Notes on Heat Transfer 3 - 1987

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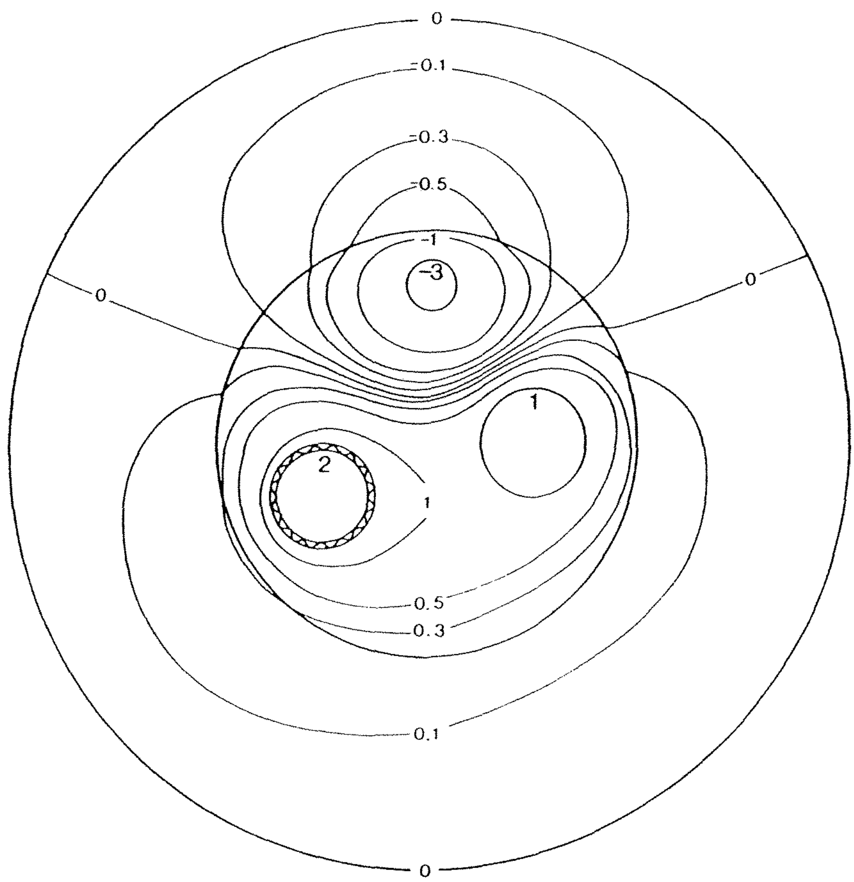

January 1987

Dep. of Building Technology and Mathematical Physics

CODEN: LUTVDG/(TVBH-7094)/1-42/(1987)

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MULTIPOLE METHOD TO COMPUTE THE CONDUCTIVE HEAT FLOWS TO AND BETWEEN PIPES IN A COMPOSITE CYLINDER

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January 1987

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

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1.INTRODUCTION

A so-called multipole method to compute the conductive heat flows to and between pipes in a cylinder is presented in [1]. The steady-state heat conduction is two-dimensional in a circular region perpendicular to the pipes. The region is homogeneous. In this paper the method is extended to the case of a composite cylinder. The pipes lie in an inner circular region, which is surrounded by an annulus of different thermal conductivity.

The applications for which the method has been developed concern so-called heat extraction boreholes and certain types of heat stores in the ground. The boreholes are used for heat extraction or as heat exchangers for heat injection/extraction. The heat carrier fluid may for example flow in a U-shaped tube in the borehole. Outside the tubes the borehole may be filled with sand. The local thermal problem in and near the borehole is quite important for the heat transfer capacity of the heat exchanger. This problem is essentially a steady-state one in the region between the fluid in the pipes and a suitably chosen cylinder around the borehole. The pipes are imbedded in a composite cylinder with an internal boundary at the borehole wall. The thermal resistance between the pipes and the outer circle is of particular interest. We are also interested in the thermal resistances between the pipes. The presented method and the computer program give a very rapid and accurate way to obtain these resistances and the complete temperature field. The method is used extensively in [2].

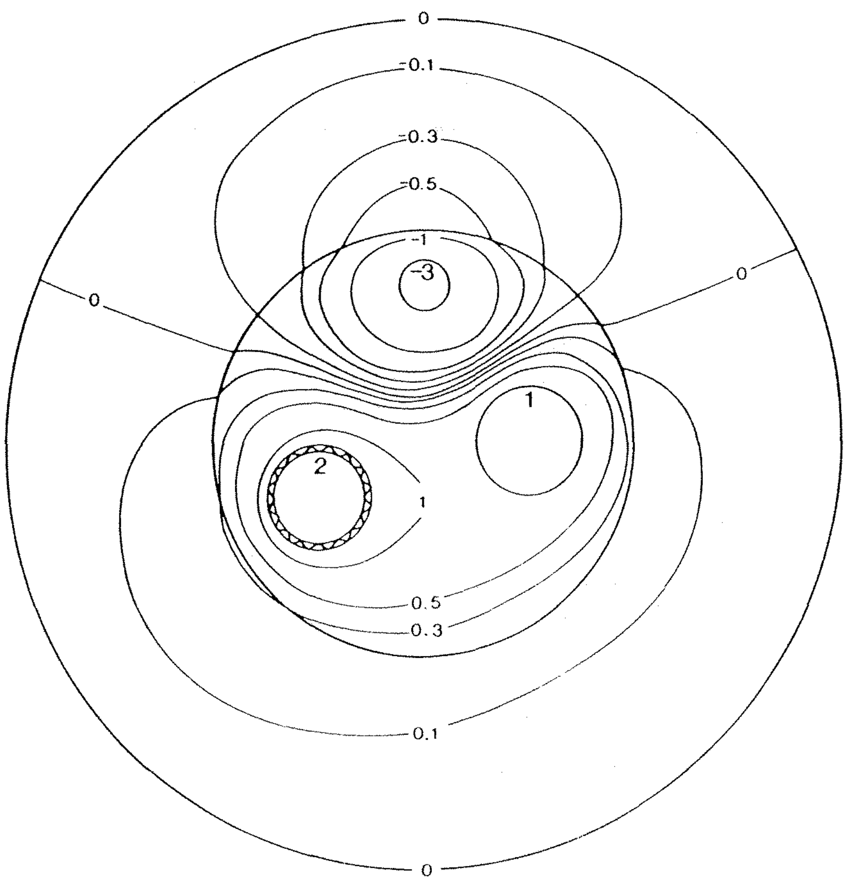

Figure 1.1. An example showing the temperature field for a case with 3 pipes. Data according to (8.12).

Figure 1.1 shows the computed temperature field for an example with three pipes. The fluid temperatures in the pipes and on the outer circle are indicated in the figure. One of the pipes has as indicated a thermal resistance layer. The complete set of data for the example is given by (8.12). The computer time for this case is only a few seconds on a main frame computer (Norsk Data ND-500) or about 40 seconds on an IBM-PC AT-3 (10 MHz) with a 80287 math coprocessor.

2. THERMAL PROBLEM

Figure 2.1 shows the considered thermal problem. There are N pipes ($N \geq 1$), which lie within the inner circular region with the radius r_b (b = borehole).The inner circle is surrounded by an annular region of another material. The outer circle has the radius r_c.

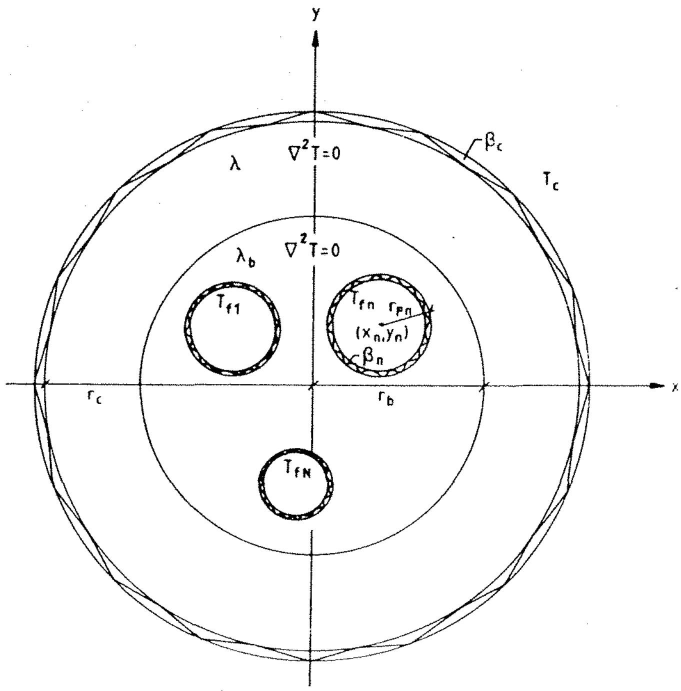

Figure 2.1 Steady-state heat conduction in a composite circular region with heat flows between the N pipes and the outer circle.

The outer radius of pipe n is r_{pn}, and its center lies at (x_n, y_n). The fluid in pipe n has the constant temperature T_{fn}, while the temperature outside the outer circle is T_c.

The annular region, $r_b < r < r_c$, is homogeneous with the thermal conductivity λ. The inner circular region outside the pipes has the thermal conductivity λ_b. The steady-state temperature $T(x, y)$ satisfies the heat conduction equation in the annular region and in the inner circle:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{2.1}$$

Cartesian, complex and polar coordinates will be used:

$$z = x + iy = re^{i\phi} \tag{2.2}$$

The center of pipe n is in complex coordinates:

$$z_n = x_n + iy_n \tag{2.3}$$

We will use the local polar coordinates ρ_n, ψ_n from the center of any pipe n. See Figure 2.2.

$$z - z_n = \rho_n e^{i\psi_n} \tag{2.4}$$

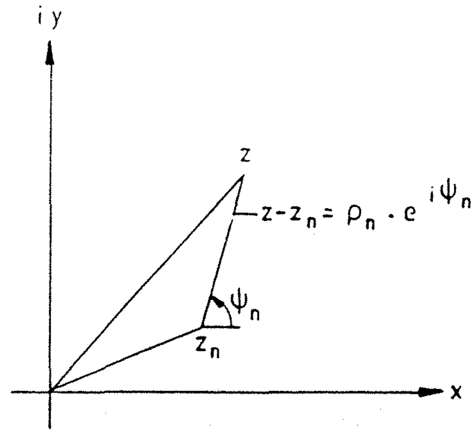

Figure 2.2. Local polar coordinates ρ_n, ψ_n from the center of pipe n.

The temperature and the radial heat flux are continuous at the inner boundary $r = r_b$:

$$T\big|_{r_b-0} = T\big|_{r_b+0}$$

$$\lambda_b \frac{\partial T}{\partial r}\bigg|_{r_b-0} = \lambda \frac{\partial T}{\partial r}\bigg|_{r_b+0} \tag{2.5}$$

The boundary conditions at the pipes and the outer circle $r = r_c$ are the same as in [1]. The boundary condition at pipe n is:

$$T - \beta_n r_{pn} \frac{\partial T}{\partial \rho_n} = T_{fn} \qquad \begin{array}{c} \rho_n = r_{pn} \\ 0 \leq \psi_n \leq 2\pi \end{array} \tag{2.6}$$

The boundary condition at the outer circle is:

$$T + \beta_c r_c \frac{\partial T}{\partial r} = T_c \qquad \begin{array}{c} r = r_c \\ 0 \leq \phi \leq 2\pi \end{array} \tag{2.7}$$

The dimensionless coefficient β_n determines the thermal resistance between the fluid in pipe n and the material just outside the pipe. This resistance is $\beta_n r_{pn}/\lambda_b$ (K/(W/m^2)). The corresponding thermal resistance R_{pn} (K/(W/m)) per unit pipe length is obtained by division with the perimeter $2\pi r_{pn}$:

$$R_{pn} = \frac{\beta_n}{2\pi\lambda_b} \qquad (K/(W/m)) \tag{2.8}$$

The thermal resistance per unit length (in the axial direction) of the outer circle is:

$$R_{pc} = \frac{\beta_c}{2\pi\lambda} \qquad (K/(W/m)) \tag{2.9}$$

The thermal resistance per unit area at the outer circle is $\beta_c r_c/\lambda = 2\pi r_c R_{pc}$ (K/(W/m^2)). The thermal resistance coefficients β_n and β_c may take any non-negative value: $0 \leq \beta_n \leq \infty, 0 \leq \beta_c \leq \infty$. The value $\beta_n = +\infty$ means zero heat flux, $\frac{\partial T}{\partial \rho_n} = 0$. The value of T_{fn} is then redundant.

3. LINE SOURCE FOR THE COMPOSITE REGION

The solution in the previous paper [1] for pipes in a homogeneous region is based on the line source solution in complex form. Suitable derivatives gave the multipoles, which were needed for the solution. We will here in the same way use the line source for a composite region.

The thermal problem for the basic line source solution is shown in Figure 3.1. The thermal conductivity is λ_b in the circle $0 \leq r < r_b$, and λ in the infinite region outside the circle, $r_b \leq r < \infty$. There is a line source with the strength $+q_n$ (W/m) at the point (x_n, y_n), which lies within the circle $(r_n < r_b)$.

The solution to this problem is given in [2].

$0 \le r \le r_b$:

$$T(x,y) = \frac{q_n}{2\pi\lambda_b}\left\{ln\left(\frac{r_b}{\sqrt{(x-x_n)^2+(y-y_n)^2}}\right) + \sigma\cdot ln\left(\frac{r_b^2/r_n}{\sqrt{(x-x_nr_b^2/r_n^2)^2+(y-y_nr_b^2/r_n^2)^2}}\right)\right\} \quad (3.1)$$

$r_b \le r < \infty$:

$$T(x,y) = \frac{q_n}{2\pi\lambda}\left\{(1-\sigma)ln\left(\frac{r_b}{\sqrt{(x-x_n)^2+(y-y_n)^2}}\right) + \sigma\cdot ln\left(\frac{r_b}{\sqrt{x^2+y^2}}\right)\right\} \quad (3.2)$$

We have introduced the notation:

$$\sigma = \frac{\lambda_b - \lambda}{\lambda_b + \lambda} \qquad (-1 < \sigma < 1) \quad (3.3)$$

The case $\sigma = 0$, i.e. $\lambda_b = \lambda$, is the one studied in [1].

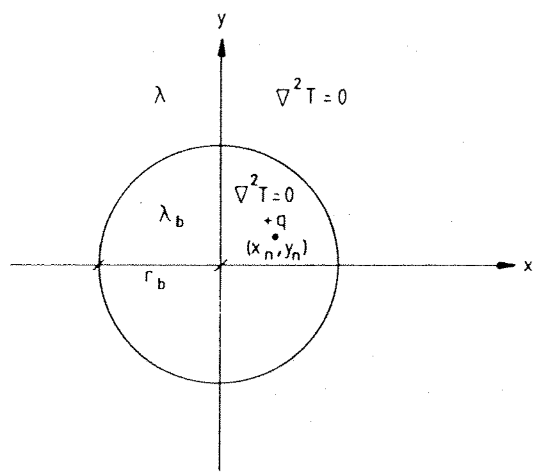

Figure 3.1. Fundamental line source problem for the composite region.

