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# HOW TO SOLVE SINGULAR DISCRETE-TIME RICCATI-EQUATIONS

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*Abstract:* There exists a solution to the discrete time algebraic Riccati equation giving closed loop eigenvalues inside or on the unit circle, assuming the system is stabilizable. This solution is always unique. Numerical methods, like the sorted generalized Schur form method and the Kleinman iteration, often fail in case of zeros on the unit circle or lack of left invertibility. It is here suggested how such singular cases could be made regular by reduction operations on the system matrix pencil  $[-zI + A, B; C, D]$ . The solution may be discontinuous with respect to parameter variation, but the reduction approach seems numerically appealing. This is demonstrated using simple illustrative examples. The solution is the largest symmetrical matrix satisfying a corresponding LMI. To obtain a feasible solution for interior-point methods it is also necessary to do a reduction, for some problems even further than what is described in this paper.

*Keywords:* discrete-time algebraic Riccati equations, linear-quadratic control, maximal solution, singularity, zeros on unit circle, left-invertibility, discontinuous solution, LMI.

## 1. INTRODUCTION

The interest in Riccati equations was revived by the theory of  $H_\infty$ -control. Actually it was found that some singular  $H_2$ -cases were not fully understood, especially in discrete time. A basic reference is Silverman (1976), while books like Kucera (1991) and Bittanti *et al.* (1990) provide a state-of-the-art survey. Some recent results are found in Trentelman and Stoorvogel (1995) and Saberi *et al.* (1996). In Hagander and Hansson (1995) it is described how there may exist optimal LQG-controllers even if the Riccati-equation based controller results in closed loop eigenvalues on the unit circle. In the classical formulation of the Riccati equation

$$S = A^T S A + Q_1 - A^T S B (B^T S B + Q_2)^{-1} B^T S A$$

the inverse does not exist in case of redundant control signals. These two singular cases are numerically diffi-

cult to solve without special care, and in this paper we give a summary of some relevant facts and suggestions for how to circumvent some of the singularity problems.

A general formulation of the Riccati equation covering singular cases as well as cross-terms would be to solve for  $S$  and  $L$  in

$$\begin{cases} S = A^T S A + Q_1 - L^T G L, \\ G = B^T S B + Q_2, \quad G L = B^T S A + Q_{12}^T \end{cases} \quad (1)$$

The equations in (1) can be summarized as

$$\begin{pmatrix} I & 0 \end{pmatrix}^T S \begin{pmatrix} I & 0 \end{pmatrix} + \begin{pmatrix} L & I \end{pmatrix}^T G \begin{pmatrix} L & I \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}^T S \begin{pmatrix} A & B \end{pmatrix} + Q \quad (2)$$

where

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}$$

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We here only discuss Riccati equations related to LQG-control and thus require that  $Q \geq 0$  or  $Q = [C, D]^T[C, D]$ . Our interest is in symmetric solutions  $S \geq 0$  giving closed loop eigenvalues inside or on the unit circle, i.e.  $|\lambda(A - BL)| \leq 1$ . In order to emphasize the relation between the Riccati equation and Schur complements – Completion of squares – we may rewrite (1) as

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (3)$$

The Riccati equation can be used to solve the stationary LQ-control problem,

$$\begin{aligned} & \inf_L \lim_{k \rightarrow \infty} EJ \\ & J = [Cx(k) + Du(k)]^T [Cx(k) + Du(k)] \\ & x(k+1) = Ax(k) + Bu(k) + v(k) \\ & u(k) = -Lx(k) \end{aligned}$$

where  $x(k) \in R^n$ ,  $u(k) \in R^m$ . The constant feedback matrix  $L$  gives a stationary state process  $x(k)$  in the limit. The disturbance  $v(k)$  consists of independent random variables with zero mean value and unit covariance. The infimal loss is  $E v^T S v = \text{trace } S$ , where  $S$  is the solution of (1) giving  $|\lambda(A - BL)| \leq 1$ .

