A gravitomagnetic thought experiment for undergraduates

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Abstract

The gravitomagnetic force is the gravitational counterpart to the Lorentz force in electromagnetics. This paper presents a thought experiment from which the gravitomagnetic force is obtained using the Newton theory of gravitation and elementary knowledge in the special theory of relativity. The example is suitable to present in an undergraduate course on basic electromagnetic theory.

1 Introduction

The gravitomagnetic effect was discovered by Joseph Lense and Hans Thirring in 1918 when they studied solutions to the Einstein field equations in a rotating system [3]. They found that a test mass that falls towards a massive rotating object experiences a force perpendicular to its motion. The force tends to drag the mass along the rotation of the object, a phenomena referred to as the Lense-Thirring effect. It was later seen that the effect appears in the linearized form of the Einstein field equations, *cf.*, [1] and [2]. The linearized version of the field equations resembles the Maxwell equations and are referred to as the gravitoelectromagnetic equations. They are only valid in regions where the space can be considered to be flat. In a local rest frame the equations are

$$\begin{aligned}
\nabla \times \boldsymbol{g} &= -\frac{1}{c}\frac{\partial \boldsymbol{b}}{\partial t} \\
\nabla \times \boldsymbol{b} &= \frac{1}{c}\left(-4\pi G \boldsymbol{J}_G + \frac{\partial \boldsymbol{g}}{\partial t}\right) \\
\nabla \cdot \boldsymbol{g} &= -4\pi G \rho_G \\
\nabla \cdot \boldsymbol{b} &= 0
\end{aligned} \tag{1.1}$$

A number of the quantities that appear in the equations need to be explained. The proper mass density is denoted $\rho_G(\boldsymbol{r})$, and the local rest-mass current density is, $\boldsymbol{J}_G(\boldsymbol{r}) = \rho_G(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r})$, where $\boldsymbol{v}(\boldsymbol{r})$ is the three velocity of the local rest mass. Furthermore, G is the gravitational constant, $\boldsymbol{g}$ is referred to as the gravitoelectric field and $\boldsymbol{b}$ as the gravitomagnetic field. The field $\boldsymbol{g}$ is the three acceleration in the Newton theory of gravitation, whereas the field $\boldsymbol{b}$ is not present in Newton's theory. In some papers on gravitomagnetics the mass density ρ_G is not the proper mass density but rather given by $\rho_G = (T_{00} + T^{ii})/2c^2$, see [2]. The linearized equations can be treated just like the Maxwell equations. Hence in a mass free region the fields $\boldsymbol{g}$ and $\boldsymbol{b}$ satisfy the homogeneous wave equation. The waves are generated from time varying mass flow densities, just as electromagnetic waves originate from time varying current densities. A review of the gravitomagnetic effect is given in [2], where also recent attempts to verify the effect are discussed.

Under the linear approximation the total force on a test mass m consists of two terms

$$\boldsymbol{F}_g = m\boldsymbol{g} + 2\frac{m}{c}\boldsymbol{v} \times \boldsymbol{b} \tag{1.2}$$

Notice the similarity between this force and the Lorentz force for a charge moving in an electromagnetic field. The second term $2\frac{m}{c}\boldsymbol{v} \times \boldsymbol{b}$ is the gravitomagnetic force. This type of force gives rise to the Lense-Thirring effect. The purpose of this paper is to present a thought experiment that can only be explained if a gravitomagnetic force is introduced. For pedagogical reasons a similar experiment is first discussed for the electromagnetic case. The electromagnetic example might be familiar to many teachers in electromagnetic theory but the gravitomagnetic example is probably novel.

2 The Lorentz force in electromagnetics

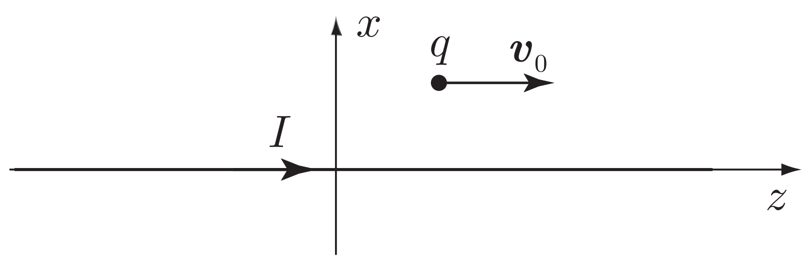

Figure 1: The conductor with current I and the point charge q

An example that gives rise to curiosity among undergraduate students in electromagnetics is the Lorentz force on a point charge q that travels parallel to a straight, infinitely long, cylindrical conductor with DC current I. The geometry is depicted in figure 1. The example leads to the paradox that the conductor has no net charge in a system that is in rest relative the conductor while in a system that is in rest relative the point charge, the conductor has a net charge. The paradox can of course be solved by the Lorentz transformation of the four vector $(c\rho, \boldsymbol{J})$. However, that theory is unknown for the students and would require a lecture on the Lorentz transformation of the electromagnetic fields. Instead it can be explained by using the Lorentz contraction formula, which most students are familiar with.

