Sommerfeld's forerunner in stratified isotropic and bi-isotropic media

Rikte, Sten

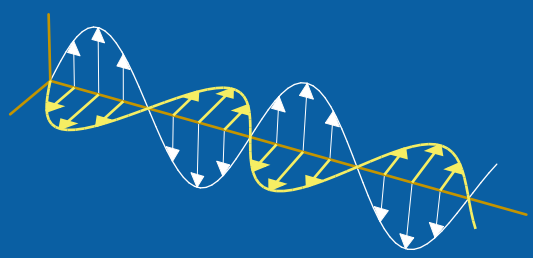
1994

CODEN:LUTEDX/(TEAT-7036)/1-25/(1994)

Sommerfeld's forerunner in stratified isotropic and bi-isotropic media

Sten Rikte

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Sten Rikte

Department of Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

Editor: Gerhard Kristensson

Abstract

Using well-established time domain methods, Sommerfeld's classical results for the forerunner (the first precursor) at distant points in a homogeneous, isotropic, and dispersive half-space are generalized to the stratified, isotropic, and bi-isotropic slab cases, with arbitrary transient, normally incident, plane wave excitation. The results hold at each point and impedance mismatch cases are treated as well. As a consequence of the analysis, it is seen that also inhomogeneities in a non-dispersive medium cause oscillations in the signal's early time behavior, similar to those in the dispersive medium.

1 Introduction and results

The early time behavior of a signal that propagates in a temporally dispersive medium is an issue that has interested physicists over the years, see Brillouin's book [3] or the original works from 1914 by Sommerfeld [32] and Brillouin [2]. The first precursor (Sommerfeld's forerunner) in the homogeneous isotropic half-space was investigated by Sommerfeld using the method of steepest descent. Sommerfeld's results hold for sinusoidal, normally incident, plane wave excitation and for media such that Lorentz' single-resonance dispersion model for the electric susceptibility kernel

$$\chi(t) = \frac{\omega_p^2}{\nu_0} e^{-\frac{\nu t}{2}} \sin(\nu_0 t)\, H(t) \tag{1.1}$$

is applicable. In this expression, $H(t)$ denotes the Heaviside step, whereas ω_p and ν are the plasma and collision frequencies of the medium, respectively, and $\nu_0^2 = \omega_0^2 - \nu^2/4$, where ω_0 is the harmonic frequency of the medium. Sommerfeld's result for the electric field strength $E(z,t)$ at a distant point $z >> 0$ in the dispersive half-space $z \in (0,\infty)$ and for small wave front times $\tau = t - z/c_0$, where c_0 is the speed of light in vacuum, reads

$$E(z,\tau) = \omega \sqrt{\frac{\tau}{a(z)}} J_1\left(2\sqrt{\tau a(z)}\right) H(\tau), \qquad a(z) = \frac{z\omega_p^2}{2c_0}. \tag{1.2}$$

In this expression, the approximation $E^i(t) = \sin(\omega t)\, H(t) \approx \omega t\, H(t)$, for small times t, of the incident sinusoidal electric field $E^i(t)$ at the impedance matched edge, $z = 0$, has been employed. The function J_1 that appears in the formula is the Bessel function of the first kind and order. The second precursor (Brillouin's forerunner) was investigated by Brillouin, also by means of asymptotic analysis. In the 1980's, the works of Sommerfeld and Brillouin were continued and improved, e.g., for multi-frequency Lorentz media [24–26, 31]. Furthermore, in the early 1990's, the first and second forerunners in a lossless, homogeneous, single-resonance Pasteur half-space (a reciprocal, bi-isotropic medium) were obtained by Engheta and Zablocky [6, 34]. Recently, also a time domain method has been used to obtain Sommerfeld's precursor in an isotropic slab, periodic with respect to depth [12].

Propagation of transient electromagnetic waves in slabs consisting of dispersive, stratified, isotropic or complex media has been investigated in a number of publications during the last decade, see, e.g., Refs. [1, 7–9, 13, 19–21, 23, 29]. As a result of this analysis, corresponding inverse scattering problems were solved as well, i.e., the characteristic parameters of such media could be reconstructed, given scattering data. Two methods of solution, both based on the wave splitting technique, are employed in the treatment of these problems: the invariant imbedding approach and the Green functions technique. In this contribution, these time domain methods are used to obtain Sommerfeld's precursor in stratified isotropic or bi-isotropic slabs. In the impedance matched case, the matrix function

$$\mathbf{P}_S(z,t) = -\sqrt{\frac{\int_0^z \mathbf{A}(z',0)\,dz'}{t}}\,J_1\Big(2\sqrt{t\int_0^z \mathbf{A}(z',0)\,dz'}\Big)\,H(t),$$

is referred to as Sommerfeld's precursor kernel. A transient, causal, and normally incident electric field $\boldsymbol{E}^i(t)$ at the front wall—a 2-dimensional vector—then generates the electric field

$$\boldsymbol{E}^+(z,t) = \mathbf{Q}^+(0,z)\{\boldsymbol{E}^i(t) + \int_0^t \mathbf{P}_S(z,t-t')\,\boldsymbol{E}^i(t')\,dt'\}.$$

inside the medium at small wavefront times t. The 2×2-matrices $\mathbf{A}(z,0)$ and $\mathbf{Q}^+(0,z)$ depend on the properties of the dispersive medium. The explicit expressions are given below. Naturally, this precursor is a good approximation to the propagating electric field only for small (positive) t.

In particular, if a Maclaurin expansion is applied to the special incident field $\boldsymbol{E}^i(t) = \sin(\omega t)\,H(t)\boldsymbol{e}_x$, the change of the order of summation and integration followed by the use of the Bessel function's equality [27]

$$\int_0^b J_1(x)\,(b^2-x^2)^m\,dx = b^{2m} - (2b)^m m! J_m(b)$$

yields the formula

$$\boldsymbol{E}^+(z,t) = \mathbf{Q}^+(0,z)\sqrt{t\left(\int_0^z \mathbf{A}(z',0)\,dz'\right)^{-1}}\times \tag{1.3}$$
$$\times\sum_{k=0}^{\infty}(-1)^k\omega^{2k+1}\Big(t\left(\int_0^z \mathbf{A}(z',0)\,dz'\right)^{-1}\Big)^k J_{2k+1}\Big(2\sqrt{t\int_0^z \mathbf{A}(z',0)\,dz'}\Big)\,H(t)\begin{pmatrix}1\\0\end{pmatrix}$$

for the first precursor. Similar results has been presented by Jackson [10] for the isotropic, dispersive half-space and by Kristensson [14] for wave guides. Sommerfeld's result referred to above is recognized as the first term in this series since the matrices $\mathbf{A}(z,0)$ and $\mathbf{Q}^+(0,z)$ are proportional to the identity matrix in isotropic cases.

The analysis in the present article holds for all stratified bi-isotropic media with sufficiently smooth material parameters. It is conjectured that the method can be

used also for other complex media. In Sections 2–9, the problem of the propagation of transient waves in the dispersive medium is formulated and systematically analyzed in the general mismatch case. In Section 10, the results in the previous sections are used to obtain both Sommerfeld's forerunner at arbitrary position inside the dispersive medium and the corresponding transient electric field measured at the back wall.

2 Definitions and basic equations

Consider a stratified, dispersive bi-isotropic medium located between the two planes $z = 0$ and $z = d$ in a right-handed Cartesian coordinate system $O(x, y, z)$, where the three basis vectors are denoted by $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$, respectively. The bi-isotropic slab is excited by a transient, normally incident plane wave with sources at $z = -\infty$, and the incident electric field at the boundary $z = 0$ at the time t is denoted by $\boldsymbol{E}^i(t)$. The medium in the half-space $z < 0$ is homogeneous, isotropic, and non-dispersive with arbitrary material constants $\epsilon(-0)$ and $\mu(-0)$. The medium in the half-space $z > d$ is also isotropic and non-dispersive with constant permittivity and permeability $\epsilon(d+0)$ and $\mu(d+0)$, respectively. For linear, causal, and time-invariant media, it is not a restriction to assume that the plane wave impinges on the surface $z = 0$ at $t = 0$. Moreover, the incident electric field is assumed to be continuously differentiable with bounded derivative at $t > 0$ except for at most finitely many points; thus, a finite number of finite jump-discontinuities are permitted for $t > 0$. The reflected electric field at $z = -0$ at time t is denoted by $\boldsymbol{E}^r(t)$. Similarly, the transmitted electric field at $z = d+0$ at time t is denoted by $\boldsymbol{E}^t(t)$. Furthermore, it is understood that the bi-isotropic medium is initially quiescent, i.e., the electric field $\boldsymbol{E}(\boldsymbol{r}, t)$ and the magnetic field $\boldsymbol{H}(\boldsymbol{r}, t)$ at the point $\boldsymbol{r} \equiv (x, y, z) \equiv x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$ at time t satisfy

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{0} \quad \text{for} \quad 0 < z < d \quad \text{when} \quad t \leq 0. \tag{2.1}$$

As usual, the electric and magnetic flux densities at the point $\boldsymbol{r}$ at time t are denoted by $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$, respectively.

