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2008

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CODEN:LUTEDX/(TEAT-7168)/1-8/(2008)

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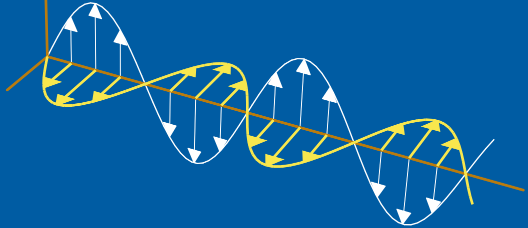

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Editor: Gerhard Kristensson

Abstract

In wave propagation problems thin structures are often replaced by boundaries of zero thickness, in order to reduce the numerical mesh. In this simplification of the geometry it is crucial that the approximate boundary condition compensates for the reduction of thickness of the structure. In this paper a boundary condition originally introduced by Mitzner is compensated for this reduction. The new compensated Mitzner condition is more general and accurate than the common impedance boundary condition. For frequency domain solvers it is as easy to implement as the impedance boundary condition. The condition is tested and compared with the impedance boundary condition for planar and cylindrical structures. The new condition is exact at normal incidence on a planar structure.

1 Introduction

Thin conductive structures are hard to handle for most numerical methods for electromagnetic wave propagation since a very fine mesh is required to resolve the fields in the structure. One way to get around this problem is to introduce an approximate boundary condition (BC) and in that way exclude the interior of the structure from the computational domain. A simple BC for a thin non-magnetic dielectric structure is the impedance BC

$$\begin{aligned} \hat{\boldsymbol{n}} \times \boldsymbol{E}^+ &= \hat{\boldsymbol{n}} \times \boldsymbol{E}^- = \hat{\boldsymbol{n}} \times \boldsymbol{E} \\ \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) &= Z_L^{-1} \hat{\boldsymbol{n}} \times (\boldsymbol{E} \times \hat{\boldsymbol{n}}) \end{aligned} \tag{1.1}$$

where $\hat{\boldsymbol{n}}$ is the normal to the upper surface of the structure and $+$ and $-$ refer to the upper and lower surface. The layer impedance $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$ is often used, *cf.*, [1], where ε and d are the complex permittivity and thickness of the layer, respectively. This BC has been analyzed in the literature, *e.g.*, [3] and [4] and is also utilized in commercial softwares. A similar BC is applicable for magnetic structures

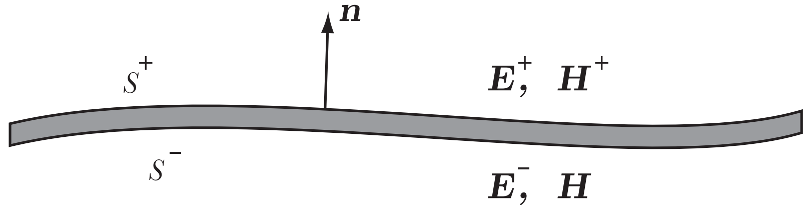

Figure 1: The thin structure.

with a complex permeabilty μ. The BC is expressed in terms of the magnetic layer admittance *cf.*, [1]. $Y_m = -i\omega(\mu - \mu_0)d$ such that

$$\begin{aligned} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^+ - \boldsymbol{E}^-) &= -Y_m \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) &= \boldsymbol{0} \end{aligned} \tag{1.2}$$

Both of the boundary conditions above assume that the attenuation depth is much larger than the thickness of the structure and that the absolute value of the wavenumber in the structure is much larger than in the surrounding media.

The approximate BC presented in this paper is valid even when the thickness is not much smaller than the attenuation depth. The condition is based on the following approximate BC for a single layer structure

$$\begin{aligned} \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) &= -i\omega\varepsilon\kappa \hat{\boldsymbol{n}} \times ((\boldsymbol{E}^+ + \boldsymbol{E}^-) \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times (\boldsymbol{E}^+ - \boldsymbol{E}^-(\boldsymbol{r})) &= i\omega\mu\kappa \hat{\boldsymbol{n}} \times ((\boldsymbol{H}^+ + \boldsymbol{H}^-) \times \hat{\boldsymbol{n}}) \end{aligned} \tag{1.3}$$

