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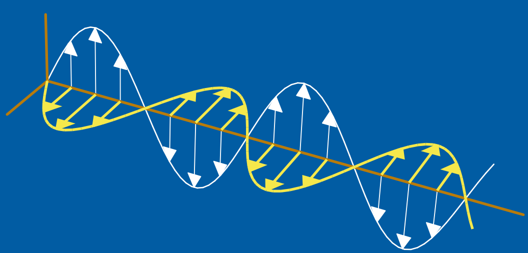

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Abstract

This paper treats the problem of calculating the macroscopic effective properties of dielectric mixtures where both the inclusions and the background medium can be anisotropic. For this homogenization process, the Maxwell Garnett -type approach is used where the inclusions are assumed to be spherical and embedded in a homogeneous background medium. The anisotropy of the background medium has to be described with a symmetric permittivity dyadic but the inclusion may be fully anisotropic, in other words the inclusion permittivity dyadic can contain an antisymmetric component. The effect of the anisotropy of the background is such that the depolarization factors of the spheres become different in different directions, even if the geometry is isotropic. This effect has to be taken into account for the calculation of the polarizability dyadic. As an example, numerical values are calculated for the case of gyrotropic spheres in anisotropic environment, both for the polarizability and effective permittivity dyadics. Finally, some thoughts are raised concerning the physical interpretation of the anisotropy effect, as well as the reciprocity of the materials and symmetry of their permittivities.

1 Introduction

Poisson, Faraday, Mossotti, Clausius, Maxwell, Lorenz, Lorentz, and Rayleigh are famous names from the 19th century that are affiliated with the dielectric and optical modeling of materials [6, 9]. The oldest mixing formula that explains dielectric permittivities, however, carries the name Maxwell Garnett and dates from the first years of the present century [2]. This Maxwell Garnett (MG) mixing rule

$$\epsilon_{\text{eff}} = \epsilon_e + 3f\epsilon_e \frac{\epsilon_i - \epsilon_e}{\epsilon_i + 2\epsilon_e - f(\epsilon_i - \epsilon_e)}$$

predicts the effective macroscopic permittivity ϵ_{eff} of a heterogeneous medium where homogeneous spheres of isotropic permittivity ϵ_i are dilutely mixed into isotropic environment with permittivity ϵ_e. The inclusions occupy a volume fraction f. This seemingly simple expression is quite capable of predicting very special material effects that a mixing process can produce, for example, the well known fact that materials may exhibit surprisingly different behavior in composites on one hand, and in bulk form on the other.

Since Garnett's times, the formula labeled after his name has been generalized in many respects. Multiphase mixtures, non-spherical inclusions, and lossy materials are but some of the more complicated problems where the MG result can be applied today. Magnetoelectric and chiral materials are also one domain into which one has succeeded in generalizing the MG rule, along with the present-day enthusiasm on exotic and complex materials.

One regime of heterogeneous media exists, however, that has been left without a correct and general Maxwell Garnett mixing formula. This is the situation when the two media composing the mixture are both anisotropic. Although an anisotropic inclusion phase can be dealt with more easily — as a matter of fact, even the bi-anisotropic inclusion case possesses an elegant solution [11] — the case when the background medium is anisotropic has not been given solution in the literature; even incorrect mixing formulas have been used for that problem. The core of the difficulties in the anisotropic-background case is that a spherical inclusion in anisotropic environment will become depolarized differently than in isotropic environment, and consequently the spherical depolarization factors cannot be used. Recently, a correct way has been presented of calculating the internal field of a sphere (that can be anisotropic) in anisotropic environment [14]. That result paves the way to the generalization of the Maxwell Garnett formula to mixtures where anisotropic spheres are embedded in another anisotropic medium, which is the direction of the present paper.

2 Theory

The aim is to try to describe the material in Figure 1 in macroscopic terms. To be able to homogenize a heterogeneous dielectric medium, some analysis is needed on the polarizability characteristics of a single inclusion of the mixture. This requires solving the internal field of the inclusion when it is exposed to an external field.

2.1 Internal field of a sphere in anisotropic medium

Consider a spherical inclusion with an arbitrary permittivity[1] dyadic $\boldsymbol{\epsilon}_i$. Let it be located in an unbounded anisotropic environment that is described by a symmetric permittivity dyadic $\boldsymbol{\epsilon}_e$. Because of the symmetry, this dyadic can be diagonalized. Choose the coordinate axes along the orthogonal eigenvectors, and the background permittivity dyadic can be written:

$$\boldsymbol{\epsilon}_e = \epsilon_{e,x}\boldsymbol{u}_x\boldsymbol{u}_x + \epsilon_{e,y}\boldsymbol{u}_y\boldsymbol{u}_y + \epsilon_{e,z}\boldsymbol{u}_z\boldsymbol{u}_z$$

where $\boldsymbol{u}_i$ is the unit vector in the i-direction, and $\epsilon_{e,i}$ is the respective eigenvalue of the dyadic.