In this paragraph, the properties of the dispersive medium are defined. The constitutive relations are assumed to be

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r}, t) = \epsilon(z)\left[\boldsymbol{E}(\boldsymbol{r}, t) + (\chi_{ee} * \boldsymbol{E})(\boldsymbol{r}, t)\right] + c(z)^{-1}(\chi_{em} * \boldsymbol{H})(\boldsymbol{r}, t), \\ \boldsymbol{B}(\boldsymbol{r}, t) = c(z)^{-1}(\chi_{me} * \boldsymbol{E})(\boldsymbol{r}, t) + \mu(z)\left[\boldsymbol{H}(\boldsymbol{r}, t) + (\chi_{mm} * \boldsymbol{H})(\boldsymbol{r}, t)\right], \end{cases} \tag{2.2}$$

where, e.g.,

$$(\chi_{ee} * \boldsymbol{E})(\boldsymbol{r}, t) = \int_0^t \chi_{ee}(z, t - t')\boldsymbol{E}(\boldsymbol{r}, t')\, dt'. \tag{2.3}$$

The value of the upper limit of integration in Eq. (2.3) is due to causality [11] and the lower limit is a direct consequence of Eq. (2.1). Thus, all time dependent functions are causal, i.e., identically zero for negative times. The functions $(0, d) \ni z \to \epsilon(z)$ and $(0, d) \ni z \to \mu(z)$ are (the non-dispersive parts of) the

permittivity and permeability of the dispersive medium, respectively. They are assumed to be continuously differentiable with bounded derivatives. Furthermore, the general mismatch case is considered, i.e., $\epsilon(-0) \neq \epsilon(+0)$ and/or $\mu(-0) \neq \mu(+0)$ in general, and similarly at the back wall $z = d$. The phase velocity at position z

$$c(z) := (\mu(z)\epsilon(z))^{-1/2}$$

has been introduced in the constitutive relations in order to simplify the analysis. At this stage, it is also appropriate to define the wave impedance of the medium

$$\eta(z) := \sqrt{\mu(z)/\epsilon(z)}$$

at position z. All the functions χ_{ee}, χ_{em}, χ_{me}, and χ_{mm} have the same unit, s^{-1}, and are known as the susceptibility kernels. The integral kernels χ_{ee} and χ_{mm} model the ordinary dispersive effects of the medium. For practical reasons, define

$$\begin{cases} G := (\chi_{ee} + \chi_{mm})/2, \\ F := (\chi_{ee} - \chi_{mm})/2. \end{cases} \tag{2.4}$$

The chirality kernel

$$K := (\chi_{em} - \chi_{me})/2 \tag{2.5}$$

and the non-reciprocity kernel

$$L := (\chi_{em} + \chi_{me})/2 \tag{2.6}$$

— the medium is reciprocal if $L = 0$, see Ref. [11] — are the characteristic properties of the bi-isotropic medium. The susceptibility kernels depend on the spatial variable z and the time t only, i.e., the slab is stratified with respect to depth z. These integral kernels and their first and second time derivatives are assumed to be bounded and continuous functions in the set $(z, t) \in (0, d) \times (0, \infty)$. Finally, it is clear, that the constitutive relations (2.2) hold throughout space if the susceptibility kernels are given the value zero outside the dispersive slab.

The electromagnetic field obeys the source-free Maxwell equations:

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \qquad \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \qquad \nabla \cdot \boldsymbol{D} = \boldsymbol{0}, \qquad \nabla \cdot \boldsymbol{B} = \boldsymbol{0}. \tag{2.7}$$

Transverse solutions to these equations, independent of the transverse coordinates (x, y), are sought, i.e., the space- and time dependence

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{e}_x E_x(z, t) + \boldsymbol{e}_y E_y(z, t), \\ \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{e}_x H_x(z, t) + \boldsymbol{e}_y H_y(z, t), \end{cases}$$

is presumed to hold (whenever these vector fields are well-defined), and similarly for the flux densities. The investigation in Ref. [28] shows that this condition can be weakened in the sense that it is not necessary to assume that the z-components of the

vector fields vanish inside the bi-isotropic medium, provided that the electromagnetic field is independent of x and y. The Maxwell equations (2.7) can now be written

$$\partial_z \boldsymbol{E} = \partial_t(\mathbf{J}\boldsymbol{B}), \qquad \partial_z(\mathbf{J}\boldsymbol{H}) = \partial_t \boldsymbol{D}, \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.8}$$

where a compact matrix notation has been introduced. Define matrix-valued susceptibility kernels by

$$\boldsymbol{\chi}_{ee} := \chi_{ee}\mathbf{I}, \qquad \boldsymbol{\chi}_{me} := \chi_{me}\mathbf{J}, \qquad \boldsymbol{\chi}_{mm} := \chi_{mm}\mathbf{I}, \qquad \boldsymbol{\chi}_{em} := \chi_{em}\mathbf{J},$$

where $\mathbf{I}$ is the 2×2 identity matrix. Elimination of the flux densities $\boldsymbol{B}$ and $\boldsymbol{D}$ in the Maxwell equations (2.8) using the constitutive relations (2.2) yields a partial integro-differential equation in the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ only. The result of this operation is

$$\begin{aligned} \partial_z \begin{pmatrix} \boldsymbol{E} \\ \eta\mathbf{J}\boldsymbol{H} \end{pmatrix} = \frac{\eta'}{\eta} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \eta\mathbf{J}\boldsymbol{H} \end{pmatrix} + \\ + c^{-1}\partial_t \left(\begin{pmatrix} \boldsymbol{\chi}_{me}* & \mathbf{I} + \boldsymbol{\chi}_{mm}* \\ \mathbf{I} + \boldsymbol{\chi}_{ee}* & -\boldsymbol{\chi}_{em}* \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ \eta\mathbf{J}\boldsymbol{H} \end{pmatrix} \right), \end{aligned} \tag{2.9}$$

where $\mathbf{0}$ is the 2×2 zero matrix. This non-local wave equation for the electromagnetic field is now to be analyzed.

3 Wave splitting

Define new dependent variables through the wave splitting

$$\begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} = \mathbf{P} \begin{pmatrix} \boldsymbol{E} \\ \eta\mathbf{J}\boldsymbol{H} \end{pmatrix}, \qquad \mathbf{P} = \frac{1}{2} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{P}^{-1} = \begin{pmatrix} \mathbf{I} & \mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}. \tag{3.1}$$

This is obviously a bijective linear map and by strict causality one has

$$\boldsymbol{E}^{\pm}(z,t) = \mathbf{0} \quad \text{for} \quad 0 < z < d \quad \text{when} \quad t < \int_0^z c(z')^{-1}\, dz', \tag{3.2}$$

see Ref. [28]. Thus, the condition (2.1) can be strengthened:

$$\boldsymbol{E}(\boldsymbol{r},t) = \mathbf{0} \quad \text{and} \quad \boldsymbol{H}(\boldsymbol{r},t) = \mathbf{0} \quad \text{for} \quad 0 < z < d \quad \text{when} \quad t < \int_0^z c(z')^{-1}\, dz',$$

and similarly for the flux densities. In the isotropic half-space $z < 0$, the vector fields $\boldsymbol{E}^{\pm}(z,t)$ coincide with the incident and reflected electric fields at the point (z,t), respectively. Analogously, $\boldsymbol{E}^+(z,t)$ is the transmitted electric field at (z,t) and $\boldsymbol{E}^-(z,t) = \mathbf{0}$ when $z > d$, since there is no incident wave from the right. A similar interpretation cannot in general be made inside the dispersive slab. However, the sum of the split vector fields is equal to the (total) electric field. For a survey of the wave splitting technique, the reader is referred to Ref. [4]. For recent contributions to the solution of direct and inverse scattering problems in complex media using the wave splitting technique, see Refs. [7, 9, 20, 21].

4 Dynamics and boundary conditions

In this section, the non-local hyperbolic equation for the split vector fields is derived. In addition, the boundary conditions relevant for the scattering and propagation problems are obtained. The wave splitting (3.1) and the continuity of (the tangential components of) the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ at the front wall imply that the relation

$$t_0\boldsymbol{E}^i(t) = \boldsymbol{E}^+(+0,t) - r_0\boldsymbol{E}^-(+0,t) \tag{4.1}$$

holds (whenever the incident electric field is well-defined), where

$$t_0 = \frac{2\eta(+0)}{\eta(+0)+\eta(-0)} \quad \text{and} \quad r_0 = \frac{\eta(-0)-\eta(+0)}{\eta(+0)+\eta(-0)}.$$

Note that t_0 is the transmission coefficient at the front wall $z=0$ for the transition from the non-dispersive medium to the dispersive medium and that r_0 is the reflection coefficient at $z=0$ viewed from the dispersive medium. Similarly, one obtains the boundary condition at the back wall:

$$\boldsymbol{E}^-(d-0,t) = r_1\boldsymbol{E}^+(d-0,t), \quad \text{where} \quad r_1 = \frac{\eta(d+0)-\eta(d-0)}{\eta(d+0)+\eta(d-0)}. \tag{4.2}$$

Clearly, r_1 is the reflection coefficient at the surface $z=d$ viewed from the dispersive medium. It is also convenient to introduce the transmission coefficient t_1 at the back wall $z=d$ for the transition from the non-dispersive medium to the dispersive medium:

$$t_1 = \frac{2\eta(d-0)}{\eta(d+0)+\eta(d-0)}.$$

Note that $r_0+t_0=1$ and $r_1+t_1=1$. The two equations (4.1) and (4.2) are relevant for the propagation of waves inside the dispersive medium. For the scattered electric fields one obtains

$$\begin{cases} \boldsymbol{E}^r(t) = (1+r_0)\boldsymbol{E}^-(+0,t) - r_0\boldsymbol{E}^i(t), \\ \boldsymbol{E}^t(t) = (1+r_1)\boldsymbol{E}^+(d-0,t). \end{cases} \tag{4.3}$$

This equation is referred to when the scattering operators are defined later on. Note that $1+r_0$ and $1+r_1$ are the transmission coefficients at the front and back walls, $z=0$ and $z=d$, respectively, for the transition from the dispersive medium to the (adequate) non-dispersive medium. Analogously, and viewed from the surrounding isotropic media, the reflection coefficients at the front and back walls are equal to $-r_0$ and $-r_1$, respectively.

The dynamic equation for the split vector fields $\boldsymbol{E}^\pm$ reads

$$\begin{pmatrix} (\partial_z + c^{-1}\partial_t)\boldsymbol{E}^+ \\ (\partial_z - c^{-1}\partial_t)\boldsymbol{E}^- \end{pmatrix} = \frac{\eta'}{2\eta}\begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix}\begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix} + \frac{1}{2c}\partial_t\left(\boldsymbol{\chi} * \begin{pmatrix} \boldsymbol{E}^+ \\ \boldsymbol{E}^- \end{pmatrix}\right), \tag{4.4}$$

where the 4×4 susceptibility kernel $\boldsymbol{\chi}$ is defined by

$$\boldsymbol{\chi} := \begin{pmatrix} -\boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & -\boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} \\ \boldsymbol{\chi}_{ee} - \boldsymbol{\chi}_{mm} + \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} & \boldsymbol{\chi}_{ee} + \boldsymbol{\chi}_{mm} - \boldsymbol{\chi}_{em} + \boldsymbol{\chi}_{me} \end{pmatrix}.$$

This equation is easily obtained from the wave equation (2.9) and the wave splitting (3.1). Recall the definitions (2.4)–(2.6) of the susceptibility kernels G, K, F, and L. By introducing the susceptibility matrices

$$\mathbf{G} := G\mathbf{I}, \qquad \mathbf{F} := F\mathbf{I}, \qquad \mathbf{K} := K\mathbf{J}, \quad \text{and} \quad \mathbf{L} := L\mathbf{J},$$

the notation for the direct and inverse scattering problems in Refs. [20, 29] is adopted. The matrix kernel $\boldsymbol{\chi}$ can now be written

$$\frac{1}{2}\boldsymbol{\chi} = \begin{pmatrix} -\mathbf{G}-\mathbf{K} & -\mathbf{F}+\mathbf{L} \\ \mathbf{F}+\mathbf{L} & \mathbf{G}-\mathbf{K} \end{pmatrix}.$$

In order to analyze the dynamic equation (4.4), it is also convenient to introduce

$$\mathbf{b}(z) \equiv \begin{pmatrix} \mathbf{b}_{++}(z) & \mathbf{b}_{+-}(z) \\ \mathbf{b}_{-+}(z) & \mathbf{b}_{--}(z) \end{pmatrix} := \frac{1}{2c}\boldsymbol{\chi}(z,0) + \frac{d}{dz}\ln\sqrt{\frac{\eta(z)}{\eta_0}} \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{pmatrix},$$

where η_0 is the wave impedance in vacuum and $\mathbf{b}_{ij}$ are 2×2-matrices. The two sub-matrices $\mathbf{b}_{\pm\pm}$ on the diagonal determine the propagation of jump-discontinuities along the directions $\pm\boldsymbol{e}_z$, respectively.