where ε and μ are the complex permittivity and permeability of the layer. The quantity κ is defined in the next section. This BC was introduced in a paper by Mitzner [2]. It seems that it has not been frequently used, even though in many cases it is superior to the conditions in Eqs. (1.1) and (1.2). A plausible explanation is that this BC is not compensated for the reduction of the structure to a boundary of zero thickness. Hence the Mitzner BC is only very accurate if the thickness of the structure is not reduced. This is in contrast to the conditions in Eqs. (1.1) and (1.2) where the compensation is done by using $\varepsilon - \varepsilon_0$ and $\mu - \mu_0$ instead of ε and μ. The merit of this paper is to compensate the BC in Eq. (1.3) for the thickness of the layer, and in that way improve the validity and accuracy of the BC. The new condition has been tested numerically for two different cases, a planar slab and a cylindical shell. In these tests it has proven to be superior to the condition in Eq. (1.1), in particular for thick structures.

2 Prerequisites

Consider a thin structure that consists of one or several layers. The structure is characterized by its complex permittivity ε, permeability μ, and thickness d. All three of these quantities may vary along the structure. The upper surface of the structure is denoted S^+ and the lower surface S^-. The outward directed normal to the upper surface is $\hat{\boldsymbol{n}}$ and the thickness is the distance along the straight line that runs in the direction of $\hat{\boldsymbol{n}}$ from a point on S^- to the corresponding point on S^+. The wavenumber in the structure is given by $k_c = \omega\sqrt{\varepsilon\mu}$. The following conditions were assumed by Mitzner, and they are also adopted here:
Condition 1 The radius of the curvatures of the structure are much larger than the attenuation length in the structure.
Condition 2 The attenuation length is much smaller than the distance to the nearest significant source.
Condition 3 The wavenumbers in the surrounding media are much smaller than the wavenumber in the structure.

Condition 4 The thickness of the structure is much smaller than the minimum of the radius of curvature of the structure.

To these conditions is added the condition that the permittivity, permeability, and thickness of the structure vary slowly in the directions tangential to the surface of the structure.

3 Single layer structure

First a structure consisting of one layer with $z-$independent permittivity and permeability is considered. The structure occupies the region $0 < z < d$. Inside the structure the electric field satisfies

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}) + k_c^2 \boldsymbol{E}(\boldsymbol{r}) = \nabla_t^2 \boldsymbol{E}(\boldsymbol{r}) + \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r})}{\partial z^2} + k_c^2 \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$$

Based on the conditions above it is assumed that the electric field varies much faster in the z-direction than in the x and y directions and hence the $\nabla_t^2 \boldsymbol{E}$ term can be neglected. That leads to

$$\frac{\partial^2 \boldsymbol{E}(\boldsymbol{r})}{\partial z^2} + k_c^2 \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0} \tag{3.1}$$

with solution

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{\alpha}(x,y) e^{ik_c z} + \boldsymbol{\beta}(x,y) e^{-ik_c z}$$

where the time-dependence $e^{-i\omega t}$ is assumed. From the Ampére and induction laws the BC used by Mitzner follows

$$\begin{aligned} \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) &= -i\omega\varepsilon\kappa \hat{\boldsymbol{n}} \times ((\boldsymbol{E}^+ + \boldsymbol{E}^-) \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times (\boldsymbol{E}^+ - \boldsymbol{E}^-) &= i\omega\mu\kappa \hat{\boldsymbol{n}} \times ((\boldsymbol{H}^+ + \boldsymbol{H}^-) \times \hat{\boldsymbol{n}}) \end{aligned} \tag{3.2}$$

where

$$\kappa = \frac{e^{ik_c d} + e^{-ik_c d} - 2}{ik_c (e^{ik_c d} - e^{-ik_c d})} = \frac{1}{k_c} \tan\left(\frac{k_c d}{2}\right)$$

An equivalent representation is

$$\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^+ \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^+ \end{pmatrix} = \frac{1}{1 + \omega^2 \varepsilon\mu\kappa^2} \begin{pmatrix} 1 - \omega^2\varepsilon\mu\kappa^2 & -2i\omega\mu\kappa \\ -2i\omega\varepsilon\kappa & 1 - \omega^2\varepsilon\mu\kappa^2 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^- \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^- \end{pmatrix} \tag{3.3}$$

