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Abstract

A method based on power series expansions of the surface charge density, with Legendre polynomials as basis functions, is introduced in this paper. With a Galerkin method, applied to the method of moment, the resulting integrals for the elements of the impedance matrix are four dimensional. The corresponding integrands are products of the static Green function and Legendre polynomials. The introduction of the Legendre polynomials leads to a reduction of the number of non-zero elements in the impedance matrix with a fast computational method as a consequence. The method is compared to standard MoM in which piecewise linear basis functions are used.

1 Introduction

The most time consuming part in the method of moment (MoM) is the time required to fill the impedance matrix. This is caused mainly by two factors: every element in the structure interacts with all of the other elements and the integral expressions have, in most cases, to be solved by some quadrature rule which can be very time comprehensive. Several methods have been proposed for reducing the time consuming calculations of the integrals and to handle the troublesome self couplings, *i.e.*, when the source and observation point coincide. Tarricone *et al.* [8] introduces a method that reduces the four-dimensional integrals to a quasi-one-dimensional numerical integration. A method based on a procedure suggested in [7] together with the Duffy transform [2] solves the self couplings for different geometries. Analytic methods for the solution of four-dimensional singular potential integrals for triangular elements have been developed in [3].

Reducing the complexity of the impedance matrix is crucial when large problems are considered. Methods like the multilevel fast multipole method [1] use advanced algorithms to reduce the number of operations from $\mathcal{O}N_{\text{it}}N^2$ to $\mathcal{O}N_{\text{it}}N\log N$. Another way to reduce the problem is to use hierarchal basis functions [9]. Jørgensen *et al.* [5] show that hierarchical Legendre basis provides a better condition number of the impedance matrix than existing interpolatory bases which imply that a fast iterative solution still can be achieved although high order basis functions are used. Higher order basis functions have the advantage of reducing the number of unknowns since larger elements can be used.

A limitation to these methods are that they first and foremost are designed for two dimensional problems solving EFIE (or MFIE) applying surface currents on PEC surfaces. For three dimensional problems further improvements have to be done.

In the quasi static regime surface currents can, for most of the cases, not be assumed which implies that the current density inside the conductive regions has to be included in the model. For thick metallic leads it is possible to assume an exponential decay of the current density, leading to a two dimensional approach, but for leads with a thickness of the order of the skin depth simple approximations are not possible. For microstrip circuits *i.e.*, transmission lines, and integrated

inductors in modern electronic systems this is the case. Since there are demands on fast and accurate EM solvers, from the circuit industry, standard methods can not be applied.

A new quasi static algorithm is under development. From the algorithm, that is based on the MoM, it will be possible to extract numerical values of the resistance, capacitance, inductance and the Q-value of a passive component, *e.g.*, an inductor. The method is based on expansion of the current density, inside the conductive regions, and the surface charge density in terms of Legendre polynomials.

To evaluate the method an investigation is presented in this paper in which the capacitive couplings between two quadratic microstrips, with a known surface charge density, are considered. The results are compared to the results achieved when standard basis functions, *i.e.*, linear splines, are used.

2 Impedance matrix terms

At quasi static conditions in homogeneous space the scalar potential, Φ, can be approximated by its static counterpart [4], that is

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon}\int_S \rho_{\mathrm{S}}(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'$$

where S is the surface on which ρ_{S} exists and $\epsilon = \epsilon_0\epsilon_{\mathrm{r}}$. To be able to solve this equation for ρ_{S}, the surface charge density is expanded into a set of basis functions, (φ_{mnk}). We find

$$\rho_{\mathrm{S}}(\boldsymbol{r}) = \sum_{mnk=0}^{\infty} c_{mnk}\varphi_{mnk}(\boldsymbol{r})$$

where $c_{mnk} \in \mathbb{C}$ are the expansion coefficients. Applying a projection procedure by using a set of test functions, (ψ_{mnk}), yields the weighted residual form

$$\int_S \psi_{mnk}(\boldsymbol{r})\Phi(\boldsymbol{r})dS = \sum_{m'n'k'} Z_{mnk,m'n'k'}c_{m'n'k'}$$

where

$$Z_{mnk,m'n'k'} = \frac{1}{4\pi\epsilon}\int_S\int_S \psi_{mnk}(\boldsymbol{r})\varphi_{m'n'k'}(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'dS$$

are the impedance (moment) matrix terms. When Galerkin's method [6] is applied the elements yield

$$Z_{mnk,m'n'k'} = \frac{1}{4\pi\epsilon}\int_S\int_S \varphi_{mnk}(\boldsymbol{r})\varphi_{m'n'k'}(\boldsymbol{r}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'dS.$$