The system  $(A, B, C, D)$  can be described by its system matrix, the pencil

$$P(z) = \begin{pmatrix} -zI + A & B \\ C & D \end{pmatrix}$$

The two types of singularity correspond to zeros on the unit circle and lack of left invertibility. The system  $(A, B, C, D)$  is left invertible when

$$\max_z \text{rank } P(z) = m + n \quad (4)$$

and there are no zeros on the unit circle when

$$\text{rank}_{|z|=1} P(z) = \max_z \text{rank } P(z) \quad (5)$$

The lack of left-invertibility is most severe, and the Riccati equation solution  $S$  is then often discontinuous w.r.t the elements of  $(A, B, C, D)$ .

EXAMPLE 1

$x(k+1) = x(k)/2 + bu(k)$ , and  $J = x(k)^2$

$$S = \begin{cases} 4/3 & \text{if } b = 0, \\ 1 & \text{otherwise} \end{cases}$$

□

EXAMPLE 2

$x_1(k+1) = u_1(k)$ ,  $x_2(k+1) = 2x_2(k) + u_2(k)$ , and  $J = [cx_1(k) + x_2(k)]^2 + u_2(k)^2$ .

$$S = \begin{pmatrix} c^2 & c \\ c & s_{22} \end{pmatrix}, \quad s_{22} = \begin{cases} 2 + \sqrt{5} & \text{if } c = 0 \\ 4 & \text{otherwise} \end{cases}$$

□

In the first example the input is redundant for  $b = 0$  and rank  $B < m$ . In the second example  $u_1$  is dynamically redundant together with  $x_1$  for  $c = 0$ . There are other perturbations of  $(A, B, C, D)$  for which the solutions  $S$  are continuous. The following example describes what happens in case of zeros on the unit circle.

EXAMPLE 3

$x(k+1) = u(k) + v(k)$ ,  $J = [x(k) - u(k)]^2$ , giving the Riccati solution  $S = 0$ ,  $L = -1$  and  $A - BL = 1$ . The infimal cost is zero, but it is only possible to come arbitrarily close to the infimum using stabilizing control.

□

## 2. EXISTENCE AND UNIQUENESS THEOREM

The basic theorems on discrete time algebraic Riccati equations are now summarized,

THEOREM 1

Assume that the system  $(A, B, C, D)$  is stabilizable and left invertible. Then there is always a real solution  $(S, L)$  to the Riccati equation (1) with symmetric  $S \geq 0$  and  $|\lambda(A - BL)| \leq 1$ . It also holds that  $G > 0$ , and both  $S$  and  $L$  are unique. Furthermore  $|\lambda(A - BL)| < 1$ , if and only if there are no zeros on the unit circle.

*Proof:* See Appendix.

□

REMARK 1

For any real symmetric solution  $S \geq 0$  and any corresponding  $A - BL$  introduce  $T = [T_-, T_0, T_+]$  with  $(A - BL)T = TJ$ ,  $J = \text{diag}(J_-, J_0, J_+)$  and  $|\lambda(J_-)| > 1$ ,  $|\lambda(J_0)| = 1$ ,  $|\lambda(J_+)| < 1$ . Then

$$(C - DL) \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0, \quad S \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$

□

REMARK 2

Assume  $\text{rank}_{|z|=1} P(z) = m + n$ . The solution  $S$  of Theorem 1 is differentiable with respect to differentiable variations in  $(A, B, C, D)$ .

□

For systems that are not left invertible it is possible to obtain a corresponding left invertible system by

elimination of redundant inputs that do not influence the system and modes (stabilized for free) not observable in the performance index. The reduction is related to the Silverman structure algorithm, e.g. Silverman (1976). Two different cases can be distinguished corresponding to examples 1 and 2 respectively. How to do the reduction will be further discussed below after a section on numerical methods. A very general theorem can then be stated.

