Encoder and Decoder Design for Signal Estimation

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***Abstract*— In this paper, we study the joint design of optimal linear encoders and decoders for filtering and transmission of a signal over an additive Gaussian noise channel subject to a real-time constraint. The objective is to minimize the variance of the estimation error at the receiving end. The design problem is nonconvex, but it is shown that a global optimum can be found by solving a related two-stage problem. The first stage consists of a mixed H_2 and H_1 norm minimization problem, where the H_2 norm corresponds to the error variance in a corresponding Wiener-Kolmogorov filtering problem and the H_1 norm is induced by the channel noise. The second stage consists of a spectral factorization. The results are illustrated by a numerical example.**

I. Introduction

Classical control theory assumes that information is perfectly communicated between different systems. This assumption is, however, invalid in many situations, for example when the systems are geographically scattered. Limitations in the communication links may constitute a factor that has to be dealt with somehow, otherwise the result may be poor control performance or instability. Therefore, a lot of research efforts in the control community have lately been aimed at problems related to communication limitations. An overview of the research on networked control systems and control with data rate constraints, as well as a thorough list of references, can be found in [4] and [11] respectively.

Communication channel requirements for stability of feedback systems was given in [18], [13] and [2], among others. Fundamental limitations originating from channel constraints have been found in [9] for feedback systems and in [10] for disturbance attenuation using side information. The problem of controller and/or encoder-decoder design was treated in [1], [5] and [15] for various architectures and channel models.

A significant problem source when using information-theoretic tools and concepts for control purposes is the fact that classical communication theory does not have a delay constraint [14]. Since time delays often have a significant negative impact on control loops, block coding may not be feasible (although this may not always be necessary, see [3]). However, there are also communication problems with real-time constraints and the problem of real-time coding has been studied with increasing interest, see [8] for an overview. We believe that developments in this area could prove to be significant for communication-limited control.

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A. General Problem Description

The block diagram in Figure 1 gives a schematic representation of the problem investigated in this paper. A signal is measured, together with some additive noise, at one location. An encoder is able to filter and encode information about the measurements and send it over a noisy communication channel to a another location, where a decoder then forms an estimate of the signal. The estimation has to occur in real-time, as dictated by the transfer function P. Besides containing a fixed time delay, P may include general dynamics that the signal passes through before it is to be estimated. The task is to design the encoder and the decoder such that the estimation error becomes as small as possible.

This setup can alternatively be interpreted as the problem of designing a feed-forward compensator with access to remote and noisy measurements of the disturbance that is to be counteracted. In this context, the encoder filters the measurements and transmits information about the disturbance to the decoder/controller, which in turn can compensate. Now, P describes the propagation of the disturbance from the remote location to the point where the controller can compensate.

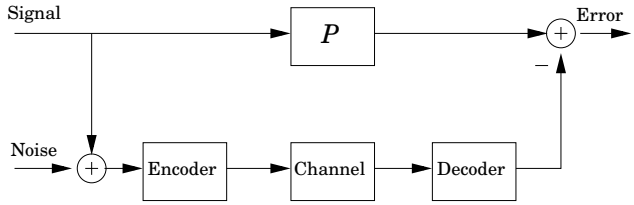

Fig. 1. Schematic illustration of the problem under consideration. The encoder and the decoder are designed to minimize the error. In the nominal case P represents a fixed time delay but more general dynamics are allowed.

B. Relations to Earlier Work

The problem setup is inspired by the work in [10], where information-theoretic tools were used to find a lower bound on the reduction of entropy rate made possible by side information communicated through a channel with given capacity. Under stationarity assumptions, this was used to derive a lower bound, which is a generalization of Bode's integral equation,

on a sensitivity-like function. Even though the problem architecture is similar in this paper, there are some important differences: The main difference is that [10] gives performance bounds for a general communication channel while this paper treats synthesis for a specific channel model. Furthermore, there are differences in the employed performance metrics: Here, the variance of the error is minimized. In [10], a lower bound is achieved on the integral of the logarithm of a sensitivity-like function. Also, in [10], a feedback controller is placed at the receiving end. The setup is generalized in this paper with the inclusion of measurement noise at the sensor, which gives an incentive to filter, as well as the possibility of general dynamics in P.