A specific case with two parallel quadratic elements of two microstrips is now considered for the comparison of two methods. Both elements have a width and length of $w = l = 30\,\mu$m and they are placed in parallel with a distance X_0 between the origins of the local coordinate systems, K_1 and K_2, which coincide with the

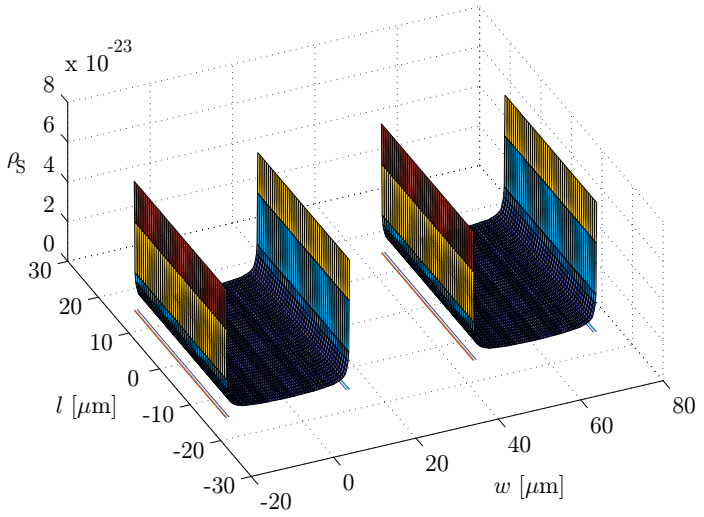


Figure 1: Part of the surface charge density, ρ_{S}, on two parallel microstrips. The two plates are separated a distance X_0 between the two centers. The charge distribution has been achieved from a FEM simulation of a microstrip at 20 GHz.

midpoints of the two elements. The surface charge densities on the two elements are assumed to vary only in the x-direction and are thus constant in the longitudinal direction (y-direction), as is illustrated in Figure 1. This corresponds to a piecewise constant distribution since the element length, l, is assumed to be small in comparison to the variations of the surface charge density along the longitudinal direction of the strip.

Two different methods are now introduced; orthogonal projection onto a finite dimensional subspace with Legendre polynomials, P_k, as basis functions and, for comparison, projection onto a finite dimensional subspace with linear splines, L_m, as basis functions. The idea behind the first method is to use large elements and orthogonal projection; the accuracy is thereby affected by the number of terms in the series expansion of the surface charge density. In the second method small elements and piecewise linear basis functions are used, which conveys that the accuracy is affected by the number of quadrature points.

Let $\boldsymbol{r} = \boldsymbol{r}_1$ be a source point on the first element and $\boldsymbol{r}' = X_0\hat{\boldsymbol{x}} + \boldsymbol{r}_2$ a source point on the second element. The impedance matrix terms of the two methods becomes

$$Z^{\mathrm{P}}_{k,k'} = \frac{1}{4\pi\epsilon}\int_{-a}^{a} dx_1 \int_{-a}^{a} dy_1 \int_{-a}^{a} dx_2 \int_{-a}^{a} dy_2 \frac{P_k(x_1/a)P_{k'}(x_2/a)}{\sqrt{(x_1 - x_2 - X_0)^2 + (y_1 - y_2)^2}}, \quad (2.1)$$

$$Z^{\mathrm{L}}_{m,m'} = \frac{1}{4\pi\epsilon}\int_{-a}^{a} dx_1 \int_{-a}^{a} dy_1 \int_{-a}^{a} dx_2 \int_{-a}^{a} dy_2 \frac{L_m(x_1)L_{m'}(x_2)}{\sqrt{(x_1 - x_2 - X_0)^2 + (y_1 - y_2)^2}} \quad (2.2)$$

where $a = w/2$, $k \in [0, K]$ is the index of the Legendre functions and $m \in [1, M]$ is the quadrature point index.