5 Propagation of jump discontinuities

Any finite jump-discontinuity

$$[\boldsymbol{E}^{\pm}(z_1,t)] := \boldsymbol{E}^{\pm}(z_1,t+0) - \boldsymbol{E}^{\pm}(z_1,t-0)$$

in $\boldsymbol{E}^{\pm}$ at (z_1,t) is attenuated and rotated as it propagates through the medium:

$$[\boldsymbol{E}^{\pm}(z_2, t \pm \int_{z_1}^{z_2} c(z')^{-1}\,dz')] = \mathbf{Q}^{\pm}(z_1,z_2)[\boldsymbol{E}^{\pm}(z_1,t)], \quad 0 < z_1,\ z_2 < d, \tag{5.1}$$

where the matrices $\mathbf{Q}^{\pm}(z_1,z_2)$ satisfy the ordinary differential equations

$$\begin{cases} \partial_{z_2}\mathbf{Q}^{\pm}(z_1,z_2) = \mathbf{b}_{\pm\pm}(z_2)\mathbf{Q}^{\pm}(z_1,z_2), & 0 < z_1,\ z_2 < d, \\ \mathbf{Q}^{\pm}(z_1,z_1) = \mathbf{I}, \qquad 0 < z_1 < d, \end{cases} \tag{5.2}$$

respectively. The solutions to these equations are

$$\mathbf{Q}^{\pm}(z_1,z_2) = e^{\int_{z_1}^{z_2}\mathbf{b}_{\pm\pm}(z')\,dz'} = \sqrt{\frac{\eta(z_2)}{\eta(z_1)}}e^{\pm a(z_1,z_2)} \begin{pmatrix} \cos\phi(z_1,z_2) & -\sin\phi(z_1,z_2) \\ \sin\phi(z_1,z_2) & \cos\phi(z_1,z_2) \end{pmatrix},$$

where the angle of rotation

$$\phi(z_1,z_2) = -\int_{z_1}^{z_2} K(z',0)c(z')^{-1}\,dz'$$

and the exponent

$$a(z_1, z_2) = -\int_{z_1}^{z_2} G(z', 0)c(z')^{-1}\, dz'.$$

Note the property

$$\partial_{z_1}\mathbf{Q}^{\pm}(z_1, z_2) = -\mathbf{b}_{\pm\pm}(z_1)\mathbf{Q}^{\pm}(z_1, z_2), \qquad 0 < z_1,\ z_2 < d. \tag{5.3}$$

The origin of the jump discontinuities in the split vector fields $\boldsymbol{E}^{\pm}$ is the jumps in the incident electric field $\boldsymbol{E}^i$. Such a jump propagates periodically back and forth through the slab. At the walls, the reflected portions are completely determined by Eqs. (4.1) and (4.2). Similarly, the transmitted fractions are given by Eq. (4.3). The process stops after a finite number of reflections if either the front or the back wall is impedance matched. Otherwise, it goes on forever.

The special case when the input is the Heaviside step function $H(t)$ is particularly important. In a general bi-an-isotropic case, both polarizations of the incident electric field are needed for a complete investigation and it is appropriate to treat these two cases simultaneously. Therefore, define the 2×2 matrix-valued functions $\mathbf{U}^{\pm}(z, t)$ such that the first columns are the solutions $\boldsymbol{E}^{\pm}(z, t)$, respectively, to the dynamic equation (4.4), subject to the the input $\boldsymbol{E}^i(t) = H(t)\boldsymbol{e}_x$, while the second columns are the corresponding solutions when the excitation is $\boldsymbol{E}^i(t) = H(t)\boldsymbol{e}_y$. Due to the axial symmetry of the bi-isotropic medium, the component form of $\mathbf{U}^{\pm}(z, t)$ is

$$\mathbf{U}^{\pm}(z, t) = \begin{pmatrix} U_1^{\pm}(z, t) & -U_2^{\pm}(z, t) \\ U_2^{\pm}(z, t) & U_1^{\pm}(z, t) \end{pmatrix}. \tag{5.4}$$

The canonical solutions $\mathbf{U}^{\pm}(z, t)$ have jump-discontinuities across the characteristics

$$t(z) = \int_0^z c(z')^{-1}\, dz' + kP \quad \text{and} \quad t(z) = \int_z^d c(z')^{-1}\, dz' + kP + P/2,$$

respectively, where

$$P := 2\int_0^d c(z')^{-1}\, dz'$$

is one roundtrip and k runs over the natural numbers. Furthermore, these functions are continuous, but not differentiable, across the families of curves

$$t(z) = \int_z^d c(z')^{-1}\, dz' + kP + P/2 \quad \text{and} \quad t(z) = \int_0^z c(z')^{-1}\, dz' + kP,$$

respectively. Elsewhere, they are well-defined and differentiable, and the entries satisfy the causality condition (3.2). The jump in $\mathbf{U}^+$ across the discontinuity curves is obtained from Eqs. (5.1), (4.1), and (4.2). The result, which has an obvious physical meaning, is

$$[\mathbf{U}^+(z, \int_0^z c(z')^{-1}\, dz' + kP)] = \mathbf{Q}^+(0, z)\big(r_0\mathbf{Q}^-(d, 0)r_1\mathbf{Q}^+(0, d)\big)^k t_0.$$

Analogously, one obtains

$$[\mathbf{U}^-(z, \int_z^d c(z')^{-1}\,dz' + kP + P/2)] = \mathbf{Q}^-(d,z)r_1\mathbf{Q}^+(0,d)(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d))^k t_0.$$

In the next section, the canonical solutions are used to obtain the solution to the general propagation problem when the input $\boldsymbol{E}^i$ has the regularity stated in Section 2.

6 Duhamel's principle

Duhamel's principle [5] for the linear, isotropic, causal, and time invariant local hyperbolic problem has been employed successively in many articles on direct and inverse scattering when scattering operators, imbedding kernels, and Green functions are defined, see, e.g., Refs. [15–18, 22]. It has also been referred to in the fundamental, non-local isotropic cases [1, 13], and in several extensions to various complex media [7–9, 19–21, 23, 29, 33]. See also Ref. [30], where Duhamel's principle is used for the more basic canonical functions in the dispersive, isotropic case. By straightforward generalization of this result to the considered vector case, the split vector fields $\boldsymbol{E}^\pm(z,t)$ at an arbitrary point z inside the dispersive medium can be related to the general excitation $\boldsymbol{E}^i(t)$ at the front wall. The result for general 2×2-systems obtained in Ref. [28] reads

$$\boldsymbol{E}^\pm(z,t) = \partial_t \int_0^{t-\int_0^z c(z')^{-1}\,dz'} \mathbf{U}^\pm(z,t-t')\boldsymbol{E}^i(t')\,dt'.$$

The matrix-valued functions $\mathbf{U}^\pm(z,t)$ are the canonical functions introduced in the foregoing section. Note that strict causality has been referred to in this expression. Evaluation of the time differentiation yields

$$\begin{aligned}\boldsymbol{E}^\pm(z,t) = &\int_0^{t-\int_0^z c(z')^{-1}\,dz'} \partial_t\mathbf{U}^\pm(z,t-t')\boldsymbol{E}^i(t')\,dt' + \\ &+ \sum_{k=k^\pm}^{\infty}[\mathbf{U}^\pm(z,\pm\int_0^z c(z')^{-1}\,dz' + kP)]\boldsymbol{E}^i(t\mp\int_0^z c(z')^{-1}\,dz' - kP),\end{aligned}$$

where $k^+ = 0$ and $k^- = 1$. Of course, ∂_t denotes the classical time derivative.

7 Green functions and scattering operators

The Green functions for the propagation of electromagnetic waves in bi-an-isotropic slabs are discussed thoroughly in Ref. [28]. Essentially, they are the classical time derivatives of the canonical solutions defined in Section 5. Here, the definition

$$\mathbf{G}^\pm(z,t) := t_0^{-1}\mathbf{Q}^+(z,0)\partial_t\mathbf{U}^\pm(z,t+\int_0^z c(z')^{-1}\,dz'). \tag{7.1}$$

is adopted. Clearly, the Green functions inherit the symmetry (5.4). By time invariance arguments, the result in the previous sections can be formulated as

$$\begin{aligned}\boldsymbol{E}^+(z,t+\int_0^z c(z')^{-1}\,dz') &= \mathbf{Q}^+(0,z)t_0\left(\mathbf{G}^+(z,\cdot)*\boldsymbol{E}^i(\cdot)\right)(t)+\\ &+\mathbf{Q}^+(0,z)\sum_{k=0}^{\infty}\Big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\Big)^k t_0\boldsymbol{E}^i(t-kP)\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\boldsymbol{E}^-(z,t+\int_0^z c(z')^{-1}\,dz') &= \mathbf{Q}^+(0,z)t_0\left(\mathbf{G}^-(z,\cdot)*\boldsymbol{E}^i(\cdot)\right)(t)+\\ &+\mathbf{Q}^-(d,z)r_1\mathbf{Q}^+(0,d)\times\\ &\times\sum_{k=0}^{\infty}\Big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\Big)^k t_0\boldsymbol{E}^i\Big(t-2\int_z^d c(z')^{-1}\,dz'-kP\Big),\end{aligned} \tag{7.3}$$

where the property $\mathbf{Q}^+(0,z)^{-1}=\mathbf{Q}^+(z,0)$ has been employed. Note that time t is measured from the wave front. The Green functions are identically zero at negative times and in the general mismatch case they both have finite jump discontinuities along the curves

$$t(z)=2\int_z^d c(z')^{-1}\,dz'+kP \quad \text{and} \quad t(z)=kP,$$

where k runs over the natural numbers. These results follow immediately from the definition (7.1) and the regularity of the function $\mathbf{U}^+(z,t)$. A discussion of the partial mismatch cases is found in Ref. [28]. The boundary values

$$\begin{cases}\mathbf{G}^+(0,t)=r_0\mathbf{G}^-(0,t), & t\neq kP\\ \mathbf{G}^-(d,t)=r_1\mathbf{G}^+(d,t), & t\neq kP\end{cases} \tag{7.4}$$

follow easily from Eqs. (4.1) and (4.2) and the definition of the Green functions. It is now easy to write down the scattering operators. Eqs. (4.3) and (7.3) immediately yield

$$\begin{aligned}\boldsymbol{E}^r(t) &= -r_0\boldsymbol{E}^i(t)+t_0(1+r_0)\left(\mathbf{R}*\boldsymbol{E}^i\right)(t)+\\ &+t_0r_1(1+r_0)\sum_{k=1}^{\infty}(r_1r_0)^{k-1}\big(\mathbf{Q}^-(d,0)\mathbf{Q}^+(0,d)\big)^k\boldsymbol{E}^i(t-kP)\end{aligned} \tag{7.5}$$

for the reflected electric field. For the transmitted electric field, the result is

$$\begin{aligned}\boldsymbol{E}^t(t+P/2) &= (1+r_1)\mathbf{Q}^+(0,d)t_0\left(\mathbf{T}(\cdot)*\boldsymbol{E}^i(\cdot)\right)(t)+\\ &+(1+r_1)\mathbf{Q}^+(0,d)t_0\sum_{k=0}^{\infty}\big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\big)^k\boldsymbol{E}^i(t-kP).\end{aligned} \tag{7.6}$$