**THEOREM 2**

Assume that  $(A, B, C, D)$  is stabilizable. There is always a real solution  $(S, L)$  to the Riccati equation (1) with symmetric  $S \geq 0$ , such that  $|\lambda(A - BL)| \leq 1$ . Here  $S$  is unique, but  $L$  may be nonunique. There exists a solution  $L$  with  $|\lambda(A - BL)| < 1$ , if and only if there are no zeros on the unit circle.  $\square$

**REMARK 3**

The modes eliminated correspond to a subspace, where  $S$  is zero.  $\square$

**3. THREE NUMERICAL METHODS**

The most common methods to solve the discrete-time algebraic Riccati equation are the ‘‘Kleinman algorithm’’ and the ‘‘Generalized Schur-form method’’. Recently it has become popular to apply ‘‘Linear Matrix Inequalities’’ (LMI).

**3.1 The Kleinman algorithm**

Solve  $S_i$  from  $S_i = A_i^T S_i A_i + C_i^T C_i$ ,  $A_i = A - BL_i$ ,  $C_i = C - DL_i$ , and then solve  $L_{i+1}$  from  $(D^T D + B^T S_i B)L_{i+1} = D^T C + B^T S_i A$ . Start with  $L_0$  such that  $A - BL_0$  is stable. From the proof of Theorem 1 in the Appendix it follows that  $S_i \rightarrow S$ , the desired solution.

The convergence is normally very fast, but it is quadratic only if (5) holds. If the system lacks left invertibility the nonunique  $L_{i+1}$  has to be chosen to be stabilizing for convergence, but that requires considerable work at each iteration.

**3.2 The Schur-form method**

First introduce the pencil

$$P_1(z) = \begin{pmatrix} 0 & -zI + A & B \\ -I + zA^T & C^T C & C^T D \\ zB^T & D^T C & D^T D \end{pmatrix}$$

which appears naturally when deriving first order optimality conditions for the LQ-problem using Lagrange multipliers. Then find orthogonal  $Q$  and  $Z$  such that  $Q^T P_1(z)Z$  is on ordered generalized real Schur form.

This means that its  $(1,1)$ -block has its zeros inside or on the unit circle. Then solve  $S$  and  $L$  from

$$\begin{pmatrix} S \\ I \\ -L \end{pmatrix} Z_{21} = \begin{pmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{pmatrix}$$

where the right-hand side is the first block-column of  $Z$ . Some matrix algebra shows that  $(S, L)$  is the desired solution of the Riccati equation (1). See e.g. Laub (1990).

The ordering has to be done with great care in the singular cases. If (4) does not hold the pencil  $P_1(z)$  is also singular, and very special software is required. Example 1 was possible to solve using a straight-forward implementation, but Example 2 was not. When the reordering is required the standard methods fail. The following theorem suggests that the structure of the pencil  $P(z)$  should be used when operating on  $P_1(z)$ .

**THEOREM 3**

For any  $|z| = 1$ ,  $\mathcal{N}(P_2(z)) = \mathcal{R}\left(\begin{pmatrix} 0 \\ Y \end{pmatrix}\right)$ , where  $\mathcal{R}(Y) = \mathcal{N}(P(z))$  and

$$P_2(z) = \begin{pmatrix} 0 & -zI + A & B \\ -z^{-1}I + A^T & C^T C & C^T D \\ B^T & D^T C & D^T D \end{pmatrix}$$

*Proof:* Utilize the structure

$$P(z) = \begin{pmatrix} N \\ M \end{pmatrix}, \quad P_2(z) = \begin{pmatrix} 0 & N \\ N^* & M^* M \end{pmatrix}$$

and rank  $N = n$  by stabilizability.  $\square$

**3.3 The LMI-method**

Solve

$$\max_{P \geq 0} \text{trace } P$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \quad (6)$$

The optimizing  $P$  is equal to  $S$ , the desired solution of the Riccati equation. An algebraic proof is given in the Appendix. There is also an immediate relation between the LMI-problem and the LQ-problem.

The most common method to solve LMI-problems is using interior point methods. Unfortunately there are sometimes no feasible interior point to the LMI-problem. This is the case in all three examples. Some insight is provided by the proofs in the Appendix.