3 Electric energy

To compare the two methods it is convenient to calculate the electric energy since it is proportional to the capacitance. In the case of surface charge densities the electric energy can be written

$$W_{\mathrm{e}} = \frac{1}{8\pi\epsilon}\int_S\int_S \frac{\rho_{\mathrm{S}}^*(\boldsymbol{r})\rho_{\mathrm{S}}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dS'dS.$$

The two projection methods convey

$$W_{\mathrm{e}}^{\mathrm{P}} = \sum_{kk'}\frac{1}{2}c_k^* Z_{k,k'}^{\mathrm{P}} c_{k'}, \quad W_{\mathrm{e}}^{\mathrm{L}} = \sum_{mm'}\frac{1}{2}d_m^* Z_{m,m'}^{\mathrm{L}} d_{m'}.$$

By introducing the integral function

$$I_{k,k'}(a,X_0,Y_0) := \int_{-a}^{a}dx_1\int_{-a}^{a}dy_1\int_{-a}^{a}dx_2\int_{-a}^{a}dy_2\frac{P_k(x_1/a)P_{k'}(x_2/a)}{\sqrt{(x_1-x_2-X_0)^2+(y_1-y_2-Y_0)^2}} \quad (3.1)$$

the electric energy based on Legendre polynomials can be written

$$W_{\mathrm{e}}^{\mathrm{P}} = \sum_{kk'}\frac{1}{8\pi\epsilon}c_k^* I_{k,k'}(a,X_0,0)c_{k'}. \quad (3.2)$$

Since the linear splines fulfill

$$d_m L_m(x) + d_{m+1}L_{m+1}(x) = q_m + p_m(x-x_m^{\mathrm{q}}), \quad x\in[x_m^{\mathrm{q}}, x_{m+1}^{\mathrm{q}}],$$

the electric energy based on linear splines becomes

$$W_{\mathrm{e}}^{\mathrm{L}} = \sum_{mm'}\frac{1}{8\pi\epsilon}\int_{x_m^{\mathrm{q}}}^{x_{m+1}^{\mathrm{q}}}dx_1\int_{-a}^{a}dy_1\int_{x_{m'}^{\mathrm{q}}}^{x_{m'+1}^{\mathrm{q}}}dx_2\int_{-a}^{a}dy_2\cdot \frac{(q_m+p_m(x_1-x_m^{\mathrm{q}}))^*(q_{m'}+p_{m'}(x_2-x_{m'}^{\mathrm{q}}))}{\sqrt{(x_1-x_2+x_m^{\mathrm{c}}-x_{m'}^{\mathrm{c}})^2+(y_1-y_2)^2}} \quad (3.3)$$

where x_m^{c} are coordinates to the center point of the local element. By coordinate translations the integrands in Eq. (3.3) can be expressed in terms of Legendre polynomials of zeroth and first order. Using Eq. (3.1) the energy can be written

$$W_{\mathrm{e}}^{\mathrm{L}} = \sum_{mm'}\frac{1}{8\pi\epsilon}\big[q_m^*q_{m'}I_{00}(\cdot) + s\big\{q_m^*p_{m'}(I_{00}(\cdot)+I_{01}(\cdot)) + p_m^*q_{m'}(I_{00}(\cdot)+I_{10}(\cdot))\big\} + s^2p_m^*p_{m'}\big\{I_{00}(\cdot)+I_{01}(\cdot)+I_{10}(\cdot)+I_{11}(\cdot)\big\}\big] \quad (3.4)$$

where $(\cdot) = (s, x_m^{\mathrm{c}}-x_{m'}^{\mathrm{c}}, 0)$ and $s = a/(N-1)$ is half the width of the local elements.