The first term in (3.1) represents a line sink $+q_n$ at (x_n, y_n) in a material with the thermal conductivity λ_b. The second term is due to the fact that the thermal conductivity is λ for $r > r_b$. This term represents a line sink with the strength σq_n situated at the mirror point $(x_nr_b^2/r_n^2, y_nr_b^2/r_n^2)$. The mirror point lies on the same radius as (x_n, y_n). The product of the distances to the center is $r_n \cdot r_b^2/r_n = r_b^2$. The temperature field (3.2) in the outer region $r \ge r_b$ consists of a line sink with the strength $q_n(1-\sigma)$ at (x_n, y_n) and another one with the remaining heat $q_n \cdot \sigma$ at the $(0,0)$. It shall be observed that the thermal conductivity is λ in (3.2) and λ_b in (3.1).

The solution (3.1-2) is rewritten in the complex form using the following expressions:

$$ln\left(\frac{r_b}{\sqrt{(x-x_n)^2+(y-y_n)^2}}\right) = Re\left[ln\left(\frac{r_b}{z-z_n}\right)\right] \quad (3.4)$$

$$ln\left(\frac{r_b}{\sqrt{x^2+y^2}}\right) = Re\left[ln\left(\frac{r_b}{z}\right)\right] \quad (3.5)$$

$$ln\left(\frac{r_b^2/r_n}{\sqrt{(x-x_nr_b^2/r_n^2)^2+(y-y_nr_b^2/r_n^2)^2}}\right)$$

$$= ln\left(\frac{r_b^2/|z_n|}{|z-z_nr_b^2/r_n^2|}\right) = ln\left(\frac{r_b^2}{|\bar{z}_n|\cdot|z-z_nr_b^2/(z_n\bar{z}_n)|}\right) \tag{3.6}$$

$$= ln\left(\frac{r_b^2}{|r_b^2-\bar{z}_nz|}\right) = ln\left(\frac{r_b^2}{|r_b^2-\bar{z}z_n|}\right) = Re\left[ln\left(\frac{r_b^2}{r_b^2-\bar{z}z_n}\right)\right]$$

In the last two lines we used that $r_n^2 = |z_n|^2 = z_n\bar{z}_n$ and $|w| = |\bar{w}|$ for any complex number w and its conjugate $\bar{w}$.

The temperature (3.1-2) may now be written in the following form:

$$T(x,y) = \frac{q_n}{2\pi\lambda_b}\cdot Re[W_{no}] \tag{3.7}$$

The complex-valued function W_{no} is from (3.1-7) defined by:

$$\begin{cases} W_{no} = ln\left(\dfrac{r_b}{z-z_n}\right) + \sigma\cdot ln\left(\dfrac{r_b^2}{r_b^2-\bar{z}z_n}\right) & r\le r_b \\ \\ W_{no} = (1+\sigma)ln\left(\dfrac{r_b}{z-z_n}\right) + \dfrac{\lambda_b}{\lambda}\sigma\cdot ln\left(\dfrac{r_b}{z}\right) & r\ge r_b \end{cases} \tag{3.8-9}$$

The factor $1+\sigma$ in (3.9) is in accordance with (3.3) equal to $\lambda_b(1-\sigma)/\lambda$. The simplicity of the expressions (3.7-9) is noteworthy.

Let us verify that (3.7-9) is indeed the solution to the fundamental line source problem for the composite region shown in Figure 3.1. The expression (3.9) is a regular (analytic) function in the region $r > r_b$, since the sources lie within the circle. Its real part $Re[W_{no}]$ satisfies the heat conduction equation $\nabla^2T = 0$. The expression (3.8) is also a regular function in $r < r_b$ except at the point $z = z_n$. It remains to verify that the boundary conditions (2.5) at $r = r_b$ are satisfied. We have at the circle $z = r_b\cdot e^{i\phi}$:

$$W_{no}|_{r_b-0} = ln\left(\frac{r_b}{r_b\cdot e^{i\phi}-z_n}\right) + \sigma\cdot ln\left(\frac{r_b^2}{r_b^2-r_be^{-i\phi}\cdot z_n}\right)$$

$$= ln\left(\frac{r_b}{r_b\cdot e^{i\phi}-z_n}\right) + \sigma\cdot ln\left(\frac{r_b\cdot e^{i\phi}}{r_be^{i\phi}-z_n}\right) \tag{3.10}$$

$$W_{no}|_{r_b+0} = (1+\sigma)ln\left(\frac{r_b}{r_b\cdot e^{i\phi}-z_n}\right) + \frac{\lambda_b}{\lambda}\sigma\cdot ln\left(e^{-i\phi}\right) \tag{3.11}$$

The difference is

$$W_{no}|_{r_b-0} - W_{no}|_{r_b+0} = \sigma\cdot i\phi - \frac{\lambda_b}{\lambda}\sigma\cdot(-i\phi)$$

,or

$$W_{no}|_{r_b-0} - W_{no}|_{r_b+0} = \frac{\lambda_b-\lambda}{\lambda}\cdot i\phi \tag{3.12}$$

This difference contains only an imaginary part. The temperature, which is given by the real part, is therefore continuous at $r = r_b$.

For the radial heat flux at $r = r_b$ we have:

$$\lambda_b\left.\frac{\partial W_{no}}{\partial r}\right|_{r_b-0} = \lambda_b\left\{\frac{-e^{i\phi}}{r_be^{i\phi}-z_n} + \sigma\cdot\frac{(-)(-z_n)e^{-i\phi}}{r_b^2-r_be^{-i\phi}z_n}\right\}$$

$$= -\frac{\lambda_b}{r_b}\cdot\frac{r_be^{i\phi}-\sigma z_n}{r_be^{i\phi}-z_n} \tag{3.13}$$

$$\lambda \left.\frac{\partial W_{no}}{\partial r}\right|_{r_b+0} = \lambda \left\{ (1+\sigma) \cdot \frac{-e^{i\phi}}{r_b e^{i\phi} - z_n} + \frac{\lambda_b}{\lambda}\sigma \cdot \frac{-e^{i\phi}}{r_b e^{i\phi}} \right\}$$

$$= \ldots = -\frac{\lambda_b}{r_b} \cdot \frac{r_b e^{i\phi} - \sigma z_n}{r_b e^{i\phi} - z_n} \tag{3.14}$$

The heat flux, i.e. the real parts of (3.13-14), is therefore continuous at $r = r_b$. This is in fact true also for the imaginary part of W_{no}:

$$\lambda_b \left.\frac{\partial W_{no}}{\partial r}\right|_{r_b-0} = \lambda \left.\frac{\partial W_{no}}{\partial r}\right|_{r_b+0} \tag{3.15}$$

4. MULTIPOLES FOR THE COMPOSITE REGION

The multipoles of order j were in complex form given by $(z - z_n)^{-j}$ in the case of a single region, [1]. They were obtained by derivatives of the complex logarithm $ln(z - z_n)$. Let W_{n1} be the derivative of W_{no}, (3.8-9), with respect to the complex variable z_n:

$$W_{n1} = \frac{\partial}{\partial z_n}(W_{no}) \tag{4.1}$$

We get from (3.8) and (3.9):

$$W_{n1} = \frac{1}{z - z_n} + \sigma \cdot \frac{\bar{z}}{r_b^2 - \bar{z} z_n} \qquad r \leq r_b \tag{4.2}$$

$$W_{n1} = (1+\sigma) \cdot \frac{1}{z - z_n} \qquad r \geq r_b \tag{4.3}$$

The function W_{n1} is regular. Thus, the real and imaginary parts, $Re(W_{n1})$ and $Im(W_{n1})$, both satisfy $\nabla^2 T = 0$, except at the point $z - z_n$. At the boundary $r = r_b$ the function W_{no} satisfies (3.12) and (3.15). The identities are valid for any z_n, and hence for the derivative with respect to z_n. But the derivative of ϕ $(= arctan(y/x))$ in (3.12) with respect to z_n is zero. Therefore we have:

$$W_{n1}|_{r_b-0} = W_{n1}|_{r_b+0} \tag{4.4}$$

$$\lambda_b \left.\frac{\partial W_{n1}}{\partial r}\right|_{r_b-0} = \lambda \left.\frac{\partial W_{n1}}{\partial r}\right|_{r_b+0} \tag{4.5}$$

This means that W_{n1} and the radial heat flux are continuous at $r = r_b$. This is true both for $Re(W_{n1})$ and $Im(W_{n1})$.

The multipole of any order j is given by the j:th derivative of W_{no} with respect to z_n. We define W_{nj} by:

$$W_{nj} = \frac{1}{(j-1)!} \cdot \frac{\partial^j}{\partial z_n^j}(W_{no}) \tag{4.6}$$

From (3.8) and (3.9) we get the neat expressions:

$$\begin{cases} W_{nj} = \dfrac{1}{(z - z_n)^j} + \sigma \cdot \left(\dfrac{\bar{z}}{r_b^2 - \bar{z} z_n} \right)^j & r \leq r_b \\ W_{nj} = (1+\sigma) \dfrac{1}{(z - z_n)^j} & r \geq r_b \end{cases} \tag{4.7-8}$$

The expressions (4.7-8) are regular functions, which implies that the real and imaginary parts satisfy $\nabla^2 T = 0$, except at $z = z_n$. The boundary conditions (4.4) and (4.5) may be derived with respect to z_n any number of times. This means that W_{nj} and the radial heat flux are continuous at $r = r_b$ for any j:

$$W_{nj}\big|_{r_b-0} = W_{nj}\big|_{r_b+0} \tag{4.9}$$

$$\lambda_b \left.\frac{\partial W_{nj}}{\partial r}\right|_{r_b-0} = \lambda \left.\frac{\partial W_{nj}}{\partial r}\right|_{r_b+0} \tag{4.10}$$

The multipole (4.7) expressed in local polar coordinates around $z = z_n$ becomes:

$$W_{nj} = \frac{1}{(\rho_n e^{i\psi_n})^j} + \sigma \cdot \left(\frac{\bar{z}_n + \rho_n e^{-i\psi_n}}{r_b^2 - r_n^2 - \rho_n e^{-i\psi_n} \cdot z_n}\right)^j \tag{4.11}$$

,or

$$W_{nj} = (cos(j\psi_n) - i \cdot sin(j\psi_n)) \cdot \rho_n^{-j} + \sigma \cdot \left(\frac{\bar{z}_n + \rho_n e^{-i\psi_n}}{r_b^2 - r_n^2 - \rho_n e^{-i\psi_n} \cdot z_n}\right)^j \qquad (\rho_n < r_b - r_n) \tag{4.12}$$

The first part represents a pure multipole behaviour, while the second part is a correction to account for the effect of the different thermal conductivity λ for $r > r_b$. The real part of W_{nj}, $Re(W_{nj})$, gives a variation $cos(j\psi_n)$ around the point $z = z_n$, and the imaginary part, $Im(W_{nj})$, gives a variation $sin(j\psi_n)$. The temperature field from the general multipole of order j at $z = z_n$ is:

$$T = Re\left(P_{nj} \cdot W_{nj}\right) = Re\left((c_{nj} + i \cdot s_{nj})W_{nj}\right) = c_{nj} \cdot Re(W_{nj}) - s_{nj} \cdot Im(W_{nj}) \tag{4.13}$$

Here P_{nj} is an arbitrary complex number.

The line sink and multipoles at z_n can be used to represent an arbitrary temperature solution outside pipe n. We need a corresponding representation for the outer boundary circle $r = r_c$. Here we need a solution in the composite region inside $r = r_c$. The solution in complex form shall vary as $e^{i \cdot j\phi}$ on $r = r_c$. We have the corresponding solutions to $\nabla^2 T = 0$ in polar coordinates in $r_b < r < r_c$:

$$r^j \cdot e^{i \cdot j\phi} = z^j \qquad r^{-j} e^{i \cdot j\phi} = \frac{1}{\bar{z}^j} \tag{4.14}$$

In the inner region $0 \le r < r_b$ we can use only the first solution z^j, since the second one is infinite at $r = 0$ or $z = 0$.

Therefore we start with the following expressions:

$$\begin{aligned} W_{cj} &= A \cdot z^j \qquad 0 \le r \le r_b \\ W_{cj} &= z^j + B/\bar{z}^j \qquad r_b \le r \le r_c \end{aligned} \tag{4.15}$$

Continuity at $z = r_b \cdot e^{i\phi}$ requires:

$$A r_b^j = r_b^j + B/r_b^j \tag{4.16}$$

The radial heat flux is also continuous at $z = r_b e^{i\phi}$:

$$\lambda_b \cdot \left.\frac{\partial W_{cj}}{\partial r}\right|_{r_b-0} = \lambda_b A \cdot j r_b^{j-1} \cdot e^{i \cdot j\phi} = \lambda \left.\frac{\partial W_{cj}}{\partial r}\right|_{r_b+0}$$

$$= \lambda \cdot j \cdot e^{i \cdot j\phi}\left(r_b^{j-1} - B/r_b^{j+1}\right) \tag{4.17}$$

The constants A and B are determined by (4.16-17). This gives the following basic expressions:

$$\begin{cases} W_{cj} = (1-\sigma) z^j & r \le r_b \\ W_{cj} = z^j - \sigma \cdot \left(r_b^2/\bar{z}\right)^j & r \ge r_b \end{cases} \tag{4.18}$$

We will call these solutions W_{cj} multipoles at infinity. The index c refers to the fact that they are needed at the outer boundary $r = r_c$. The multipole of order j can represent the variation $e^{i \cdot j\phi}$ or in real form any combination of $cos(j\phi)$ and $sin(j\phi)$ at $r = r_c$.

5. GENERAL EXPRESSION FOR THE TEMPERATURE

The general expression for the temperature field uses all the line sinks and the multipoles of all orders at the pipes and at infinity. We have the following general expression:

$$T = T_o + Re\left[\sum_{n=1}^{N} P_n \cdot W_{no} + \sum_{n=1}^{N}\sum_{j} P_{nj} \cdot r_{pn}^j \cdot W_{nj} + \sum_{j} P_{cj} r_c^{-j} W_{cj}\right] \tag{5.1}$$

Here W_{no} is given by (3.8-9), W_{nj} by (4.7-8) and W_{cj} by (4.18). The factors r_{pn}^j and r_c^{-j} are introduced for dimensional reasons. The dimension of P_n, P_{nj}, and P_{cj} is that of a temperature.

The strength of the line source q_n is in accordance with (3.7) related to P_n by:

$$P_n = \frac{q_n}{2\pi\lambda_b} \tag{5.2}$$

The multipole factors P_{nj} and P_{cj} are complex numbers, while P_n of course is real. The temperature T_o is an arbitrary constant. The summation in j runs from 1 to infinity for an exact solution, and from 1 to J in the truncated approximate solution. The multipoles are not used in the case $J = 0$.