It can be noted that in the case where P is a pure time delay and the communication channel is perfect, then this problem is equivalent to the classical situation treated by Wiener-Kolmogorov filtering theory [6]. On the other hand, when there is no measurement noise and the channel is noisy, a finite time delay in P gives a real-time coding problem. The difficulty of the present problem comes from the fact that it is necessary to both filter and encode the measured signal at the same time, under a finite time constraint, in order to remove noise from measurements and channel respectively.

C. Main Result and Organization

The main result of this paper is that the joint design of an optimal linear encoder-decoder pair for a Gaussian channel can be formulated as a convex optimization problem followed by a spectral factorization. Specifically, it takes the form of a mixed H_2 and H_1 norm minimization problem, with the relative weight of the two norms determined by the channel capacity.

The rest of this paper is organized as follows: After some comments on notation, the problem formulation is given in Section II. The solution is presented in Section III and examples are found in Section IV. Some extensions to the problem are discussed in Section V. Concluding remarks are given in Section VI. We use two theorems from complex analysis, which are included in the appendix as a convenience to the reader.

D. Notation

For $1 < p \leq \infty$, we define the Lebesgue spaces $\mathbf{L_p}$ and the Hardy spaces $\mathbf{H_p}$, over the unit circle, as well as their corresponding norms $||\cdot||_p$ in the usual manner. For more details, consult a standard textbook such as [12].

To shorten notation, we omit the argument $e^{i\omega}$ of transfer functions when it is clear from the context.

II. Problem Description

The structure of the problem is shown in Figure 2. The input signals w_1, w_2, w_3 are mutually independent scalar white noise sequences with zero mean and unit variance. Every block in the figure represents a linear, time-invariant, single-input, single-output system described by a transfer function. We assume that $F, G, P \in \mathbf{H}_\infty$ and that $C, D \in \mathbf{H_2}$. These transfer functions may be rational or not. Furthermore, we assume that F and G have no common zeros on the unit circle and that the whole system is in stationarity.

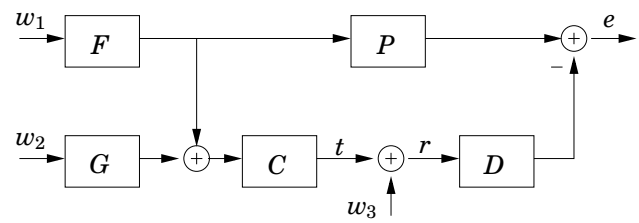

Fig. 2. Structure of the system. With F, G and P given, the objective is to design C and D such that $\mathbf{E}(e^2)$ is minimized.

The transfer functions F and G are shaping filters for the signal and the measurement noise respectively. P represents the dynamics that the signal undergoes between the points where it is measured and where it is to be estimated. Typically, P consists of a fixed time delay, but may contain more general dynamics. The encoder C and the decoder D are the design variables.

The communication channel is modeled as an additive white Gaussian noise (AWGN) channel.[1] That is,

$$r(k) = t(k) + w_3(k),$$

where $t(k)$ is the transmitted variable, $r(k)$ is the received variable, and $w_3(k)$ is the channel noise at time k. The power of the transmission is limited by a constant σ^2, that is:

$$\mathbf{E}(t(k)^2) \leq \sigma^2 \quad \text{for } k \in \mathbb{Z}.$$

The Shannon Capacity of this channel is

$$C_G = \frac{1}{2}\log_2(1+\sigma^2).$$

The objective is to minimize $\mathbf{E}(e^2)$, the variance of the estimation error. This objective, as well as the channel input constraint, can be expressed in the frequency domain as

$$\mathbf{E}(e^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |(P-DC)F|^2 + |DCG|^2 + |D|^2\, d\omega \quad (1)$$

and

$$\mathbf{E}(t^2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |C|^2 \left(|F|^2 + |G|^2\right) d\omega \leq \sigma^2. \quad (2)$$

The problem is to find $C, D \in \mathbf{H_2}$ that minimizes (1) subject to the channel constraint (2). The search is restricted to $\mathbf{H_2}$ since we are only interested in stable and causal solutions. We expect to find the optimal linear C, D, but make no claim that linear solutions are optimal per se.

