$, $U_h(z, t) = U_0 P(z_1, z, z_2) P(t_1, t, t_2)$,

we have

$$\begin{aligned} Q_h(z, t) &= \frac{1}{2} \partial_t (CU_h^2) + GU_h^2 \\ &= U_0^2 (C[\delta(t - t_1) - \delta(t - t_2)] + GP(t_1, t, t_2)) P(t_1, t, t_2) P^2(z_1, z, z_2) \\ &= U_0^2 (C[\delta(t - t_1) - \delta(t - t_2)] + GP(t_1, t, t_2)) P(t_1, t, t_2) P(z_1, z, z_2). \end{aligned}$$

It is seen that the condition $Q_h(z, t) \geq 0$ is not satisfied at $t = t_2$, because at this moment power is absorbed by the sources.

4.2 Lossy transmission line

In the case of lossy transmission line ($G \neq 0$, $R \neq 0$), uniqueness of the inverse source problem does not follow from the requirement that the source not absorb energy. If the energy supplied by an NR source is absorbed by the losses within the source region, no power, and hence no fields, can be detected outside. This can be illustrated with a simple example.

Take $I_h(z, t) = 0$ and $U_h(z, t) = U_0 H(t, 0) e^{-Gt/C} P(z_1, z, z_2)$ where the step function $H(t, t_1)$ is zero for $t < t_1$ and unity for $t \geq t_1$. Now, using (3.2), we have for the sources

$$\begin{cases} u_h(z, t) = U_0 H(t, 0) e^{-Gt/C} [\delta(z - z_1) - \delta(z - z_2)] \\ i_h(z, t) = CU_0 \delta(t) P(z_1, z, z_2) \end{cases}$$

and for the power (per unit length) supplied by the source:

$$\begin{aligned} Q_h(z, t) &= CU_0^2 P^2(z_1, z, z_2) \delta(t) H(t, 0) e^{-Gt/C} \\ &= CU_0^2 P(z_1, z, z_2) \delta(t) \end{aligned}$$

which is positive at the starting moment $t = 0$ and zero after that. There is no absorption by the source; however, it is non-radiating.

5 Discussion and conclusion

Non-uniqueness of sources in electromagnetic field problems is the topic of the present paper. Two different current distributions, confined within a certain domain, may produce exactly the same radiation fields at every point outside this domain. From the linearity of Maxwell equations, it is then obvious that the difference of these two current distributions is a non-radiating source with respect to the external region. A corollary of this is that the problem of determining the source currents from externally measured fields is difficult. In fact, the inverse source problem is ill-posed. Some additional a priori information is needed about the source if one wants to have a unique solution for this problem. Moreover, the problem of inverse scattering is related to the inverse source problem and the existence of non-radiating sources. In inverse scattering problems, one measures radiated fields that are generated by incident waves, and from this information one likes to infer

information about the scatterer. This problem has large similarities to the inverse source problem due to the presence of equivalent induced sources in the scatterer.

The aim of the present paper has been to present a method to construct non-radiating source distributions for transmission-line problems. The inverse source problem does not seem to have been studied previously in transmission-line theory. An application of the partial differential operator L of Equation (3.1) on a localized function yields source functions that are bounded in space and which do not generate any voltage and current waves traveling along the transmission line. These sources cannot be detected with any voltage or current measurements on the transmission line, external to the source region.

With simple generating functions, such NR sources could be created that can be intuitively understood as being non-radiating. Two basic examples of such type with both voltage and current sources were discussed. The idea behind these source combinations was that the current (voltage) was confined within the source region by a certain source, and to extinguish the transients resulting from the onset and offset of this source, voltage (current) flashes of opposite polarity have to be added. Power conditions were also discussed because energy balance leads to certain requirements for the character of non-radiating sources. If no power is flowing out of the source region, the power emitted by the source either has to be absorbed by another part of the same source, or be dissipated in the losses of the transmission line. This would suggest that a non-absorbing and non-radiating source cannot exist in a lossless transmission line.

The paper also presents a way of constructing NR sources that consist of only voltage functions, or only current functions. In practical applications, this type of source description may be more useful. The forced voltage-only source could be thought as a slot in the wall of a waveguide which an external plane wave is illuminating. The amplitude can be controlled by the width of the opening. A pure current source would be the result of surface currents induced on an unshielded microstrip line under a similar exposure to an incident field. The present analysis of non-radiating transmission line sources is hopefully useful for EMC applications, eliminating interference, and the design of microwave equipment in difficult electromagnetic environments.

6 Acknowledgments

The present study was inspired by the PhD thesis of Dr. Jonas Lundstedt from the Department of Electromagnetic Theory at the Royal Institute of Technology, Stockholm, Sweden. Helpful discussions with Professor Staffan Ström and Dr. Sailing He from the same department are acknowledged. Also, we thank Dr. W. Ross Stone and Professor Karl J. Langenberg for advice concerning the existing literature of non-radiating sources.

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