The general temperature (5.1) becomes with the explicit formulas (3.8-9), (4.7-8) and (4.18):

$r \le r_b$:

$$\begin{aligned} T = T_o + Re\Bigg[&\sum_{n=1}^{N} P_n \left\{ ln\left(\frac{r_b}{z - z_n}\right) + \sigma \cdot ln\left(\frac{r_b^2}{r_b^2 - \bar{z} z_n}\right)\right\} \\ &+ \sum_{n=1}^{N}\sum_{j=1} P_{nj}\left\{\left(\frac{r_{pn}}{z - z_n}\right)^j + \sigma\left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}}\right)^j\right\} \\ &+ \sum_{j=1} P_{cj}(1-\sigma)\left(\frac{z}{r_c}\right)^j\Bigg] \end{aligned} \tag{5.3}$$

$r_b \le r \le r_c$:

$$\begin{aligned} T = T_o + Re\Bigg[&\sum_{n=1}^{N} P_n \left\{(1+\sigma) \cdot ln\left(\frac{r_b}{z - z_n}\right) + \frac{\lambda_b}{\lambda}\sigma \cdot ln\left(\frac{r_b}{z}\right)\right\} \\ &+ \sum_{n=1}^{N}\sum_{j=1} P_{nj}(1+\sigma)\left(\frac{r_{pn}}{z - z_n}\right)^j \\ &+ \sum_{j=1} P_{cj}\left\{\left(\frac{z}{r_c}\right)^j - \sigma \cdot \left(\frac{r_b^2}{r_c\bar{z}}\right)^j\right\}\Bigg] \end{aligned} \tag{5.4}$$

The general expressions (5.1) or (5.3-4) satisfy the heat conduction equation (2.1) everywhere except at the points $z = z_n$ for any choice of the real constants T_o, P_n and of the complex ones, P_{nj} and P_{cj}. Each term satisfies the boundary conditions (2.5) at $r = r_b$. The constants T_o and P_{cj} are chosen so that the boundary condition (2.7) at the outer circle is fulfilled. This is done in the next chapter, while Chapter 7 deals with the boundary conditions at the pipes. We will get a set of equations for T_o, P_n, P_{nj} and P_{cj}.

The method of solution presented in this paper and in [1] is essentially to use Fourier series expansions; one for each pipe and one for the outer circle. The problem is to transform the coordinates of the different components into polar coordinates for each pipe and for the outer circle. A basic idea in the present method is to use the complex form, which greatly facilitates the derivation of the equation system for the boundary conditions.

6. EQUATIONS FROM THE BOUNDARY CONDITION AT THE OUTER CIRCLE

The expressions (5.3) and (5.4) shall satisfy the boundary conditions (2.6-7). In the previous study [1], each type of term was analysed separately (in Chapter 5 of [1]). We will here use a slightly different approach. The temperature (5.3) is first expressed directly in the local polar coordinates of the considered pipe. Then the expression is inserted in the boundary condition (2.6), and we get an equation system that determines the line sinks P_n and the multipoles P_{nj}. This is done in the next chapter.

In this chapter we consider the boundary condition (2.7) at the outer circle $r = r_c$. Our first goal is to express the temperature (5.4) for the outer region $r_b \leq r \leq r_c$ in polar coordinates: $T = T(r, \phi)$. This is achieved by putting $z = re^{i\phi}$ in (5.4). We will need to distinguish between different powers $(e^{i\phi})^k$ or z^k in the subsequent analysis. The functions of z in (5.4) are therefore expanded in Taylor series.

For the logarithm of (5.4) we have:

$$ln\left(\frac{r_b}{z - z_n}\right) = ln\left(\frac{r_b}{z}\right) + ln\left(\frac{1}{1 - z_n/z}\right) \tag{6.1}$$

We need the Taylor series:

$$ln\left(\frac{1}{1-s}\right) = \sum_{k=1}^{\infty} \frac{1}{k} s^k \qquad |s| < 1 \tag{6.2}$$

Then we have:

$$ln\left(\frac{r_b}{z - z_n}\right) = ln\left(\frac{r_b}{z}\right) + \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{z_n}{z}\right)^k \qquad |z_n| < |z| \tag{6.3}$$

We also need a Taylor expansion of the multipole term $(r_{pn}/(z - z_n))^j$ in (5.4). We have:

$$\left(\frac{r_{pn}}{z - z_n}\right)^j = \left(\frac{r_{pn}}{z}\right)^j \cdot \sum_{k'=0}^{\infty} \binom{j + k' - 1}{j - 1} \left(\frac{z_n}{z}\right)^{k'} \qquad |z_n| < |z| \tag{6.4}$$

Here we have used a Taylor series given in [1]:

$$\frac{1}{(1-s)^j} = \sum_{k=0}^{\infty} \binom{j + k - 1}{j - 1} s^k \qquad |s| < 1 \tag{6.5}$$

The multipoles (6.4) are summed over j in (5.4). We need to rearrange the double sum in the following way:

$$\begin{aligned} \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z - z_n}\right)^j &= \sum_{j=1}^{\infty}\sum_{k'=0}^{\infty} P_{nj} \left(\frac{r_{pn}}{z}\right)^j \binom{j + k' - 1}{j - 1} \left(\frac{z_n}{z}\right)^{k'} \\ &= [j + k' = k] = \sum_{k=1}^{\infty}\sum_{j=1}^{k} P_{nj} \binom{k - 1}{j - 1} \frac{r_{pn}^j z_n^{k-j}}{z^k} \end{aligned} \tag{6.6}$$

The change of summation from j, k' to k, j means that j varies between 1 and k for $k = 1, 2 \ldots$.

We insert (6.3) and (6.6) in (5.4) and put $z = re^{i\phi}$. Consider first the terms containing $ln(z)$. For these we have from (5.4) and (6.3):

$$
\begin{aligned}
&Re\left[\sum_{n=1}^{N} P_n\left\{(1+\sigma)\cdot ln\left(\frac{r_b}{re^{i\phi}}\right)+\frac{\lambda_b}{\lambda}\sigma\cdot ln\left(\frac{r_b}{re^{i\phi}}\right)\right\}\right]\\
&=\sum_{n=1}^{N} P_n\left(1+\sigma+\frac{\lambda_b}{\lambda}\sigma\right)\cdot ln\left(\frac{r_b}{r}\right)=\sum_{n=1}^{N} P_n\frac{\lambda_b}{\lambda}\cdot ln\left(\frac{r_b}{r}\right)
\end{aligned}
\tag{6.7}
$$

In the last line we used the definition (3.3) for σ. The temperature as a function of r and ϕ is now using (5.4), (6.3), (6.6) and (6.7) with z equal to $re^{i\phi}$:

$$
\begin{aligned}
T(r,\phi)=T_o&+\sum_{n=1}^{N} P_n\frac{\lambda_b}{\lambda}\cdot ln\left(\frac{r_b}{r}\right)+Re\left[\sum_{n=1}^{N}\sum_{k=1}^{\infty} P_n(1+\sigma)\frac{1}{k}\left(\frac{z_n}{r}\right)^k e^{-i\cdot k\phi}\right.\\
&+\sum_{n=1}^{N}\sum_{k=1}^{\infty}\sum_{j=1}^{k} P_{nj}(1+\sigma)\binom{k-1}{j-1}\frac{r_{pn}^j z_n^{k-j}}{r^k}\cdot e^{-i\cdot k\phi}\\
&\left.+\sum_{k=1}^{\infty} P_{ck}\left\{\left(\frac{r}{r_c}\right)^k-\sigma\cdot\left(\frac{r_b^2}{r_c r}\right)^k\right\}e^{i\cdot k\phi}\right]\qquad (r_b\le r\le r_c)
\end{aligned}
\tag{6.8}
$$

The summation index is in the last line changed from j to k.

The temperature (6.8) shall satisfy the boundary condition (2.7). For the radial derivative of any power of r we have:

$$
\begin{aligned}
&\beta_c r_c\frac{\partial}{\partial r}\left(r^k\right)\Big|_{r=r_c}=\beta_c r_c\cdot k\cdot r_c^{k-1}=k\beta_c\cdot r_c^k\\
&\beta_c r_c\frac{\partial}{\partial r}\left(r^{-k}\right)\Big|_{r=r_c}=-k\beta_c\cdot r_c^{-k}
\end{aligned}
\tag{6.9}
$$

We will separate different k, i.e. different orders in $e^{-i\cdot k\phi}$. The last line of (6.8) has a positive exponent, which is changed by complex conjugation:

$$
Re\left[P_{ck}\cdot e^{i\cdot k\phi}\right]=Re\left[\bar{P}_{ck}\cdot e^{-i\cdot k\phi}\right]
\tag{6.10}
$$

The boundary condition (2.7) may now be written:

$$
\begin{aligned}
T_c=\left(T+\beta_c\frac{\partial T}{\partial r}\right)\Big|_{r=r_c}&=T_o+\sum_{n=1}^{N} P_n\frac{\lambda_b}{\lambda}\left\{ln\left(\frac{r_b}{r_c}\right)+\beta_c r_c\cdot\frac{-1}{r_c}\right\}\\
&+\sum_{k=1}^{\infty} Re\left[e^{-i\cdot k\phi}\cdot\left\{\sum_{n=1}^{N} P_n(1+\sigma)\frac{1}{k}(1-k\beta_c)\left(\frac{z_n}{r_c}\right)^k\right.\right.\\
&+\sum_{n=1}^{N}\sum_{j=1}^{k} P_{nj}(1+\sigma)\binom{k-1}{j-1}(1-k\beta_c)\frac{r_{pn}^j z_n^{k-j}}{r_c^k}\\
&\left.\left.+\bar{P}_{ck}\left((1+k\beta_c)\cdot\left(\frac{r_c}{r_c}\right)^k-\sigma(1-k\beta_c)\left(\frac{r_b^2}{r_c\cdot r_c}\right)^k\right)\right\}\right]\qquad 0\le\phi\le 2\pi
\end{aligned}
\tag{6.11}
$$

Equation (6.11) is valid for $0\le\phi\le 2\pi$. The constant part (independent of ϕ) must vanish. This determines T_o:

$$
T_o=T_c+\sum_{n=1}^{N} P_n\frac{\lambda_b}{\lambda}\left(ln\left(\frac{r_c}{r_b}\right)+\beta_c\right)
\tag{6.12}
$$

This equation relates the temperature level T_o to the temperature T_c at the outer circle and the line sources P_n.

The remaining part of (6.11) is an equation of the following type:

$$
\begin{aligned}
0 &= \sum_{k=1}^{\infty} Re\left[e^{-i\cdot k\phi}\cdot Z_k\right] \\
&= \sum_{k=1}^{\infty} \{cos(k\phi)\cdot Re(Z_k) + sin(k\phi)\cdot Im(Z_k)\} \qquad 0 \le \phi \le 2\pi
\end{aligned}
\tag{6.13}
$$

All coefficients before $cos(k\phi)$ and $sin(k\phi)$ must vanish. This means that the complex factor Z_k is zero for all k. We have the following equations:

$k = 1, 2, \ldots$:

$$
\begin{aligned}
&P_{ck}\cdot\left\{1 - \sigma\frac{1-k\beta_c}{1+k\beta_c}\left(\frac{r_b}{r_c}\right)^{2k}\right\} \\
&+ (1+\sigma)\frac{1-k\beta_c}{1+k\beta_c}\cdot\left\{\sum_{n=1}^{N} P_n\frac{1}{k}\left(\frac{z_n}{r_c}\right)^k + \sum_{n=1}^{N}\sum_{j=1}^{k} P_{nj}\binom{k-1}{j-1}\frac{r_{pn}^j z_n^{k-j}}{r_c^k}\right\} = 0
\end{aligned}
\tag{6.14}
$$

This equation relates the value of a multipole P_{ck} at infinity to the values of line sources P_n and the multipoles P_{nj} at the pipes up to order $j = k$.

7. EQUATIONS FROM THE BOUNDARY CONDITION AT THE PIPES

The boundary condition at pipe m is from (2.6):

$$
T_{fm} = \left(T - \beta_m r_{pm}\frac{\partial T}{\partial \rho_m}\right)\Bigg|_{\rho_m = r_{pm}} \qquad 0 \le \psi_m \le 2\pi \tag{7.1}
$$

The temperature is given by (5.3). We will first express this temperature in the local polar coordinates ρ_m, ψ_m of pipe m, which are given by (2.4):

$$
z - z_m = \rho_m e^{i\psi_m} \tag{7.2}
$$

We want to separate different powers $(z - z_m)^k$ or $(e^{i\psi_m})^k$. We therefore expand the various terms of (5.3) in Taylor series in $z - z_m$.

We will use the following expressions.

A. $n \neq m$

$$
\begin{aligned}
ln\left(\frac{r_b}{z - z_n}\right) &= ln\left(\frac{r_b}{z - z_m + z_m - z_n}\right) \\
&= ln\left(\frac{r_b}{z_m - z_n}\right) + ln\left(\frac{1}{1 - \frac{z - z_m}{z_n - z_m}}\right)
\end{aligned}
\tag{7.3}
$$

The second logarithm is expanded in the Taylor series (6.2).

B.

$$
\begin{aligned}
ln\left(\frac{r_b^2}{r_b^2 - z_n\bar{z}}\right) &= ln\left(\frac{r_b^2}{r_b^2 - z_n\bar{z}_m - z_n(\bar{z} - \bar{z}_m)}\right) \\
&= ln\left(\frac{r_b^2}{r_b^2 - z_n\bar{z}_m}\right) + ln\left(\frac{1}{1 - \frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}}\right)
\end{aligned}
\tag{7.4}
$$

The second logarithm is expanded in the Taylor series (6.2).

C. $n \neq m$

$$\left(\frac{r_{pn}}{z - z_n}\right)^j = \left(\frac{r_{pn}}{z_m - z_n}\right)^j \cdot \frac{1}{\left(1 - \frac{z - z_m}{z_n - z_m}\right)^j} \tag{7.5}$$

The second factor is expanded with the Taylor series (6.5).