In the case of a dielectric layer for which $|k_c d| \ll 1$ this leads to

$$\begin{aligned} \hat{\boldsymbol{n}} \times (\boldsymbol{H}^+ - \boldsymbol{H}^-) &= -i\omega\varepsilon d \hat{\boldsymbol{n}} \times (\times \boldsymbol{E} \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{E}^+ = \hat{\boldsymbol{n}} \times \boldsymbol{E}^- &= \hat{\boldsymbol{n}} \times \boldsymbol{E} \end{aligned} \tag{3.4}$$

In order to replace the structure with a layer of zero thickness one needs to modify Eq. (3.4) and that leads to Eq. (1.1) with the impedance $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$. In the next section the corresponding modification of Eq. (3.3) is presented.

3.1 Reduction to a boundary of zero thickness

In Eq. (3.3) the original layer has been replaced by a BC that excludes the layer from the computational domain, but keeps the thickness d of the structure. It is often convenient to replace the structure with a layer of zero thickness. One has to be aware of that the replacement alters the geometry and therefore affects the numerical solution. It is possible to reduce the errors by compensating for the change in geometry. In this paper the compensation is done by replacing the original structure with a layer of thickness d with the same material parameters as in the region above S^+ and an interface of zero thickness at $z = 0$. Denote the electric and magnetic fields on the side $z = 0^+$ of the zero thickness layer $\boldsymbol{E}_1$ and $\boldsymbol{H}_1$. To find the condition between $\hat{\boldsymbol{n}} \times \boldsymbol{E}_1$, $\hat{\boldsymbol{n}} \times \boldsymbol{H}_1$, $\hat{\boldsymbol{n}} \times \boldsymbol{E}^-$, and $\hat{\boldsymbol{n}} \times \boldsymbol{H}^-$ one introduces the two matrices

$$\boldsymbol{A} = \frac{1}{1+\omega^2\varepsilon\mu\kappa^2}\begin{pmatrix} 1-\omega^2\varepsilon\mu\kappa^2 & -2i\omega\mu\kappa \\ -2i\omega\varepsilon\kappa & 1-\omega^2\varepsilon\mu\kappa^2 \end{pmatrix}$$
$$\boldsymbol{A}_1 = \frac{1}{1+\omega^2\varepsilon_1\mu_1\kappa_1^2}\begin{pmatrix} 1-\omega^2\varepsilon_1\mu_1\kappa_1^2 & -2i\omega\mu_1\kappa_1 \\ -2i\omega\varepsilon_1\kappa_1 & 1-\omega^2\varepsilon_1\mu_1\kappa_1^2 \end{pmatrix}$$

where ε_1, μ_1 and $\kappa_1 = k_1^{-1}\tan(k_1 d/2)$ are the quantities in the region $z > d$. It is now seen that

$$\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^1 \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^1 \end{pmatrix} = \boldsymbol{A}_1^{-1}\boldsymbol{A}\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^- \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^- \end{pmatrix} \tag{3.5}$$

This BC is reduced to the impedance BC in Eq. (1.1) with the impedance $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$ when $k_c d \ll 1$. If instead the zero thickness boundary is placed at a position $0 < z_0 \leq d$ then the condition reads

$$\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^1 \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^1 \end{pmatrix} = \boldsymbol{A}_{\text{above}}^{-1}\boldsymbol{A}\boldsymbol{A}_{\text{below}}^{-1}\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}^2 \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}^2 \end{pmatrix} \tag{3.6}$$

where in this case $\boldsymbol{E}^1$, $\boldsymbol{H}^1$ are the fields at $z = z_0^+$ and $\boldsymbol{E}^2$, $\boldsymbol{H}^2$ are the fields at $z = z_0^-$. Here $\boldsymbol{A}_{\text{above}}$ is the matrix corresponding to $\boldsymbol{A}_1$ with parameters for the region $z > d$ and thickness $d - z_0$ and $\boldsymbol{A}_{\text{below}}$ is the corresponding matrix with parameters in the region $z < 0$ and thickness z_0.

