# Analysis of Real-Time Control Systems with Time Delays

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# Analysis of Real-Time Control Systems with Time Delays *

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## Abstract

We discuss modeling and analysis of real-time control systems subject to random time delays in the communication network. A new method for analysis of given control schemes is presented. The state of the network is modeled by a Markov chain and Lyapunov equations for the expected LQG performance are presented. An example that illustrates the results is given.

## 1. Introduction

Real-time control systems are increasingly often implemented as distributed control systems, where control loops are closed over a communication network. The communication network is shared by different processors, each having different priorities and computational loads. There will inevitably be time delays in the communication of information between different units. Computational delays can also be time-varying. The length of the time delays are often hard to predict and are here modeled as being random.

Different control schemes for systems with time-varying delays have been suggested. One interesting possibility that we will analyze here is to use so called time-stamps on control and measurement signals. We present a method to evaluate the performance of such control schemes. Our analysis generalizes the approach taken in Nilsson *et al.* (1996) in that we use a Markov chain to model the communication network. Section 2 describes three different models of the network delays. In Section 3 we give Lyapunov recursions for the expected LQG-performance and present an example that illustrates the results.

The analysis is based on techniques from jump linear systems, see e.g. Wonham (1971), Chizeck *et al.* (1986) , Mariton (1990), Ji and Chizeck (1990), Ji *et al.* (1991), Gajic and Qureshi (1995). Our system model is, however, more general. We allow for the probability distribution of the system matrices to be generated by the Markov chain. Previous references assumed the system matrices being given directly by the state of the Markov chain.

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## 2. Modeling of Network Delays

Network delays, or network transfer times, have different characteristics depending on the network hardware and software. In order to analyze control systems with network delays in the loop we have to model these. The network delay is typically varying due to varying network load, scheduling policies in the network and the nodes, and due to network failures. We will use three models of the network delay:

- Constant delay
- Random delay, which is independent from transfer to transfer
- The distribution of the delay is governed by an underlying Markov chain

The control loop usually also contains computational delays. The effect of these can be embedded in the network delays, see Section 3.

### 2.1 Network Modeled as Constant Delay

The simplest model of the network delay is to model it as being constant for all transfers in the communication network. This can be a good model even if the network has varying delays, for instance if the time scale in the process is much larger than the delay introduced by the communication.

One way to achieve constant delays is by introduction of timed buffers after each transfer. By making these buffers longer than the worst case delay time the transfer time can be seen as being constant.

This method to make the communication delays constant was proposed in Luck and Ray (1990). A drawback with this method is that the delay time often is longer than necessary, which can lead to decreased performance as shown in Nilsson *et al.* (1996).

### 2.2 Network Modeled as Consecutive Delays Being Independent

To take the randomness of the network delays into account, the time delays can be modeled as being taken from a probabilistic distribution. To keep the model simple to analyze one can assume the transfer delay to be independent of previous delay times, see Nilsson *et al.* (1996).

### 2.3 Network Modeled Using Markov Chain

In a real communication system the transfer time will usually be correlated with the last transfer delay. For example, the network load, which is one of the factors affecting the delay, is typically varying at a slower time scale than the sampling period in a control system, i.e. the time between two transfers. One way to model dependence between samples is by letting the distribution of the network delay be governed by the state of an underlying Markov chain. Effects such as varying network load can be modeled by making the Markov chain do a transition every time a transfer is done in the communication network.

EXAMPLE 1—SIMPLE NETWORK MODEL

To get a simple network model we can let the network have three states, one for low network load, one for medium network load, and one for high network load. In Figure 1 the transitions between different states in the communication network are modeled as a Markov chain. Together with every state in the

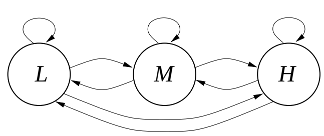


**Figure 1** An example of a Markov chain modeling the state in a communication network. Here $L$ is the state for low network load, $M$ the state for medium network load, and $H$ is the state for high network load. The arrows show possible transitions in the system.