D.

$$\begin{aligned}\left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}}\right)^j &= \left(\frac{r_{pn}(\bar{z}_m + \bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m - z_n(\bar{z} - \bar{z}_m)}\right)^j \\ &= \left(\frac{r_{pn}}{r_b^2 - z_n\bar{z}_m}\right)^j \cdot (\bar{z}_m + \bar{z} - \bar{z}_m)^j \cdot \frac{1}{\left(1 - \frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^j}\end{aligned} \tag{7.6}$$

The second factor in expanded as a binominal. The third factor is expanded in the series (6.5):

$$(\bar{z}_m + \bar{z} - \bar{z}_m)^j = \sum_{j'=0}^{j} \binom{j}{j'} (\bar{z} - \bar{z}_m)^{j'} \cdot \bar{z}_m^{j-j'} \tag{7.7}$$

$$\frac{1}{\left(1 - \frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^j} = \sum_{k'=0}^{\infty} \binom{j + k' - 1}{j - 1} \left(\frac{z_n(\bar{z} - \bar{z}_m)}{r_b^2 - z_n\bar{z}_m}\right)^{k'} \tag{7.8}$$

The product of (7.7) and (7.8) is a double sum containing the powers $(\bar{z} - \bar{z}_m)^{k'+j'}$. We change the summation from k' and j' to $k = k' + j'$ and j'. The index k will vary from 0 to infinity. The values of j' run from 0 to min (j, k). We have the expansion:

$$\begin{aligned}\left(\frac{r_{pn}\bar{z}}{r_b^2 - z_n\bar{z}}\right)^j &= \sum_{k=0}^{\infty} \sum_{j'=0}^{min(j,k)} \binom{j}{j'} \binom{j + k - j' - 1}{j - 1} \\ &\cdot \frac{r_{pn}^j \bar{z}_m^{j-j'} z_n^{k-j'}}{(r_b^2 - z_n\bar{z}_m)^{j+k-j'}} \cdot (\bar{z} - \bar{z}_m)^k\end{aligned} \tag{7.9}$$

E.

$$\begin{aligned}\sum_{j=1}^{\infty} P_{cj}(1 - \sigma) \left(\frac{z}{r_c}\right)^j &= \sum_{j=1}^{\infty} P_{cj}(1 - \sigma) \left(\frac{z_m + z - z_m}{r_c}\right)^j \\ &= \sum_{j=1}^{\infty} \sum_{k=0}^{j} P_{cj}(1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} \cdot (z - z_m)^k}{r_c^j}\end{aligned} \tag{7.10}$$

The summation order is changed. Then we have:

$$\sum_{j=1}^{\infty} \sum_{k=0}^{j} \dots = \sum_{k=0}^{\infty} \sum_{j=max(1,k)}^{\infty} \dots \tag{7.11}$$

The temperature (5.3) may now with the use of (7.2-11) be written in the following way:

$$
\begin{aligned}
T = {} & T_o + P_m ln\left(\frac{r_b}{\rho_m}\right) \\
& + Re\left[\sum_{\substack{n=1\\n\neq m}}^{N} P_n\left\{ln\left(\frac{r_b}{z_m - z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\rho_m}{z_n - z_m}\right)^k \cdot e^{i\cdot k\psi_m}\right\}\right. \\
& + \sum_{n=1}^{N} P_n\sigma\left\{ln\left(\frac{r_b^2}{r_b^2 - \bar{z}_m z_n}\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_n\rho_m}{r_b^2 - \bar{z}_m z_n}\right)^k \cdot e^{-i\cdot k\psi_m}\right\} \\
& + \sum_{j=1}^{\infty} P_{mj}\left(\frac{r_{pm}}{\rho_m}\right)^j \cdot e^{-i\cdot j\psi_m} \\
& + \sum_{\substack{n=1\\n\neq m}}^{N}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} P_{nj}\left(\frac{r_{pn}}{z_m - z_n}\right)^j \binom{j+k-1}{j-1}\left(\frac{\rho_m}{z_n - z_m}\right)^k \cdot e^{i\cdot k\psi_m} \\
& + \sum_{n=1}^{N}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\sum_{j'=0}^{min(j,k)} P_{nj}\sigma\binom{j}{j'}\binom{j+k-j'-1}{j-1}\cdot\frac{r_{pn}^j \bar{z}_m^{j-j'} z_n^{k-j'}\cdot\rho_m^k}{(r_b^2 - z_n\bar{z}_m)^{j+k-j'}}\cdot e^{-i\cdot k\psi_m} \\
& \left. + \sum_{k=0}^{\infty}\sum_{j=max(1,k)}^{\infty} P_{cj}(1-\sigma)\binom{j}{k}\frac{z_m^{j-k}\cdot\rho_m^k}{r_c^j} e^{i\cdot k\psi_m}\right]
\end{aligned} \tag{7.12}
$$

The summation index on the fourth line (concerning P_{mj}) is changed to k. The dependence on ψ_m lies in the exponents $e^{i\cdot k\psi_m}$ and $e^{-i\cdot k\psi_m}$. The latter terms may be changed to the positive exponent $e^{i\cdot k\psi_m}$ by taking the complex conjugate as in (6.10). Expression (7.12) is with these modifications inserted in the boundary condition (7.1) for pipe m. The derivatives of powers of ρ_m are simple. We have as in (6.9):

$$
\begin{aligned}
-\beta_m r_{pm}\frac{\partial}{\partial\rho_m}\left(\rho_m^k\right)\Big|_{\rho_m=r_{pm}} &= -k\beta_m r_{pm}^k \\
-\beta_m r_{pm}\frac{\partial}{\partial\rho_m}\left(\rho_m^{-k}\right)\Big|_{\rho_m=r_{pm}} &= k\beta_m r_{pm}^{-k}
\end{aligned} \tag{7.13}
$$

We finally get the following expression for the boundary condition at pipe m:

$$
\begin{aligned}
T_{fm} &= \left(T - \beta_m r_{pm} \frac{\partial T}{\partial \rho_m}\right)\Bigg|_{\rho_m = r_{pm}} \\
&= T_o + P_m \left\{ ln\left(\frac{r_b}{r_{pm}}\right) - \beta_m r_{pm} \cdot \frac{-1}{r_{pm}} \right\} \\
&+ \sum_{\substack{n=1 \\ n \neq m}}^{N} P_n ln\left(\frac{r_b}{r_{mn}}\right) + \sum_{n=1}^{N} P_n \sigma \cdot ln\left(\frac{r_b^2}{|r_b^2 - \bar{z}_m z_n|}\right) \\
&+ Re\left[\sum_{\substack{n=1 \\ n \neq m}}^{N} \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z_m - z_n}\right)^j + \sum_{n=1}^{N}\sum_{j=1}^{\infty} P_{nj}\sigma \left(\frac{r_{pn}\bar{z}_m}{r_b^2 - z_n \bar{z}_m}\right)^j \right. \\
&\left. + \sum_{j=1}^{\infty} P_{cj}(1-\sigma)\left(\frac{z_m}{r_c}\right)^j \right] \\
&+ Re\left[\sum_{k=1}^{\infty} e^{i \cdot k\psi_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^{N} P_n(1 - k\beta_m)\frac{1}{k}\left(\frac{r_{pm}}{z_n - z_m}\right)^k \right. \right. \\
&+ \sum_{n=1}^{N} P_n\sigma(1 - k\beta_m)\frac{1}{k}\left(\frac{\bar{z}_n r_{pm}}{r_b^2 - z_m \bar{z}_n}\right)^k \\
&+ \bar{P}_{mk}(1 + k\beta_m)\left(\frac{r_{pm}}{r_{pm}}\right)^k \\
&+ \sum_{\substack{n=1 \\ n \neq m}}^{N}\sum_{j=1}^{\infty} P_{nj}(1 - k\beta_m)\binom{j+k-1}{j-1}\left(\frac{r_{pn}}{z_m - z_n}\right)^j \left(\frac{r_{pm}}{z_n - z_m}\right)^k \\
&+ \sum_{n=1}^{N}\sum_{j=1}^{\infty}\sum_{j'=0}^{min(j,k)} \bar{P}_{nj}\sigma(1 - k\beta_m)\binom{j}{j'}\binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\
&\left.\left. + \sum_{j=k}^{\infty} P_{cj}(1-\sigma)(1 - k\beta_m)\binom{j}{k}\frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\}\right] \qquad 0 \leq \psi_m \leq 2\pi
\end{aligned}
\tag{7.14}
$$

The first four lines on the right-hand side give the constant part; i.e. the part independent of ψ_m. The length r_{mn} denotes the distance between the centers of pipes m and n:

$$r_{mn} = |z_m - z_n| = \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} \tag{7.15}$$

Equation (7.14) is valid for all ψ_m around the pipe. The last six lines of (7.14) contain the part that depends on ψ_m. It must be equal to zero. The expression is of the same type as (6.13). The complex factor for each component $e^{i \cdot k\psi_m}$ must vanish. We get the following equations, which determine the multipoles P_{nj}:

$m = 1, ...N \quad ; \quad k = 1, 2, ...:$

$$\bar{P}_{mk} + \frac{1 - k\beta_m}{1 + k\beta_m} \left\{ \sum_{\substack{n=1 \\ n \neq m}}^{N} P_n \frac{1}{k} \left(\frac{r_{pm}}{z_n - z_m} \right)^k + \sum_{n=1}^{N} P_n \sigma \frac{1}{k} \left(\frac{r_{pm} \bar{z}_n}{r_b^2 - z_m \bar{z}_n} \right)^k \right.$$
$$+ \sum_{\substack{n=1 \\ n \neq m}}^{N} \sum_{j=1}^{\infty} P_{nj} \binom{j+k-1}{j-1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j \left(\frac{r_{pm}}{z_n - z_m} \right)^k$$
$$+ \sum_{n=1}^{N} \sum_{j=1}^{\infty} \sum_{j'=0}^{min(j,k)} \bar{P}_{nj} \sigma \binom{j}{j'} \binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'} \bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \tag{7.16}$$
$$\left. + \sum_{j=k}^{\infty} P_{cj} (1 - \sigma) \binom{j}{k} \frac{z_m^{j-k} r_{pm}^k}{r_c^j} \right\} = 0$$

The constant part of (7.14) must also vanish. This means that the first four lines on the right-hand side are equal to T_{fm}. We can eliminate T_o from (6.12). Then we get the following equations:

$m = 1, ...N$:

$$T_{fm} - T_c = \sum_{n=1}^{N} q_n \cdot R_{mn}^o$$
$$+ Re \left[\sum_{\substack{n=1 \\ n \neq m}}^{N} \sum_{j=1}^{\infty} P_{nj} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^{N} \sum_{j=1}^{\infty} P_{nj} \sigma \left(\frac{r_{pn} \bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j \right. \tag{7.17}$$
$$\left. + \sum_{j=1}^{\infty} P_{cj} (1 - \sigma) \left(\frac{z_m}{r_c} \right)^j \right]$$

The first line involves on the right-hand side the line sources P_n. We have returned to q_n via (5.2).

The coefficients R_{mn}^o are given by:

$$R_{mm}^o = \frac{1}{2\pi\lambda_b} \left\{ \beta_m + ln\left(\frac{r_b}{r_{pm}} \right) + \sigma \cdot ln\left(\frac{r_b^2}{r_b^2 - r_m^2} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ ln\left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m = 1, ...N$$
$$R_{mn}^o = \frac{1}{2\pi\lambda_b} \left\{ ln\left(\frac{r_b}{r_{mn}} \right) + \sigma \cdot ln\left(\frac{r_b^2}{|r_b^2 - z_n \bar{z}_m|} \right) \right\} + \frac{1}{2\pi\lambda} \left\{ ln\left(\frac{r_c}{r_b} \right) + \beta_c \right\} \quad m \neq n \quad m, n = 1, ...N \tag{7.18}$$

8. FINAL EQUATIONS AND ITERATIVE SOLUTION

The final equations for q_n (or P_n), P_{nj} and P_{cj} are (7.17-18), (7.16) and (6.14). Multipoles of all orders are needed in an exact solution. In the numerical one we truncate the equation system and consider multipoles up to order J at each pipe and at infinity. The sine- and cosinevariation around the pipes and around the outer circle can then be satisfied up to order J only. The truncation error is discussed in the next chapter.

We have from (7.17), (7.16), (5.2) and (6.14) the following final equations:
$m = 1, ..., N$:

$$T_{fm} - T_c = \sum_{n=1}^{N} q_n \cdot R^o_{mn} + Re\left[\sum_{\substack{n=1\\n\neq m}}^{N}\sum_{j=1}^{J} P_{nj}\left(\frac{r_{pn}}{z_m - z_n}\right)^j + \sum_{n=1}^{N}\sum_{j=1}^{J} P_{nj}\sigma\left(\frac{r_{pn}\bar{z}_m}{r_b^2 - z_n\bar{z}_m}\right)^j + \sum_{j=1}^{J} P_{cj}(1-\sigma)\left(\frac{z_m}{r_c}\right)^j\right] \tag{8.1}$$

$m = 1, ...N \quad ; \quad k = 1, ...J$:

$$\bar{P}_{mk} + \frac{1 - k\beta_m}{1 + k\beta_m}\left\{\sum_{\substack{n=1\\n\neq m}}^{N} \frac{q_n}{2\pi\lambda_b}\frac{1}{k}\left(\frac{r_{pm}}{z_n - z_m}\right)^k + \sum_{n=1}^{N}\frac{q_n}{2\pi\lambda_b}\sigma\frac{1}{k}\left(\frac{r_{pm}\bar{z}_n}{r_b^2 - z_m\bar{z}_n}\right)^k \right.$$
$$+ \sum_{\substack{n=1\\n\neq m}}^{N}\sum_{j=1}^{J} P_{nj}\binom{j+k-1}{j-1}\left(\frac{r_{pn}}{z_m - z_n}\right)^j\left(\frac{r_{pm}}{z_n - z_m}\right)^k$$
$$+ \sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{j'=0}^{min(j,k)} \bar{P}_{nj}\sigma\binom{j}{j'}\binom{j+k-j'-1}{j-1} \cdot \frac{r_{pn}^j r_{pm}^k z_m^{j-j'}\bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \tag{8.2}$$
$$\left. + \sum_{j=k}^{J} P_{cj}(1-\sigma)\binom{j}{k}\frac{z_m^{j-k} r_{pm}^k}{r_c^j}\right\} = 0$$

$k = 1, ...J$:

$$\bar{P}_{ck} \cdot \left\{1 - \sigma\frac{1 - k\beta_c}{1 + k\beta_c}\left(\frac{r_b}{r_c}\right)^{2k}\right\}$$
$$+ (1+\sigma)\frac{1 - k\beta_c}{1 + k\beta_c} \cdot \left\{\sum_{n=1}^{N}\frac{q_n}{2\pi\lambda_b}\frac{1}{k}\left(\frac{z_n}{r_c}\right)^k + \sum_{n=1}^{N}\sum_{j=1}^{k} P_{nj}\binom{k-1}{j-1}\frac{r_{pn}^j z_n^{k-j}}{r_c^k}\right\} = 0 \tag{8.3}$$

The thermal resistances R^o_{mn} are given by (7.18):

$$R^o_{mm} = \frac{1}{2\pi\lambda_b}\left\{\beta_m + ln\left(\frac{r_b}{r_{pm}}\right) + \sigma \cdot ln\left(\frac{r_b^2}{r_b^2 - r_m^2}\right)\right\} + \frac{1}{2\pi\lambda}\left\{ln\left(\frac{r_c}{r_b}\right) + \beta_c\right\} \qquad m = 1, ...N$$
$$R^o_{mn} = \frac{1}{2\pi\lambda_b}\left\{ln\left(\frac{r_b}{r_{mn}}\right) + \sigma \cdot ln\left(\frac{r_b^2}{|r_b^2 - z_n\bar{z}_m|}\right)\right\} + \frac{1}{2\pi\lambda}\left\{ln\left(\frac{r_c}{r_b}\right) + \beta_c\right\} \qquad m \neq n \tag{8.4}$$
$$m, n = 1, ...N$$

There are N real-valued equations (8.1) and $N \cdot J + J$ complex-valued ones (8.2-3). This corresponds to the N line sources q_n and the $N \cdot J + J$ complex-valued multipoles P_{nj} and P_{cj}.