The compensated Mitzner BC is exact at normal incidence. When the angle of incidence increases there will be an error in the phase of the reflected and transmitted fields. However these errors are always smaller than the corresponding errors for the Mitzner BC and the impedance BC.

4 Approximate boundary conditions for multilayered structures

Consider a thin structure with N layers. The prerequisites are otherwise the same as before. The lower surface of the structure is at $z = z_0 = 0$ and the upper at

$z = z_N = d$. The thickness of layer number n is d_n and thus $d = \sum_{n=1}^{N} d_n$. From Eq. (3.3) the BC for the multilayered structure is

$$\begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}_T^+ \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}_T^+ \end{pmatrix} = \prod_{n=1}^{N} P_n \begin{pmatrix} \hat{\boldsymbol{n}} \times (\boldsymbol{E}_T^- \times \hat{\boldsymbol{n}}) \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}_T^- \end{pmatrix}$$

where the index n refers to the layer number n and

$$P_n = \frac{1}{1+\omega^2 \varepsilon_n \mu_n \kappa_n^2} \begin{pmatrix} 1 - \omega^2 \varepsilon_n \mu_n \kappa_n^2 & -2i\omega\mu_n\kappa_n \\ -2i\omega\varepsilon_n\kappa_n & 1-\omega^2\varepsilon_n\mu_n\kappa_n^2 \end{pmatrix}$$

$$\kappa_n = \frac{1}{k_n} \tan\left(\frac{k_n d_n}{2}\right)$$

Equations (3.5) and (3.6) are valid if the matrix $\boldsymbol{A}$ is replaced by

$$\boldsymbol{A} = \prod_{n=1}^{N} P_n$$

5 Numerical examples

Two numerical examples are given here. In both examples the thin structure is non-magnetic $\mu = \mu_0$. The conductivity of the structures is σ and the relative permittivity ε_r. The frequency is 1 GHz in all examples. Results referring to different boundary conditions are denoted by **M** for the BC by Mitzner, Eq. (3.2), **CM** for the compensated Mitzner BC, Eq. (3.5), and **IMP** for the impedance BC in Eq. (1.1).

5.1 Plane wave incidence at a planar layer

The layer occupies $0 < z < d$ and the equivalent boundary is placed at $z = 0$. A plane wave $\boldsymbol{E}(x, y, z) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ is incident on the structure. The angle of incidence is defined as the angle between $\boldsymbol{k}$ and $\hat{\boldsymbol{z}}$, *i.e.*, $\alpha = \arccos(\hat{\boldsymbol{n}} \cdot \boldsymbol{k}/|\boldsymbol{k}|)$. The incident plane wave is either a TE-wave with the electric field parallel to the surface, or a TM-wave with the magnetic field parallel to the surface. In all examples the impedance BC is quite good for very thin layers as long as $|k_c|d \ll 1$ but fails when $|k_c|d > 1$. At normal incidence ($\alpha = 0$) the compensated Mitzner BC gives the correct values regardless of thickness d, whereas the Mitzner BC gives reflection and transmission coefficients, R and T, with correct absolute values but erroneous phases.

The relative error in the absorbed power is defined by

$$\frac{|P_{\rm bc} - P_{\rm exact}|}{P_{\rm exact}}$$

where $P_{\rm bc}$ is the absorbed power calculated from one of the approximate boundary conditions and $P_{\rm exact}$ is the exact absorbed power calculated from the full problem.

The relative error is calculated for the three boundary conditions in Eqs. (1.1), with $Z_L^{-1} = -i\omega(\varepsilon - \varepsilon_0)d$, (3.3), and (3.5) at different angles of incidence, thicknesses d, permittivities ε_r and conductivities σ. The relative errors are given in table 1 and 2 for two cases. The compensated Mitzner BC and the Mitzner BC gives the same result for this case. However, the transmission coefficient has large errors in the phase in the Mitzner BC, even at normal incidence. The corresponding error is very small in the compensated Mitzner BC. If there had been a source, or reflecting structure in the region $z > d$ then the Mitzner BC would give large errors due to this phase shift. At very high values of σ the compensated Mitzner BC give very small relative errors for the absorbed power, for all angles of incidence, whereas the impedance BC gives very large relative errors. The same is true when the thickness of the layer is greater than the attenuation distance.