#### 4. REDUCTION ALGORITHMS

For systems that are not left invertible it is possible to obtain a corresponding left invertible system by elimination of redundant inputs:

1. Elimination of inputs until  $\begin{pmatrix} B \\ D \end{pmatrix}$  has full rank.
2. Elimination of the modes and inputs of the maximal controllability subspace from free inputs.

We describe two alternative reduction methods. In the first one the states and inputs removed represent the exact non-uniqueness of the original problem. The second one is easier to implement, and it may result in a larger reduction than necessary.

##### 4.1 Algorithm 1

First introduce a stabilizing feedback and input/output transformations giving

$$\begin{pmatrix} qx \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \bar{A} & B_1 & B_2 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 & 0 \\ C_3 & D_1 & D_2 & 0 & I \end{pmatrix} \begin{pmatrix} x \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

where  $\bar{A}$  is stable and  $[B_1, B_2]$  has full column rank. Here  $q$  denotes the forward shift operator. Now  $u_3$  can be arbitrary, so it is eliminated. The inputs  $u_1$  and  $u_4$  can be chosen uniquely to make  $z_1 = 0$  and  $z_3 = 0$ . By state coordinate changes, state feedback and  $u_2$  transformations it is found that

$$\begin{pmatrix} -zI + \bar{A} & B_2 \\ C_2 & 0 \end{pmatrix} \sim \begin{pmatrix} -zI + A_{11} & 0 & 0 & B_{11} & 0 \\ A_{21} & -zI + A_{22} & 0 & 0 & 0 \\ A_{31} & A_{32} & -zI + A_{33} & 0 & B_{32} \\ C_{21} & 0 & 0 & 0 & 0 \end{pmatrix}$$

where  $\sim$  denotes strong equivalence, and where  $[A_{33}, B_{32}]$  corresponds to the maximal controllability subspace in the nullspace of  $C_2$ . The corresponding control signals have to be stabilizing, but they are otherwise arbitrary and can be removed together with the corresponding states. It follows that the rank loss in  $P(z)$  equals the number of eliminated arbitrary inputs, i.e. the rank deficiency of  $[B; D]$  plus the rank loss in  $[-zI + A_{33}, B_{32}]$ , i.e. the number of columns in  $B_{32}$ . After the reduction the remaining system satisfies the left-invertibility condition (4).

##### 4.2 Algorithm 2

First make the input transformation

$$\begin{pmatrix} B \\ D \end{pmatrix} W = \begin{pmatrix} B_1 & 0 \\ D_1 & 0 \end{pmatrix}$$

where  $\text{rank} \begin{pmatrix} B \\ D \end{pmatrix} = \text{rank} \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} = m_1$ , and

$$P_1(z) = \begin{pmatrix} -zI + A & B_1 \\ C & D_1 \end{pmatrix}$$

If  $\text{rank} P_1(0) = n + m_1$ , solve the problem using Theorem 1. Otherwise derive its nullspace basis  $\begin{pmatrix} X \\ U \end{pmatrix}$ . There exists  $L$  with  $-LX = U$ , since  $X$  is full rank. Thus

$$\begin{pmatrix} A - B_1L \\ C - D_1L \end{pmatrix} X = 0$$

and the  $X$ -modes are unobservable in  $C - D_1L$  and can be removed. The procedure is repeated until  $\text{rank} P_j(0) = n + m_j$ .

##### 4.3 Almost singular pencils

For a system that is only marginally left invertible very large feedback- $L$ 's may be needed to obtain the optimal  $S$ , as seen in examples 1 and 2. In a practical situation we would like not to utilize such control signals. The reduction should thus be performed also for marginally left invertible system. Actually these ideas are compatible with numerical methods for the transformation of a pencil to Kronecker canonical form using reducing subspaces.