The equation system (8.1-4) is solved iteratively as in [1]. Let K^o_{mn} be the elements of the inverse matrix of the resistance matrix with the elements R^o_{mn}:

$$(K^o_{mn}) = (R^o_{mn})^{-1} \tag{8.5}$$

Inversion of (8.1) gives the heat fluxes:

$$\begin{aligned} q_{m'} = \sum_{m=1}^{N} K^o_{m'm} \Bigg\{ T_{fm} - T_c \\ - Re\left[\sum_{\substack{n=1\\ n\neq m}}^{N} \sum_{j=1}^{J} P_{nj} \left(\frac{r_{pn}}{z_m - z_n}\right)^j + \sum_{n=1}^{N}\sum_{j=1}^{J} P_{nj}\sigma \left(\frac{r_{pn}\bar{z}_m}{r_b^2 - z_n\bar{z}_m}\right)^j \right. \\ \left. + \sum_{j=1}^{J} P_{cj}(1-\sigma)\left(\frac{z_m}{r_c}\right)^j \right] \Bigg\} \end{aligned} \tag{8.6}$$

Let q_n^ν, P_{nj}^ν and P_{cj}^ν denote the values of our variables for iteration step ν. We start for $\nu = 0$ with the values:

$$\begin{aligned} P^o_{nj} = 0 \qquad\qquad P^0_{cj} = 0 \\ q^o_{m'} = \sum_{m=1}^{N} K^o_{m'm}\,(T_{fm} - T_c) \end{aligned} \tag{8.7}$$

This means that we start without multipoles and compute the heat fluxes with (8.6).
For $\nu = 1, 2 \ldots$ we use the following recursive formulas in accordance with (8.2), (8.3) and (8.6):
$m = 1, \ldots N \quad ; \quad k = 1, \ldots J$:

$$\begin{aligned} \bar{P}^{\nu+1}_{mk} = -\frac{1 - k\beta_m}{1 + k\beta_m} \Bigg\{ \sum_{\substack{n=1\\ n\neq m}}^{N} \frac{q_n^\nu}{2\pi\lambda_b}\frac{1}{k}\left(\frac{r_{pm}}{z_n - z_m}\right)^k + \sum_{n=1}^{N} \frac{q_n^\nu}{2\pi\lambda_b}\sigma\frac{1}{k}\left(\frac{r_{pm}\bar{z}_n}{r_b^2 - z_m\bar{z}_n}\right)^k \\ + \sum_{\substack{n=1\\ n\neq m}}^{N}\sum_{j=1}^{J} P^\nu_{nj}\binom{j+k-1}{j-1}\left(\frac{r_{pn}}{z_m - z_n}\right)^j\left(\frac{r_{pm}}{z_n - z_m}\right)^k \\ + \sum_{n=1}^{N}\sum_{j=1}^{J}\sum_{j'=0}^{min(j,k)} \bar{P}^\nu_{nj}\sigma\binom{j}{j'}\binom{j+k-j'-1}{j-1}\cdot\frac{r^j_{pn} r^k_{pm} z_m^{j-j'}\bar{z}_n^{k-j'}}{(r_b^2 - \bar{z}_n z_m)^{j+k-j'}} \\ + \sum_{j=k}^{J} P^\nu_{cj}(1-\sigma)\binom{j}{k}\frac{z_m^{j-k} r^k_{pm}}{r_c^j} \Bigg\} \end{aligned} \tag{8.8}$$

$k = 1, \ldots J$:

$$\begin{aligned} \bar{P}^{\nu+1}_{ck}\cdot\left\{1 - \sigma\frac{1 - k\beta_c}{1 + k\beta_c}\left(\frac{r_b}{r_c}\right)^{2k}\right\} = \\ -(1+\sigma)\frac{1 - k\beta_c}{1 + k\beta_c}\cdot\left\{\sum_{n=1}^{N}\frac{q_n^\nu}{2\pi\lambda_b}\frac{1}{k}\left(\frac{z_n}{r_c}\right)^k + \sum_{n=1}^{N}\sum_{j=1}^{k} P^{\nu+1}_{nj}\binom{k-1}{j-1}\frac{r^j_{pn} z_n^{k-j}}{r_c^k}\right\} \end{aligned} \tag{8.9}$$

$m' = 1, \dots N$:

$$q_{m'}^{\nu+1} = \sum_{m=1}^{N} K^o_{m'm} \Bigg\{ T_{fm} - T_c - Re\left[\sum_{\substack{n=1\\n\neq m}}^{N} \sum_{j=1}^{J} P_{nj}^{\nu+1} \left(\frac{r_{pn}}{z_m - z_n} \right)^j + \sum_{n=1}^{N} \sum_{j=1}^{J} P_{nj}^{\nu+1} \sigma \left(\frac{r_{pn}\bar{z}_m}{r_b^2 - z_n \bar{z}_m} \right)^j + \sum_{j=1}^{J} P_{cj}^{\nu+1} (1-\sigma) \left(\frac{z_m}{r_c} \right)^j \right] \Bigg\} \tag{8.10}$$

The iterative solution (8.7-10) is physically reasonable. Consider an iteration step ν. There is an approximate solution with line sinks and multipoles. The equations (8.1-3) are not satisfied exactly. The equations (8.8) and (8.9) mean that we change the multipoles so that the variation vanishes exactly (up to order J) at the considered boundary circle. But the difference $P_{mk}^{\nu+1} - P_{mk}^{\nu}$ will induce a multipole behaviour at the other pipes. These secondary disturbances are however damped with factors of the type:

$$\left(\frac{z_n}{r_c} \right)^k \quad ; \quad \left(\frac{r_{pm}}{r_c} \right)^{k'} \quad ; \quad \left(\frac{r_{pm}}{z_n - z_m} \right)^{k''}$$

The convergence of the iterative procedure is therefore rapid.

The iteration procedure is quite robust. It has worked without any problems for all cases that we have tested. An example is given below.

The iterations are performed until the following criterion is satisfied.

$$\begin{aligned} &\frac{\left| P_{nj}^{\nu+1} - P_{nj}^{\nu} \right|}{\max\limits_{1 \le k \le j} |P_{nk}|} < \epsilon \qquad \text{for all } n \text{ and all } j \; (j \le J) \text{ with } |P_{nj}| \neq 0 \\ &\frac{\left| P_{cj}^{\nu+1} - P_{cj}^{\nu} \right|}{\max\limits_{1 \le k \le j} |P_{cj}|} < \epsilon \qquad \text{for all } j \; (j \le J) \text{ with } |P_{cj}| \neq 0 \end{aligned} \tag{8.11}$$

Here ϵ is a measure of the iteration accuracy. The multipole differences are divided by the largest multipole up to the same order j according to the expression in the denominators. Normally, this is the multipole of first order. This will mean that ϵ gives the accuracy relative to the magnitude of the strongest multipole.

Example 8.1. Three pipes with different temperatures. We take the following data:

$$\begin{array}{lllll} N = 3 & \lambda_b = 0.6 & \lambda = 3.6 & & \\ r_b = 2 & r_c = 4 & \beta_c = 0 & T_c = 0 & \\ x_1 = 1 & y_1 = 0 & \beta_1 = 0 & T_{f1} = 1 & r_{p1} = 0.5 \\ x_2 = 0 & y_2 = 1.5 & \beta_2 = 0 & T_{f2} = -3 & r_{p2} = 0.25 \\ x_3 = -1 & y_3 = -0.5 & \beta_3 = 0.5 & T_{f3} = 2 & r_{p3} = 0.5 \end{array} \tag{8.12}$$

The temperature field of this case is shown in Figure 1.1 (and on the cover).

The number of iterations in order to obtain the required accuracy ϵ is given in Table 8.1 for different J and ϵ.

	ϵ=10^{-4}	10^{-5}	10^{-6}
J=1	6	7	8
2	6	7	8
3	6	8	9
5	6	8	9
10	6	8	9
15	6	8	9
20	6	8	9

Table 8.1. Required number of iterations in example 8.1.

We have found as in [1] that the value of ϵ is not critical. One may use $\epsilon = 10^{-4}$ or 10^{-5}.

9. REQUIRED NUMBER OF MULTIPOLES

The boundary conditions at the pipes and the outer circle are satisfied up to the order J in Fourier terms i.e. up to the order $cos(J\psi_m)$ and $sin(J\psi_m)$. The approximation becomes better, when J is increased, but the computational effort and the execution time also increase. It is an important question what value of J to choose in any particular case. One can always increase J, until the solution does not change.

The accuracy as a function of J will be studied and illustrated in this chapter with a few examples.

9.1 Error on the boundary circles

The example shown in Figure 9.1.1. is used to illustrate the error on the boundary circles.

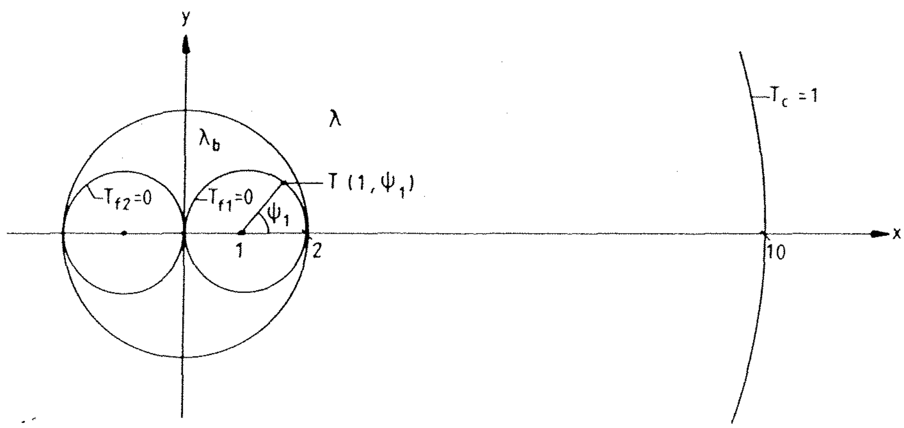

Figure 9.1.1. Example with two pipes.

The following data are used:

$$
\begin{array}{lllll}
N=2 & \lambda_b=1 & \lambda=5 & \epsilon=10^{-5} & \\
r_b=2 & r_c=10 & \beta_c=0 & T_c=1 & \\
x_1=1 & y_1=0 & \beta_1=0 & T_{f1}=0 & r_{p1}=1 \\
x_2=-1 & y_2=0 & \beta_2=0 & T_{f2}=0 & r_{p2}=1
\end{array}
\tag{9.1.1}
$$

The heat fluxes q_1 and q_2 are equal. The computed values for different J are given in Table 9.1.1. The computer model described in the following chapters is used. We see that $J = 1$ gives an error for q_1 of 2

%, and it is 0.3% for $J = 3$. The first order, $J = 1$, is sufficient in this case in order to calculate the heat fluxes.

J	0	1	2	3	4	5	10
q_1	-4.8560	-6.4986	-6.5596	-6.6044	-6.6174	-6.6206	-6.6247
$q_1(J)/q_1(10)$	0.73	0.981	0.990	0.997	0.9989	0.9994	1

Table 9.1.1. Calculated heat fluxes for example (9.1.1) for different J.

The polar coordinates of pipe 1 are ρ_1, ψ_1. The temperature on the uninsulated pipe is $T(1, \psi_1)$. In an exact solution $T(1, \psi_1)$ is equal to $T_{f1} = 0$, but for a finite J there will be a certain error or variation around the pipe. This boundary temperature is shown in Figure 9.1.2 for $J = 10$. The calculations are made with the computer program described in Chapters 10-11. The temperature $T(1, \psi_1)$ is obtained from the general formula (5.1) with summation in j up to $J = 10$.

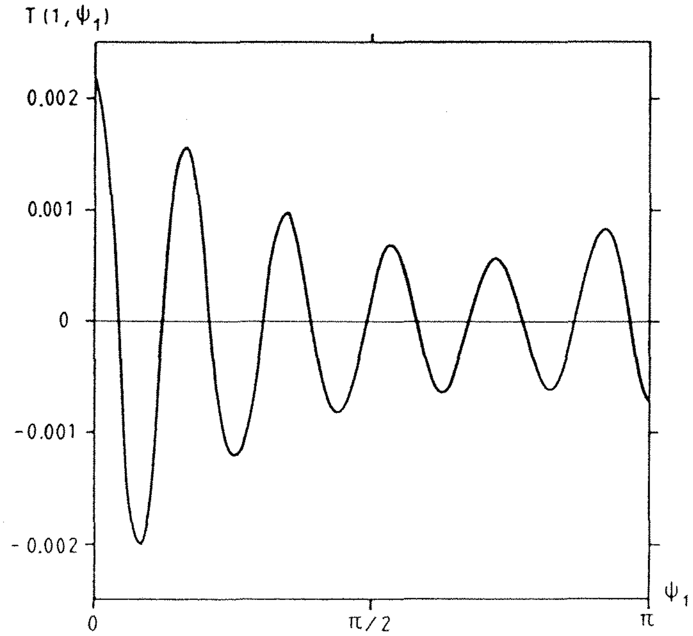

Figure 9.1.2. Temperature variation on pipe 1. Data according to (9.1.1) with $J = 10$.

The largest deviation from $T_{f1} = 0$ is 0.002. The largest error of boundary temperature is therefore 0.2% (of the temperature difference $T_c - T_{f1}$).

9.2 A THREE-PIPE PROBLEM

As a second example we take the three-pipe problem of example 8.1. The data are given by (8.12). The temperature field of this case is shown in Figure 1.1. Table 9.2 gives the calculated heat fluxes and the temperatures in a few points for different J.

J	0	1	2	3	5	10	15
q_1	3.702	3.747	3.775	3.776	3.776	3.776	3.776
q_2	-8.121	-8.644	-8.685	-8.688	-8.689	-8.689	-8.689
q_3	4.570	4.766	4.792	4.792	4.792	4.792	4.792
T(0.5,1)	-1.7209	-0.8213	-0.8090	-0.8080	-0.8079	-0.8079	-0.8079
T(0,0)	0.7851	0.7343	0.7267	0.7245	0.7243	0.7243	0.7243
T(0,-2)	1.2410	0.1452	0.1724	0.1704	0.1706	0.1706	0.1706

Table 9.2.1. Heat flows and temperatures in a few points for different J. Data according to (8.12).

The table shows that the error in heat fluxes is at most 7% for $J = 0$ and 0.8% for $J = 1$. The temperatures differ with up to a factor 2 for $J = 0$, while the largest error is 0.03 temperature units for $J = 1$. This difference decreases to 0.0024 for $J = 2$. The value $J = 1$ should be sufficient in this example.