The precursor is evaluated from the first term on the right side of the latter equation for small t. The properties

$$\begin{cases}\mathbf{R}(t)=\mathbf{G}^-(0,t),\\ \mathbf{T}(t)=\mathbf{G}^+(d,t)\end{cases} \tag{7.7}$$

are called the physical reflection and transmission kernels at the time t, respectively.

The Green functions equations are now derived. Straightforward differentiation of Eqs. (7.2) and (7.3) using Eq. (5.2) and the knowledge of the location of the jump discontinuities of the Green functions yields

$$\begin{aligned}
&\partial_z \boldsymbol{E}^+(z, t + \int_0^z c(z')^{-1}\, dz') = \mathbf{b}_{++}(z)\mathbf{Q}^+(0,z)t_0 \left(\mathbf{G}^+(z,\cdot) * \boldsymbol{E}^i(\cdot)\right)(t) + \\
&+ \mathbf{Q}^+(0,z)t_0 \left((\partial_z \mathbf{G}^+(z,\cdot)) * \boldsymbol{E}^i(\cdot)\right)(t) + \\
&+ \mathbf{Q}^+(0,z)t_0 \frac{2}{c(z)} \sum_{k=0}^{\infty} [\mathbf{G}^+(z, 2\int_z^d c(z')^{-1}\, dz' + kP)] \boldsymbol{E}^i(t - 2\int_z^d c(z')^{-1}\, dz' - kP) + \\
&+ \mathbf{b}_{++}(z)\mathbf{Q}^+(0,z) \sum_{k=0}^{\infty} \Big(r_0 \mathbf{Q}^-(d,0) r_1 \mathbf{Q}^+(0,d)\Big)^k t_0 \boldsymbol{E}^i(t - kP)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\partial_z - \frac{2}{c(z)}\partial_t\right) \boldsymbol{E}^-(z, t + \int_0^z c(z')^{-1}\, dz') = \\
&\qquad = \mathbf{b}_{++}(z)\mathbf{Q}^+(0,z)t_0 \left(\mathbf{G}^-(z,\cdot) * \boldsymbol{E}^i(\cdot)\right)(t) + \\
&\qquad + \mathbf{Q}^+(0,z)t_0 \left(\left(\left(\partial_z - \frac{2}{c(z)}\partial_t\right) \mathbf{G}^-(z,\cdot)\right) * \boldsymbol{E}^i(\cdot)\right)(t) + \\
&\qquad + \mathbf{Q}^+(0,z)t_0 \left(-\frac{2}{c(z)}\right) \sum_{k=0}^{\infty} [\mathbf{G}^-(z,kP)] \boldsymbol{E}^i(t - kP) + \\
&\qquad + \mathbf{b}_{--}(z)\mathbf{Q}^-(d,z) r_1 \mathbf{Q}^+(0,d) \times \\
&\qquad\quad \times \sum_{k=0}^{\infty} \Big(r_0 \mathbf{Q}^-(d,0) r_1 \mathbf{Q}^+(0,d)\Big)^k t_0 \boldsymbol{E}^i(t - 2\int_z^d c(z')^{-1}\, dz' - kP).
\end{aligned}$$

Note that ∂_z denotes the total derivative in these two formulas. Notice also, that in the derivation of the first expression, the jumps $[\mathbf{G}^+(z,kP)]$ do not contribute, since the differentiation is along the discontinuity lines $t = kP$. Similarly, the jumps $[\mathbf{G}^-(z, 2\int_z^d c(z')^{-1}\, dz' + kP)]$ do not enter the second formula. Observe that the obtained expressions equal $(\partial_z \pm c(z)^{-1}\partial_t)\boldsymbol{E}^\pm(z,t)$ (partial derivatives), respectively, evaluated at the time $t + \int_0^z c(z')^{-1}\, dz'$. The Green functions equations are now obtained by substituting the dynamic equation (4.4) into these equations, followed by identification of terms. The result is

$$\begin{aligned}
&\begin{pmatrix} \partial_z \mathbf{G}^+(z,t) \\ \left(\partial_z - \frac{2}{c(z)}\partial_t\right) \mathbf{G}^-(z,t) \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{b}_{+-}(z) \\ \mathbf{b}_{-+}(z) & \mathbf{b}_{--}(z) - \mathbf{b}_{++}(z) \end{pmatrix} \begin{pmatrix} \mathbf{G}^+(z,t) \\ \mathbf{G}^-(z,t) \end{pmatrix} + \\
&+ \frac{1}{c(z)} \begin{pmatrix} \partial_t(-\mathbf{G} - \mathbf{K})(z,\cdot) & \partial_t(-\mathbf{F} + \mathbf{L})(z,\cdot) \\ \partial_t(\mathbf{F} + \mathbf{L})(z,\cdot) & \partial_t(\mathbf{G} - \mathbf{K})(z,\cdot) \end{pmatrix} * \begin{pmatrix} \mathbf{G}^+(z,\cdot) \\ \mathbf{G}^-(z,\cdot) \end{pmatrix}(t) + \\
&+ \frac{1}{c(z)} \sum_{k=0}^{\infty} \begin{pmatrix} \partial_t(-\mathbf{G} - \mathbf{K})(z, t - kP) & \partial_t(-\mathbf{F} + \mathbf{L})(z, t - 2\int_z^d c(z')^{-1}\, dz' - kP) \\ \partial_t(\mathbf{F} + \mathbf{L})(z, t - kP) & \partial_t(\mathbf{G} - \mathbf{K})(z, t - 2\int_z^d c(z')^{-1}\, dz' - kP) \end{pmatrix} \times \\
&\quad \times \begin{pmatrix} \big(r_0 \mathbf{Q}^-(d,0) r_1 \mathbf{Q}^+(0,d)\big)^k \\ r_1 \mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\big(r_0 \mathbf{Q}^-(d,0) r_1 \mathbf{Q}^+(0,d)\big)^k \end{pmatrix}.
\end{aligned}$$

The point (z,t) is of course assumed to be located off the discontinuity lines. In addition, one obtains the (cross-) jump relations

$$\begin{aligned}[\mathbf{G}^+(z, 2\int_z^d c(z')^{-1}\,dz' + kP)] = \\ = \mathbf{b}_{+-}(z)\frac{c(z)}{2} r_1 \mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\big)^k\end{aligned} \tag{7.8}$$

and

$$[\mathbf{G}^-(z,kP)] = -\mathbf{b}_{-+}(z)\frac{c(z)}{2}\big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\big)^k. \tag{7.9}$$

Alternatively, the Green functions equations can be obtained from the definition (7.1) by differentiating the canonical functions equation with respect to time as was done in Ref. [28]. The canonical functions equation is just the dynamic equation with the vector fields $\boldsymbol{E}^\pm$ replaced by the matrix functions $\mathbf{U}^\pm$, respectively. By straightforward analysis of the Green functions equations and the use of the (cross-) jump relations (7.8)–(7.9), the (co-) jump relations

$$[\mathbf{G}^+(z,kP)] = -\big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\big)^k\int_0^z \mathbf{a}^+(z')\,dz' + \lim_{z\to+0}[\mathbf{G}^+(z,kP)] \tag{7.10}$$

and

$$\begin{aligned}&[\mathbf{G}^-(z, 2\int_z^d c(z')^{-1}\,dz' + kP)] = \mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\times \\ &\left\{-r_1\left(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\right)^k\int_z^d \mathbf{a}^-(z')\,dz' + \lim_{z\to d-0}[\mathbf{G}^-(z, 2\int_z^d c(z')^{-1}\,dz' + kP)]\right\}\end{aligned} \tag{7.11}$$

are obtained, where

$$\mathbf{a}^\pm(z) = \frac{c(z)\mathbf{b}_{+-}(z)\mathbf{b}_{-+}(z)}{2} - \frac{\partial_t\mathbf{G}(z,0)\pm\partial_t\mathbf{K}(z,0)}{c(z)}.$$

At the boundaries, the jumps are related to one another by the equations

$$[\mathbf{G}^+(0,kP)] = r_0[\mathbf{G}^-(0,kP)], \qquad [\mathbf{G}^-(d,kP)] = r_1[\mathbf{G}^+(d,kP)], \tag{7.12}$$

where k runs over the natural numbers. This follows immediately from Eq. (7.4). Each jump at the boundary is the sum of two jumps (except at $(0,0)$). Below, special need arises for the Green functions equation for $\mathbf{G}^+$ during the first roundtrip, i.e., when $0<t<P$. It reads

$$\begin{aligned}\partial_z\mathbf{G}^+ = -\frac{\eta'}{2\eta}\mathbf{G}^- - \frac{1}{c}(\mathbf{I}+\mathbf{G}^+*)\partial_t(\mathbf{G}+\mathbf{K}) - \frac{1}{c}\partial_t\left\{(\mathbf{F}-\mathbf{L})*\mathbf{G}^-\right\} + \\ - \frac{1}{c}r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\partial_t\left(\mathbf{F}-\mathbf{L}\right)_{t\to t-2\int_z^d c(z')^{-1}\,dz'}.\end{aligned} \tag{7.13}$$

This equation is evaluated at (z,t) except for the second line which is evaluated at $(z, t-2\int_z^d c(z')^{-1}\,dz')$.