Table 1: Relative error absorbed power. $\sigma = 10$ S/m, $\varepsilon_r = 5$, $\alpha = 30°$, $f = 1$ GHz. The skin depth is $\delta = 5$ mm and the impedance BC fails completely for $d > \delta$.

Polarization	Method	d=0.001 m	d=0.005 m	d=0.01 m	d=0.1 m
TE	CM	$4.9 \cdot 10^{-4}$	$3.9 \cdot 10^{-4}$	$0.63 \cdot 10^{-4}$	$0.13 \cdot 10^{-4}$
TE	IMP	$7 \cdot 10^{-4}$	0.09	0.47	0.95
TM	CM	$1 \cdot 10^{-4}$	$15 \cdot 10^{-4}$	$5 \cdot 10^{-4}$	$6 \cdot 10^{-4}$
TM	IMP	$4 \cdot 10^{-4}$	0.092	0.41	0.95

Table 2: Relative error absorbed power. $\sigma = 1$ S/m, $\varepsilon_r = 5$, $\alpha = 45°$, $f = 1$ GHz. The skin depth is $\delta = 16$ mm and the impedance BC fails completely for $d > \delta$.

Polarization	Method	d=0.001 m	d=0.005 m	d=0.01 m	d=0.1 m
TE	CM	$7 \cdot 10^{-4}$	$2.9 \cdot 10^{-4}$	0.0026	0.0049
TE	IMP	0.0011	0.0001	0.022	0.83
TM	CM	0.004	0.012	0.025	0.0044
TM	IMP	$4 \cdot 10^{-4}$	0.012	0.050	0.81

5.2 Scattering from circular cylindrical shell

The cylinder consists of a shell with inner radius a, outer radius $a+d$, conductivity σ and relative permittivity ε_r. There is vacuum for $r > a+d$ and $r < a$. The incident plane wave is either a TE-wave, *i.e.*, with the E-field along the symmetry axis, or a TM-wave with the H-field along the symmetry axis. The boundary conditions have been tested for a large number of cases and in general the compensated Mitzner BC gives the best results. Both the Mitzner BC and the impedance BC are much less robust and sometimes give large errors. The examples, see Figs. 2–5, in this paper are chosen since similar examples are used in the papers [1] and [3]. The exact resuts were calculated using Mie theory. For large values of the conductivity the compensated Mitzner BC and the Mitzner BC give very accurate results for both TE and TM whereas the impedance BC gives accurate result for TM but not as accurate for TE. The examples show the radar cross section (RCS), or equivalently back scattering cross section. In general the composite Mitzner BC gives an accurate near field results when the backscattering cross section are accurate.