9.3 Test of Maxwell's reciprocity theorem

We do not have analytical solutions to compare with in more complicated cases. We will therefore use a general theorem due to J.C. Maxwell (and others). Consider a steady-state heat conduction problem. The region is bounded by a number of boundary surfaces S_1, S_2, S_3.... The temperature is zero on all surfaces except one. Two cases are considered:

$$\begin{array}{lllllllll} A: & T=1 & \text{on} & S_1 & T=0 & \text{on} & S_2 & (\text{and} & S_3,...) \\ B: & T=1 & \text{on} & S_2 & T=0 & \text{on} & S_1 & (\text{and} & S_3,...) \end{array} \quad (9.3.1)$$

Let q_2^A be the heat flux from surface 2 in case A, and q_1^B the heat flux from surface 1 in case B. The reciprocity theorem states that these two fluxes are equal:

$$q_2^A = q_1^B \quad (9.3.2)$$

We have found that the reciprocity theorem is valid for our truncated problem (8.1-3) as well. (We have not taken the trouble to try to prove this).

In a test of the reciprocity theorem we use example (8.1). The data (8.12) are valid except for the boundary temperatures. We use:

$$T_c = 0 \qquad T_{f1} = 0 \qquad J = 5 \quad (9.3.3)$$

and

$$\text{case} \quad A: \quad T_{f2} = 1 \qquad T_{f3} = 0 \quad (9.3.3A)$$

$$\text{case} \quad B: \quad T_{f2} = 0 \qquad T_{f3} = 1 \quad (9.3.3B)$$

The calculated heat fluxes became equal with seven digits:

$$\begin{aligned} q_3^A &= -0.1752401 \\ q_2^B &= -0.1752401 \end{aligned} \quad (9.3.4)$$

Another example concerns a case with a single region ($\lambda_b = \lambda$). The first five pipes of the example in section 9.3 in [1] are used. The following data are valid:

$$
\begin{array}{lllll}
N = 5 & \lambda = \lambda_b = 1 & J = 5 & & \\
T_c = 0 & r_c = 10 & \beta_c = 0 & & \\
x_1 = 1 & y_1 = 0 & r_{p1} = 0.5 & \beta_1 = 0 & \\
x_2 = 2 & y_2 = 1 & r_{p2} = 0.5 & \beta_2 = 0 & T_{f2} = 0 \\
x_3 = 1 & y_3 = 2 & r_{p3} = 0.5 & \beta_3 = 0 & T_{f3} = 0 \\
x_4 = 0 & y_4 = 2 & r_{p4} = 0.5 & \beta_4 = 0 & T_{f4} = 0 \\
x_5 = 2 & y_5 = -1 & r_{p5} = 0.5 & \beta_5 = 0.3 & \\
\text{case A}: & T_{f1} = 1 & T_{f5} = 0 & & \\
\text{case B}: & T_{f1} = 0 & T_{f5} = 1 & &
\end{array}
\tag{9.3.5}
$$

The computed reciprocal heat fluxes became again equal with 7 digits:

$$
\begin{aligned}
q_5^A &= -1.890565 \\
q_1^B &= -1.890565
\end{aligned}
\tag{9.3.6}
$$

9.4 An example with 15 pipes

We use the example with 15 pipes in [1], section 9.3. The data according to (9.3.1) in [1] are valid. The corresponding temperature field is shown on the cover of [1]. The outer radius r_c is equal to 10. We take r_b equal to this value and add an outer region out to $r_c = 20$. The thermal conductivity is taken to be ten times higher than in the inner region. The data of (9.3.1) in [1] are supplemented with:

$$
r_b = 10 \qquad \lambda_b = 1 \qquad r_c = 20 \qquad \lambda = 10 \tag{9.4.1}
$$

Table 9.4.1 gives the computed heat fluxes $q_n, 1 \leq n \leq 15$, for different J. The values do not differ much from those in [1], (Table 9.3.1). We see that $J = 1$ is not acceptable, while $J = 5$ is quite sufficient.

	J=0	1	5	10
q_1	1.1330	2.3566	2.0842	2.0843
q_2	-0.51959	-5.4018	-4.9154	-4.9020
q_3	5.1286	6.1679	6.2540	6.2540
q_4	-11.816	-10.516	-11.340	-11.353
q_5	-2.7569	-1.0451	-1.0552	-1.0552
q_6	11.743	12.983	12.116	12.116
q_7	-6.1510	-6.7283	-6.6958	-6.6958
q_8	-5.8188	-9.0225	-9.247	-9.247
q_9	10.321	17.765	20.958	20.988
q_{10}	20.065	24.527	24.925	24.926
q_{11}	10.803	12.387	12.632	12.632
q_{12}	-13.252	-14.507	-14.727	-14.727
q_{13}	1.0835	1.7532	1.7310	1.7310
q_{14}	-7.3037	-14.710	-15.793	-15.813
q_{15}	-9.8105	-12.163	-14.086	-14.098

Table 9.4.1. Computed heat fluxes q_n for different J for a case with 15 pipes.

10. COMPUTER MODEL. EXECUTION TIMES

The computer model calculates the line sources and the multipoles up to the given order J, and it gives the corresponding temperature field. The next chapter is a manual for the computer program. The source code is given in appendix 1.

The input data of the model are:

$$\begin{array}{llllll} \lambda, & \lambda_b, & N, & J & & \\ r_b, & r_c, & \beta_c, & T_c & & \\ x_n, & y_n, & r_{pn}, & \beta_n, & T_{fn} \quad \text{for} & n=1,\ldots N \end{array} \tag{10.1}$$

There are the following restrictions on the input variables:

$$\begin{array}{llll} \lambda > 0 & \lambda_b > 0 & N = 1,2,\ldots & J = 0,1,\ldots \\ 0 < r_b < r_c & \beta_c \geq 0 & & \\ r_{pn} > 0 & \beta_n \geq 0 \quad \text{for} & n = 1,\ldots N & \\ r_n = \sqrt{x_n^2 + y_n^2} \leq r_b - r_{pn} \quad \text{for} & n = 1,\ldots N & & \\ r_{mn} \geq r_{pm} + r_{pn} \quad \text{for} \quad m \neq n & & & \\ r_{mn} = r_{pm} + r_{pn} \quad \text{only if} & \beta_m + \beta_n > 0 \quad \text{or} & T_{fm} = T_{fn} & \end{array} \tag{10.2}$$

The condition $r_{mn} \geq r_{pm} + r_{pn}$ ensures that the pipes do not cover each other. They may touch each other, if there is a thermal insulation ($\beta_m + \beta_n > 0$) or if the temperatures T_{fm} and T_{fn} are equal, so that the heat flux between the pipes remains finite.

The first step after input and test of the restrictions (10.2) is to calculate auxiliary variables and the resistance matrix (R^o_{mn}) with the elements (8.4).The inverse matrix (K^o_{mn}) is then calculated. The initial values of line sources and multipoles for $\nu = 0$ are given by (8.7). Formulas (8.8-10) are used in successive iterations, until the ϵ-criterion (8.11) is met.

The output is the values of the line sources and multipoles and, if requested, the temperature field $T(x,y)$, which is obtained from (5.3-4) with summation in j up to J.

The steady-state heat conduction problem may be solved numerically with finite difference or finite element methods. The present method is however more rapid, and it is simpler to get a high and controlled accuracy. We will give the execution time for a number of cases in order to show this.

Consider the case with 15 pipes in section 9.4. In the first examples we use the first 5 pipes ($N = 5$), then the first 10 pipes ($N = 10$) and finally all 15 pipes. The accuracy ϵ is 10^{-4} and 10^{-5}, while J is 5 and 10. The execution times in CPU-seconds on a ND-500 computer from Norsk Data are given in Table 10.1.

		$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
N = 5	J = 5	13 s	18 s
	J = 10	140s	200 s
N = 10	J = 5	44 s	60 s
	J = 10	460 s	620 s
N = 15	J = 5	110 s	150 s
	J = 10	1200 s	1700 s

Table 10.1. Execution time in CPU-seconds on ND-500 for 5 to 15 pipes and multipole order $J = 5, 10$.

Table 10.1 shows that the CPU-time is below a few minutes even for 15 pipes, if the order of multipoles is kept below $J \leq 5$. The CPU-time for a more moderate case with $N \leq 3$, $J \leq 2$ is only a few seconds. The execution times for an IBM-PC AT-3 (10 MHz) with a 80287 math co-processor are about 60 times longer.

11. MANUAL FOR THE COMPUTER CODE

The program has been adapted to run on IBM-PC and other compatible computers. The source code is written in FORTRAN77. It has been compiled with the MS-FORTRAN77 compiler V3.3.

The executable code supplied on the disk may be used for cases with a maximum of 15 pipes. The maximum order of multipoles is 10. The size of the program is then 133K. These restrictions can be removed by changing the dimensions in the PARAMETER statement at the beginning of the main program. The maximum number of pipes is IW and the maximum order of multipoles is IJ.

The basic version of the source code is listed in Appendix 1. The current version of the program differs from the basic version according to the changes listed on the file README.DOC. This file may also contain important information about changes in this manual. Be sure to read this file before you run the program.

Files on the Disk.

A brief description of the files on the enclosed disk is given below.

MPC.EXE	Executable code
MPC.FOR	Source code for the current version of the program
SAMPLE.DAT	Input data file for the example. See below
SAMPLE.OUT	Output file for the example. See below
MPC.BAT	Batch file
SIGNAL.COM	Program that generates a signal when the job is completed
README.DOC	Contains updates of this manual and the basic version of the source code.

Using the Program.

To run the program MPC (MultiPoleComposite) , insert the disk into a drive, make that drive the default drive, and type the program name:

MPC

The program is intended for interactive use. The user is prompted for the names of an input data file (optional) and the output file. The input data may be entered interactively on the screen or read directly from a disk file. If the input data is entered interactively it may be saved on a disk file.

Input Data.

The input data must satisfy the restrictions (10.2). The program indicates any violation of these restrictions. The input data list is read in free format, i.e. the values must be separated by one or more contiguous blanks or a comma. The input data records are specified below.

1.	λ_b	Thermal conductivity in the inner region, (W/mK)
	λ	Thermal conductivity in the outer region, (W/mK)
	N	Number of pipes
	J	Maximal order of multipoles
2.	r_b	Radius of the inner region, (m)
	r_c	Radius of the outer region, (m)
	β_c	Thermal resistance coefficient at the outer circle, (-)
	T_c	Temperature at outer circle, (°C)

The following record must be repeated for each pipe, i.e. N times.

3.	x_i	x-coordinate of the center of pipe i, (m)
	y_i	y-coordinate of the center of pipe i, (m)
	r_{pi}	Radius of pipe i, (m)
	β_i	Thermal resistance coefficient of pipe i, (-)
	T_{fi}	Fluid temperature in pipe i, (°C)
4.	ϵ	Iteration accuracy. See section 8. A suitable value is 10^{-4}
	ITMAX	Maximum number of iterations. A suitable value is 500

The temperatures are calculated in a rectangular grid within a rectangular region with the corners determined by XMIN, XMAX, YMIN, and YMAX. The number of grid points in the X- and Y-direction are NX and NY respectively. If a grid point lies within the circle of one of the pipes or outside the outer boundary, then there will be no temperature calculation in that point. There is no calculation of temperatures if NX≤0 or NY≤0.

5.	XMIN	Minimum x-coordinate of rectangular region for temperature calculation, (m)
	XMAX	Maximum x-coordinate of rectangular region for temperature calculation, (m)
	YMIN	Minimum y-coordinate of rectangular region for temperature calculation, (m)
	YMAX	Maximum y-coordinate of rectangular region for temperature calculation, (m)
	NX	Number of grid points in the x-direction
	NY	Number of grid points in the y-direction

The program is the same as in [1] except for the necessary modifications. The source code is given in appendix 1.

AN EXAMPLE

The example 8.1 with 3 pipes is used. The complete set of data is given by (8.12). The accuracy ϵ is set to 10^{-4} and ITMAX to 500. The corner points of the region of temperature calculation are XMIN = -2., YMIN =-2., XMAX = 2., and YMAX = 2. The number of grid points in the x-direction, NX, and the number of grid points in the y-direction, NY, are both 5. This gives the following indata list:

```
0.6  3.6  3  10
2.0  4.0  0.0  0.0
1.0    0.0  0.5   0.0   1.0
0.0    1.5  0.25  0.0  -3.0
-1.0  -0.5  0.5   0.5   2.0
1.0E-4  500
-2.0  2.0  -2.0  2.0   5  5
```

The output is given on the next two pages. The line sources and multipoles are denoted in the following way: q_n = q(n), P_{nk} = P(n,k), P_{ck} = PC(k).

Correction on p.28, line 35-36:

Number of grid points along the x-axis	5
Number of grid points along the y-axis	5

```
MULTIPOLE METHOD - Pipes in a composite cylinder
================================================

INPUT DATA

Input file  a:sample.dat
Output file a:sample.out

Thermal conductivity in inner region         .600 W/(m*K)
Thermal conductivity in outer region        3.600 W/(m*K)
Number of pipes                    3
Order of multipoles               10

Radius of inner region               2.000 m
Radius of outer region               4.000 m
Outer boundary
  Thermal resistance coefficient      .000E+00
  Temperature                            .000 C

 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
 Pipe *    x(n)    *    y(n)    *    rp(n)   *   beta(n)  *   Temp C  *
 * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
   1  *   1.000    *    .000    *     .500   *  .000E+00  *    1.000  *
   2  *    .000    *   1.500    *     .250   *  .000E+00  *   -3.000  *
   3  *  -1.000    *   -.500    *     .500   *  .500E+00  *    2.000  *

Iteration accuracy                   .10E-03
Maximum number of iterations           500

TEMPERATURE OUTPUT: Definition of rectangular area
  Minimum x-value                 -2.000
  Maximum x-value                  2.000
  Minimum y-value                 -2.000
  Maximum y-value                  2.000
  Number of grid points along x-axis          4
  Number of grid points along y-axis          4

****************************************************************************

 Pipe  Initial values for q(n) (Order of multipole = 0)
   1      .3701710E+01
   2     -.8120926E+01
   3      .4570385E+01

Number of iterations:     6


 Pipe          q(n)
   1      .3775983E+01
   2     -.8688749E+01
   3      .4791657E+01

 Pipe     Order            P(n,k)
   1        1      .85094E-01   .26170E+00
   1        2     -.30722E-01  -.72742E-01
   1        3      .18398E-01   .80464E-02
   1        4     -.27537E-02   .11994E-02
```