In order to be able to calculate the polarizability of the anisotropic inclusion, the electrostatic problem of this inclusion in a uniform field has to be solved. One can follow the classical treatment of the corresponding isotropic problem [3, Sec. 4.4], and it turns out [1] that the anisotropy of the inclusion does not create any complications. However, the anisotropy of the environment is a problem because it causes the effect that the potential outside the sphere does not obey the Laplace equation which is the starting point for the polarizability calculations of classical dielectrics. One needs to resort to an affine transformation to the external medium [7, Sec. 4.3]. Then the Laplace equation holds for the potential in the transformed space. As a consequence of the affine transformation of the space, the spherical surface of the inclusion has become ellipsoidal. Thus the internal field $\boldsymbol{E}_i$ due to an external field $\boldsymbol{E}_e$ can be

[1] The permittivities are absolute permittivities, carrying units As/Vm.

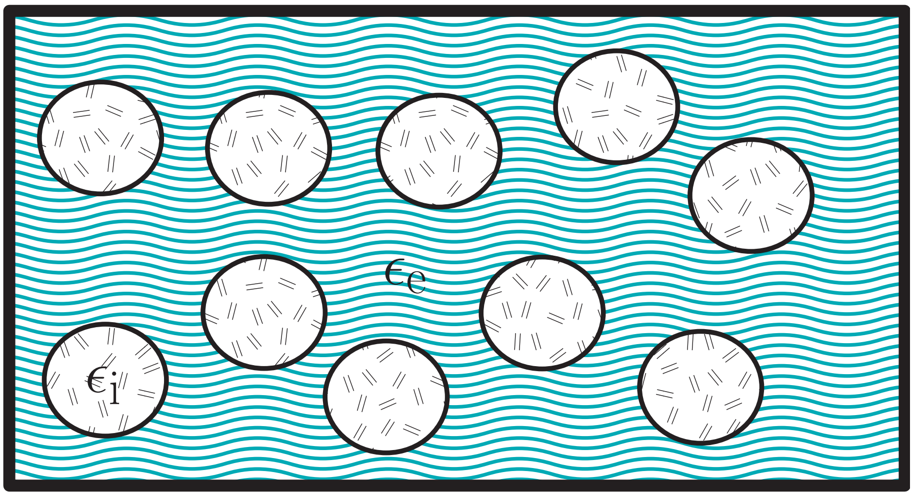

Figure 1: The homogenization problem: dielectrically anisotropic spheres embedded in anisotropic environment.

written [14]:

$$\boldsymbol{E}_i = [\boldsymbol{\epsilon}_e + \boldsymbol{N} \cdot (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e)]^{-1} \cdot \boldsymbol{\epsilon}_e \cdot \boldsymbol{E}_e \tag{2.1}$$

where $\boldsymbol{N}$ is the depolarization dyadic of the ellipsoid:

$$\boldsymbol{N} = \sum_{i=x,y,z} N_i \boldsymbol{u}_i \boldsymbol{u}_i \tag{2.2}$$

The well-known depolarization factors [15, Sec. 3.27] are

$$N_x = \frac{a_x a_y a_z}{2} \int_0^\infty \frac{ds}{(s + a_x^2)\sqrt{(s + a_x^2)(s + a_y^2)(s + a_z^2)}} \tag{2.3}$$

for an ellipsoid with semiaxes a_x, a_y, and a_z.[2] For the ellipsoid in question, the semiaxes are determined by the affine transformation:

$$a_x = \frac{a}{\sqrt{\epsilon_{e,x}/\epsilon_0}}, \qquad a_y = \frac{a}{\sqrt{\epsilon_{e,y}/\epsilon_0}}, \qquad a_z = \frac{a}{\sqrt{\epsilon_{e,z}/\epsilon_0}} \tag{2.4}$$

where ϵ_0 is the free-space permittivity and a is the radius of the original sphere.

Another form for the depolarization dyadic (2.2) is

$$\boldsymbol{N} = \frac{a^3}{2} \int_0^\infty ds \frac{(s\boldsymbol{I} + a^2 \epsilon_0 \boldsymbol{\epsilon}_e^{-1})^{-1}}{\sqrt{\det(s\boldsymbol{\epsilon}_e/\epsilon_0 + a^2 \boldsymbol{I})}} \tag{2.5}$$

It is easy to see that for the case of isotropic environment ($\boldsymbol{\epsilon}_e = \epsilon_e \boldsymbol{I}$), the depolarization dyadic[3] becomes the familiar one-third multiple of the unit dyadic: $\boldsymbol{N} = \frac{1}{3}\boldsymbol{I}$.

[2]For N_y and N_z, change a_x in the first term of the denominator of the integrand to a_y and a_z, respectively.

[3]In fact, the radius a is superfluous in Equation (2.5) and can be dropped.

2.2 Generalized polarizability dyadic

As the inclusion material differs from its environment, it creates a perturbation to the incident field. This "scattered" field due to an electrically small particle can be thought as a result of a static dipole. In classical dielectric studies of the polarizability behavior of particles, the dipole moment amplitude can be directly written down from the expression of the scattered field. Alternatively, the amplitude can be enumerated by integrating the polarization density $\boldsymbol{P}$ over the inclusion volume V.[4]

In the present anisotropic case, the latter approach is simpler because then there is no need for treating the dipole perturbation field in an anisotropic medium. The integration is easy because the internal field is constant. The dipole moment $\boldsymbol{p}$ is

$$\boldsymbol{p} = \int\limits_{\text{inclusion}} \boldsymbol{P}\, dV = \int\limits_{\text{inclusion}} (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e) \cdot \boldsymbol{E}_i\, dV = V\, (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e) \cdot \boldsymbol{E}_i \tag{2.6}$$

and because the polarizability dyadic $\boldsymbol{\alpha}$ is the linear relation between the dipole moment and the external field,

$$\boldsymbol{p} = \boldsymbol{\alpha} \cdot \boldsymbol{E}_e$$

we can write, using (2.1) and (2.6):

$$\boldsymbol{\alpha} = V\, (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e) \cdot \left[\boldsymbol{\epsilon}_e + \boldsymbol{N} \cdot (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e)\right]^{-1} \cdot \boldsymbol{\epsilon}_e \tag{2.7}$$

Note that this result differs from the expected results for spherical inclusions [8] by the appearance of the non-isotropic depolarization dyadic.