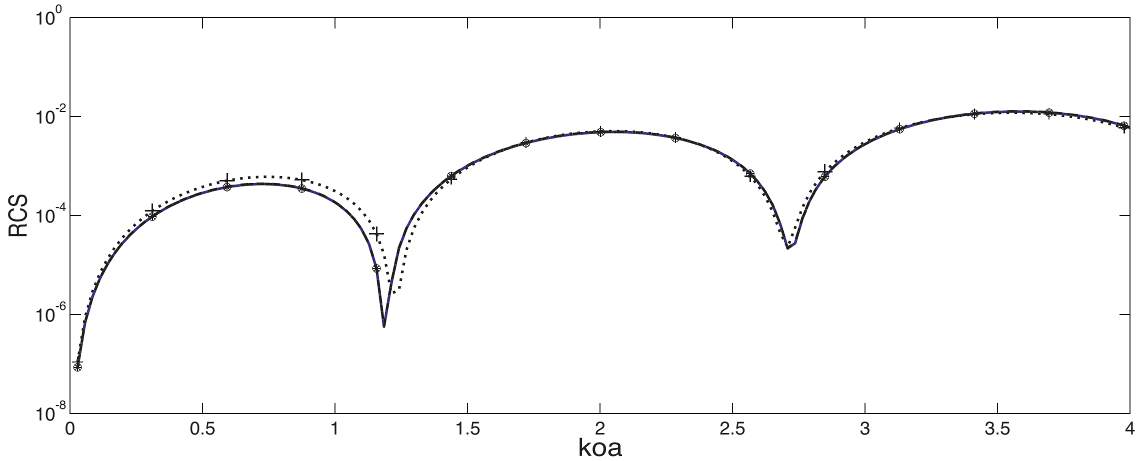

Figure 2: RCS from a circular cylinder shell with radius a, thickness $d = 0.01a$, permittivity $\varepsilon_r = 5$ and $\sigma = 0$. k_0 is the vacuum wavenumber. The incident plane wave is a TE-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) and impedance BC (dash-dotted line with $*$) give almost the exact values. The Mitzner BC (dotted curve with $+$) is slightly off.

6 Conclusions and comments

When approximate boundary conditions for thin structures are used it is often assumed that the structure can be replaced by a layer of zero thickness. This is the case for the most common impedance boundary conditions. A BC for thin structures was suggested by Mitzner [2] that was more general than the impedance BC. However, the condition does not take into account a reduction to zero thickness and that leads to errors that increases with thickness of the structure. In this paper the Mitzner BC is modified such that the structure can be replaced by a layer of zero thickness. This makes the Mitzner BC more general than the traditional impedance BC, *cf.*, [1]. Numerical examples show that the compensated Mitzner BC is almost always superior to the Mitzner BC and the traditional impedance BC. The BC is simple and should be straightforward to implement in numerical programs. It is possible to take into account the curvature of the structure for the Mitzner BC. The technique is discussed in [2].

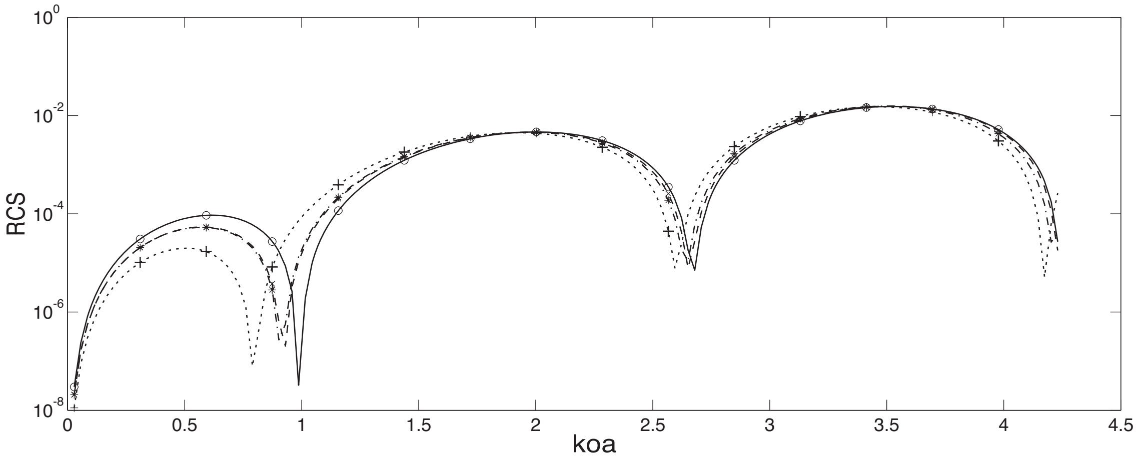

Figure 3: RCS from a circular cylinder shell with radius a, thickness $d = 0.01a$, permittivity $\varepsilon_r = 5$ and $\sigma = 0$. k_0 is the vacuum wavenumber. The incident plane wave is a TM-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) and impedance BC (dash-dotted line with $*$) give almost the same results. The error is larger for the Mitzner BC (dotted curve with $+$).