[1]For the purposes of this paper, it actually doesn't matter if w_3 is Gaussian or not.

III. Solution

Due to the product DC in the integrand, the objective function (1) is clearly not convex. Also, there may be multiple minimizing solutions even for simple configurations for which the minimum can be calculated analytically (for an example, see section IV). However, we will show that it is possible to make this problem convex by solving it in two stages.

The idea is to consider the product DC as given and then to find the optimal factorization. It is shown that this problem can be solved by a spectral factorization. The solution gives a cost in terms of the product, which may then be optimized. Given the optimal product, the factorization can be applied to find optimal C and D.

The solution to the encoder-decoder factorization problem is given in Lemma 1. To understand the meaning of this lemma, note that if $B = DC$ is assumed to be fixed, the two first terms in (1) are constant. It is therefore sufficient to consider the third term. The channel input constraint is rewritten using a function H, which will be defined later.

Lemma 1: Consider $\sigma > 0$, $B \in \mathbf{H_1}$ and $H \in \mathbf{H}_\infty$ with $H^{-1} \in \mathbf{H}_\infty$. Then the minimum

$$\min_{C,D\in\mathbf{H_2}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |D|^2 d\omega \tag{3}$$

subject to the constraints

$$B = DC, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |CH|^2 d\omega \le \sigma^2 \tag{4}$$

is attained. The minimum value is

$$\frac{1}{\sigma^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |BH| d\omega\right)^2. \tag{5}$$

Moreover, if $B = 0$, then the minimum is achieved by $D = 0$ and any function $C \in \mathbf{H_2}$ that satisfies (4). Otherwise, C and D are optimal if and only if $C \in \mathbf{H_2}$, $D = BC^{-1} \in \mathbf{H_2}$ and

$$|C|^2 = \lambda^{-1}|BH^{-1}| \tag{6}$$

almost everywhere on the unit circle, with λ given by

$$\lambda = \frac{1}{2\pi\sigma^2}\int_{-\pi}^{\pi} |BH|\, d\omega.$$

Proof: If $B = 0$ the proof is trivial, so assume that B is not identically zero. Then C is not identically zero and $D = BC^{-1}$. Cauchy-Schwarz's inequality gives

$$\begin{aligned}\left(\int_{-\pi}^{\pi} |BH| d\omega\right)^2 &\le \int_{-\pi}^{\pi} |BC^{-1}|^2 d\omega \int_{-\pi}^{\pi} |CH|^2 d\omega \\ &\le 2\pi\sigma^2 \int_{-\pi}^{\pi} |BC^{-1}|^2 d\omega.\end{aligned}$$

In particular, (5) is a lower bound on (3). Equality holds if and only if $|BC^{-1}|$ and $|CH|$ are proportional almost everywhere on the unit circle and

$$\int_{-\pi}^{\pi} |CH|^2 d\omega = 2\pi\sigma^2. \tag{7}$$

Thus, the minimum value (5) is achieved if and only if $C \in \mathbf{H_2}$, $D = BC^{-1} \in \mathbf{H_2}$ and (6) holds almost everywhere on the unit circle.

It remains to show existence of such C and D. Note that $BH^{-1} \in \mathbf{H_1}$ is not identically zero. Hence, by Theorem 2 (in appendix), $\log|BH^{-1}| \in \mathbf{L_1}$. It follows by Theorem 3 (in appendix) that there exists a function $C \in \mathbf{H_2}$, with $C(z) \neq 0$ for $|z| > 1$, that satisfies (6) almost everywhere on the unit circle. Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |BC^{-1}|^2 d\omega = \frac{\lambda^2}{2\pi}\int_{-\pi}^{\pi} |CH|^2 d\omega = \lambda^2\sigma^2 < \infty$$

so $BC^{-1} \in \mathbf{L_2}$. Since $B \in \mathbf{H_1}$ and C is analytic and nonzero for all $|z| > 1$ it follows that $BC^{-1} \in \mathbf{H_2}$. ■

If the product of an encoder and a decoder is given, Lemma 1 shows how to factorize that product so that the third term in (1) is minimized. Inserting the minimum cost (5), the objective (1) becomes convex in the product and the problem described in Section II can be solved using convex optimization techniques. This result is shown in the following theorem, which is the main result of this paper.