2.3 Maxwell Garnett model and the effective permittivity dyadic

To construct the Maxwell Garnett model for the mixture in Figure 1 where anisotropic spherical inclusion particles are embedded in the anisotropic environment, the concept of the Lorentzian field [16] has to be carefully regarded. Now that the inclusions are — because of their anisotropy — not symmetric with respect to rotation, their orientation distribution is an important parameter affecting the effective permittivity. Let us assume in the following that all the inclusions are aligned, in other words, they have the same permittivity dyadic, and the eigenvectors of the symmetric part and the antisymmetric axis are the same for all inclusions.

Now a macroscopic consideration takes the medium homogeneous: an external field $\boldsymbol{E}_e$ subjected on the mixture creates a background polarization density $\boldsymbol{\epsilon}_e \cdot \boldsymbol{E}_e$ and in addition an average inclusion polarization density $\boldsymbol{P} = n\boldsymbol{p}$ where n is the number density of the inclusions. But because of the permeated polarization, the field $\boldsymbol{E}_L$ exciting the dipole moment into an inclusion is greater than the external field:

$$\boldsymbol{E}_L = \boldsymbol{E}_e + \boldsymbol{\epsilon}_e^{-1} \cdot \boldsymbol{N} \cdot \boldsymbol{P} \tag{2.8}$$

[4] These two approaches have been called as the internal and external field methods [13].

This expression differs from the classical case of isotropic media [6] in that the Lorentzian correction term due to surrounding polarization density $\boldsymbol{P}$ is anisotropic. This is again a consequence of the affine transformation to reduce the environment isotropic whence the spherical surface becomes ellipsoidal.[5]

The dipole moment of an inclusion surrounded by the average polarization is

$$\boldsymbol{p} = \boldsymbol{\alpha} \cdot \boldsymbol{E}_L \tag{2.9}$$

Multiplying the dipole moment vector with the number density of the inclusions n gives the polarization density $\boldsymbol{P}$ and using (2.8), we can write the polarization density in terms of the polarizability dyadic. Define the effective permittivity dyadic by

$$\boldsymbol{D} = \boldsymbol{\epsilon}_e \cdot \boldsymbol{E}_e + \boldsymbol{P} = \boldsymbol{\epsilon}_{\text{eff}} \cdot \boldsymbol{E}_e \tag{2.10}$$

where $\boldsymbol{D}$ is the electric flux density, and we can write

$$\boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{\epsilon}_e + \left[\boldsymbol{I} - n\boldsymbol{\alpha} \cdot \boldsymbol{\epsilon}_e^{-1} \cdot \boldsymbol{N}\right]^{-1} \cdot n\boldsymbol{\alpha} \tag{2.11}$$

This generalized Clausius–Mossotti relation can be also written into a generalized Maxwell Garnett form where the permittivities of the mixture components appear instead of polarizabilities, because the polarizability dyadic (2.7) is known:

$$\boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{\epsilon}_e + f\,(\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e) \cdot \left[\boldsymbol{\epsilon}_e + (1-f)\boldsymbol{N} \cdot (\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e)\right]^{-1} \cdot \boldsymbol{\epsilon}_e \tag{2.12}$$

Here $f = nV$ is the dimensionless quantity: the volume fraction of the inclusions. In deriving this formula, one needs to be careful with dyadic algebra: the inclusion permittivity dyadic $\boldsymbol{\epsilon}_i$ does not commute in general with the other dyadics in the expression. However, $\boldsymbol{N}$ and $\boldsymbol{\epsilon}_e$ commute since both are symmetric dyadics with the same eigenvectors.

3 Example: gyrotropic spheres in anisotropic background

As an example to illustrate the theory presented above, let us treat the problem of a gyrotropic sphere in an anisotropic environment and, furthermore, a mixture where this type of spheres occupy random positions in the background medium.

The permittivity dyadics

Let us assume the gyrotropic dyadic of the inclusions to be of the following form where the z-axis of the coordinate system is assumed to align with the gyrotropy axis of the sphere:

$$\boldsymbol{\epsilon}_i = \epsilon_{i,t}\boldsymbol{I}_t + \epsilon_{i,z}\boldsymbol{u}_z\boldsymbol{u}_z + g\boldsymbol{u}_z \times \boldsymbol{I} \tag{3.1}$$

[5] In spite of the similarities of Equations (2.1) and (2.8), the field concepts are different: here we treat a cavity in which the field is defined to be the local field that excites the dipole moment, whereas in the case when the field ratio (2.1) for a single inclusion was calculated, it was the true field within the anisotropic inclusion material that was solved. Nevertheless, the same depolarization dyadic applies in both cases.

where the two-dimensional unit dyadic in the xy-plane is denoted by $\boldsymbol{I}_t = \boldsymbol{I} - \boldsymbol{u}_z\boldsymbol{u}_z$. Here g is the amplitude of the gyrotropy, and $\epsilon_{e,t}$ and $\epsilon_{e,z}$ are the permittivities in the transversal and axial directions, respectively.