4 Results

The surface charge density, ρ_S, is expanded in the Legendre basis using the projection method. The values of the expansion coefficients are:

$$c_0 = 4.7454 \cdot 10^{-24}, \quad c_4 = 6.6313 \cdot 10^{-24}, \quad c_8 = 6.7803 \cdot 10^{-24}, \quad c_{12} = 6.3031 \cdot 10^{-24},$$
$$c_2 = 6.2969 \cdot 10^{-24}, \quad c_6 = 6.8852 \cdot 10^{-24}, \quad c_{10} = 6.6259 \cdot 10^{-24}, \quad c_{14} = 5.9598 \cdot 10^{-24}.$$

All odd coefficients are zero since the surface charge density is an even function. The results of the projection method are shown in Figure 2, where the surface charge density has been projected onto Legendre polynomials for five different values of the truncation constant K. The results show how powerful orthogonal projection really

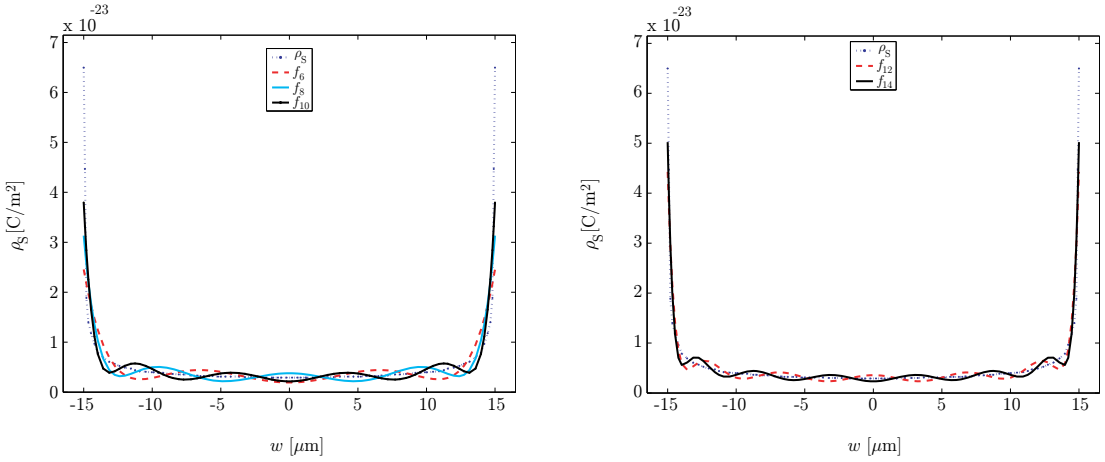
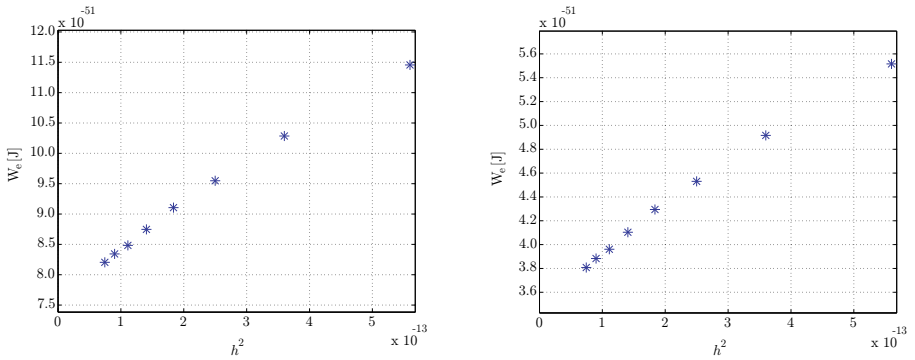


Figure 2: The results from the orthogonal projections of the surface charge density of the microstrip. The projections have been performed for five different values of the truncation constant, K, in the series expansions. The dot curve, in Figure (a) and (b), corresponds to the surface charge density of the microstrip and has been included as a reference.