Markov chain we have a corresponding delay distribution modeling the delay for that network state. These distributions could typically look like the probability distributions in Figure 2. □

## 3. Analysis of Control Laws

In Figure 3 the control system is illustrated in a block diagram. We will analyze given linear control

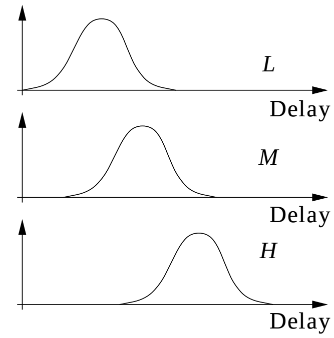


**Figure 2** The delay distributions corresponding to the states of the Markov chain in Figure 1. Here $L$ is the state for low network load, $M$ the state for medium network load, and $H$ is the state for high network load.

laws. We will assume that the sensor node is sampled regularly at a constant sampling period $h$. The actuator node is assumed to be event driven, i.e. the control signal will be used as soon as it arrives. We will analyze several different models for the communication network. The controlled process is

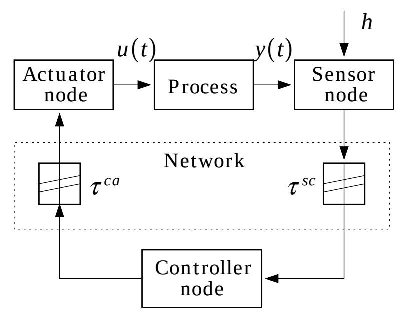


**Figure 3** Distributed digital control system with induced delays, $\tau^{sc}$ and $\tau^{ca}$.

assumed to be

$$\frac{dx}{dt} = Ax(t) + Bu(t) + v(t), \qquad (1)$$

where $x(t) \in R^n$, $u(t) \in R^m$ is the controlled input and $v(t) \in R^n$ is white noise with zero mean and covariance $R_v$. We will assume that the delay from sensor to actuator is less than the sampling period $h$, i.e. $\tau_k^{sc} + \tau_k^{ca} < h$. If this condition is not satisfied control signals may arrive at the actuator in corrupted order, which makes the analysis much harder. The influence from the network is collected in the variable $\tau_k$. For instance $\tau_k$ can be a vector with the delays in the loop, i.e. $\tau_k = [\tau_k^{sc}, \tau_k^{ca}]^T$. Discretizing (1) in the sampling instants, see Åström and Wittenmark (1990), gives

$$x_{k+1} = \Phi^p x_k + \Gamma_0^p(\tau_k)u_k + \Gamma_1^p(\tau_k)u_{k-1} + v_k. \qquad (2)$$

The output equation is

$$y_k = C^p x_k + w_k, \tag{3}$$

where $y_k \in R^p$. The stochastic processes $v_k$ and $w_k$ are uncorrelated white noise with zero mean and covariance matrices $R_1$ and $R_2$ respectively. A linear controller for this system can be written as

$$x^c_{k+1} = \Phi^c(\tau_k)x^c_k + \Gamma^c(\tau_k)y_k \tag{4}$$
$$u_k = C^c(\tau_k)x^c_k + D^c(\tau_k)y_k, \tag{5}$$

where appearance of $\tau_k$ in $\Phi^c$, $\Gamma^c$, $C^c$ or $D^c$, captures that the controller knows the network delays completely or partly. For a discussion of this see Nilsson *et al.* (1996). Examples of such controllers are given in Krtolica *et al.* (1994), Ray (1994), and Nilsson *et al.* (1996).

From (2) – (5) we see that the closed loop system can be written as

$$z_{k+1} = \Phi(\tau_k)z_k + \Gamma(\tau_k)e_k, \tag{6}$$

where

$$z_k = \begin{bmatrix} x_k \\ x^c_k \\ u_{k-1} \end{bmatrix}, \quad \text{and} \quad e_k = \begin{bmatrix} v_k \\ w_k \end{bmatrix}. \tag{7}$$

The matrices $\Phi(\tau_k)$ and $\Gamma(\tau_k)$ can be derived from (2)-(7). The variance of $e_k$ is $R = \mathrm{diag}(R_1, R_2)$.

The rest of this section investigates properties of the closed loop system (6). The analysis is made for the network models described in Section 2.

### 3.1 Network Modeled as Constant Delay

If we make the assumption that $\tau_k$ in (6) is constant for all $k$, we can use standard tools from the theory of linear time-invariant discrete time systems to analyze stability, variances of signals etc., see Åström and Wittenmark (1990). One way to make the closed loop system time invariant is to introduce buffers as discussed in Section 2.

### 3.2 Network Modeled as Consecutive Delays Being Independent

As described in Section 2, communication delays in a data network usually vary from transfer to transfer. In this situation the standard methods from linear time-invariant discrete time systems cannot be applied. There are examples where the closed loop system is stable for all constant delays, but give instability when the delay is varying. This section develops some analysis tools for systems where consecutive delays are random and independent.