Consider the two inertial systems A and B, where the system A is at rest relative the conductor and the system B is at rest relative the point charge. In A the charge density of the conductor is zero and the current can be viewed as two line charges ρ_l and $-\rho_l$ moving with the velocity $\boldsymbol{v}^+ = v_0\hat{\boldsymbol{z}}$ and $\boldsymbol{v}^- = -v_0\hat{\boldsymbol{z}}$, respectively. The resulting current in the conductor is $I = 2\rho_l v_0$. The point charge is moving with the constant velocity $\boldsymbol{v} = v_1\hat{\boldsymbol{z}}$ at a radial distance r_c from the $z-$axis. The speeds v_0 and v_1 are considered to be much smaller than the speed of light. Standard calculations give the Lorentz force on the point charge in the system A

$$\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B} = -q\mu_0 v_1 \frac{I}{2\pi r_c}\hat{\boldsymbol{r}}_c \tag{2.1}$$

where $\hat{\boldsymbol{r}}_c$ is radial unit vector and μ_0 is the permeability in vacuum.

Now move to system B and check out the force on q. Needless to say, the force on q is the same in systems A and B. In B the charge has no velocity and the Lorentz force is zero and hence the force must be an electric force $\boldsymbol{F} = q\boldsymbol{E}(r_c)$ where

$$\boldsymbol{E}(r_c) = -\mu_0 v_1 \frac{\rho_l v_0}{\pi r_c} \hat{\boldsymbol{r}}_c \tag{2.2}$$

According to the divergence theorem this corresponds to a linear charge density

$$\rho_l^B = -2\rho_l \frac{v_0 v_1}{c^2} = -\frac{v_0 I}{c^2} \tag{2.3}$$

along the $z-$axis. In a course on basic electromagnetic theory one can ask the students if they have any clue from where this charge is coming. Giving some hints a discussion can start where eventually the problem is solved by using the special theory of relativity. Most students have heard of that lengths and time are transformed between inertial systems and they easily accept the formulas for the transformation of these quantities. The formula for length contraction implies that the line charge density ρ_l is different in system B . Neglecting higher order terms than $(v_0 \pm v_1)^2/c^2$ the Lorentz transformation to system B gives the density

$$\rho_l^+ = \frac{\rho_l}{\sqrt{1 - \left(\frac{v_1 - v_0}{c}\right)^2}} = \rho_l \left(1 + \frac{v_1^2 + v_0^2}{2c^2} - \frac{v_0 v_1}{c^2}\right) + \text{h.o.t} \tag{2.4}$$

Accordingly, the line charge $-\rho_l$ is transformed to

$$\rho_l^- = -\frac{\rho_l}{\sqrt{1 - \left(\frac{v_1 + v_0}{c}\right)^2}} = -\rho_l \left(1 + \frac{v_1^2 + v_0^2}{2c^2} + \frac{v_0 v_1}{c^2}\right) + \text{h.o.t} \tag{2.5}$$

Thus, in system B the total charge density of the conductor is

$$\rho_l^+ + \rho_l^- = -2\rho_l \frac{v_0 v_1}{c^2} + \text{h.o.t} \tag{2.6}$$

which is the same result as in Eq. (2.3). A similar thought experiment is now performed for gravitation.

3 The gravitomagnetic force

This gravitational thought experiment only assumes basic knowledge of the Newton theory of gravitation and the special theory of relativity. When the experiment has been discussed one can show that the outcome of it is in accordance with the linearized equations (1.1) and the gravitomagnetic force in Eq. (1.2).