Theorem 1: Assume that $\sigma^2 > 0$, that $F, G, P \in \mathbf{H}_\infty$, where F and P are not identically zero, and that

$$0 < \varepsilon \le |F(e^{i\omega})|^2 + |G(e^{i\omega})|^2, \ \forall \omega \in [-\pi, \pi). \tag{8}$$

The minimum

$$\min_{C,D\in\mathbf{H_2}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |(P - DC)F|^2 + |DCG|^2 + |D|^2\, d\omega \tag{9}$$

subject to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |C|^2 \left(|F|^2 + |G|^2\right) d\omega \le \sigma^2 \tag{10}$$

is attained and is equal to the minimum of the convex optimization problem

$$\begin{aligned}\min_{B\in\mathbf{H_2}} &\frac{1}{2\pi}\int_{-\pi}^{\pi} |(P - B)F|^2 + |BG|^2\, d\omega \\ &+ \frac{1}{\sigma^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |B|\sqrt{|F|^2 + |G|^2} d\omega\right)^2,\end{aligned} \tag{11}$$

which is attained by a unique minimizer.

Further, suppose $B \in \mathbf{H_2}$ minimizes (11). If $B = 0$, then (9) subject to (10) is minimized by $D = 0$ and any function $C \in \mathbf{H_2}$ that satisfies (10). Otherwise, C and D minimize (9) subject to (10) if and only if $C \in \mathbf{H_2}$, $D = BC^{-1} \in \mathbf{H_2}$ and

$$|C|^2 = \frac{|B|}{\lambda\sqrt{|F|^2 + |G|^2}} \tag{12}$$

almost everywhere on the unit circle, with λ given by

$$\lambda = \frac{1}{2\pi\sigma^2}\int_{-\pi}^{\pi} |B|\sqrt{|F|^2 + |G|^2}\, d\omega. \tag{13}$$

Proof: We start by rewriting the the channel input constraint (10). By (8) and Theorem 3 (in appendix),

there exists a function $H \in \mathbf{H_2}$ that satisfies $H(z) \neq 0$ for $|z| > 1$ and

$$|H|^2 = |F|^2 + |G|^2 \tag{14}$$

almost everywhere on the unit circle. Since $F, G \in \mathbf{H}_\infty$ it follows that $H \in \mathbf{H}_\infty$. Moreover, it follows from (8) that $H^{-1} \in \mathbf{H}_\infty$. With this H, (10) is equivalent to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |CH|^2 \, d\omega \leq \sigma^2. \tag{15}$$

Now, define the sets

$$\Theta = \{(C, D) : C, D \in \mathbf{H_2},\ (15)\}$$
$$\Theta_B = \{(C, D) : C, D \in \mathbf{H_2},\ (15),\ B = DC\}$$

and the functional

$$\varphi(C, D) = ||(P - DC)F||_2^2 + ||DCG||_2^2 + ||D||_2^2.$$

The infimum of (9) subject to (10) can be written

$$\begin{aligned}
&\inf_{C,D\in\Theta} \varphi(C, D) = \inf_{B\in\mathbf{H_1}} \inf_{C,D\in\Theta_B} \varphi(C, D) \\
&= \inf_{B\in\mathbf{H_1}} \left(||(P - B)F||_2^2 + ||BG||_2^2 + \inf_{C,D\in\Theta_B} ||D||_2^2 \right) \\
&= \inf_{B\in\mathbf{H_1}} ||(P - B)F||_2^2 + ||BG||_2^2 + \frac{1}{\sigma^2}||BH||_1^2 \\
&= \inf_{B\in\mathbf{H_1}} \psi(B)
\end{aligned} \tag{16}$$

The first equality comes from the fact that a product of two functions in $\mathbf{H_2}$ is in $\mathbf{H_1}$, and that any function in $\mathbf{H_1}$ can be written as a product of two functions in $\mathbf{H_2}$. In the third equality, Lemma 1 was applied to perform the inner minimization.