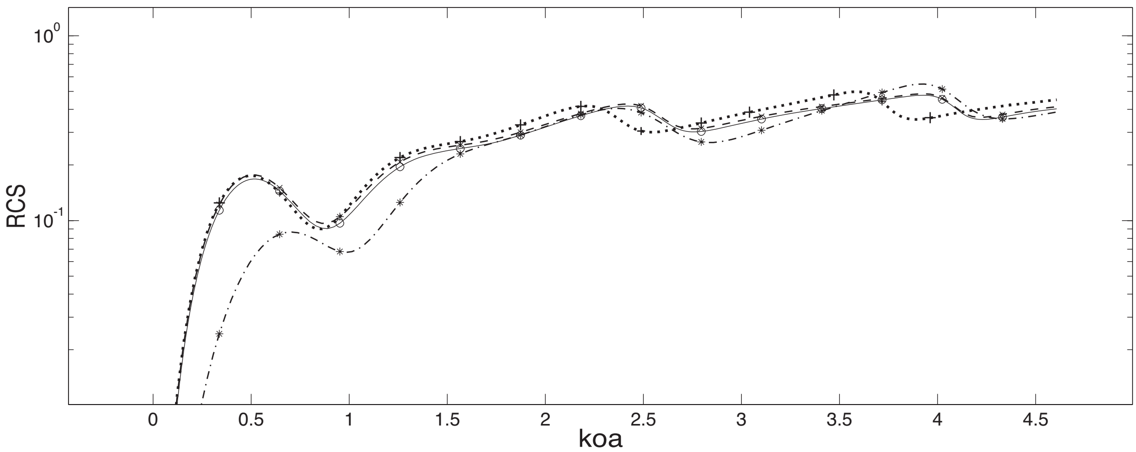

Figure 4: RCS from a circular cylinder shell with radius a, thickness $d = 0.1a$, permittivity $\varepsilon_r = 2.56$ and $\sigma = 1$ S/m. k_0 is the vacuum wavenumber. The incident plane wave is a TE-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) gives very good results, the Mitzner BC (dotted curve with $+$) is almost as good. The impedance BC (dash-dotted line with $*$) gives quite large errors

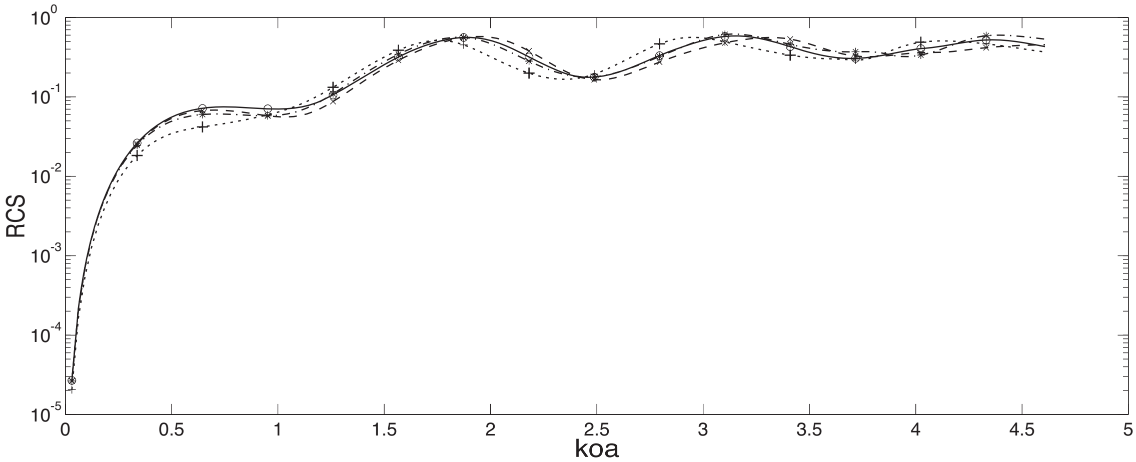

Figure 5: RCS from a circular cylinder shell with radius a, thickness $d = 0.1a$, permittivity $\varepsilon_r = 2.56$ and $\sigma = 1$ S/m. k_0 is the vacuum wavenumber. The incident plane wave is a TM-wave with frequency 1 GHz. The exact values are given by the solid line with o. The compensated Mitzner (dashed line with x) and the impedance BC (dash-dotted line with $*$) gives quite accurate results. The Mitzner BC (dotted curve with +) gives slightly larger errors.

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