1	5	-.12744E-04	-.50169E-03
1	6	.14905E-03	.25332E-04
1	7	-.32846E-04	.30497E-04
1	8	.43431E-06	-.11177E-04
1	9	.17959E-05	.15383E-05
1	10	-.46264E-06	.19077E-06
2	1	-.19320E-01	-.20056E+00
2	2	.50384E-01	.47747E-02
2	3	.57549E-03	.63122E-02
2	4	-.91680E-03	-.21108E-04
2	5	.16576E-05	-.20578E-03
2	6	.33463E-04	-.14848E-05
2	7	-.18595E-06	.70751E-05
2	8	-.13949E-05	.40628E-07
2	9	.14387E-08	-.28067E-06
2	10	.59501E-07	-.37228E-09
3	1	-.60168E-01	.10666E-01
3	2	.00000E+00	.00000E+00
3	3	.18779E-02	.12627E-02
3	4	.23297E-03	.46381E-03
3	5	-.93677E-04	.14950E-03
3	6	-.21800E-04	.14709E-05
3	7	-.41423E-05	-.31219E-05
3	8	.39756E-06	-.36033E-06
3	9	.12760E-07	.21779E-07
3	10	-.28358E-07	.25599E-07

Order	PC(k)	
1	.15866E-01	-.24283E+00
2	-.63526E-01	.12837E-01
3	-.10585E-02	.97228E-02
4	.32898E-02	.67845E-03
5	-.14272E-03	-.11445E-02
6	-.32200E-03	.14539E-04
7	-.11403E-04	.11161E-03
8	.38088E-04	-.51801E-06
9	-.34440E-06	-.12229E-04
10	-.43815E-05	-.15054E-06

****

TEMPERATURES (Deg C)

x	y	Temp
-2.00000	-2.00000	.14027
-2.00000	-1.00000	.26004
-2.00000	.00000	.20526
-2.00000	1.00000	-.00343
-2.00000	2.00000	-.09997
-1.00000	-2.00000	.20032
-1.00000	-1.00000	1.37087
-1.00000	.00000	1.31407
-1.00000	1.00000	-.31126
-1.00000	2.00000	-.35746
.00000	-2.00000	.17064
.00000	-1.00000	.82908
.00000	.00000	.72430

115	.00000	1.00000	-1.55455
116	.00000	2.00000	-.83812
117	1.00000	-2.00000	.12871
118	1.00000	-1.00000	.55357
119	1.00000	1.00000	-.06704
120	1.00000	2.00000	-.33614
121	2.00000	-2.00000	.08151
122	2.00000	-1.00000	.14928
123	2.00000	.00000	.18674
124	2.00000	1.00000	.03184
125	2.00000	2.00000	-.08219

REFERENCES

1. J. Claesson and J. Bennet, Multipole Method to Compute the Conductive Heat Flows between Pipes in a Cylinder; Notes on Heat Transfer 2-1987, Dep. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987.

2. J. Claesson and G. Hellström , Thermal Resistances to and between Pipes in a Composite Cylinder, Notes on Heat Transfer 1987, Dep. of Building Technology and Mathematical Physics, University of Lund, Box 118, S-221 00 Lund, Sweden, 1987. (To be published)