***Evaluation of Covariance*** Let the closed loop system be given by (6), where $\{\tau_k\}$ is a random process independent of $\{e_k\}$. We assume that $\tau_k$ has known stationary distribution, and that $\tau_k$ is independent from sample to sample. To keep track of the noise processes we collect the random components up to time $k$ in the set

$$Y_k = \{\tau_0, ..., \tau_k, e_0, ..., e_k\}.$$

Introduce the state covariance $P_k$ as

$$P_k = \underset{Y_{k-1}}{\mathrm{E}}(z_k z_k^T), \tag{8}$$

where the expectation is calculated with respect to noise in the process and randomness in the communication delays. By iterating (8) we get

$$\begin{aligned} P_{k+1} &= \underset{Y_k}{\mathrm{E}}(z_{k+1}z_{k+1}^T) \\ &= \underset{Y_k}{\mathrm{E}}\left((\Phi(\tau_k)z_k z_k^T \Phi(\tau_k)^T + \Gamma(\tau_k)e_k e_k^T \Gamma(\tau_k)^T\right) \\ &= \underset{\tau_k}{\mathrm{E}}\left((\Phi(\tau_k)P_k\Phi(\tau_k)^T + \Gamma(\tau_k)R\Gamma(\tau_k)^T\right). \end{aligned}$$

Here we have used that $\tau_k$, $z_k$ and $e_k$ are independent, and that $e_k$ has mean zero. This is crucial for the applied technique to work and indirectly requires that $\tau_k$ and $\tau_{k-1}$ are independent. Using Kronecker products the iteration can be written as

$$\begin{aligned} \mathrm{vec}(P_{k+1}) &= \underset{\tau_k}{\mathrm{E}}(\Phi(\tau_k)\otimes\Phi(\tau_k))\,\mathrm{vec}(P_k) \\ &+ \mathrm{vec}\,\underset{\tau_k}{\mathrm{E}}(\Gamma(\tau_k)R\Gamma(\tau_k)^T) = A\,\mathrm{vec}(P_k) + G, \end{aligned} \tag{9}$$

where

$$A = \underset{\tau_k}{\mathrm{E}}(\Phi(\tau_k)\otimes\Phi(\tau_k)), \quad G = \underset{\tau_k}{\mathrm{E}}(\Gamma(\tau_k)\otimes\Gamma(\tau_k))\,\mathrm{vec}(R).$$

From (9) we see that stability in the sense of $\mathrm{E}(z_k^T z_k) < \infty$, i.e. second moment stability, is guaranteed if $\rho(\mathrm{E}(\Phi(\tau_k)\otimes\Phi(\tau_k))) < 1$, where $\rho(A)$ denotes the spectral radius of a matrix $A$. For a discussion of the connection between second moment stability and other stability concepts such as mean square stability, stochastic stability and exponential mean square stability see Ji *et al.* (1991).

***Calculation of Stationary Covariance*** If the recursion (9) is stable, $\rho(\mathrm{E}(\Phi(\tau_k)\otimes\Phi(\tau_k))) < 1$, the stationary covariance

$$P^\infty = \lim_{k\to\infty} P_k \tag{10}$$

can be found from the unique solution of the linear equation

$$\begin{aligned} \mathrm{vec}(P^\infty) &= \mathrm{E}(\Phi(\tau_k)\otimes\Phi(\tau_k))\,\mathrm{vec}(P^\infty) \\ &+ \mathrm{vec}\,\mathrm{E}(\Gamma(\tau_k)R\Gamma(\tau_k)^T). \end{aligned} \tag{11}$$

***Calculation of Quadratic Cost Function*** In LQG-control it is of importance to evaluate quadratic cost functions like $\mathrm{E}\, z_k^T S(\tau_k) z_k$. This can be done as

$$\underset{Y_k}{\mathrm{E}}\, z_k^T S(\tau_k) z_k = \mathrm{tr}\, \underset{Y_k}{\mathrm{E}}\, z_k^T S(\tau_k) z_k = \mathrm{tr}(\underset{\tau_k}{\mathrm{E}}\, S(\tau_k) \underset{Y_{k-1}}{\mathrm{E}}\, z_k z_k^T), \tag{12}$$

which as $k \to \infty$ gives

$$\lim_{k\to\infty} \mathrm{E}\, z_k^T S(\tau_k) z_k = \mathrm{tr}(\underset{\tau_k}{\mathrm{E}}\, S(\tau_k) P^\infty). \tag{13}$$

This quantity can now be calculated using (11).