#### 5. RECOMMENDATIONS

In this paper we have reviewed some results about the discrete-time Riccati-equation with implications on the way numerical Riccati-solvers should be implemented. We suggest that before using a standard Riccati-solver, such as the one in the Matlab Control System Toolbox, the problem should be reduced using e.g. the nullspace of  $P(0)$ . It should be stressed that this is definitely not the only reduction that has to be done in order to use interior point methods for LMI:s. For the Schur-form method and the Kleinman method it may be sufficient, but we believe that reducing also the parts corresponding to the nullspace of  $P(z)$  for  $|z| = 1$  would be a good idea. It remains to find a good algorithm that implements this reduction, but it would definitely yield a more robust Riccati-solution procedure.

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## 7. APPENDIX

### LEMMA 1

Assume  $S \geq 0$  satisfies  $S = A^T S A + Q$ ,  $Q \geq 0$ . Then  $Q \neq 0$ , only if  $A$  has an eigenvalue  $|\lambda| < 1$ .

*Proof:* Assume first that  $A$  has a complete set of eigenvectors. There is then for  $Q \neq 0$  an eigenvector  $x$  with  $x^* Q x > 0$ , and  $x^* S x = |\lambda|^2 x^* S x + x^* Q x$  then requires  $|\lambda| < 1$ . For defective  $A$  the proof is extended using generalized eigenvectors.  $\square$

Remark 1 now follows using stabilizability and

$$T^T S T = J^T T^T S T J + T^T (C - D L)^T (C - D L) T$$

### 7.1 Proof of Theorem 1

A straight-forward proof, cf. Kucera (1991), can be made using the Kleinman recursion

$$A_i = A - B L_i, \quad C_i = C - D L_i \quad (7a)$$

$$S_i = A_i^T S_i A_i + C_i^T C_i \quad (7b)$$

$$G_i = D^T D + B^T S_i B \quad (7c)$$

$$G_i L_{i+1} = D^T C + B^T S_i A \quad (7d)$$

for  $i = 0, 1, \dots$  with initial value  $L_0$  such that  $A_0$  is stable. It will first be shown that the sequence of  $L_i$  is well defined, and then the question about convergence will be investigated. Assume that  $A_i$  is stable. Then there exists a unique  $S_i \geq 0$  that solves (7b), since it

is a Lyapunov-equation, and there exists an  $L_{i+1}$  that solves (7d), since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} S_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0$$

If it can be concluded that  $A_{i+1}$  is stable, it thus follows by induction that  $A_i$  is stable for all  $i \geq 0$ . Assume that  $A_{i+1}$  is not stable. Then there exist  $\lambda$  and  $x$  such that  $|\lambda| \geq 1$  and  $A_{i+1} x = x \lambda$ . Now use  $\Delta_i = (L_i - L_{i+1})^T G_i (L_i - L_{i+1})$  in (7) to obtain

$$S_i = A_{i+1}^T S_i A_{i+1} + C_{i+1}^T C_{i+1} + \Delta_i \quad (8)$$

and

$$(1 - |\lambda|^2) x^* S_i x = x^* C_{i+1}^T C_{i+1} x + x^* \Delta_i x$$

From  $|\lambda| \geq 1$  and  $S_i \geq 0$  follows that  $x^* \Delta_i x = 0$ . Thus  $L_i x = L_{i+1} x$  provided  $G_i > 0$ , and hence the contradiction that  $\lambda$  is also an eigenvalue of  $A_i$ . To show that  $G_i > 0$ , rewrite (7) and (8) as

$$\begin{bmatrix} I & 0 \\ L_{i+1} & I \end{bmatrix}^T \begin{bmatrix} S_i - \Delta_i & 0 \\ 0 & G_i \end{bmatrix} \begin{bmatrix} I & 0 \\ L_{i+1} & I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} S_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (9)$$

Let  $\Psi(z) = (zI - A)^{-1} B$ , and let  $H(z) = C \Psi(z) + D$ . Notice that  $A \Psi(z) + B = -z \Psi(z)$ . Thus by multiplying (9) by  $\begin{bmatrix} \Psi(z) \\ I \end{bmatrix}$  from the right and its adjoint from the left the following equality is obtained

$$H^*(z) H(z) + \Psi^*(z) \Delta_i \Psi(z) = [I + L_{i+1} \Psi(z)]^* G_i [I + L_{i+1} \Psi(z)] \quad (10)$$

Now the condition (4) implies that  $\text{rank } H(z) = m$  for some  $z$ , which by (10) and  $\Delta_i \geq 0$  implies that  $G_i > 0$ . Thus it is proven that the sequence of  $L_i$  is well defined and  $A_i$  is stable for all  $i \geq 0$ .