Let us further assume for simplicity that the host medium permittivity $\boldsymbol{\epsilon}_e$ is uniaxial, with the optical axis coinciding with the gyrotropy axis of the inclusion:

$$\boldsymbol{\epsilon}_e = \epsilon_{e,t}\boldsymbol{I}_t + \epsilon_{e,z}\boldsymbol{u}_z\boldsymbol{u}_z$$

with the two permittivity components $\epsilon_{e,t}$ and $\epsilon_{e,z}$.

Operations on gyrotropic dyadics

Using the results of [7, Sec. 2.8.4], operations on gyrotropic dyadics can be used in the following form. Denote by $\boldsymbol{G}$ a gyrotropic dyadic with arbitrary coefficients:

$$\boldsymbol{G}(\alpha, \beta, \gamma) = \alpha\boldsymbol{I}_t + \beta\boldsymbol{u}_z\boldsymbol{u}_z + \gamma\boldsymbol{u}_z \times \boldsymbol{I}$$

The determinant of a gyrotropic dyadic is

$$\det\{\boldsymbol{G}(\alpha, \beta, \gamma)\} = \beta(\alpha^2 + \gamma^2)$$

and the inverse of a gyrotropic dyadic is another gyrotropic dyadic

$$\boldsymbol{G}^{-1}(\alpha, \beta, \gamma) = \boldsymbol{G}\left(\frac{\alpha}{\alpha^2+\gamma^2}, \frac{1}{\beta}, \frac{-\gamma}{\alpha^2+\gamma^2}\right)$$

The dot product between two gyrotropic dyadics is again another gyrotropic dyadic:

$$\boldsymbol{G}(\alpha_1, \beta_1, \gamma_1) \cdot \boldsymbol{G}(\alpha_2, \beta_2, \gamma_2) = \boldsymbol{G}\left(\alpha_1\alpha_2 - \gamma_1\gamma_2,\ \beta_1\beta_2,\ \alpha_1\gamma_2 + \gamma_1\alpha_2\right)$$

From this expression it is easy to see that two gyrotropic dyadics commute with each other.

The polarizability dyadic

And now, since the permittivity dyadic (3.1) of the gyrotropic sphere is

$$\boldsymbol{\epsilon}_i = \boldsymbol{G}(\epsilon_{e,t}, \epsilon_{e,z}, g)$$

we can write the polarizability of this sphere in anisotropic medium with permittivity (3) as

$$\boldsymbol{\alpha} = \boldsymbol{G}(\alpha_{\rm pol}, \beta_{\rm pol}, \gamma_{\rm pol}) \tag{3.2}$$

with

$$\alpha_{\rm pol} = V\epsilon_{e,t}\frac{(\epsilon_{i,t} - \epsilon_{e,t})[N_t\epsilon_{i,t} + (1-N_t)\epsilon_{e,t}] + N_t g^2}{[N_t\epsilon_{i,t} + (1-N_t)\epsilon_{e,t}]^2 + N_t^2 g^2} \tag{3.3}$$

$$\beta_{\rm pol} = V\epsilon_{e,z}\frac{\epsilon_{i,z} - \epsilon_{e,z}}{N_z\epsilon_{i,z} + (1-N_z)\epsilon_{e,z}} \tag{3.4}$$

$$\gamma_{\rm pol} = V\epsilon_{e,t}\frac{\epsilon_{e,t}\ g}{[N_t\epsilon_{i,t} + (1-N_t)\epsilon_{e,t}]^2 + N_t^2 g^2} \tag{3.5}$$

using (2.7). Note the appearance of the depolarization factors N_t and N_z in the expression. These come from the dyadic (2.2), which is uniaxial in the present example.

The depolarization factors

The depolarization factors in the polarizability expression are those of an oblate spheroid for the case of positive uniaxiality of the background medium, in other words, when $\epsilon_{e,z} > \epsilon_{e,t}$. Then we have

$$N_z = \frac{1+e^2}{e^3}(e - \arctan e), \quad \text{where} \quad e = \sqrt{\epsilon_{e,z}/\epsilon_{e,t} - 1}$$

and $N_t = (1 - N_z)/2$.

In case of negative uniaxiality, the depolarization factors are those of a prolate spheroid. Then $\epsilon_{e,z} < \epsilon_{e,t}$, and

$$N_z = \frac{1-e^2}{2e^3}\left(\ln\frac{1+e}{1-e} - 2e\right), \quad \text{where now} \quad e = \sqrt{1 - \epsilon_{e,z}/\epsilon_{e,t}}$$

and, again, $N_t = (1 - N_z)/2$.

The macroscopic permittivity

Using the anisotropic Maxwell Garnett mixing formula (2.12) and the gyrotropic dyadic operations, the effective permittivity dyadic can be written for a mixture where gyrotropic spheres with the permittivity dyadic (3.1) occupy a volume fraction f in a heterogeneous medium. In the following, all spheres are assumed to be oriented such that their gyrotropy axis aligns with the optical axis of the environment.