Normally we want to calculate a cost function on the form $\mathrm{E}(x_k^T S_{11} x_k + u_k^T S_{22} u_k)$. As $u_k$ is not an element of $z_k$, see (7), this cost function can not always directly be cast into the formalism of (12). A solution to this problem is to rewrite $u_k$ of (5) using the output equation (3) as

$$\begin{aligned} u_k &= C^c(\tau_k) x_k^c + D^c(\tau_k)(C^p x_k + w_k) \\ &= \begin{bmatrix} D^c(\tau_k) C^p & C^c(\tau_k) & 0 \end{bmatrix} z_k + D^c(\tau_k) w_k. \end{aligned}$$

Noting that $\tau_k$ and $w_k$ are independent, and that $w_k$ has zero mean, the cost function can be written as

$$\mathrm{E}(x_k^T S_{11} x_k + u_k^T S_{22} u_k) = \mathrm{E}(z_k^T S(\tau_k) z_k) + J_1,$$

where

$$\begin{aligned} S(\tau_k) &= \begin{bmatrix} S_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &+ \begin{bmatrix} (D^c(\tau_k) C^p)^T \\ C^c(\tau_k)^T \\ 0 \end{bmatrix} S_{22} \begin{bmatrix} D^c(\tau_k) C^p & C^c(\tau_k) & 0 \end{bmatrix} \\ J_1 &= \mathrm{tr}\left( \mathrm{E}\{ D^c(\tau_k)^T S_{22} D^c(\tau_k) \} R_2 \right), \end{aligned}$$

where the first part again is on the form of (12).

### 3.3 Network Modeled Using Markov Chain

As described in Section 2 a more realistic model for communication delays in data networks is to model the delays as being random with the distribution selected from an underlying Markov process. In this section some analysis tools for these systems are developed. Variances of signals and stability of the closed loop is studied for a system with a Markov chain which makes one transition every sample. These results can be generalized to the case when the Markov chain makes two transitions every sample, this to allow for the state of the Markov chain to change between sending measurement and control signals. For details see Nilsson (1996).

***Evaluation of Covariance*** Let the closed loop system be described by (6), where $\tau_k$ is a random variable with probability distribution given by the state of a Markov chain. The Markov chain has the state $r_k \in \{1, \ldots, s\}$ when $\tau_k$ is generated. The Markov chain makes a transition between samples $k$ and $k+1$. The transition matrix for the Markov chain is $Q = \{q_{ij}\}$, $i, j \in \{1, \ldots, s\}$, where

$$q_{ij} = P(r_{k+1} = j \mid r_k = i).$$

The Markov chain is assumed to be stationary and regular, see Elliot *et al.* (1995). Introduce the Markov state probability

$$\pi_i(k) = P(r_k = i), \tag{14}$$

and the Markov state distribution

$$\pi(k) = \begin{bmatrix} \pi_1(k) & \pi_2(k) & \ldots & \pi_s(k) \end{bmatrix}.$$

The probability distribution for $r_k$ is given by the recursion

$$\begin{aligned} \pi(k+1) &= \pi(k) Q \\ \pi(0) &= \pi^0, \end{aligned}$$

where $\pi^0$ is the probability distribution for $r_0$. The state noise $e_k$ is assumed to be white with unit variance. The random components up to time $k$ are collected in

$$Y_k = \{e_0, \ldots, e_k, \tau_0, \ldots, \tau_k, r_0, \ldots, r_k\}.$$

Introduce the conditional state covariance as

$$P_i(k) = \underset{Y_{k-1}}{\mathrm{E}} \left( z_k z_k^T \mid r_k = i \right),$$

and

$$\widetilde{P}_i(k) = \pi_i(k) P_i(k).$$

The following relationship now holds for the state covariance $P(k)$

$$P(k) = \sum_{i=1}^{s} \pi_i(k) P_i(k) = \sum_{i=1}^{s} \widetilde{P}_i(k). \tag{15}$$

The following theorem gives an algorithm to evaluate $\widetilde{P}_i(k)$.