It will now be shown that the sequence  $S_i$  converges to some limit  $S$ . Further manipulations show that the following Lyapunov-equation holds

$$S_i - S_{i+1} = A_{i+1}^T (S_i - S_{i+1}) A_{i+1} + \Delta_i \quad (11)$$

Since  $A_{i+1}$  is stable and since  $\Delta_i \geq 0$  it follows that  $S_i - S_{i+1} \geq 0$ . Thus it holds that  $0 \leq S_{i+1} \leq S_i$ , which implies that  $S_i \rightarrow S \geq 0$ . The equation (7c) implies that  $G_i \rightarrow G = D^T D + B^T S B$ , and there exists  $L$  such that  $GL = D^T C + B^T S A$ , since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \geq 0$$

Further  $A_{i+1}^T(S_i - S_{i+1})A_{i+1} \rightarrow 0$  and  $\Delta_i \rightarrow 0$ , since both matrices are positive semidefinite. Thus it is proven that the limit  $S$  solves the algebraic Riccati equation (1). Similarly to (10) it also holds that

$$H^*(z)H(z) = [I + L\Psi(z)]^*G[I + L\Psi(z)] \quad (12)$$

Now the rank condition (4) implies that  $G > 0$ , and hence  $L$  is a unique solution. The sequence  $L_i$  therefore converges to  $L$ , and since the eigenvalues of  $A_i$  are inside the unit circle, it follows that in the limit the eigenvalues of  $A_c = A - BL$  are inside or on the unit circle.

Now the right-hand side of (12) loses rank on the unit circle when  $z$  is a closed loop eigenvalue on the unit circle, since  $G > 0$  and

$$\begin{pmatrix} -zI + A & 0 \\ L & I + L\Psi(z) \end{pmatrix} = \begin{pmatrix} -zI + A_c & B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} I & \Psi(z) \\ 0 & I \end{pmatrix}$$

Similarly the left-hand side loses rank on the unit circle when  $P(z)$  does, since

$$P(z) = \begin{pmatrix} -zI + A & 0 \\ C & H(z) \end{pmatrix} \begin{pmatrix} I & -\Psi(z) \\ 0 & I \end{pmatrix}$$

It can here be assumed that  $A$  is stable, since an initial stabilizing feedback  $L_0$  just corresponds to multiplying the Riccati equation (3) by  $\begin{pmatrix} I & 0 \\ -L_0 & I \end{pmatrix}$  from the right and its transpose from the left. It is thus proven that  $L$  is stabilizing if and only if  $\text{rank}_{|z|=1} P(z) = n + m$ .

To show the uniqueness of  $S$  consider two solutions  $S_1$  and  $S_2$  with corresponding closed loop matrices  $A_1 = A - BL_1$  and  $A_2 = A - BL_2$ . Then it holds that

$$A_2^T(S_1 - S_2)A_1 = S_1 - S_2$$

Let  $T_1 = [T_{10}, T_{1+}]$  and  $T_2 = [T_{20}, T_{2+}]$  be spectrum-splitting transformations with

$$\begin{aligned} A_1 T_1 &= T_1 \text{diag}(J_{10}, J_{1+}) \\ A_2 T_2 &= T_2 \text{diag}(J_{20}, J_{2+}) \end{aligned}$$

where  $J_{1+}$  and  $J_{2+}$  are the blocks with eigenvalues inside the unit circle. It then holds that for  $k \rightarrow \infty$

$$\begin{aligned} (S_1 - S_2)T_{1+} &= (A_2^T)^k (S_1 - S_2)T_{1+}J_{1+}^k \rightarrow 0 \\ T_{2+}^T(S_1 - S_2) &= (J_{2+}^T)^k T_{2+}^T(S_1 - S_2)A_1^k \rightarrow 0 \end{aligned}$$

Furthermore  $S_1 T_{10} = 0$  and  $S_2 T_{20} = 0$  by Remark 1, so  $S_1 = S_2$ .