8 Reflection imbedding equations

In this section, the reflection kernel $\mathbf{R}(z,t)$ for a subsection $[z,d]$, $0 < z < d$, of the physical slab $[0,d]$ is considered. These kernels are the physical reflection kernels for the subslab problem $[z,d]$, with impedance matched front wall $z = z$, i.e., $\eta(\zeta) = \eta(z)$ for all $\zeta < z$. This problem can be viewed as an imbedding problem in which a one-parameter family $\mathbf{R}(z,t)$, $0 < z < d$, of reflection kernels is studied. The integral kernel $\mathbf{R}(z,t)$ is called the reflection imbedding kernel for the dispersive slab $[z,d]$.

The representation of the reflection imbedding kernel $\mathbf{R}(z,t)$ is obtained as the special impedance matched case ($r_0 = 0$, $t_0 = 1$) of Eq. (7.5).

$$\begin{aligned} \boldsymbol{E}^-(z,t) = \big(\mathbf{R}(z,\cdot) * \boldsymbol{E}^+(z,\cdot)\big)(t) + \\ + r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\boldsymbol{E}^+(z, t - 2\int_z^d c(z')^{-1}\,dz'). \end{aligned} \tag{8.1}$$

The reflection imbedding kernel has the same symmetry, (5.4), as the canonical functions and the Green functions. As an immediate consequence of Eq. (4.2) and the definition Eq. (7.5), the boundary value

$$\mathbf{R}(d-0,t) = \mathbf{0} \tag{8.2}$$

is obtained. Furthermore, repeated use of Eqs. (7.2), (7.3), and (8.1) yields a close relation between the imbedding approach and the Green functions formulation:

$$\begin{aligned} \mathbf{G}^-(z,t) = \mathbf{R}(z,t) + \big(\mathbf{R}(z,\cdot) * \mathbf{G}^+(z,\cdot)\big)(t) + \\ + \sum_{k=1}^{\infty}\big(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\big)^k\mathbf{R}(z,t-kP) + \\ + r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\mathbf{G}^+(z, t - 2\int_z^d c^{-1}(z')\,dz'). \end{aligned} \tag{8.3}$$

This is a relation between the Green functions for the general mismatch case and the reflection kernels of the imbedded impedance matched subslabs $[z,d]$. From the relation (8.3), it is clear that the imbedding kernels have only jump discontinuities across the three curves

$$t(z) = 0, \qquad t(z) = 2\int_z^d c(z')^{-1}\,dz', \quad \text{and} \quad t(z) = 4\int_z^d c(z')^{-1}\,dz'.$$

To prove this, Eqs. (7.8) and (7.9) are needed. The following jump relations are also obtained:

$$[\mathbf{R}(z,0)] = [\mathbf{G}^-(z,0)] = \frac{c(z)\eta'(z)}{4\eta(z)}\mathbf{I} - \frac{1}{2}(\mathbf{F}(z,0) + \mathbf{L}(z,0)),$$

$$[\mathbf{R}(z, 2\int_z^d c(z')^{-1}\,dz')] = [\mathbf{G}^-(z, 2\int_z^d c(z')^{-1}\,dz')] - r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)[\mathbf{G}^+(z,0)],$$

$$[\mathbf{R}(z, 4\int_z^d c(z')^{-1}\,dz')] = -r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)[\mathbf{G}^+(z, 2\int_z^d c(z')^{-1}\,dz')].$$

Note that from the boundary condition (7.12), the sum of these jumps in the limit z tends to d is equal to zero in agreement with (8.2). The specific values are given by the jump relations (7.8)–(7.11).

Naturally, there is a close relationship between the physical reflection kernel $\mathbf{R}(t)$ in the general mismatch case, i.e., the function $\mathbf{G}^-(0,t)$, and the reflection imbedding kernel $\mathbf{R}(+0,t)$ of the matched subsection problem. From the continuity of the electric and magnetic fields at the boundary and the definitions (7.5) and (8.1) of the particular reflection kernels, it follows that this relation is a delayed Volterra equation of the second kind.

$$\begin{aligned}\mathbf{0} = \mathbf{R}(t) - \mathbf{R}(+0,t) - r_0\{\mathbf{R}(\cdot) * \mathbf{R}(+0,\cdot)\}(t) + \\ - r_0 r_1 \big(\mathbf{R}(t-P) + \mathbf{R}(+0,t-P)\big)\mathbf{Q}^-(d,0)\mathbf{Q}^+(0,d) + \\ - \sum_{k=2}^{\infty} \big(r_0\mathbf{Q}^-(d,0) r_1 \mathbf{Q}^+(0,d)\big)^k \mathbf{R}(+0,t-kP).\end{aligned} \tag{8.4}$$

In particular, $\mathbf{R}(t) = \mathbf{R}(+0,t)$ when the front wall is impedance matched ($r_0 = 0$). Note that the result above also is obtained by setting $z = 0$ in Eq. (8.3) and using the boundary condition (7.4).

Straightforward differentiation of Eq. (8.1) yields in combination with Eqs. (5.2)–(5.3) and the knowledge of the location of the jump discontinuities of the imbedding kernel

$$\begin{aligned}\left(\partial_z - \frac{1}{c(z)}\partial_t\right)\boldsymbol{E}^-(z,t) = \left(\left(\left(\partial_z - \frac{2}{c(z)}\partial_t\right)\mathbf{R}(z,\cdot)\right) * \boldsymbol{E}^+(z,\cdot)\right)(t) + \\ + \left(\mathbf{R}(z,\cdot) * \left(\partial_z + \frac{1}{c(z)}\partial_t\right)\boldsymbol{E}^+(z,\cdot)\right)(t) - \frac{2}{c(z)}\mathbf{R}(z,0)\boldsymbol{E}^+(z,t) + \\ + \frac{2}{c(z)}[\mathbf{R}(z, 4\int_z^d c(z')^{-1}\,dz')]\boldsymbol{E}^+(z, t - 4\int_z^d c(z')^{-1}\,dz') + \\ + (\mathbf{b}_{--}(z) - \mathbf{b}_{++}(z)) r_1 \mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\boldsymbol{E}^+(z, t - 2\int_z^d c(z')^{-1}\,dz') + \\ + r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\left(\left(\partial_z + \frac{1}{c(z)}\partial_t\right)\boldsymbol{E}^+(z,t)\right)|_{t\to t-2\int_z^d c(z')^{-1}\,dz'}.\end{aligned}$$

The notation

$$((\partial_z + c(z)^{-1}\partial_t)\boldsymbol{E}^+(z,t))|_{t\to t-2\int_z^d c(z')^{-1}\,dz'}$$

means that $(\partial_z + c(z)^{-1}\partial_t)\boldsymbol{E}^+(z,t)$ (partial derivatives) is evaluated at the time $t - 2\int_z^d c(z')^{-1}\,dz'$. By repeated substitution of the dynamics (4.4) into this equation and another application of Eq. (8.1), the vector field $\boldsymbol{E}^-$ can be eliminated.

Identification of terms then yields the reflection imbedding equation:

$$\left(\partial_z - \frac{2}{c}\partial_t\right)\mathbf{R} = \frac{\eta'}{2\eta}\mathbf{R}*\mathbf{R} + \frac{1}{c}\partial_t\left\{\mathbf{F}+\mathbf{L}+2\mathbf{G}*\mathbf{R}+(\mathbf{F}-\mathbf{L})*\mathbf{R}*\mathbf{R}\right\}+$$
$$+ r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)2\big(\frac{\eta'}{2\eta}\mathbf{R}+\frac{1}{c}\partial_t\left\{\mathbf{G}+(\mathbf{F}-\mathbf{L})*\mathbf{R}\right\}\big)\big|_{t\to t-2\int_z^d c(z')^{-1}\,dz'}+$$
$$+\frac{1}{c(z)}\big(r_1\mathbf{Q}^-(d,z)\mathbf{Q}^+(z,d)\big)^2\big(\partial_t\left\{\mathbf{F}-\mathbf{L}\right\}\big)\big|_{t\to t-4\int_z^d c(z')^{-1}\,dz'}. \tag{8.5}$$

The notation means that the first line is evaluated at the point (z,t), the second at $(z, t-2\int_z^d c(z')^{-1}\,dz')$, and the last at $(z, t-4\int_z^d c(z')^{-1}\,dz')$, respectively.

9 Partial solution to the propagation problem

Recently, an analytic expression of the reflection kernel in an absolutely and uniformly convergent series has been obtained in the homogeneous bi-isotropic case [29]. In this section, it is assumed that the imbedding equation () has been solved and it is shown that the solution to the propagation problem, i.e., the Green functions $\mathbf{G}^\pm(z,t)$, can be expressed in the reflection imbedding kernels $\mathbf{R}(z,t)$ from the left, the corresponding reflection imbedding kernels from the right, and the material parameters of the dispersive medium. Obviously, it suffices to obtain the expression for $\mathbf{G}^+(z,t)$ since $\mathbf{G}^-(z,t)$ is given by Eq. (8.3) once $\mathbf{G}^+(z,t)$ is known.

Assume first that the back wall is impedance matched, i.e., $r_1 = 0$. In order to separate this problem from the general one, lower-case letters are used for the physical and imbedding reflection kernels, $\mathbf{r}(t)$ and $\mathbf{r}(z,t)$, respectively, and the Green functions $\mathbf{g}^\pm(z,t)$. Eq. (8.3) then reads

$$\mathbf{g}^-(z,t) = \mathbf{r}(z,t) + \big(\mathbf{r}(z,\cdot)*\mathbf{g}^+(z,\cdot)\big)(t) \tag{9.1}$$

and the boundary condition (8.4) at the front wall becomes

$$\mathbf{r}(t) - \mathbf{r}(+0,t) - r_0\{\mathbf{r}(\cdot)*\mathbf{r}(+0,\cdot)\}(t) = \mathbf{0}. \tag{9.2}$$

Recall also the boundary condition (Eqs. (7.4) and (7.7))

$$\mathbf{g}^+(+0,t) = r_0\mathbf{r}(t). \tag{9.3}$$

By Eq. (9.1), it is possible to eliminate $\mathbf{g}^-$ in Eq. (7.13) (which, in this case, holds in each bounded time interval except on a finite number of discontinuity lines) to obtain an equation in $\mathbf{g}^+$ and the material parameters only:

$$\partial_z\mathbf{g}^+ = -(\mathbf{I}+\mathbf{g}^+*)\mathbf{A}, \tag{9.4}$$

where

$$\mathbf{A}(z,t) = \frac{\eta'(z)}{2\eta(z)}\mathbf{r}(z,t)+ \tag{9.5}$$
$$+\frac{1}{c(z)}\partial_t\big(\mathbf{G}(z,t)+\mathbf{K}(z,t)+\{(\mathbf{F}-\mathbf{L})(z,\cdot))*\mathbf{r}(z,\cdot)\}(t)\big).$$

Since all the matrices commute, this equation has the solution

$$\mathbf{g}^+(z,t) = \mathbf{g}^+(+0,t) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Big(\big(\int_0^z \mathbf{A}(z',\cdot)\, dz' * \big)^{n-1} \int_0^z \mathbf{A}(z',\cdot)\, dz' \Big)(t) + \\ + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Big(\big(\int_0^z \mathbf{A}(z',\cdot)\, dz' * \big)^{n} \mathbf{g}^+(+0,\cdot) \Big)(t), \tag{9.6}$$

where the boundary value $\mathbf{g}^+(+0,t)$ is related to $\mathbf{r}(+0,t)$ by Eqs. (9.3) and (9.2). Note that this result also holds in the general mismatch case in the region bounded by the curves $t = 0$, $z = 0$, and $t = 2\int_z^d c(z')^{-1}\, dz'$. Note also that the series (9.6) converges uniformly and absolutely in this region by Weierstrass' comparison test since the terms consist of convolutions of causal functions.