No wonder that the effective permittivity turns out to be a gyrotropic dyadic, too:

$$\boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{G}(\epsilon_{\text{eff},t}, \epsilon_{\text{eff},z}, g_{\text{eff}}) \tag{3.6}$$

with

$$\epsilon_{\text{eff},t} = \epsilon_{e,t} + f\epsilon_{e,t}\frac{(\epsilon_{i,t} - \epsilon_{e,t})[\epsilon_{e,t} + (1-f)N_t(\epsilon_{i,t} - \epsilon_{e,t})] + (1-f)N_t g^2}{[\epsilon_{e,t} + (1-f)N_t(\epsilon_{i,t} - \epsilon_{e,t})]^2 + (1-f)^2 N_t^2 g^2} \tag{3.7}$$

$$\epsilon_{\text{eff},z} = \epsilon_{e,z} + f\epsilon_{e,z}\frac{\epsilon_{i,z} - \epsilon_{e,z}}{\epsilon_{e,z} + (1-f)N_z(\epsilon_{i,z} - \epsilon_{e,z})} \tag{3.8}$$

$$g_{\text{eff}} = f\epsilon_{e,t}\frac{\epsilon_{e,t}\, g}{[\epsilon_{e,t} + (1-f)N_t(\epsilon_{i,t} - \epsilon_{e,t})]^2 + (1-f)^2 N_t^2 g^2} \tag{3.9}$$

A first look at the effective permittivity expression (and also at the polarizability expression) would suggest that the axial and transversal components of the dyadics are decoupled from each other. Only the z-components of the permittivities of the phases appear in the z-component of the mixture permittivity. However, this is only illusory because the depolarization factors N_z and N_t both depend, very non-linearly, on the uniaxiality ratio of the background medium, which leads to the fact that the mixture permittivity components cannot be calculated independently. This non-linear dependence is weak if the uniaxial anisotropy is small ($\epsilon_{e,t}/\epsilon_{e,z} \approx 1$).

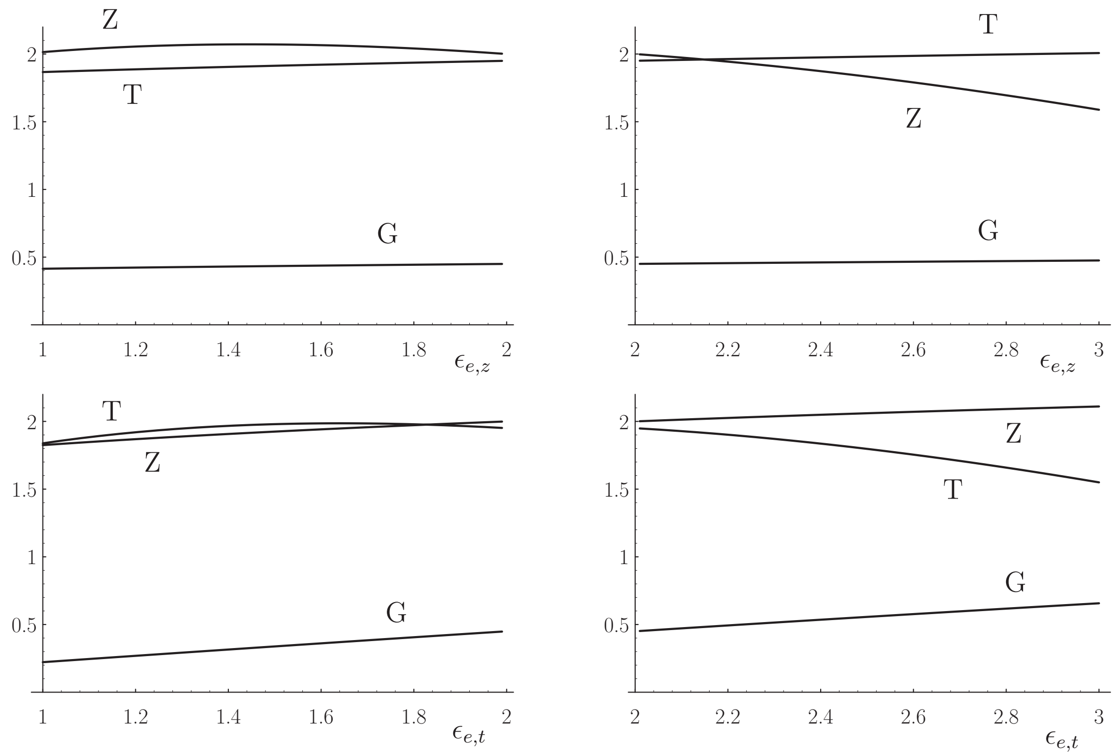

Figure 2: The three components (T: transversal ($\alpha_{\rm pol}$), Z: axial ($\beta_{\rm pol}$), and G: antisymmetric ($\gamma_{\rm pol}$)) of the polarizability dyadic (3.2) of a gyrotropic sphere as functions of the uniaxiality ratio of the background medium. The parameters of the sphere are: $\epsilon_{i,t} = \epsilon_{i,z} = 5\epsilon_0$, and $g = {\rm j}\epsilon_0$. In the upper figures $\epsilon_{e,t} = 2\epsilon_0$ and $\epsilon_{e,z}$ varies; in the lower ones, $\epsilon_{e,z} = 2\epsilon_0$ and $\epsilon_{e,t}$ varies. The polarizability components are normalized with the magnitude $V\epsilon_0$.