We will now show that the minimum in (16) is attained by a unique $B \in \mathbf{H_2}$. To this end, perform a completion of squares:

$$\begin{aligned}
\psi(B) &= ||(P - B)F||_2^2 + ||BG||_2^2 + \frac{1}{\sigma^2}||BH||_1^2 \\
&= ||BH - H^{-*}F^*FP||_2^2 + \frac{1}{\sigma^2}||BH||_1^2 + \text{const.}
\end{aligned}$$

Let $X = BH \in \mathbf{H_1}$ and $R = H^{-*}F^*FP \in \mathbf{L}_\infty$. Minmizing $\psi(B)$ is then equivalent to minimizing

$$\rho(X) = ||R - X||_2^2 + \frac{1}{\sigma^2}||X||_1^2$$

over $X \in \mathbf{H_1}$. However, since we want to minimize ρ it is enough to consider X with $\rho(X) \leq \rho(0) = ||R||_2^2$. Hence,

$$\begin{aligned}
||X||_2 &\leq ||R - X||_2 + ||R||_2 \\
&\leq \sqrt{\rho(X)} + ||R||_2 \leq 2||R||_2 = r.
\end{aligned}$$

Now, in the weak topology, $\rho(X)$ is lower semicontinuous on L_2 and the set $\{X : ||X||_2 \leq r\}$ is compact. This proves the existence of a minimum. Moreover, $\rho(X)$ is strictly convex, and thus the minimum is unique.

Moreover, since $||X||_2 \leq r$, we can restrict the search to $X \in \mathbf{H_2}$ without loss of generality. From $H^{-1} \in \mathbf{H}_\infty$ it follows that $B = H^{-1}X \in \mathbf{H_2}$ and that (16) is equal to (11).

Since the minimum is attained in (16), this is also true for the minimum (9) subject to (10), since they are equal. The optimality condition (12) follows from the application of Lemma 1, using that $|H| = \sqrt{|F|^2 + |G|^2}$ almost everywhere on the unit circle. ■

Theorem 1 shows that it is possible to solve the problem described in section II by the following procedure: First, minimize (11). In practice, this is done approximately using a finite basis representation of B and sum approximations of the integrals. This minimization can then be cast a quadratic program with second-order cone constraints. Then calculate λ according to (13) and perform a spectral factorization to find $H \in \mathbf{H}_\infty$ with $H^{-1} \in \mathbf{H}_\infty$ that satisfies (14). Finally, apply another spectral factorization to determine $C \in \mathbf{H_2}$ with $C(z) \neq 0$ for $|z| > 1$ such that

$$|C|^2 = \lambda^{-1}|BH^{-1}|$$

holds almost everywhere on the unit circle, and set $D = BC^{-1}$.

The cost function (11) consists of two terms, which can be given the following interpretations: The first is equal to the cost in the situation where the channel is noise-free and has unlimited capacity. Hence, this is the error variance in the Wiener-Kolmogorov filtering problem [6]. The second term is the error induced by the channel noise. It is interesting to note that the first term is a 2-norm function of the decision variable B, while the second term is a weighted 1-norm of B. Thus, the problem is a mixed norm minimization program with the parameter σ^2 determining the relative importance of the two terms.

Rewriting the weight in terms of the channel's Shannon Capacity C_G, the problem becomes

$$\min_{B\in\mathbf{H_2}} ||R - BH||_2^2 + \frac{1}{2^{2C_G} - 1}||BH||_1^2$$

with $R = H^{-*}F^*FP \in \mathbf{L}_\infty$. It is noted that the problem approaches the Wiener-Kolmogorov filtering problem as $C_G \to \infty$. On the other hand, as $C_G \to 0$, the weight grows large and the optimal B approaches 0. This is because the channel noise dominates the transmitted signal.

