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A Generalization of the Predictable Degree Property to Rational Convolutional Encoding Matrices[1]

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Abstract — **The predictable degree property was introduced by Forney [1] for polynomial convolutional encoding matrices. In this paper two generalizations to rational convolutional encoding matrices are discussed.**

I. Introduction

The predictable degree property, introduced by Forney [1], is a useful analytic tool when we study the structural properties of convolutional encoding matrices.

Let $G(D)$ be a rate $R = b/c$ binary polynomial encoding matrix with ν_i as the constraint length of the i-th row. For any polynomial input $\underline{u}(D)$ the output $\underline{v}(D) = \underline{u}(D)G(D)$ is also polynomial. We have

$$\begin{aligned}\deg \underline{v}(D) &= \deg \underline{u}(D)G(D) = \deg \textstyle\sum_{i=1}^{b} u_i(D)\underline{g}_i(D) \\ &\leq \max_{1\leq i\leq b}\{\deg u_i(D) + \nu_i\}. \end{aligned} \tag{1}$$

Definition 1 A polynomial encoding matrix $G(D)$ is said to have the *predictable degree property* if for all polynomial inputs $\underline{u}(D)$ we have equality in (1).

Let $[G(D)]_h$ be the $(0,1)$-matrix with 1 in the position (i,j) where $\deg g_{ij}(D) = \nu_i$ and 0, otherwise. Then we have

Theorem 1 *Let $G(D)$ be a polynomial encoding matrix. Then $G(D)$ has the predictable degree property if and only if $[G(D)]_h$ is of full rank.*

Since a basic encoding matrix is minimal-basic if and only if $[G(D)]_h$ is of full rank ([1] [4]) we have the following theorem which is due to Forney [1]:

Theorem 2 *Let $G(D)$ be a basic encoding matrix. Then $G(D)$ has the predictable degree property if and only if it is minimal-basic.*

In [4] we gave an example of a basic encoding matrix that is minimal but not minimal-basic. That minimal encoding matrix does *not* have the predictable degree property.

II. The Predictable Degree Property for Rational Encoding Matrices

Let $\underline{g}(D) = (g_1(D), \ldots, g_c(D))$, where $g_1(D), \ldots, g_c(D) \in \mathbb{F}_2(D)$. Denote by

$$\mathcal{P}^* = \{p(D) \in \mathbb{F}_2[D] \mid p(D) \text{ is irreducible}\} \cup \{D^{-1}\}. \tag{2}$$

For any $p \in \mathcal{P}^*$ we define

$$e_p(\underline{g}(D)) = \min\{e_p(g_1(D)), \ldots, e_p(g_c(D))\}, \tag{3}$$

where $e_p(g_i(D))$ is an exponential valuation of $g_i(D)$ [2] [3].

For any rational input $\underline{u}(D)$ the output $\underline{v}(D)$ is also rational. We have

$$\begin{aligned} e_p(\underline{v}(D)) &= e_p\left(\textstyle\sum_{i=1}^{b} u_i(D)\underline{g}_i(D)\right) \\ &\geq \min_{1\leq i\leq b}\{e_p(u_i(D)) + e_p(\underline{g}_i(D))\}. \end{aligned} \tag{4}$$

Definition 2 A rational encoding matrix $G(D)$ is said to have the *predictable degree property* if for $p = D^{-1}$ and all rational inputs $\underline{u}(D)$ we have equality in (4).

Let $G(D)$ be a rational encoding matrix. As a counterpart to $[G(D)]_h$ for polynomial encoding matrices, for any $p \in \mathcal{P}^*$ we introduce the $b \times c$ matrix $[G(D)]_h(p)$ to be a matrix whose element in the position (i,j) is equal to the coefficient of the lowest term of $g_{ij}(D)$, written as a Laurent series of p, if $e_p(g_{ij}(D)) = e_p(\underline{g}_i(D))$, and equal to 0, otherwise.

Then we can prove

Theorem 3 *Let $G(D)$ be a rational encoding matrix. Then $G(D)$ has the predictable degree property if and only if $[G(D)]_h(D^{-1})$ has full rank.*

III. The Predictable Exponential Valuation Property

Definition 3 A rational encoding matrix $G(D)$ is said to have the *predictable exponential valuation property* if we have equality in (4) for all $p \in \mathcal{P}^*$.

Theorem 4 *Let $G(D)$ be a rational encoding matrix. Then $G(D)$ has the predictable exponential valuation property if and only if $[G(D)]_h(p)$ mod p has full rank for all $p \in \mathcal{P}^*$.*

A rational encoding matrix is said to be *canonical* if it can be realized with a minimal number of delay elements in controller canonical form [5].

Theorem 5 *Let $G(D)$ be a rational encoding matrix and assume that $e_p(\underline{g}_i(D)) \leq 0$, $1 \leq i \leq b$, for all $p \in \mathcal{P}^*$. Then $G(D)$ has the predictable exponential valuation property if and only if $G(D)$ is canonical.*

The predictable exponential valuation property is *not* equivalent to being canonical.

References

[1] Forney, G.D., Jr. (1970). Convolutional codes I: Algebraic structure. *IEEE Trans. Inform. Theory*, IT-16:720–738.

[2] Jacobson, N. (1989). *Basic Algebra II*, 2nd ed. Freeman, New York.

[3] Forney, G.D., Jr. (1991). Algebraic structure of convolutional codes, and algebraic system theory. In *Mathematical System Theory*, A.C. Antoulas, Ed., Springer-Verlag, Berlin, 527–558.

[4] Johannesson, R. and Wan, Z. (1993). A linear algebra approach to minimal convolutional encoders, *IEEE Trans. Inform. Theory*, IT-39:1219–1233.

[5] Johannesson, R. and Wan, Z. (1994). On canonical encoding matrices and the generalized constraint lengths of convolutional codes. To appear in a book published by Kluwer on the occasion of J.L. Massey's 60^{th} birthday.

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