Illustrations

Let us next illustrate the behavior of the polarizability and effective permittivity of the gyrotropic spheres in uniaxial background medium.

Polarizability Figure 2 displays the dependence of normalized polarizability components of a gyrotropic sphere. The parameters used in the calculations are $\epsilon_{i,t} = \epsilon_{i,z} = 5\epsilon_0$, and $g = {\rm j}\epsilon_0$, where ϵ_0 is the free-space permittivity. Note that the gyrotropy parameter g is chosen pure imaginary; this leads to a lossless character for the material.[6] Shown are the components of (3.2): $\alpha_{\rm pol}/(V\epsilon_0), \beta_{\rm pol}/(V\epsilon_0)$, and ${\rm Im}\{\gamma_{\rm pol}\}/(V\epsilon_0)$, as functions of the axial ratio of the uniaxial permittivity of the environment. One component is kept at $2\epsilon_0$ while the other varies between ϵ_0 and $3\epsilon_0$.

From the upper curves of Figure 2 one can see that changing the axial component of the background permittivity component will have the strongest effect on the axial

[6]For imaginary g and real $\epsilon_{i,t}$, $\epsilon_{i,z}$, the permittivity dyadic (3.1) becomes Hermitian, which means that the medium is dielectrically lossless [4, p. 51].

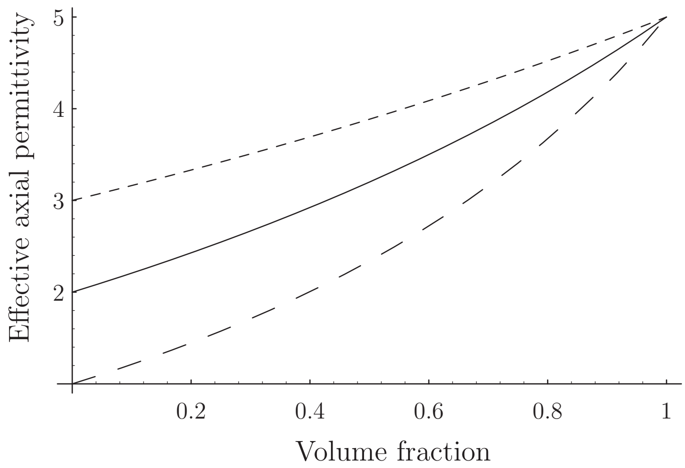

Figure 3: The axial component of the relative effective permittivity dyadic (3.6) of mixture with gyrotropic spheres embedded in uniaxial background medium. The parameters of the spheres are: $\epsilon_{i,t} = \epsilon_{i,z} = 5\epsilon_0$, and $g = 3\mathrm{j}\epsilon_0$. The background permittivities are $\epsilon_{e,t} = 2\epsilon_0$ and the curves represent different values of $\epsilon_{e,z}$. Solid curve: $\epsilon_{e,z} = 2\epsilon_0$, long-dashed: $\epsilon_{e,z} = \epsilon_0$, and short-dashed: $\epsilon_{e,z} = 3\epsilon_0$.

polarizability. As the background permittivity component $\epsilon_{e,z}$ increases, the polarizability component β_{pol} decreases because the contrast between the inclusion and the environment decreases. A similar effect can be noticed in the lower curves between the transversal components of the background permittivity and polarizability.

On the other hand, the effect of the axial permittivity on the transversal polarizability (and transversal permittivity on axial polarizability) is small. However, as discussed above, these are not independent but a slight increase can be noted.

The antisymmetric component of the polarizability is proportional to the gyrotropy parameter of the sphere, and its expression contains explicitly the transversal permittivity components. Also the figure testifies the notable effect of $\epsilon_{e,t}$ on γ_{pol}. But also a very slight effect of the axial permittivity $\epsilon_{e,z}$ can be seen, which is due to the anisotropy effect through the depolarization factors differing from the sphere values 1/3.

Effective permittivity The effective permittivity of gyrotropic spheres aligned in uniaxial environment is also gyrotropic. The components of the permittivity dyadic can be calculated using Equation (3.6), and are shown in Figures 3–5 for a mixture where the inclusions have parameters $\epsilon_{i,t} = \epsilon_{i,z} = 5\epsilon_0$, and $g = 3\mathrm{j}\epsilon_0$ and the background $\epsilon_{e,t} = 2\epsilon_0$, with the axial permittivity having three different values: $\epsilon_{e,z}/\epsilon_0 = 1$, 2, and 3. The curves are given as functions of the volume fraction of the inclusions, and the obvious property of a Maxwell Garnett prediction can be seen from all three figures that for $f = 0$, the curves start from the background values, stopping at the inclusion values for $f = 1$.

Figure 3 shows the rather uninteresting behaviour of the axial effective permittivity values, for different values of the inclusion axial permittivity component.