Before the investigation of Sommerfeld's forerunner (the first precursor), it is necessary to examine the difference between the signal in a medium that is impedance matched at the back wall and the general mismatch case.

In Ref. [17], it is shown, that the discontinuities in the permittivity at the front and back walls can be removed with so called Redheffer products. This leads to solving Volterra equations of the second kind. In this paper, however, a different, but related, approach is adopted. It is obvious that the solution $\boldsymbol{E}^{\pm}(z,t)$ of Eq. (4.4), subject to the general boundary conditions Eq. (4.1) and Eq. (4.2), can be obtained as the superposition of the solutions to the following two problems:

- **Problem 1.** Find the solution $\boldsymbol{E}^{\pm}_{\text{match}}(z,t)$ to Eq. (4.4) subject to the boundary conditions
$$\begin{cases} t_0 \boldsymbol{E}^i(t) = \boldsymbol{E}^+(+0,t) - r_0 \boldsymbol{E}^-(+0,t), \\ \boldsymbol{E}^-(d-0,t) = \mathbf{0}. \end{cases}$$
This is the impedance matched case $r_1 = 0$ discussed above. The incident electric field $\boldsymbol{E}^i(t)$ is the same as for the considered full mismatch problem.

- **Problem 2.** Find the solution $\boldsymbol{E}^{\pm}_{\text{right}}(z,t)$ to Eq. (4.4) subject to the boundary conditions
$$\begin{cases} \boldsymbol{E}^+(+0,t) = r_0 \boldsymbol{E}^-(+0,t), \\ \boldsymbol{E}^-(d-0,t) = t_1 \boldsymbol{E}^i_{\text{right}}(t) + r_1 \boldsymbol{E}^+(d-0,t), \end{cases}$$
where the incident electric field $\boldsymbol{E}^i_{\text{right}}(t)$ from the right is
$$\boldsymbol{E}^i_{\text{right}}(t) = \frac{r_1}{t_1} \boldsymbol{E}^+_{\text{match}}(d-0,t). \tag{9.7}$$
The function $\boldsymbol{E}^+_{\text{match}}(d-0,t)$ is of course obtained from Problem 1. This is the full mismatch case. However, the slab is now excited from the right and not from the left.

Attention is now paid to the (total) transmitted electric field $\boldsymbol{E}^t(t)$. Naturally, similar results for the split vector fields $\boldsymbol{E}^{\pm}(z,t)$ inside the dispersive slab can be

obtained in the same vein as $\boldsymbol{E}^t(t)$. However, as far as the observation of the precursor is concerned, these results are not of interest, and they are not discussed further in this article. Consecutive use of Eq. (4.3), Eq. (4.2), the continuity of the total electric field at the back wall, Eq. (9.7), and the identity $r_1 + t_1 = 1$ yields

$$\begin{aligned}\boldsymbol{E}^t(t+P/2) &= (1+r_1)\boldsymbol{E}^+(d-0,t+P/2) = \\ &= \boldsymbol{E}^-_{\text{right}}(d-0,t+P/2) + \boldsymbol{E}^+_{\text{right}}(d-0,t+P/2) + \boldsymbol{E}^+_{\text{match}}(d-0,t+P/2) = \\ &= \boldsymbol{E}^r_{\text{right}}(t+P/2) + \boldsymbol{E}^i_{\text{right}}(t+P/2) + \boldsymbol{E}^+_{\text{match}}(d-0,t+P/2) = \\ &= \boldsymbol{E}^r_{\text{right}}(t+P/2) + \frac{1}{t_1}\boldsymbol{E}^+_{\text{match}}(d-0,t+P/2).\end{aligned} \tag{9.8}$$

The vector $\boldsymbol{E}^r_{\text{right}}(t+P/2)$ is, of course, the reflected electric field at the surface $z = d$ at time $t+P/2$ in Problem 2. In complete analogy with Eq. (7.5), a physical reflection kernel $\mathbf{R}_{\text{right}}(t)$ can be defined in this case. Consequently, the notation $\mathbf{R}_{\text{left}}(t)$ for the reflection kernel $\mathbf{R}(t)$ defined in Eq. (7.5) is adopted, i.e., $\mathbf{R}_{\text{left}}(t) \equiv \mathbf{R}(t)$. The expression (9.8) for the (total) transmitted electric field now becomes

$$\begin{aligned}\boldsymbol{E}^t(t+P/2) &= \frac{1}{t_1}\boldsymbol{E}^+_{\text{match}}(d-0,t+P/2) - r_1\boldsymbol{E}^i_{\text{right}}(t+P/2) + \\ &+ t_1(1+r_1)\left(\mathbf{R}_{\text{right}}(\cdot) * \boldsymbol{E}^i_{\text{right}}(\cdot+P/2)\right)(t) + \\ &+ r_0 t_1(1+r_1)\sum_{k=1}^{\infty}(r_1 r_0)^{k-1}\left(\mathbf{Q}^-(d,0)\mathbf{Q}^+(0,d)\right)^k \boldsymbol{E}^i_{\text{right}}(t+P/2-kP).\end{aligned}$$

Using Eq. (9.7), $\boldsymbol{E}^t(t+P/2)$ can be expressed in the vector field $\boldsymbol{E}^+_{\text{match}}(d-0,t+P/2)$ only:

$$\begin{aligned}\boldsymbol{E}^t(t+P/2) &= (1+r_1)\boldsymbol{E}^+_{\text{match}}(d-0,t+P/2) + \\ &+ r_1(1+r_1)\left(\mathbf{R}_{\text{right}}(\cdot) * \boldsymbol{E}^+_{\text{match}}(d-0,\cdot+P/2)\right)(t) + \\ &+ (1+r_1)\sum_{k=1}^{\infty}\left(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\right)^k \boldsymbol{E}^+_{\text{match}}(d-0,t+P/2-kP).\end{aligned} \tag{9.9}$$

The solution $\boldsymbol{E}^\pm_{\text{match}}(z,t)$ to Problem 1 is easily obtained by putting $r_1 = 0$ in Eqs. (7.2) and (7.3). The result for the propagating field is

$$\boldsymbol{E}^+_{\text{match}}(d,t+P/2) = \mathbf{Q}^+(0,d)t_0\boldsymbol{E}^i(t) + \mathbf{Q}^+(0,d)t_0\left(\mathbf{t} * \boldsymbol{E}^i\right)(t),$$

where $\mathbf{t}(t) \equiv \mathbf{g}^+(d-0,t)$ is the transmission kernel (in Problem 1). Insertion of this equation in Eq. (9.9) and comparison with Eq. (7.6) yields a relation between the transmission kernel $\mathbf{T}(t)$ for the full problem, the transmission kernel $\mathbf{t}(t)$ (in Problem 1), and the (physical) reflection kernel $\mathbf{R}_{\text{right}}(t)$ (in Problem 2). The result is

$$\begin{aligned}\mathbf{T}(t) &= \mathbf{t}(t) + r_1\mathbf{R}_{\text{right}}(t) + r_1\big(\mathbf{R}_{\text{right}} * \mathbf{t}\big)(t) + \\ &+ \sum_{k=1}^{\infty}\left(r_0\mathbf{Q}^-(d,0)r_1\mathbf{Q}^+(0,d)\right)^k \mathbf{t}(t-kP).\end{aligned} \tag{9.10}$$

Note that there is only a finite number of terms on the right side of this equation for each fixed (finite) time interval. By Eqs. (9.6) and (9.3), it follows that

$$\begin{aligned}\mathbf{t}(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Big(\big(\int_0^d \mathbf{A}(z',\cdot)\, dz' * \big)^{n-1} \int_0^d \mathbf{A}(z',\cdot)\, dz' \Big)(t) + \\ + r_0 \mathbf{r}(t) + r_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Big(\big(\int_0^d \mathbf{A}(z',\cdot)\, dz' * \big)^{n} \mathbf{r}(\cdot) \Big)(t),\end{aligned} \tag{9.11}$$

Recall that $\mathbf{A}(z,t)$ defined by Eq. (9.5) depends on $\mathbf{r}(z,t)$. During the first roundtrip, Eq. (9.10) reads

$$\mathbf{T}(t) = \mathbf{t}(t) + r_1 \mathbf{R}_{\text{right}}(t) + r_1 \big(\mathbf{R}_{\text{right}} * \mathbf{t} \big)(t), \qquad 0 < t < P,$$

or

$$\mathbf{T}(t) = \mathbf{f}(t) + \mathbf{g}(t) + (\mathbf{f} * \mathbf{g})(t), \qquad 0 < t < P, \tag{9.12}$$

where

$$\mathbf{f}(t) = r_0 \mathbf{R}_{\text{left}}(t) + r_1 \mathbf{R}_{\text{right}}(t) + r_0 r_1 (\mathbf{R}_{\text{right}} * \mathbf{R}_{\text{left}})(t) \tag{9.13}$$

and

$$\mathbf{g}(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Big(\big(\int_0^d \mathbf{A}(z',\cdot)\, dz' * \big)^{n-1} \big(\int_0^d \mathbf{A}(z',\cdot)\, dz' \big) \Big)(t).$$

Note that the fact, that the reflection kernel from the left, $\mathbf{R}_{\text{left}}(t)$, is independent of the conditions at the rear wall during the first roundtrip, i.e., $\mathbf{R}_{\text{left}}(t) = \mathbf{r}(t)$ when $0 < t < P$, has been used to obtain this expression. Finally, note that it is possible to obtain approximations to the (physical) reflection kernels for small t from the reflection imbedding kernels at the walls, see Eq. (9.2):

$$\begin{cases} \mathbf{R}_{\text{left}}(t) = \mathbf{R}_{\text{left}}(+0,+0) e^{r_0 \mathbf{R}_{\text{left}}(+0,+0) t}, \\ \mathbf{R}_{\text{right}}(t) = \mathbf{R}_{\text{right}}(d-0,+0) e^{r_1 \mathbf{R}_{\text{right}}(d-0,+0) t}, \end{cases} \tag{9.14}$$

where

$$\begin{cases} \mathbf{R}_{\text{left}}(z,0) = -\dfrac{c(z)\mathbf{b}_{-+}(z)}{2} = \dfrac{c(z)\eta'(z)}{4\eta(z)} \mathbf{I} - \dfrac{(\mathbf{F}(z,0) + \mathbf{L}(z,0))}{2}, \\ \mathbf{R}_{\text{right}}(z,0) = \dfrac{c(z)\mathbf{b}_{+-}(z)}{2} = -\dfrac{c(z)\eta'(z)}{4\eta(z)} \mathbf{I} - \dfrac{\mathbf{F}(z,0) - \mathbf{L}(z,0)}{2}. \end{cases} \tag{9.15}$$

Obviously, for small t, the function $\mathbf{f}(t)$ is a linear combination of exponentials. It is identically zero if $r_0 = r_1 = 0$. In the next section, the precursor inside the dispersive medium is investigated with the aid of the above results.