It was noted earlier that the solution is not unique. To clarify, the optimal B is unique but there are multiple factorizations of B into C and D that achieve the optimal value. For example, a second solution is trivially found by changing the sign of both C and D. Moreover, in the proof of Lemma 1, C is always chosen to be nonzero for every $|z| > 1$. However, if B has time delays or zeroes outside the unit circle, it is possible to put these in either C or D.

IV. Examples

A. Static Case

Suppose $P = z^{-d}$, $F = \sigma_F > 0$, $G = \sigma_G > 0$. This simple case can be solved analytically. Optimal solutions are given by

$$C(z) = (-1)^k \sqrt{\frac{\sigma^2}{\sigma_F^2 + \sigma_G^2}} z^{-m}$$

$$D(z) = (-1)^k \frac{\sigma_F^2}{\sigma^2 + 1} \sqrt{\frac{\sigma^2}{\sigma_F^2 + \sigma_G^2}} z^{-n}$$

where $k \in \{0,1\}$, $m,n \in \mathbb{Z}$ and $m + n = d$. The minimum value is

$$\frac{\sigma_F^2}{\sigma^2 + 1} + \frac{\sigma^2 \sigma_F^2 \sigma_G^2}{(\sigma^2 + 1)(\sigma_F^2 + \sigma_G^2)}.$$

B. Numerical Example

Suppose $P = z^{-2} + 0.5z^{-7}$, $F = \frac{1}{z-0.5}$, $G = 1$ and $\sigma = 1$. We parameterize $X = BH$ as an FIR filter:

$$X(z) = \sum_{k=0}^{N_x} x_k z^{-k}$$

where $N_x = 30$ seems to be sufficiently large for this example. The minimization is implemented in Matlab using Yalmip [7]and SeDuMi [16], with a grid distance of 0.001 used for numerical computation of the integrals. Figure 3 shows the resulting impulse response of B. Note that the two peaks in the impulse response corresponds to the non-zero coefficients of P.

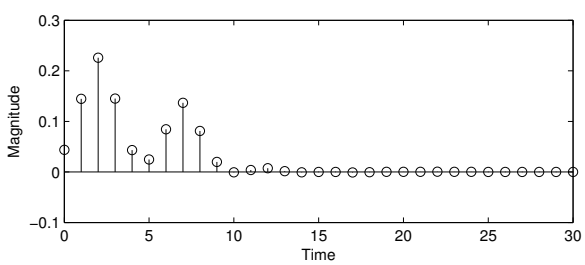

Fig. 3. Impulse response of B, the product of the encoder and the decoder, in the numerical example.

λ is calculated using (13) and a spectral factorization is performed to determine H according to (14). The spectrum of C is parameterized as

$$|C|^2(\omega) = \tilde{c}_0 + \sum_{k=1}^{N_c} \tilde{c}_k \left(e^{ki\omega} + e^{-ki\omega}\right)$$

where $N_c = 30$ seems to be enough for this example. The coefficients are found by solving a least-squares problem (in the form of an overdetermined equation system corresponding to (12)). Finally, C is obtained through spectral factorization and $D = BC^{-1}$. The impulse responses of C and D are shown in Figures 4 and 5, respectively. The minimum value for this problem is 1.00.

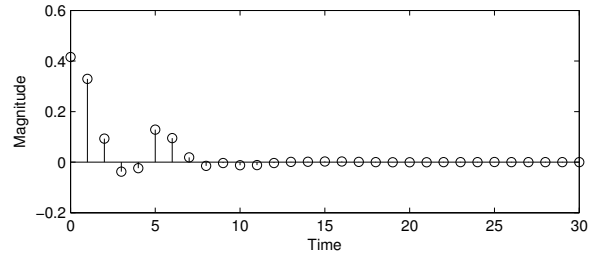

Fig. 4. Impulse response of the encoder C in the numerical example.