APPENDIX 1. LISTING OF COMPUTER CODE

```
C-----------------------------------------------------------------------
C                               PROGRAM MPC
C                 Multipole method  to compute the heat flows
C                 and temperatures to  and between  insulated
C                 circular pipes in a composite cylinder with
C                   two concentric regions with different
C                           thermal conductivities
C
C Authors: Johan Bennet, Johan Claesson, Goran Hellstrom
C          Departments of Building Technology and Mathematical Physics
C          Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden
C Date:    1987-03-25
C Reference: Notes on Heat Transfer 3 - 1987
C-----------------------------------------------------------------------
      PARAMETER (IW=15,IJ=10,IW2=IW*IW)
      COMPLEX*8 Z(IW),P(IW,IJ),P2(IW,IJ),PC(IJ),ZRC(IW,IJ),TERM
     +  ,RPZMN(IW,IW,IJ),RPMZN(IW,IW,IJ),PMK,PRB,ZPR,RPZ(IW,IJ)
      DIMENSION RP(IW),BETA(IW),TF(IW),QBEG(IW)
     +  ,Q(IW),RRR(IW),QM(IW)
      DOUBLE PRECISION RKO(IW,IW),RKOVEC(IW2),MC(IW),MD(IW)
      CHARACTER*16 READFI,OUTFI
      CHARACTER ANS
      LOGICAL INFIL,OK
      DATA INFIL/.FALSE./
c********** OPEN INPUT AND OUTPUT FILES **********
      WRITE(*,441)
      WRITE(*,*)
    2 WRITE(*,*)'Input data from file ? (Y/N)'
      READ(*,5) ANS
      IF((ANS.EQ.'Y').OR.(ANS.EQ.'y')) THEN
          WRITE(*,*) 'Name of input file ?'
          READ(*,10) READFI
          INQUIRE(FILE=READFI,EXIST=OK)
          IF (OK) THEN
              OPEN(UNIT=5,FILE=READFI)
              INFIL=.TRUE.
            ELSE
              WRITE(*,*)' *** ERROR ***    ',READFI,' File not found'
              GO TO 2
          ENDIF
      ENDIF
      WRITE(*,*) 'Name of output file ?'
      READ(*,10) OUTFI
      INQUIRE(FILE=OUTFI,EXIST=OK)
      IF (OK) THEN
          OPEN(UNIT=6,FILE=OUTFI)
        ELSE
          OPEN(UNIT=6,FILE=OUTFI,STATUS='NEW')
      ENDIF
    5 FORMAT(A1)
   10 FORMAT(A16)
C********** READ INPUT DATA **********
      IF(INFIL) THEN
          READ(5,*) RLAMB,RLAM,NW,J
```

```
        READ(5,*) RB,RC,BETAC,TC
        DO 15 M=1,NW
          READ(5,*) ZRE,ZIM,RP(M),BETA(M),TF(M)
          Z(M)=CMPLX(ZRE,ZIM)
15      CONTINUE
        READ(5,*) EPS,ITMAX
        READ(5,*) XMIN,XMAX,YMIN,YMAX,NX,NY
      ELSE
        WRITE(*,*)'Thermal conductivity in the inner region ?'
        READ(*,*) RLAMB
        WRITE(*,*)'Thermal conductivity in the outer region ?'
        READ(*,*) RLAM
        WRITE(*,*)'Number of pipes ?'
        READ(*,*) NW
        WRITE(*,*)'Order of multipoles ?'
        READ(*,*) J
        WRITE(*,*)'Radius of inner region ?'
        READ(*,*) RB
        WRITE(*,*)'Radius of outer region ?'
        READ(*,*) RC
        WRITE(*,*)
        WRITE(*,*)'OUTER BOUNDARY'
        WRITE(*,*)'  Thermal resistance coefficient ?'
        READ(*,*) BETAC
        WRITE(*,*)'  Temperature ?'
        READ(*,*) TC
        DO 16 M=1,NW
          WRITE(*,443) M
          WRITE(*,*)'  x-value ?'
          READ(*,*) ZRE
          WRITE(*,*)'  y-value ?'
          READ(*,*) ZIM
          Z(M)=CMPLX(ZRE,ZIM)
          WRITE(*,*)'  Pipe radius ?'
          READ(*,*) RP(M)
          WRITE(*,*)'  Thermal resistance coefficient ?'
          READ(*,*) BETA(M)
          WRITE(*,*)'  Temperature ?'
          READ(*,*) TF(M)
16      CONTINUE
        WRITE(*,*)
        WRITE(*,*)'Iteration accuracy ?'
        READ(*,*) EPS
        WRITE(*,*)'Maximum number of iterations ?'
        READ(*,*) ITMAX
        WRITE(*,*)
        WRITE(*,*)'TEMPERATURE OUTPUT: Define rectangular area'
        WRITE(*,*)'  Minimum x-value ?'
        READ(*,*) XMIN
        WRITE(*,*)'  Maximum x-value ?'
        READ(*,*) XMAX
        WRITE(*,*)'  Minimum y-value ?'
        READ(*,*) YMIN
        WRITE(*,*)'  Maximum y-value ?'
        READ(*,*) YMAX
        WRITE(*,*)'  Number of grid points along the x-axis ?'
        READ(*,*) NX
```

```
          WRITE(*,*)'  Number of grid points along the y-axis ?'
          READ(*,*) NY
          WRITE(*,*)
          IF(.NOT.INFIL) THEN
              WRITE(*,*)'Save input data on file ? (Y/N)'
              READ(*,5) ANS
              IF((ANS.EQ.'Y').OR.(ANS.EQ.'y')) THEN
                  WRITE(*,*) 'Name of input file ?'
                  READ(*,10) READFI
                  INQUIRE(FILE=READFI,EXIST=OK)
                  IF (OK) THEN
                      OPEN(UNIT=5,FILE=READFI)
                    ELSE
                      OPEN(UNIT=5,FILE=READFI,STATUS='NEW')
                  ENDIF
                  WRITE(5,701) RLAMB,RLAM,NW,J
                  WRITE(5,702) RB,RC,BETAC,TC
                  DO 17 M=1,NW
                    ZRE=REAL(Z(M))
                    ZIM=AIMAG(Z(M))
                    WRITE(5,702) ZRE,ZIM,RP(M),BETA(M),TF(M)
   17             CONTINUE
                  WRITE(5,703) EPS,ITMAX
                  WRITE(5,704) XMIN,XMAX,YMIN,YMAX,NX,NY
                  CLOSE(UNIT=5)
              ENDIF
          ENDIF
      ENDIF
C********** CONSISTENCY TESTS OF INPUT DATA **********
      IF(RLAMB.LE.0.) STOP ' RLAMB < 0.'
      IF(RLAM.LE.0.) STOP ' RLAM < 0.'
      IF(NW.LT.1) STOP ' NW < 1'
      IF(J.LT.0) STOP ' J < 0'
      IF(RB.LE.0.) STOP ' RB < 0.'
      IF(RC.LE.0.) STOP ' RC < 0.'
      IF(BETAC.LT.0.) STOP ' BETAC < 0.'
      DO 20 N=1,NW
        IF(RB.LT.CABS(Z(N))+RP(N)-1.E-6) STOP ' RB < R(N)+RP(N)'
        IF(BETA(N).LT.0.) STOP ' BETA(N) < 0'
        IF(RP(N).LE.0.) STOP ' RP(N) < 0.'
        DO 20 M=1,NW
          IF(M.NE.N) THEN
              IF(CABS(Z(N)-Z(M)).LT.RP(N)+RP(M)-1.E-6)
     +          STOP ' RMN < RP(N)+RP(M)'
              IF(CABS(Z(N)-Z(M))-RP(N)-RP(M).LE.1.E-6) THEN
                  IF(BETA(M)+BETA(N).EQ.0..AND.TF(N).NE.TF(M))
     +STOP ' RMN=RP(M)+RP(N) AND BETA(M)+BETA(N)=0. AND TF(M) NE TF(N)'
              END IF
          END IF
   20     CONTINUE
      IF(EPS.LE.0.) STOP ' EPS LE 0.'
C********** LISTING OF INPUT DATA **********
      WRITE(*,*)
      WRITE(*,441)
      WRITE(*,450) RLAMB,RLAM,NW,J
      WRITE(*,460) RB,RC,BETAC,TC
      WRITE(*,470) (M,Z(M),RP(M),BETA(M),TF(M),M=1,NW)
```

```
      WRITE(*,480) EPS,ITMAX
      WRITE(*,490) XMIN,XMAX,YMIN,YMAX,NX,NY
      WRITE(6,441)
      WRITE(6,442) READFI,OUTFI
      WRITE(6,450) RLAMB,RLAM,NW,J
      WRITE(6,460) RB,RC,BETAC,TC
      WRITE(6,470) (M,Z(M),RP(M),BETA(M),TF(M),M=1,NW)
      WRITE(6,480) EPS,ITMAX
      WRITE(6,490) XMIN,XMAX,YMIN,YMAX,NX,NY
      WRITE(6,410)
C********** SET CONSTANTS **********
      IIT=0
      PI=3.14159265
      PILAMB=1./(2.*PI*RLAMB)
      PILAM=1./(2.*PI*RLAM)
      SIGMA=(RLAMB-RLAM)/(RLAMB+RLAM)
      ALBETC=BETAC+ALOG(RC/RB)
C********** CALCULATION OF MATRICE RKO **********
C**********    AND AUXILIARY ARRAYS    **********
      DO 60 M=1,NW
        RBM=RB**2/(RB**2-CABS(Z(M))**2)
        RKO(M,M)=PILAMB*(ALOG(RB/RP(M))+BETA(M)+SIGMA*ALOG(RBM))
     +    +PILAM*ALBETC
        IF(J.GE.1) THEN
            DO 30 K=1,J
              ZRC(M,K)=(0.,0.)
              RPZ(M,K)=(0.,0.)
              RPMZN(M,M,K)=(RP(M)*CONJG(Z(M))*RBM/RB**2)**K
              IF(CABS(Z(M)).NE.0.) THEN
                  ZRC(M,K)=(Z(M)/RC)**K
                  RPZ(M,K)=(RP(M)/Z(M))**K
                END IF
   30         CONTINUE
          END IF
        DO 50 N=1,NW
          IF(M.NE.N) THEN
              PMK=Z(N)-Z(M)
              RMN=CABS(PMK)
              RBM=RB**2/CABS(RB**2-Z(N)*CONJG(Z(M)))
              PRB=RP(M)*CONJG(Z(N))/(RB**2-CONJG(Z(N))*Z(M))
              RKO(M,N)=PILAMB*(ALOG(RB/RMN)+SIGMA*ALOG(RBM))+
     +          PILAM*ALBETC
              IF(J.GE.1) THEN
                  DO 40 K=1,J
                    RPZMN(M,N,K)=(RP(M)/PMK)**K
                    RPMZN(M,N,K)=PRB**K
   40               CONTINUE
                END IF
            END IF
   50     CONTINUE
   60   CONTINUE
C********** CONVERT MATRIX RKO TO VECTOR RKOVEC **********
      K=0
      DO 64 JJ=1,NW
        DO 62 I=1,NW
          K=K+1
          RKOVEC(K)=RKO(I,JJ)
```

```
   62     CONTINUE
   64     CONTINUE
C********** INVERSION OF MATRIX RKO **********
       CALL INV1(RKOVEC,MC,MD,NW,DET)
C********** CONVERT VECTOR RKOVEC TO MATRIX RKO
       K=0
       DO 68 JJ=1,NW
         DO 66 I=1,NW
           K=K+1
           RKO(I,JJ)=RKOVEC(K)
   66     CONTINUE
   68     CONTINUE
C******************** INITIAL VALUES OF ENERGY FLOWS *******
C********************        AND MULTIPOLES          *******
       DO 80 M=1,NW
         QBEG(M)=0.
         DO 70 N=1,NW
           QBEG(M)=QBEG(M)+RKO(M,N)*(TF(N)-TC)
   70     CONTINUE
         Q(M)=QBEG(M)
   80   CONTINUE
       WRITE(*,430)(N,Q(N),N=1,NW)
       WRITE(6,430)(N,Q(N),N=1,NW)
       IF(J.EQ.0) GO TO 280
       DO 90 M=1,NW
         DO 90 K=1,J
           P(M,K)=(0.,0.)
   90     CONTINUE
       DO 100 K=1,J
         PC(K)=(0.,0.)
  100   CONTINUE
C****************** START OF ITERATION LOOP ****************
       WRITE(*,495) EPS
       DO 270 IIT=1,ITMAX
         EPSMAX=0.
C****************** MULTIPOLES AT THE PIPES ****************
         DO 160 M=1,NW
           PMMAX=0.
           DO 150 K=1,J
             PMK=(0.,0.)
             DO 130 N=1,NW
               PRB=1./(RB**2-CONJG(Z(N))*Z(M))
               KFAK=1
               DO 120 JJ=1,J
                 IF(N.NE.M) PMK=PMK+P(N,JJ)*RPZMN(N,M,JJ)
     +             *RPZMN(M,N,K)*KFAK
                 JPEND=MIN0(JJ,K)
                 KFAK1=KFAK
                 KFAK2=1
                 DO 110 JPRIM=0,JPEND
                   JJPRIM=JJ-JPRIM
                   KJPRIM=K-JPRIM
                   TERM=(1.,0.)
                   IF(JJPRIM.GE.1) TERM=CONJG(RPMZN(N,M,JJPRIM))
                   IF(KJPRIM.GE.1) TERM=TERM*RPMZN(M,N,KJPRIM)
                   PMK=PMK+CONJG(P(N,JJ))*SIGMA*TERM*(RP(M)*RP(N)
     +               *PRB)**JPRIM*KFAK1*KFAK2
```

```
                    IF(JPRIM.NE.JPEND) THEN
                      KFAK1=KFAK1*KJPRIM/(K+JJ-1-JPRIM)
                      KFAK2=KFAK2*JJPRIM/(JPRIM+1)
                    END IF
  110             CONTINUE
                KFAK=KFAK*(K+JJ)/JJ
  120           CONTINUE
              IF(N.NE.M) PMK=PMK+Q(N)*PILAMB*RPZMN(M,N,K)/K
              PMK=PMK+Q(N)*PILAMB*SIGMA*RPMZN(M,N,K)/K
  130         CONTINUE
            KFAK=1
            DO 140 JJ=K,J
              PMK=PMK+PC(JJ)*(1.-SIGMA)*RPZ(M,K)*ZRC(M,JJ)*KFAK
              KFAK=KFAK*(JJ+1)/(JJ+1-K)
  140       CONTINUE
            PMK=CONJG(PMK)*(BETA(M)*K-1.)/(BETA(M)*K+1.)
            PMMAX=AMAX1(CABS(PMK),PMMAX)
            IF(CABS(PMK).GT.1.E-7) EPSMAX=
     +        AMAX1(EPSMAX,CABS(PMK-P(M,K))/PMMAX)
            P2(M,K)=PMK
  150     CONTINUE
  160   CONTINUE
C****************** NEW MULTIPOLES ARE ASSIGNED ************
        DO 170 M=1,NW
          DO 170 K=1,J
            P(M,K)=P2(M,K)
  170     CONTINUE
C****************** CALCULATION OF MULTIPOLES **************
C******************   AT THE OUTER CIRCLE      **************
        PMMAX=0.
        DO 200 K=1,J
          PMK=(0.,0.)
          DO 190 M=1,NW
            KFAK=1
            DO 180 JJ=1,K
              PMK=PMK+P(M,JJ)*ZRC(M,K)*RPZ(M,JJ)*KFAK
              KFAK=KFAK*(K-JJ)/JJ
  180       CONTINUE
            PMK=PMK+ZRC(M,K)*Q(M)*PILAMB/K
  190     CONTINUE
          XX=(1.-BETAC*K)/(1.+BETAC*K)
          PMK=CONJG(PMK)*XX*(SIGMA+1.)/(SIGMA*XX*(RB/RC)**(2*K)-1.)
          PMMAX=AMAX1(CABS(PMK),PMMAX)
          IF(CABS(PMK).GT.1.E-7) EPSMAX=
     +      AMAX1(EPSMAX,CABS(PMK-PC(K))/PMMAX)
          PC(K)=PMK
  200   CONTINUE
C****************** CALCULATION OF NEW ENERGY FLOWS ********
        DO 240 M=1,NW
          QQQ=0.
          DO 220 N=1,NW
            DO 210 JJ=1,J
              IF(M.NE.N) QQQ=QQQ+REAL(RPZMN(N,M,JJ)*P(N,JJ))
              QQQ=QQQ+SIGMA*REAL(P(N,JJ)*RPMZN(N,M,JJ))
  210       CONTINUE
  220     CONTINUE
          DO 230 JJ=1,J
```

```
              QQQ=QQQ+(1.-SIGMA)*REAL(PC(JJ)*ZRC(M,JJ))
  230         CONTINUE
            QM(M)=QQQ
  240       CONTINUE
          DO 260 M=1,NW
            QQQQ=0.
            DO 250 N=1,NW
              QQQQ=QM(N)*RKO(M,N)+QQQQ
  250         CONTINUE
            Q(M)=QBEG(M)-QQQQ
  260       CONTINUE
C****************** TEST WHETHER THE ACCURACY **************
C******************  CONDITION IS FULFILLED   **************
          WRITE(*,500)  IIT,EPSMAX
          IF(EPSMAX.LT.EPS) GO TO 280
  270   CONTINUE
C****************** OUTPUT OF ENERGY FLOWS *****************
C******************      AND MULTIPOLES      *****************
  280 WRITE(6,370) IIT
      WRITE(*,370) IIT
      WRITE(6,380)(N,Q(N),N=1,NW)
      WRITE(*,380)(N,Q(N),N=1,NW)
      IF(J.GE.1) THEN
          WRITE(6,390)((N,K,P(N,K),K=1,J),N=1,NW)
          WRITE(6,420)(K,PC(K),K=1,J)
        END IF
      WRITE(6,440)
      WRITE(6,395)
C****************** CALCULATION AND OUTPUT *****************
C******************     OF TEMPERATURES      *****************
      DO 360 MPR=0,NX
      DO 360 NPR=0,NY
        ZPR=CMPLX((XMAX-XMIN)*MPR/NX+XMIN,(YMAX-YMIN)*NPR/NY+YMIN)
C       DO 360 MPR=0,100
C         ZPR=(1.,0.)+CEXP(CMPLX(0.,MPR*PI/100.))
        DO 290 N=1,NW
          IF(CABS(ZPR-Z(N)).LT.RP(N)-1.E-5)  GO TO 360
  290     CONTINUE
        IF(CABS(ZPR).GT.RC) GO TO 360
        PMK=(0.,0.)
        IF(CABS(ZPR).LE.RB) THEN
C****************** CALCULATION OF TEMPERATURES ***********
C******************  WHEN ZPR IS LESS THEN RB    ***********
            DO 310 N=1,NW
              IF(J.GE.1) THEN
                  PRB=1./(RB**2-Z(N)*CONJG(ZPR))
                  DO 300 K=1,J
                    PMK=PMK+P(N,K)*((RP(N)/(ZPR-Z(N)))**K+
     +                SIGMA*(RP(N)*CONJG(ZPR)*PRB)**K)
  300               CONTINUE
                END IF
              RON=CABS(ZPR-Z(N))
              PMK=PMK+Q(N)*(PILAM*ALBETC+
     +          PILAMB*(ALOG(RB/RON)+SIGMA*ALOG(RB**2*CABS(PRB))))
  310         CONTINUE
            IF(J.GE.1) THEN
                DO 320 K=1,J
```

```
                  PMK=PMK+(1.-SIGMA)*PC(K)*(ZPR/RC)**K
 320              CONTINUE
              END IF
          ELSE
C****************** CALCULATION OF TEMPERATURES ************
C*****************  WHEN IS ZPR GREATER THEN RB ************
            DO 340 N=1,NW
              IF(J.GE.1) THEN
                  DO 330 K=1,J
                    PMK=PMK+P(N,K)*(1.+SIGMA)*(RP(N)/(ZPR-Z(N)))**K
 330                CONTINUE
                END IF
              RON=CABS(ZPR-Z(N))
              PMK=PMK+Q(N)*(PILAM*(ALBETC+SIGMA*
     +          ALOG(RB/CABS(ZPR)))+PILAMB*(1.+SIGMA)*ALOG(RB/RON))
 340          CONTINUE
            IF(J.GE.1) THEN
                DO 350 K=1,J
                  PMK=PMK+PC(K)*((ZPR/RC)**K-SIGMA*
     +              (RB**2/(RC*CONJG(ZPR)))**K)
 350              CONTINUE
               END IF
           END IF
        TPR=REAL(PMK)+TC
        X=REAL(ZPR)
        Y=AIMAG(ZPR)
        WRITE(6,400) X,Y,TPR
 360    CONTINUE
      WRITE(*,510) OUTFI
 370 FORMAT(/' Number of iterations:  ',I4/)
 380 FORMAT(/'  Pipe         q(n)'/(I5,3X,E14.7))
 390 FORMAT(/'  Pipe',4X,'Order',11X,'P(n,k)'/(I5,I8,3X,2E12.5))
 395 FORMAT(/8X,'x',13X,'y',12X,'Temp',/)
 400 FORMAT(2X,F10.5,4X,F10.5,4X,F10.5)
 410 FORMAT(/,75('*'))
 420 FORMAT(/'  Order',13X,'PC(k)'/(I5,4X,2E12.5))
 430 FORMAT(/'  Pipe  Initial values for q(n) (Order of multipole = 0)'
     +      /(I5,E17.7))
 440 FORMAT(/1X,75('*')/ ' TEMPERATURES (Deg C)')
 441 FORMAT(' MULTIPOLE METHOD - Pipes in a composite cylinder'/,
     +  1X,48(1H=),//,' INPUT DATA')
 442 FORMAT(/,' Input file  ',A16,/,' Output file ',A16,/)
 443 FORMAT(/,' Pipe n=',I2)
 450 FORMAT(' Thermal conductivity in inner region',
     +  3X,F10.3,' W/(m*K)'/' Thermal conductivity in outer region',
     +  3X,F10.3,' W/(m*K)'/' Number of pipes',8X,I10/
     +  ' Order of multipoles',4X,I10)
 460 FORMAT(/' Radius of inner region',9X,F10.3,' m'/
     +  ' Radius of outer region',9X,F10.3,' m'/
     +  ' Outer boundary',/,
     +  '   Thermal resistance coefficient  ',2X,E10.3,/,
     +  '   Temperature',23X,F10.3,' C')
 470 FORMAT(/1X,34(' *')/
     +  ' Pipe *    x(n)   *    y(n)   *    rp(n)  *   beta(n)',
     +  ' *   Temp C  *'/1X,34(' *')/
     +  (I5,'  *',3(F9.3,'  *'),E10.3,'  *',F9.3,'  *'))
 480 FORMAT(/' Iteration accuracy',13X,E10.2/
```

```
     +  ' Maximum number of iterations',2X,I10)
  490 FORMAT(/' TEMPERATURE OUTPUT: Definition of rectangular area',/,
     +  '    Minimum x-value',10X,F10.3/
     +  '    Maximum x-value',10X,F10.3/
     +  '    Minimum y-value',10X,F10.3/
     +  '    Maximum y-value',10X,F10.3/
     +  '    Number of grid points along x-axis',I11/
     +  '    Number of grid points along y-axis',I11)
  495 FORMAT(/' Starting iterations  Goal: Iteration accuracy ',E10.2,/)
  500 FORMAT(' Number of iterations',I4,'   Iteration accuracy ',E10.2)
  510 FORMAT(//' Job ended. Output written to ',A16)
  701 FORMAT(1X,2(E12.5,2X),2I5)
  702 FORMAT(1X,5(E12.5,2X))
  703 FORMAT(1X,E12.5,I7)
  704 FORMAT(1X,4(E12.5,2X),2I5)
      END
      SUBROUTINE INV1(B,MC,MD,N,D)
C-----------------------------------------------------------------------
C        Calculate inverse of a matrix
C
C        Author: Anders Peterson
C        Date: 1982-05-28
C-----------------------------------------------------------------------
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION B(1),MC(1),MD(1)
C
      DATA C1/1.0D0/
      DATA C2/1.0D50/
C
      D=C1
      NK=-N
      DO 80 K=1,N
      NK=NK+N
      MC(K)=K
      MD(K)=K
      KK=NK+K
      BIGA=B(KK)
      DO 20 J=K,N
      IZ=N*(J-1)
      DO 20 I=K,N
      IJ=IZ+I
      R1=B(IJ)
      IF(R1.LT.0.0D0) R1=-R1
      R2=BIGA
      IF(R2.LT.0.0D0) R2=-R2
      R2=R2-R1
   10 IF(R2.GT.0.0D0) GO TO 20
      BIGA=B(IJ)
      MC(K)=I
      MD(K)=J
   20 CONTINUE
      J=MC(K)
      IF(J-K.LE.0) GO TO 35
      KI=K-N
      DO 30 I=1,N
      KI=KI+N
      HOLD=-B(KI)
```

```
      JI=KI-K+J
      B(KI)=B(JI)
   30 B(JI) =HOLD
   35 I=MD(K)
      IF(I-K.LE.0) GO TO 45
      JP=N*(I-1)
      DO 40 J=1,N
      JK=NK+J
      JI=JP+J
      HOLD=-B(JK)
      B(JK)=B(JI)
   40 B(JI) =HOLD
   45 IF(ABS(BIGA).GT.0.0D0) GO TO 48
   46 D=0.0
      RETURN
   48 DO 55 I=1,N
      IF(I-K.EQ.0) GO TO 55
      IK=NK+I
      B(IK)=B(IK)/(-BIGA)
   55 CONTINUE
      DO 65 I=1,N
      IK=NK+I
      HOLD=B(IK)
      IJ=I-N
      DO 65 J=1,N
      IJ=IJ+N
      IF(I-K.EQ.0) GO TO 65
      IF(J-K.EQ.0) GO TO 65
      KJ=IJ-I+K
      B(IJ)=HOLD*B(KJ)+B(IJ)
   65 CONTINUE
      KJ=K-N
      DO 75 J=1,N
      KJ=KJ+N
      IF(J-K.EQ.0) GO TO 75
      B(KJ)=B(KJ)/BIGA
   75 CONTINUE
C
C     UNDEFINED VALUE OF D IF ABS(D).GT.C2
C
      IF(D.LT.C2 .AND.D.GT.-C2)  D=D*BIGA
C
      B(KK)=C1/BIGA
   80 CONTINUE
      K=N
  100 K=(K-1)
      IF(K.LE.0) GO TO 150
      I=MC(K)
      IF(I-K.LE.0) GO TO 120
  108 JQ=N*(K-1)
      JR=N*(I-1)
      DO 110 J=1,N
      JK=JQ+J
      HOLD=B(JK)
      JI=JR+J
      B(JK)=-B(JI)
  110 B(JI) =HOLD
```

```
  120 J=MD(K)
      IF(J-K.LE.0) GO TO 100
  125 KI=K-N
      DO 130 I=1,N
      KI=KI+N
      HOLD=B(KI)
      JI=KI-K+J
      B(KI)=-B(JI)
  130 B(JI) =HOLD
      GO TO 100
  150 CONTINUE
      RETURN
      ENDæ
```