According to Newton the force between two point like masses m and M is given by

$$\boldsymbol{F}_m = -\boldsymbol{F}_M = G\frac{mM}{r^2}\hat{\boldsymbol{r}} \tag{3.1}$$

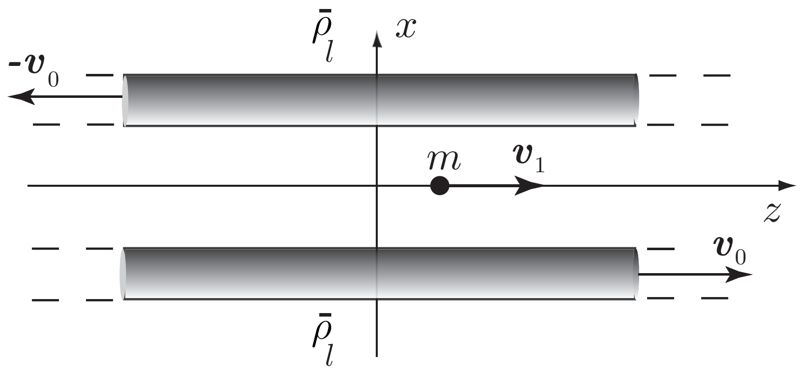

Figure 2: The two axisymmetric cylinders and the point-like test mass m seen from system A. In this case the two cylinders have the same mass per unit length and according to Newton's theory the gravitational force on the mass m is zero .

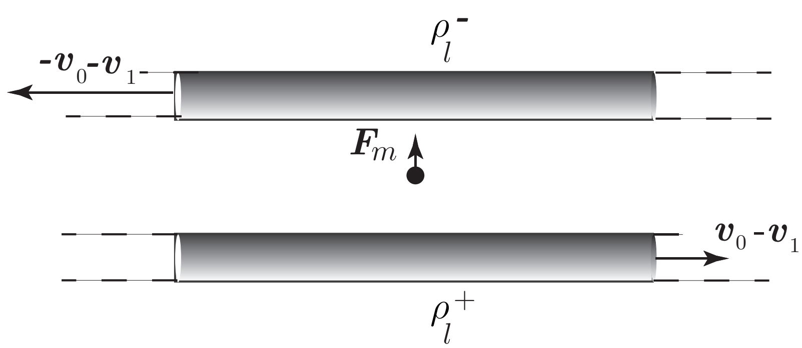

Figure 3: The two axisymmetric cylinders and the point-like test mass m seen from system B. In this case the upper cylinder has a larger mass per unit length than the lower one and there is a resulting gravitational force $\boldsymbol{F}_m$ on the mass m.

where r is the distance between the masses and $\hat{\boldsymbol{r}}$ is the unit vector directed from mass m towards mass M. According to the relation $E = mc^2$ we interpret mass as $m = E/c^2$. The vector $\boldsymbol{g} = \boldsymbol{F}_m/m$ is introduced as the counterpart to the electric field. It satisfies Poisson's equation

$$\nabla \cdot \boldsymbol{g}(\boldsymbol{r}) = -4\pi G \rho_G(\boldsymbol{r}) \tag{3.2}$$

where $\rho_G(\boldsymbol{r})$ is the mass density. According to the divergence theorem the field $\boldsymbol{g}$ outside an infinitely long axisymmetric cylinder with mass ρ_l per unit length and with the $z-$axis as symmetry axis is

$$\boldsymbol{g}(r_c) = -2G\frac{\rho_l}{r_c}\hat{\boldsymbol{r}}_c \tag{3.3}$$

Next consider two infinitely long axially symmetric circular cylinders with radius a, as depicted in figure 2. Each cylinder has a rest mass per unit length of ρ_l. One cylinder has its symmetry axis along the line $(x, y) = (-x_0, 0)$, where $x_0 > a$, and is moving with the velocity $\boldsymbol{v}_0 = v_0\hat{\boldsymbol{z}}$ relative a fixed coordinate system and the other cylinder has its symmetry axis along the line $(x, y) = (x_0, 0)$ and is moving with the velocity $-v_0\hat{\boldsymbol{z}}$. A point like test mass m is moving with speed $\boldsymbol{v}_1 = v_1\hat{\boldsymbol{z}}$ along the $z-$axis. Again two inertial systems are introduced: system A which is at rest relative the coordinate system and system B which is moving with the same velocity as the test mass. According to Newton's theory the gravitational force on the test mass is zero in system A, due to symmetry. In system B the mass per unit length of the two cylinders is altered due to length contraction and the Lorentz transformation of mass[1]. The mass per unit length of the two cylinders seen from system B are

$$\rho_l^+ = \frac{\rho_l}{1 - \left(\frac{v_0 - v_1}{c}\right)^2} \tag{3.4}$$

for the cylinder moving with velocity $v_1\hat{\boldsymbol{z}}$ and

$$\rho_l^- = \frac{\rho_l}{1 - \left(\frac{v_0 + v_1}{c}\right)^2} \tag{3.5}$$

for the other cylinder. On the $z-$axis the corresponding gravitational fields are

$$\begin{aligned} \boldsymbol{g}^+ &= -2G\frac{\rho_l^+}{x_0}\hat{\boldsymbol{x}} \\ \boldsymbol{g}^- &= 2G\frac{\rho_l^-}{x_0}\hat{\boldsymbol{x}} \end{aligned} \tag{3.6}$$