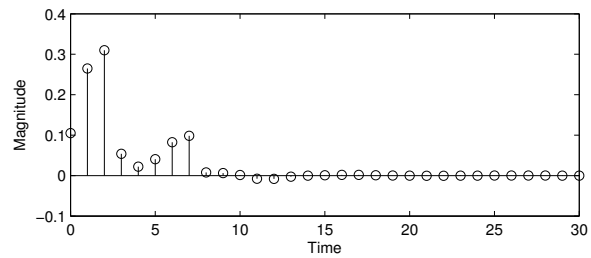

Fig. 5. Impulse response of the decoder D in the numerical example.

V. Extensions

Consider the more general problem setup depicted in Figure 6. The difference from the original problem is that the channel noise is not assumed to be white and that the error signal is frequency weighted. The solution in section III is easily modified to handle these extensions, as long as it is assumed that $N \in \mathbf{H}_\infty$ and $S \in \mathbf{H}_\infty$ with $S^{-1} \in \mathbf{H}_\infty$. Note that S could represent a frequency weighting of the error or, in a feed-forward context, the sensitivity function of the closed-loop system that is affected by the disturbance.

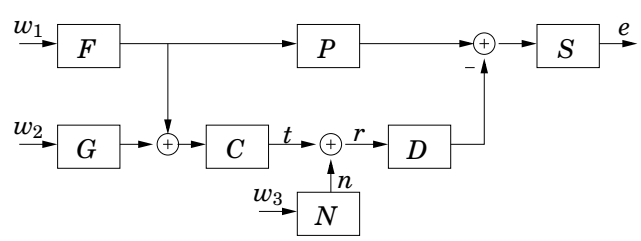

Fig. 6. Extended problem setup. The channel noise is shaped by N and the error is frequency weighted by S.

VI. Conclusions

In this paper, we have studied the joint design of optimal linear encoder and decoder pairs for filtering and transmitting a signal over a Gaussian additive noise channel. The objective of the design is to minimize the variance of the estimation error. The main result is that this problem can be formulated as a convex optimization problem. More specifically, it is a mixed $\mathbf{H_1}$ and $\mathbf{H_2}$ problem.

The results given in [10] are more general in the sense that they hold for any communication channel,

although this paper generalizes the problem setting to include measurement noise as well as general dynamics in P. An important difference between the two frameworks is the performance metric. Another difference is the inclusion of feedback in the receiving end. Using this feedback, the authors of [10] were also able to construct (and achieve) a lower bound on the integral of the square of the sensitivity-like function. It would be interesting to see how the use of feedback in a similar manner could be used to decrease the variance further than what was achieved here. Comparing with the results presented in [10] for the Gaussian channel (with the feedback not taken into consideration) we note that the results coincide when $G = 0$ and F is constant. When F is not constant, the solutions are different, due to the different performance metrics.

This work provides many topics for further research, that we plan to investigate in the future:

- Can the error be further decreased (the disturbance be further attenuated) by the use of feedback at the receiving end?
- Under what conditions are linear solutions optimal? If they are not, is there a good method of finding (sub)optimal nonlinear solutions?
- If P, F and G are rational, will the optimal C and D also be rational?
- Is the method used in this paper applicable to other structures, such as feedback loops?

Appendix

This appendix contains two theorems from complex analysis. The first consists of one of the results stated in Theorem 17.17 in [12].

Theorem 2: Suppose $0 < p \leq \infty$, $X \in \mathbf{H_p}$, and X is not identically zero. Define

$$\bar{X}(e^{i\omega}) = \lim_{r \to 1^+} X(re^{i\omega}).$$

Then $\log|\bar{X}| \in \mathbf{L_1}$.

The following theorem is a generalization of the Fejér-Riesz Theorem and can be found in [17].

Theorem 3 (Szegő): Suppose that $f(\omega)$, defined for $\omega \in [-\pi, \pi]$, is a non-negative function that is integrable in Lebesgue's sense and that

$$\int_{-\pi}^{\pi} \log f(\omega)\, d\omega > -\infty.$$

Then there exists $X \in \mathbf{H_2}$ such that $X(z) \neq 0$ for $|z| > 1$ and for almost all $\omega \in [-\pi, \pi]$ it holds that

- $X(e^{i\omega}) = \lim_{r \to 1^+} X(re^{i\omega})$ exists
- $f(\omega) = |X(e^{i\omega})|^2$.

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