## 7.2 Some LMI results

### LEMMA 2

Assume  $|\lambda(A)| = 1$  and  $[A, B]$  controllable. The only  $P \geq 0$  satisfying the LMI (6) with  $C = 0$  is  $P = 0$ .

*Proof:*  $P$  satisfies (6) with  $C = 0$ , so there is an  $R_1 \geq 0$  such that  $A^T P A = L^T G L + P + R_1$ ,  $G = B^T P B + D^T D$ , and  $G L = B^T P A$ , or

$$\begin{cases} P = A^{-T} P A^{-1} + R_2 \\ G M = G L A^{-1} = B^T P \\ R_2 = A^{-T} R_1 A^{-1} + M^T G M \end{cases}$$

Thus  $R_2 = 0$  by Lemma 1, so  $G M = B^T P = 0$ . From the controllability there is  $L_0$  with  $A - B L_0$  stable, so  $A^{-T} P = P A = P(A - B L_0)$  gives  $P = 0$ .  $\square$

### THEOREM 4

For any  $P \geq 0$  satisfying the LMI (6) it holds that  $P \leq S$ , where  $S$  is the unique solution of the Riccati equation (1) allowing closed loop eigenvalues  $|\lambda(A - B L)| \leq 1$ . Furthermore  $P = S$  satisfies the LMI (6), so  $S$  is its maximal solution.

*Proof:* Use  $A_c = A - B L$  to rewrite (6) as

$$\begin{aligned} [C - D L, D]^T [C - D L, D] \\ + [A_c, B]^T P [A_c, B] \geq \text{diag}(P, 0) \end{aligned} \quad (13)$$

The (1,1)-block of (13) means that there exists  $\Delta \geq 0$  such that

$$(C - D L)^T (C - D L) + A_c^T P A_c = P + \Delta \quad (14)$$

With  $A_c [X_+, X_0] = [X_+, X_0] \text{diag}(A_+, A_0)$  it holds as in Remark 1 that  $(C - D L)X_0 = 0$  and  $S X_0 = 0$  giving

$$\begin{aligned} A_0^T P_0 A_0 &= P_0 + \Delta_0 \\ A_0^T P_{0+} A_+ &= P_{0+} + \Delta_{0+} \end{aligned}$$

where  $X_0^T P X_0 = P_0$ ,  $X_0^T \Delta X_0 = \Delta_0$ ,  $X_0^T P X_+ = P_{0+}$ , and  $X_0^T \Delta X_+ = \Delta_{0+}$ . It first follows that  $\Delta_0 = 0$  and thus  $\Delta_{0+} = 0$ . Then  $P_{0+} = 0$  by stability of  $A_+$ . Subtracting (14) from the Riccati equation  $(C - D L)^T (C - D L) + A_c^T S A_c = S$  gives  $A_+^T (S_+ - P_+) A_+ + \Delta_+ = S_+ - P_+$ , where  $X_+^T P X_+ = P_+$ ,  $X_+^T S X_+ = S_+$ , and  $X_+^T \Delta X_+ = \Delta_+$ . Now  $S_+ - P_+ \geq 0$  by stability of  $A_+$ . It only remains to show that  $P_0 = 0$ . From (13) it follows that

$$\begin{aligned} [0, D]^T [0, D] + [0, B_+]^T P_+ [0, B_+] + \\ [A_0, B_0]^T P_0 [A_0, B_0] \geq [I, 0]^T P_0 [I, 0] \end{aligned} \quad (15)$$

where  $[X_+, X_0] \begin{pmatrix} B_+ \\ B_0 \end{pmatrix} = B$ . Now  $P_0 = 0$  by controllability of  $[A_0, B_0]$  and Lemma 2.  $\square$