10 Sommerfeld's precursor

Sommerfeld's precursor kernel $\mathbf{P}_S(z,t)$ inside the dispersive medium is defined as the solution to Eq. (9.4) subject to the boundary condition given by (9.3) and (9.2)

with linearized input, i.e., the solution to

$$\begin{cases} \partial_z \mathbf{P}_S(z,t) = -\big(\mathbf{I} + \int_0^t \mathbf{P}_S(z,t')\,dt'\big)\mathbf{A}(z,0), \quad z>0,\ t>0 \\ \mathbf{P}_S(+0,t) = r_0 \mathbf{R}_{\text{left}}(+0,+0) e^{r_0 \mathbf{R}_{\text{left}}(+0,+0)t}, \quad t>0. \end{cases} \tag{10.1}$$

Of course, the precursor kernel is valid as a solution to the propagation problem only for small (positive) times t such that $t < 2\int_z^d c(z')^{-1}\,dz'$. It is obtained in the same way as Eq. (9.6):

$$\mathbf{P}_S(z,t) = \mathbf{P}_S(+0,t) + \mathbf{h}(z,t) + \big(\mathbf{P}_S(+0,\cdot) * \mathbf{h}(z,\cdot)\big)(t),$$

where

$$\mathbf{h}(z,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{t^{n-1}}{(n-1)!} \Big(\int_0^z \mathbf{A}(z',0)\,dz'\Big)^n H(t),$$

and

$$\mathbf{A}(z,0) = \frac{\big(\frac{c(z)\eta'(z)}{2\eta(z)}\mathbf{I} + \mathbf{F}(z,0) - \mathbf{L}(z,0)\big)\mathbf{R}_{\text{left}}(z,+0) + \partial_t \mathbf{G}(z,0) + \partial_t \mathbf{K}(z,0)}{c(z)} =$$
$$= \frac{\big(\frac{c(z)\eta'(z)}{2\eta(z)}\big)^2 \mathbf{I} - \frac{c(z)\eta'(z)}{\eta(z)}\mathbf{L}(z,0) - \mathbf{F}(z,0)^2 + \mathbf{L}(z,0)^2 + 2\partial_t \mathbf{G}(z,0) + 2\partial_t \mathbf{K}(z,0)}{2c(z)}.$$

By the familiar Bessel function expansion

$$\frac{J_1(2\sqrt{z})}{\sqrt{z}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{z^m}{(m+1)!}, \tag{10.2}$$

the matrix-valued function $\mathbf{h}(z,t)$ is expressed as

$$\mathbf{h}(z,t) = -\sqrt{\frac{\int_0^z \mathbf{A}(z',0)\,dz'}{t}}\, J_1\Big(2\sqrt{t\int_0^z \mathbf{A}(z',0)\,dz'}\Big)\, H(t). \tag{10.3}$$

(Recall that $H(t)$ is the Heaviside step.) Note that the series expansion of $\mathbf{h}(z,t)$ in Eq. (10.3) is well-defined by the fact that the radius of convergence of the series (10.2) is infinite. The choice of branch-cut for the square root function is irrelevant since both sides of Eq. (10.2) are even functions of $\sqrt{z}$. As mentioned before, the observed electric field at the back wall differs from the precursor inside the medium. The necessary modification to this case is provided by Eq. (9.12):

$$\mathbf{T}_{\text{precursor}}(t) = \mathbf{f}_1(t) + \mathbf{h}(d-0,t) + \big(\mathbf{f}_1(\cdot) * \mathbf{h}(d-0,\cdot)\big)(t),$$

where $\mathbf{f}_1$ is the approximation of the function $\mathbf{f}$ for small (positive) t given by Eqs. (9.13), (9.14), and (9.15).

Sommerfeld's precursor inside the medium at the position z is defined as (c.f. Eq. (7.2))

$$\boldsymbol{E}^+(z,t) = t_0 \mathbf{Q}^+(0,z)\{\boldsymbol{E}^i(t) + \int_0^t \mathbf{P}_S(z,t-t')\,\boldsymbol{E}^i(t')\,dt'\}.$$

The expression is valid for small wave front times t such that $t < 2\int_z^d c(z')^{-1}\,dz'$. At the boundary $z = d$, the expression of Sommerfeld's precursor is (c.f. Eq. (7.6))

$$(1 + r_1)\mathbf{Q}^+(0, d)t_0\{\boldsymbol{E}^i(t) + \big(\mathbf{T}_{\text{precursor}}(\cdot) * \boldsymbol{E}^i(\cdot)\big)(t)\}.$$

Inside the impedance matched, semi-infinite medium, the incident signal is canceled by Sommerfeld's precursor. Specifically, $\mathbf{h}(z, t) = \mathbf{P}_S(z, t) \to -\delta(t)\mathbf{I}$ as $z \to +\infty$ in the space $\mathcal{S}'$ of temperated distributions. This is seen in a straightforward manner from the relation $J_0'(t) = -J_1(t)$ for the Bessel functions $J_0(t)$ and $J_1(t)$ of the first kind, an integration by parts, and, finally, from a substitution of variables such that the Hankel transform of order zero is recognized. (Recall that the Hankel transform of order zero is closely related to the Fourier transformation in two variables. Since the Fourier transformation is an isomorphism on the Schwartz class $\mathcal{S}$ of rapidly decreasing functions, the proposition follows from the Riemann-Lebesgue lemma.)

Sommerfeld's precursor kernel, $\mathbf{P}_S(z, t)$, is now derived in the isotropic, stratified, and impedance matched case, with $\chi_{em} = \chi_{em} = \chi_{em} = 0$, $\epsilon(z)\chi_{ee}(z, t) = \epsilon_0\chi(z, t)$, and $\mu(z) = \mu_0$ throughout space. Specifically, the constitutive relations are

$$\boldsymbol{D}(z, t) = \epsilon_0\big\{\epsilon_r(z)\boldsymbol{E}(z, t) + \big(\chi(z, \cdot) * \boldsymbol{E}(z, \cdot)\big)(t)\big\}, \qquad \boldsymbol{B}(z, t) = \mu_0\boldsymbol{H}(z, t),$$

where the relative permittivity $\epsilon_r(z)$ is continuous throughout space and and continuously differentiable inside the dispersive medium. As usual, the permittivity and permeability of vacuum are denoted by ϵ_0 and μ_0, respectively. Sommerfeld's precursor kernel is given by Eq. (10.3), where

$$\mathbf{A}(z, 0) = \frac{c'(z)^2 + 4c_r(z)^2\partial_t\chi(z, 0) - c_r(z)^4\chi(z, 0)^2}{8c(z)}\mathbf{I}$$

and $c_r(z) = c(z)/c_0 = 1/\sqrt{\epsilon_r(z)}$ is the relative phase velocity of the dispersive medium at position z. This is a generalization of the result for the periodic medium given in Ref. [12]. Note, in particular, that also inhomogeneities in a non-dispersive medium cause oscillations in the propagating and transmitted fields. If the medium is homogeneous, then

$$\mathbf{A} := \mathbf{A}(z, 0) = \frac{4c_r^2\chi'(0) - c_r^4\chi(0)^2}{8c(0)}\mathbf{I}.$$

Naturally, the homogeneous medium is also a special case of the periodic medium treated by Karlsson and Stewart [12]. The result for the single-resonance Lorentz' kernel (1.1) is $\mathbf{A} = c_r^2\omega_p^2/(2c)$. By Eq. (1.3), Sommerfeld's result (1.2) is easily obtained ($c_r = 1$). In the general homogeneous, impedance matched case, the precursor kernel is

$$\mathbf{P}_S(z, t) = \mathbf{h}(z, t) = -\sqrt{\frac{z(4c_r^2\chi'(0) - c_r^4\chi(0)^2)}{8tc(0)}}\,J_1\Big(\sqrt{\frac{zt(4c_r^2\chi'(0) - c_r^4\chi(0)^2)}{2c(0)}}\Big)\,H(t).$$

A similar situation occurs in the propagation of transient electromagnetic waves in waveguides, see Ref. [14].