Neglecting higher order terms than $(v_0 \pm v_1)^2/c^2$ the force on the test mass in system B is given by

$$\boldsymbol{F}_m = m(\boldsymbol{g}^+ + \boldsymbol{g}^-) = 8mG\rho_l\frac{v_0 v_1}{x_0 c^2}\hat{\boldsymbol{x}} + \text{h.o.t.} \tag{3.7}$$

[1] It is convenient to use the Lorentz transformation of mass. The mass is then not the proper mass but defined by $m = E/c^2$, where E is the total energy.

Now the same force must act on the test mass in system A, but, as mentioned earlier, in A that force is not caused by Eq. (3.1) and must be interpreted as some other kind of force. The force is proportional to the speed of the test mass and is perpendicular to the velocity of the test mass. Hence it is a vector product between the velocity $\boldsymbol{v}_1$ and a vector field. This gives

$$\boldsymbol{F}_m = 2\frac{m}{c}\boldsymbol{v}_1 \times \boldsymbol{b} \tag{3.8}$$

where

$$\boldsymbol{b} = -4G\rho_l \frac{v_0}{x_0 c}\hat{\boldsymbol{y}} \tag{3.9}$$

The field $\boldsymbol{b}$ is the sum of the magnetic type fields from each cylinder. The conclusion is that an infinitely long axisymmetric cylinder with the $z-$axis as symmetry axis and mass ρ_l per unit length moving with the velocity $\boldsymbol{v} = v_0\hat{\boldsymbol{z}}$ generates a gravitomagnetic field

$$\boldsymbol{b} = -2G\frac{v_0}{c}\frac{\rho_l}{r_c}\hat{\boldsymbol{\varphi}} \tag{3.10}$$

where $\hat{\boldsymbol{\varphi}}$ is the azimuthal unit vector, *cf.*, figure 4 and r_c is the distance from the symmetry axis. Notice that the gravitomagnetic field is in opposite direction to the corresponding magnetic field from a line current. It is easy to verify that the result in Eq. (3.10) is in accordance with the second equation in Eq. (1.1).

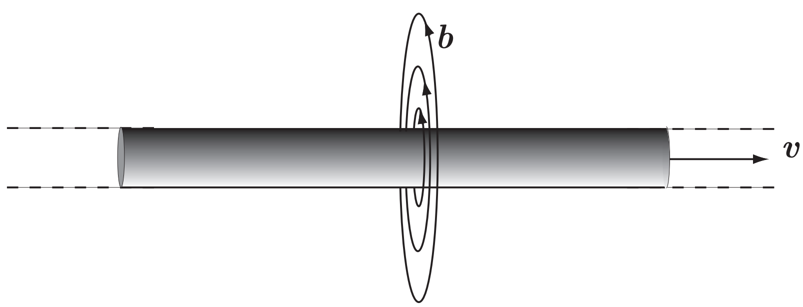

Figure 4: A long cylinder moving with a velocity $\boldsymbol{v}$ along its axis of symmetry creates a circular gravitomagnetic field $\boldsymbol{b}$. The field $\boldsymbol{b}$ is a consequence of the Newton theory of gravitation and the special theory of relativity.

4 Discussion

Without any knowledge in general relativity theory it is possible to derive the gravitomagnetic field from a long cylinder moving with constant velocity along its symmetry axis. It is of interest to have discussions with students on the magnitude of the gravitomagnetic force and how it can be experimentally verified. The ratio between the force $m\boldsymbol{g}$ and the gravitomagnetic force is $v_0 v_1/c^2$, an incredibly small number at normal speeds v_0 and v_1. Even so, there seems to be successful verifications using satellites [2]. A relevant question is why the difference between the Newton gravitational force and the gravitomagnetic force is so huge, when the Lorentz force easily can be made larger than the electric force? As one proceed in a basic course on electromagnetics one can get back to the equations (1.1) and compare them with the Maxwell equations. In the context of electromagnetic waves and antennas it is interesting to discuss gravitational waves and antennas for gravitational waves. There are still no experiments that prove that gravitational waves can be generated and detected. This lack of experimental proofs might start creative discussions among the students.

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