THEOREM 1
The vectorized state covariance matrix $\widetilde{\mathbf{P}}(k)$ satisfies the recursion

$$\begin{aligned} \widetilde{\mathbf{P}}(k+1) = &(Q^T \otimes I)\, \mathrm{diag}(A_i) \widetilde{\mathbf{P}}(k) \\ &+ (Q^T \otimes I)(\mathrm{diag}(\pi_i(k)) \otimes I)\mathbf{G}. \end{aligned} \tag{16}$$

where

$$
\begin{aligned}
A_i &= \mathop{\mathrm{E}}_{\tau_k} \left(\Phi(\tau_k) \otimes \Phi(\tau_k) \mid r_k = i\right), \\
G_i &= \mathop{\mathrm{E}}_{\tau_k} \left(\Gamma(\tau_k) R \Gamma^T(\tau_k) \mid r_k = i\right), \\
\widetilde{\mathbf{P}}(k) &= \begin{bmatrix} \mathrm{vec}\, \widetilde{P}_1(k) \\ \mathrm{vec}\, \widetilde{P}_2(k) \\ \vdots \\ \mathrm{vec}\, \widetilde{P}_s(k) \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathrm{vec} G_1 \\ \mathrm{vec} G_2 \\ \vdots \\ \mathrm{vec} G_s \end{bmatrix}.
\end{aligned}
$$

□

The proof of Theorem 1 is given in Nilsson (1996).

From (16) it is seen that the closed loop will be stable, in the sense that the covariance is finite, if the matrix $(Q^T \otimes I)\, \mathrm{diag}(A_i)$ has all its eigenvalues in the unit circle.

This result generalize the results in Ji *et al.* (1991) and Gajic and Qureshi (1995) in the sense that we let the Markov chain postulate the distribution of $\Phi(\tau_k)$ and $\Gamma(\tau_k)$, while Ji *et al.* (1991) and Gajic and Qureshi (1995) let the Markov chain postulate a deterministic $\Phi(\tau_k)$ and $\Gamma(\tau_k)$ for every Markov state. The results in Gajic and Qureshi (1995) are for the continuous time case.

***Calculation of Stationary Covariance*** In the stable case the recursion (16) will converge as $k \to \infty$,

$$
\widetilde{\mathbf{P}}^\infty = \lim_{k \to \infty} \widetilde{\mathbf{P}}(k).
$$

As the Markov chain is irreducible the stationary distribution $\pi^\infty$ is given uniquely by $\pi^\infty Q = \pi^\infty$. Since (16) is a stable linear difference equation it follows that $\widetilde{\mathbf{P}}^\infty$ will be the unique solution of

$$
\begin{aligned}
\widetilde{\mathbf{P}}^\infty = {} & (Q^T \otimes I)\, \mathrm{diag}(A_i) \widetilde{\mathbf{P}}^\infty \\
& + (Q^T \otimes I)(\mathrm{diag}(\pi_i^\infty) \otimes I)\mathbf{G}.
\end{aligned}
$$

The stationary value of $\mathrm{E}\, z_k z_k^T$ is given by

$$
\begin{aligned}
P^\infty &= \lim_{k \to \infty} \mathrm{E}(z_k z_k^T) \\
&= \lim_{k \to \infty} \sum_{i=1}^{s} \mathrm{E}(z_k z_k^T \mid r_k = i) P(r_k = i) = \sum_{i=1}^{s} \widetilde{P}_i^\infty,
\end{aligned}
$$

where $\widetilde{P}_i^\infty$ is the corresponding part of $\widetilde{\mathbf{P}}^\infty$.

EXAMPLE 2—VARIABLE DELAY
Consider the closed loop system in Figure 4. Assume

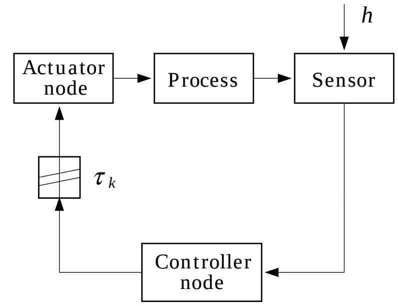


**Figure 4** Digital control system with induced delay. The probability distribution of the time-delay $\tau_k$ is determined by the state $r_k$ of the Markov chain in Figure 5

that the distribution of the communication delay $\tau_k$ from controller to actuator is given by

$$
\tau_k = \begin{cases} 0 & \text{if } r_k = 1, \\ \mathrm{rect}(d - a, d + a) & \text{if } r_k = 2, \end{cases} \tag{17}
$$

where $r_k$ is the state of the Markov chain in Figure 5, and $\mathrm{rect}(d - a, d + a)$ denotes a uniform distribution on the interval $[d - a, d + a]$. It is also assumed that $d - a > 0$ and $d + a < h$. The controlled process is

$$
\begin{cases} \dot{x} = x + u + e \\ y = x \end{cases}
$$

Let the control strategy be given by $u_k = -Lx_k$.