The present investigation is completed by deriving the precursor kernel in the homogeneous, reciprocal, and impedance matched bi-isotropic half-space discussed in Refs. [6, 34]. The specific medium has a single resonance and the collision frequency is negligible. The counterpart in the time domain to the time-harmonic constitutive relations of Post-type used in these references is

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon\Big\{\boldsymbol{E}(\boldsymbol{r},t) + (\mathcal{G} * \boldsymbol{E})(\boldsymbol{r},\sqcup) + \rfloor(\mathcal{K} * \boldsymbol{B})(\boldsymbol{r},\sqcup)\Big\}, \\ \boldsymbol{H}(\boldsymbol{r},t) = \epsilon\Big\{c(\mathcal{K} * \boldsymbol{E})(\boldsymbol{r},\sqcup) + \rfloor^{\in}\big[\boldsymbol{B}(\boldsymbol{r},\sqcup) + (\mathcal{F} * \boldsymbol{B})(\boldsymbol{r},\sqcup)\big]\Big\}, \end{cases}$$

where

$$\begin{cases} \mathcal{G} = \dfrac{\overset{\omega^{\in}}{\overset{\surd}{}}}{\omega_{\rfloor}} \sin(\omega_{\rfloor}\sqcup)\,\mathcal{H}(\sqcup), \\ \mathcal{F} = -\dfrac{\omega^{\in}_{\Updownarrow}}{\sqrt{\omega^{\in}_{\rfloor} + \omega^{\in}_{\Updownarrow}}} \sin\big(\sqrt{\omega^{\in}_{\rfloor} + \omega^{\in}_{\Updownarrow}}\,\sqcup\big)\,\mathcal{H}(\sqcup), \\ \mathcal{K} = -\alpha_{\rfloor} \cos(\omega_{\rfloor}\sqcup)\,\mathcal{H}(\sqcup). \end{cases}$$

The relation between this set of constitutive relations and the set of constitutive relations (2.2)–(2.6)—recall that $L \equiv 0$ in this reciprocal case—is given in terms of the resolvent of the kernel $\mathcal{F}(\boldsymbol{r},\sqcup)$, see Kristensson and Rikte [21]:

$$G - F + \mathcal{F} + (\mathcal{G} - \mathcal{F}) * \mathcal{F} = \prime$$

This is a Volterra equation of the second kind and therefore uniquely solvable. The relation between the kernels $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$ and G, F, K is easily found.

$$\begin{cases} K = \mathcal{K} + (\mathcal{G} - \mathcal{F}) * \mathcal{K}, \\ G + F = \mathcal{G} - \mathcal{K} * \mathcal{K}. \end{cases}$$

These equations are used to transform the kernels $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$ into the kernels G, F, K. Conversely, they are also used to transform the kernels G, F, K into the kernels $\mathcal{G}$, $\mathcal{F}$, $\mathcal{K}$, by solving suitable Volterra equations of the second kind.

The initial values of the susceptibility kernels $G(t)$, $F(t)$, $K(t)$ and the (time) derivatives $G'(t)$ and $K'(t)$ are

$$\begin{cases} G(0) = 0, \\ F(0) = 0, \\ K(0) = -\alpha_c, \\ G'(0) = (\mathcal{G}'(\prime) - \mathcal{F}'(\prime) - \mathcal{K}'(\prime)^{\in})/\in = (\omega^{\in}_{\surd} + \omega^{\in}_{\Updownarrow} - \alpha^{\in}_{\rfloor})/\in, \\ K'(0) = 0, \end{cases}$$

respectively. These values are used in the evaluation of the two relevant matrices $\mathbf{A}(z,0)$ and $\mathbf{Q}^+(0,z)$. The result is

$$\begin{cases} \mathbf{A}(z,0) = \dfrac{G'(0)}{c}\mathbf{I} = \dfrac{\omega_p^2 + \omega_m^2 - \alpha_c^2}{2c}\mathbf{I}, \\ \mathbf{Q}^+(0,z) = \begin{pmatrix} \cos\phi(0,z) & -\sin\phi(0,z) \\ \sin\phi(0,z) & \cos\phi(0,z) \end{pmatrix}, \qquad \phi(0,z) = -\dfrac{K(0)z}{c} = \dfrac{\alpha_c z}{c}. \end{cases}$$

If these matrices are inserted in Eq. (1.3), the leading edge result in Refs. [6, 34] is obtained. Clearly, the precursor has rotated the angle $\alpha_c z/c$ during the travel through the medium from $z = 0$ to $z = z$ in agreement with the heuristic picture of optical rotatory power. This simple interpretation is mainly due to the fact that $K'(0) = 0$. The more realistic case with, e.g., a damping term in the expression for $\mathcal{K}(\sqcup)$ is much more complicated. However, with the method presented in this paper, this problem can be solved numerically.

11 Conclusion

In this paper, a generalization of Sommerfeld's results for the first precursor in homogeneous Lorentz media is presented by applying time domain methods. Explicit results of the first precursor can be obtained

- in stratified isotropic and bi-isotropic media with smooth but otherwise arbitrary dispersion models,
- for finite slabs with impedance mismatch at the boundaries,
- with arbitrary (sufficiently smooth) excitation.

In addition, it is shown that also inhomogeneities cause oscillations in the early time response. These oscillations are quite similar to those caused by, e.g., a Lorentz medium. Furthermore, it is believed that the results presented in this article can be extended to other complex media, e.g., an-isotropic media, although the analysis becomes more complicated in these cases.

12 Acknowledgment

The work reported in this article is supported by a grant from the Swedish Research Council for Engineering Sciences, and its support is gratefully acknowledged.

References

[1] R.S. Beezley and R.J. Krueger. An electromagnetic inverse problem for dispersive media. *J. Math. Phys.*, **26**(2), 317–325, 1985.

[2] L. Brillouin. Über die Fortpflanzung des Lichtes in dispergierenden Medien. *Ann. Phys.*, **44**, 203–240, 1914.

[3] L. Brillouin. *Wave propagation and group velocity.* Academic Press, New York, 1960.

[4] J.P. Corones, G. Kristensson, P. Nelson, and D.L. Seth, editors. *Invariant Imbedding and Inverse Problems.* SIAM, 1992.

[5] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume 2. Interscience, New York, 1962.

[6] N. Engheta and P.G. Zablocky. Effect of chirality on the transient signal wave front. *Opt. Lett.*, **16**, 1924–1926, 1991.

[7] J Fridén. Inverse scattering for anisotropic mirror image symmetric media. *Inverse Problems*, **10**(5), 1133–1144, 1994.

[8] J Fridén. Inverse scattering for the homogenous dispersive anisotropic slab using transient electromagnetic fields. Technical Report LUTEDX/(TEAT-7031)/1–23/(1994), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1994.

[9] J. Fridén, G. Kristensson, and R.D. Stewart. Transient electromagnetic wave propagation in anisotropic dispersive media. *J. Opt. Soc. Am. A*, **10**(12), 2618–2627, 1993.

[10] J.D. Jackson. *Classical Electrodynamics.* John Wiley & sons, New York, second edition, 1975.

[11] A. Karlsson and G. Kristensson. Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain. *J. Electro. Waves Applic.*, **6**(5/6), 537–551, 1992.

[12] A. Karlsson and R. Stewart. Wave propagators for transient waves in periodic media. Technical Report LUTEDX/(TEAT-7035)/1–24/(1994), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1994.

[13] G. Kristensson. Direct and inverse scattering problems in dispersive media—Green's functions and invariant imbedding techniques. In Kleinman R., Kress R., and Martensen E., editors, *Direct and Inverse Boundary Value Problems*, Methoden und Verfahren der Mathematischen Physik, Band 37, pages 105–119, Mathematisches Forschungsinstitut Oberwolfach, FRG, 1991.

[14] G. Kristensson. Transient electromagnetic wave propagation in wave guides. Technical Report LUTEDX/(TEAT-7026)/1–22/(1993), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1993.

[15] G. Kristensson and R.J. Krueger. Direct and inverse scattering in the time domain for a dissipative wave equation. Part 1: Scattering operators. *J. Math. Phys.*, **27**(6), 1667–1682, 1986.

[16] G. Kristensson and R.J. Krueger. Direct and inverse scattering in the time domain for a dissipative wave equation. Part 2: Simultaneous reconstruction of dissipation and phase velocity profiles. *J. Math. Phys.*, **27**(6), 1683–1693, 1986.

[17] G. Kristensson and R.J. Krueger. Direct and inverse scattering in the time domain for a dissipative wave equation. Part 3: Scattering operators in the presence of a phase velocity mismatch. *J. Math. Phys.*, **28**(2), 360–370, 1987.

[18] G. Kristensson and R.J. Krueger. Direct and inverse scattering in the time domain for a dissipative wave equation. Part 4: Use of phase velocity mismatches to simplify inversions. *Inverse Problems*, **5**(3), 375–388, 1989.

[19] G. Kristensson and S. Rikte. Scattering of transient electromagnetic waves in reciprocal bi-isotropic media. *J. Electro. Waves Applic.*, **6**(11), 1517–1535, 1992.

[20] G. Kristensson and S. Rikte. The inverse scattering problem for a homogeneous bi-isotropic slab using transient data. In L. Päivärinta and E. Somersalo, editors, *Inverse Problems in Mathematical Physics*, pages 112–125. Springer, Berlin, 1993.

[21] G. Kristensson and S. Rikte. Transient wave propagation in reciprocal bi-isotropic media at oblique incidence. *J. Math. Phys.*, **34**(4), 1339–1359, 1993.

[22] R.J. Krueger and R.L. Ochs, Jr. A Green's function approach to the determination of internal fields. *Wave Motion*, **11**, 525–543, 1989.

[23] H. Otterheim. *Time domain direct and inverse scattering for gyrotropic media.* PhD thesis, Royal institute of technology, Stockholm, Sweden, 1993.

[24] K.E. Oughstun and S. Shen. Velocity of energy transport for a time-harmonic field in a multiple-resonance Lorentz medium. *J. Opt. Soc. Am. B*, **5**(11), 2395–2398, 1988.

[25] K.E. Oughstun and G.C. Sherman. Propagation of electromagnetic pulses in a linear dispersive medium with absorption (the Lorentz medium). *J. Opt. Soc. Am. B*, **5**(4), 817–849, 1988.

[26] K.E. Oughstun and G.C. Sherman. Uniform asymptotic description of electromagnetic pulse propagation in a linear dispersive medium with absorption (the Lorentz medium). *J. Opt. Soc. Am. A*, **6**(9), 1394–1420, 1989.

[27] A.P. Prudnikov, Y.A. Brychkov, and O.I. Marichev. *Integrals and Series*, volume 2: Special functions. Gordon and Breach Science Publishers, New York, 1986.

[28] S. Rikte. Causality theorems and Green functions for transient wave propagation problems in stratified complex media. Technical Report LUTEDX/(TEAT-7029)/1–17/(1994), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1994.

[29] S. Rikte. Reconstruction of bi-isotropic material parameters using transient electromagnetic fields. Technical Report LUTEDX/(TEAT-7033)/1–22/(1994), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-211 00 Lund, Sweden, 1994.

[30] T.M. Roberts. Causality theorems. In J.P. Corones, G. Kristensson, P. Nelson, and D.L. Seth, editors, *Invariant Imbedding and Inverse Problems*. SIAM, 1992.

[31] S. Shen and K.E. Oughstun. Dispersive pulse propagation in a double-resonance Lorentz medium. *J. Opt. Soc. Am. B*, **6**(5), 948–963, 1989.

[32] A. Sommerfeld. Über die Fortpflanzung des Lichtes in dispergierenden Medien. *Ann. Phys.*, **44**, 177–202, 1914.

[33] R.D. Stewart. *Transient electromagnetic scattering on anisotropic media*. PhD thesis, Iowa State University, Ames, Iowa, 1989.

[34] P.G. Zablocky and N. Engheta. Transients in chiral media with single-resonance dispersion. *J. Opt. Soc. Am. A*, **10**(4), 740–758, 1993.