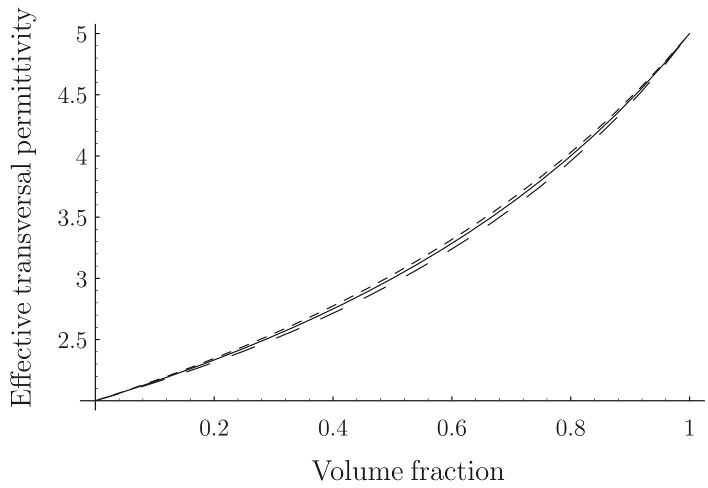

Figure 4: The same as Figure 3, for the transversal component of the effective permittivity dyadic.

The strong direct effect between the axial permittivities masks all subtleties of the non-spherical depolarizabilities. On the other hand, these finer structures display themselves more visibly in Figure 4 which shows the effective transversal permittivity curves for three different values of the axial inclusion permittivity. For all three curves, the background and inclusion transversal permittivities are the same, and indeed all three curves are very close to one another. However, the small anisotropy effect can be seen: the effective transversal permittivity is slightly smaller if the axial inclusion permittivity is small (the curve with $\epsilon_{e,z} = \epsilon_0$ is the lowest one), and slightly higher if the axial inclusion permittivity is large (the curve with $\epsilon_{e,z} = 3\epsilon_0$ is the highest).

The anisotropy effect that comes from the depolarization factors explains the small differences of the three curves in Figure 5, too. There the gyrotropic component of the effective permittivity dyadic is shown for the three different values of the axial component of the inclusion permittivity dyadic. Equation (3.9) testifies that the antisymmetric component of the effective permittivity does not explicitely depend on the axial components of the permittivities of the two phases of the mixture. The axial effect comes only through the depolarization factors N_t and N_z which depend on the anisotropy of the background material. Again, the effect of increasing the axial permittivity of the inclusion is an increase in the macroscopic gyrotropy, albeit a small one.

4 Discussion

Let us discuss in some detail the anisotropy effect and the symmetries of the different material dyadics treated above.

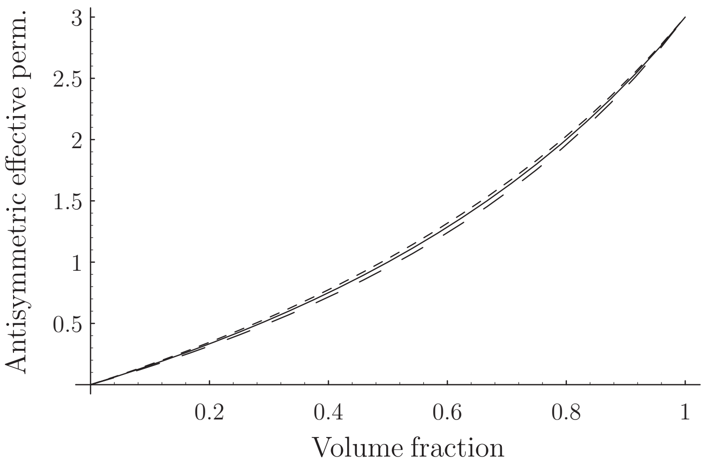

Figure 5: The same as Figure 3, for the antisymmetric component of the effective permittivity dyadic.

4.1 Physical interpretation of the anisotropy effect

The illustrations in the previous section have shown that the effective permittivities in the axial and transversal directions cannot be calculated independently of the knowledge of the other components, even in the case when the anisotropy axes of the spherical inclusions and the environment coincide. The "anisotropy effect" means that the anisotropy of the environment leads to the use of such depolarization factors in the calculations that differ from the sphere values 1/3 [12]. Let us consider the physical polarizability behavior of this effect in the case of a simple example: let the background be uniaxial and the inclusion isotropic. Consider the following two cases separately: electrically heavy spheres in uniaxial background with small permittivity components (raisin pudding model), and electrically light spheres in uniaxial background with large permittivity components (Swiss cheese model).

- **Raisin pudding model**. The spheres are of higher permittivity than the environment. Consider the permittivity component in the direction of the optical axis, and assume positive uniaxiality: the axial component of the environment permittivity is higher than in the transversal one. Then the anisotropy effect is to lower the effective permittivity in this direction.

  This can be understood in the following sense: the anisotropy effect flattens the spherical shape into an oblate ellipsoid in the transformed space (see Equations (2.4)). And we are studying field excitation in the direction of the axis of revolution of this ellispoid. The effect of flattening is to *decrease* the internal field (it is known that the field ratio for a sphere is $3\epsilon_e/(\epsilon_i + 2\epsilon_e)$ and for a flat disk it is ϵ_e/ϵ_i, which is smaller than the sphere ratio because $\epsilon_e < \epsilon_i$). Therefore the dipole moment component of the inclusion decreases and the increase of the effective permittivity due to the polarizable inclusions is not as large as it were, had it been calculated with the sphere depolarizabilities.