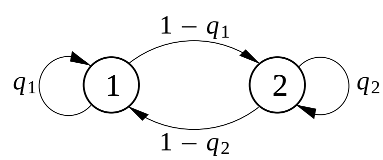


**Figure 5** Markov chain with two states. State 1 corresponds to no delay, and state 2 corresponds to a time-delay in the interval $[d - a, d + a]$, see equation (17).

Discretizing the process in the sampling instants determined by the sensor we get

$$
x_{k+1} = \Phi x_k + \Gamma_0(\tau_k) u_k + \Gamma_1(\tau_k) u_{k-1} + \Gamma_e e_k,
$$

where $\Phi = e^{Ah} = e^h$, and

$$
\Gamma_0(\tau_k) = \begin{cases} \displaystyle\int_0^h e^{As} ds B = e^h - 1, & \text{if } r_k = 1, \\ \displaystyle\int_0^{h-\tau_k} e^{As} ds B = e^{h-\tau_k} - 1, & \text{if } r_k = 2, \end{cases}
$$

$$
\Gamma_1(\tau_k) = \begin{cases} 0, & \text{if } r_k = 1, \\ \displaystyle\int_{h-\tau_k}^h e^{As} ds B = e^{h-\tau_k}(e^{\tau_k} - 1), & \text{if } r_k = 2. \end{cases}
$$

The closed loop system can then be written as

$$z_{k+1} = A(\tau_k)z_k + \Gamma(\tau_k)e_k,$$

where $z_k = \begin{bmatrix} x_k^T & u_{k-1}^T \end{bmatrix}^T$, and

$$A(\tau_k) = \begin{bmatrix} \Phi - \Gamma_0(\tau_k)L & \Gamma_1(\tau_k) \\ -L & 0 \end{bmatrix}, \quad \Gamma(\tau_k) = \begin{bmatrix} \Gamma_e \\ 0 \end{bmatrix}.$$

Stability of the closed loop system is determined by the spectral radius of $(Q^T \otimes I)\,\mathrm{diag}(A_i)$, where

$$\begin{aligned} A_1 &= A(0) \otimes A(0), \\ A_2 &= \mathop{\mathrm{E}}_{\tau_k}\{A(\tau_k) \otimes A(\tau_k) | r_k = 2\}, \end{aligned}$$

and the transition matrix for the Markov chain is

$$Q = \begin{bmatrix} q_1 & 1-q_1 \\ 1-q_2 & q_2 \end{bmatrix}.$$

Figure 6 shows the stability region in the $q_1 - q_2$ space for $h = 0.3$, $d = 0.8h$, $a = 0.1h$ and $L = 4$. This corresponds to a control close to deadbeat for the nominal case. In Figure 6 the upper left corner ($q_1 = 1$ and $q_2 = 0$) corresponds to the nominal system, i.e. a system without delay. The lower right corner ($q_1 = 0$ and $q_2 = 1$) corresponds to the system with a delay uniformly distributed on $[d-a, d+a]$. As seen from Figure 6 the controller does not stabilize the process in this case. When $q_1 = q_2$ the stationary distribution of the state in the Markov chain is $\pi_1 = \pi_2 = 0.5$. In Figure 6 this is a line from the lower left corner to the upper right corner. Note that if the Markov chain stays a too long or a too short short time in the states ( $q_1 = q_2$ near one or $q_1 = q_2$ near zero) the closed loop is not stable, but for a region in between the closed loop is stable. □

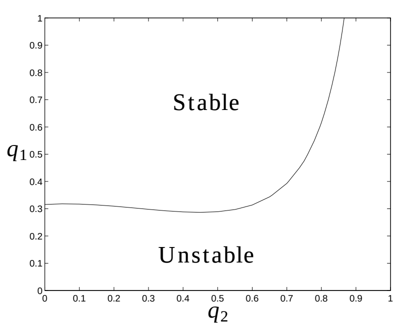


**Figure 6** Stability region in $q_1 - q_2$ space.

## 4. Conclusions and Future Work

We have used techniques from jump linear systems to analyze the performance of control systems with randomly varying time-delays. We have shown how to analyze the performance improvement given by so called time-stamps of control signals. Future work will include studies of

- Optimal controllers when the distributions of the network delays are generated from a Markov chain.
- Experimental verification of the theoretical results for systems with network delays.

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