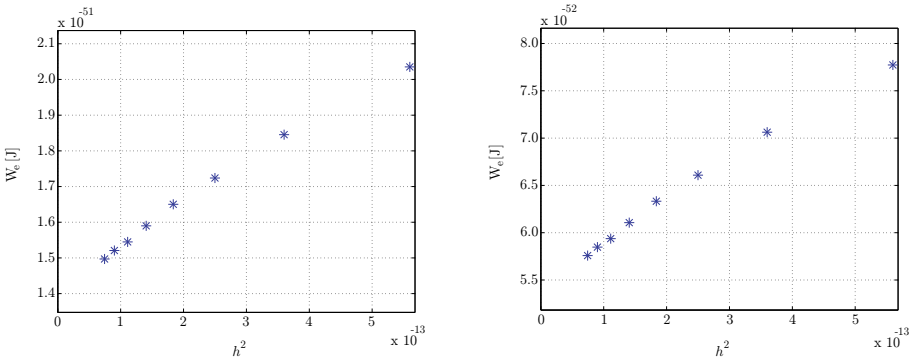
is; already at $K = 6$ a good approximation is achieved. Thus only a few terms are necessary to approximate the surface charge density sufficiently.

The convergence study of the two integration techniques is considered next. The technique based on linear splines has a slow convergence speed as can be seen in Figure 3 where the electric energy is considered. The energy has been calculated for the values $M = \{40, 50, \ldots, 110\}$, where M is the number of quadrature points. The calculations have been performed for different values of X_0.

Even at $M = 110$ the method has not converged to a satisfactory level; the error is several percent. The main reason is that the surface charge density has a gradient that goes to infinity at the edges, which conveys that the number of quadrature points has to be very large in this region to increase the convergence (a rule of thumb is that the total charge on each element should be the same). Since an



(a) $X_0 = 0$ (b) $X_0 = w$



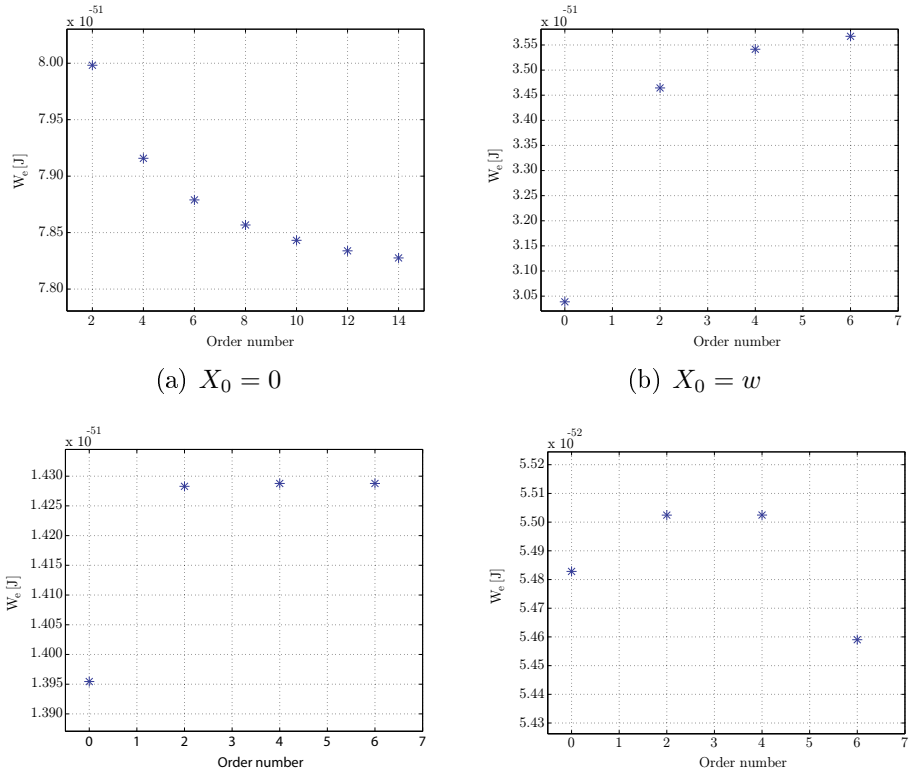
(c) $X_0 = 2w$ (d) $X_0 = 5w$

Figure 3: Convergence study of the integration technique based on linear splines. The electric energy is plotted as a function of the square of the element size, $h = w/(M-1)$, for different values of X_0. It has been calculated for the values $M = \{40, 50, \ldots, 110\}$. Figure (a) corresponds to the self coupling and the other figure to mutual coupling.