- **Swiss cheese model**. The spheres are of lower permittivity than the environment. Consider again the permittivity component in the direction of the optical axis (the axial permittivity component of the environment is, as before, higher than the component in the transversal direction). What is now the anisotropy effect? The effect is also in this case to lower the effective permittivity in this direction!

  Just like above, the anisotropy effect flattens the spherical shape into an oblate ellipsoid in the transformed space. The effect of flattening is now to *increase* the internal field (because the field ratio for a flat disk is now larger than for a sphere, see the above ratios). Therefore the dipole moment component of the inclusion increases in amplitude but is negative (because the inclusions had a lower permittivity than the environment). Therefore the effective permittivity component is smaller than it would be in the case of using the (incorrect) spherical depolarization rule.

The above reasoning about increases and decreases can be turned around if we treat the transversal effective permittivity components, and also in the case of negative uniaxiality (axial permittivity of the environment is smaller than the transversal) of the anisotropy of the environment.

4.2 Symmetry and reciprocity

Reciprocity is a concept of electromagnetic materials that has received increased attention recently [5, 10]. Reciprocity is a certain manifestation of symmetry with respect to the interchange of transmitter and receiver [4, Sec. 5.5]. The permittivity dyadic of a reciprocal material is symmetric, although it can be anisotropic. It is natural to accept that also a mixture composed of reciprocal materials must display reciprocal electromagnetic behavior, in other words, the effective permittivity dyadic should be symmetric:

$$\boldsymbol{\epsilon}_e = \boldsymbol{\epsilon}_e^T \;\text{ and }\; \boldsymbol{\epsilon}_i = \boldsymbol{\epsilon}_i^T \quad \Longrightarrow \quad \boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{\epsilon}_{\text{eff}}^T \tag{4.1}$$

where T denotes the transpose operation. From the expression (2.12) for the effective permittivity this symmetry property is not obvious because $\boldsymbol{\epsilon}_i$ does not commute with $\boldsymbol{\epsilon}_e$ or $\boldsymbol{N}$, even if it is symmetric.[7] However, the property (4.1) can be more easily seen to hold if the effective permittivity dyadic (2.12) is rewritten in the following form:

$$\boldsymbol{\epsilon}_{\text{eff}} = \boldsymbol{\epsilon}_e + f\left[(\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_e)^{-1} + (1-f)\boldsymbol{\epsilon}_e^{-1}\cdot\boldsymbol{N}\right]^{-1} \tag{4.2}$$

This form of the Maxwell Garnett equation may be more practical than (2.12) in some applications. The reciprocity of the mixture for reciprocal components is obvious after using the fact that the inverse and transpose operations on a dyadic commute.

[7] The eigenvector directions of $\boldsymbol{\epsilon}_i$ may be different from those of $\boldsymbol{\epsilon}_e$ and $\boldsymbol{N}$.

5 Conclusions

Dielectric anisotropy and its effect on macroscopic properties of mixtures has been the topic of the present paper. Although the Maxwell Garnett formula has been known for over 90 years, its correct form for anisotropic components, especially for the case of anisotropic background media, has remained hidden. The important feature in the correct form of the MG mixing formula, very much emphasized in the present paper, is the so-called "anisotropy effect," which means that in a certain sense the spherical inclusions have to be treated like ellipsoids when their polarizability characteristics are calculated. The deformation of the spheres is a consequence of the affine transformation that has to be executed to render the environment isotropic. The semiaxes of the sphere-deformed-into-ellipsoid are proportional to the inverse of the square root of the permittivity components of the environment.

The results show that the anisotropy effect is not very large for small anisotropies of the environment. This is a consequence of the above-mentioned square-root dependence on the permittivity components, and the fact that the ellipsoidal depolarization factors are quite slow functions of the eccentricity near the spherical case. Nevertheless, the numerical curves in the previous sections showed visible effects of this anisotropy as the uniaxiality ratio of the environment was allowed to range between 1/2 and 3/2.

What are the limitations of the present model? As always for quasistatic mixing formulas, the results apply for wavelengths much larger than the size of the scatterers. These are low-frequency approximations. The anisotropies of the problem, however, are allowed to be quite general: there are no limitations for the inclusion permittivity, and the environment anisotropy is described by an arbitrary symmetric permittivity dyadic.

Concerning losses, the permittivity components of dissipative materials are described by complex numbers in the frequency domain. Therefore the polarizability and effective permittivity expressions retain their form also in the lossy case, now only complex values have to be used, and losses are contained in the imaginary parts of the results. The starting phases of the analysis in the paper made use of the affine transformation that was determined by the permittivity dyadic of the environment, and one may ask how that procedure is affected if the environment is lossy. The first step was to diagonalize the symmetric permittivity dyadic of the background medium. Then the eigenvalues become complex. This causes no problems for the enumeration of the results using the expression in the paper. In general, the eigenvectors are complex, too. But these complex vectors are orthogonal, and we have to choose a complex orthonormal coordinate system. Then it may hurt intuition to use the familiar equations for the ellipsoidal depolarization factors (2.3) in complex domain. However, there should be no reason why the analytical form for the depolarization dyadic (2.5) could not be used, even if the permittivity dyadic $\boldsymbol{\epsilon}_e$ is an arbitrary complex but symmetric dyadic.

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