equidistant mesh is used, the total number of quadrature points must be very large to assure that regions with large gradients are provided with a dense mesh, with a slow convergence as a consequence. An interesting observation, though, is that for increasing values on X_0 the error decreases. The explanation to this phenomenon is that the energy expression, Eq. (3.4), contains polynomials of different orders in the integrands. Since the interaction between integral expressions containing higher order polynomials (HOP) are more sensitive to a change in the distance between the elements, in comparison to the expressions containing low order polynomials (LOP), the error at far distances is mainly caused by the integration of LOP. Since the errors to the expressions containing LOP are smaller, due to smoother integrands, the error of the total integral expression is expected to decrease as the distance between the elements increases.

The results from the second method, in which Legendre polynomials are used, are presented in Figure 4. In the first case, regarding the interaction due to self coupling (Figure (a)), the convergence is rather fast; the error at the expansion

truncated at order four is around 1% and at twelve less than 0.1%. In the second



(a) $X_0 = 0$ (b) $X_0 = w$

(c) $X_0 = 2w$ (d) $X_0 = 5w$

Figure 4: Convergence study of the integration technique based on Legendre polynomial expansions. The electric energy is plotted as a function of the truncation coefficient, K, in the series expansion for different values of X_0. w is the width of the strip. Figure (a) corresponds to the self coupling and the other figures to mutual coupling. Observe the cancelation effect in the series expansion of order six in figure (d).

case, the interaction due to the mutual coupling (Figure (b)-(d)), it can be seen that the convergence becomes faster when the value of X_0 increases. Already at short distances the convergence is extremely fast; at $X_0 = 2w$ an expansion up to order two is sufficient to achieve results with an error less than 0.1%. At distances $X_0 \geq 2w$ terms of order three and higher can be neglected. The deviation in the results of $K = 6$, in Figure (d), is caused by cancellation effects which can appear when analytic solutions to Eq. (3.1) are used. To prevent this phenomenon the number of digits has to be increased. This has a drawback of slowing down the computational speed and increasing the memory needed. In the calculations a precision of 16 digits have been used.

The different terms in the polynomial series expansion couples to the other elements in a way that resembles the interaction of multipoles. Terms of zeroth order correspond to the monopoles and terms of first order correspond to dipoles and so

on. The electric field for a multipole of order ℓ decreases as $1/r^{\ell+2}$. Since all of the coefficient are of the same magnitude the Legendre polynomials of higher orders only affect the elements in a close neighborhood to the element.

5 Conclusions

A comparison between two different sets of basis functions in the method of moments has been presented. In the first method an orthogonal projection on a base of Legendre polynomials is used and in the second method the projection is done on a base of linear splines.

We have shown that when the coupling between two elements at short distances are calculated, only a few terms of Legendre functions are necessary to achieve good accuracy. Thus the time for filling the impedance matrix will be much shorter in comparison to the method with linear splines. Legendre functions lead to a coarse matrix for which the number of operations to solve the system of equations is considerable reduced.

The use of Legendre polynomials leads to a significant reduction of the number of unknowns. In the example coefficients up to order fourteen have been used. Since the coefficients of odd order numbers are zero (due to case that the function that represents the surface charge density is even) the number of unknowns in the element is seven.[1] This should be compared to one hundred unknowns when using quadrature points. In the case when the surface charge density can not be described by an even function, coefficients of odd order must be included, with an increase in the number of unknowns as a consequence.

The comparison has been done for an equidistant grid (for the linear splines), which is the case when the mesh has been produced automatically by a mesh generator. For a problem adjusted grid, *i.e.*, an adaptive grid, the convergence will be significantly improved. Regions with a large gradient is then supplied with a large amount of quadrature points whereas the other regions can be rather coarse.

Since it becomes very cumbersome to produce analytic solutions to all combinations of elements and all different combinations of Legendre polynomials, additional methods have to be applied. By using the procedure suggested in [7] and the Duffy transform [2], the four dimensional integrals can be solved for more general cases.

In the full problem there is coupling between charge densities and current densities, via the continuity equation. Applying the same algorithm to the current densities the advantage of the current method is even more evident.

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