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PROPAGATION INSIDE A BIANISOTROPIC WAVEGUIDE AS AN EVOLUTION PROBLEM

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Abstract

The free source Maxwell system for the bianisotropic medium, in a fixed frequency $\omega \geqslant 0$ and with time convention $\mathrm{e}^{-\mathrm{i}\omega t}$, is represented by the equation

$$\nabla \times \mathsf{J}\mathsf{e} = \mathrm{i}\omega \mathsf{M}\mathsf{e} \tag{0.1}$$

where $\mathsf{e} := (\boldsymbol{E}, \boldsymbol{H})^{\mathrm{T}}$ is the electromagnetic (E/M) field; it is defined in a domain $\Omega \subset \mathbb{R}^3$, depend on ω and take values in $\mathbb{C}^6$. We denote

$$\mathsf{J} := \begin{bmatrix} 0 & -I_3 \\ I_3 & 0 \end{bmatrix}$$

The matrix

$$\mathsf{M} := \begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{bmatrix}$$

characterizes the medium inside Ω and its entries are complex functions of the frequency ω and the position $\boldsymbol{r} \in \Omega$. The Gauss law implies that

$$\nabla \cdot \mathsf{M}\mathsf{e} = 0 \tag{0.2}$$

Assume that the boundary $\Gamma := \partial\Omega$ is smooth enough; usually Lipschitz is sufficient for most of the applications. Let $\hat{\boldsymbol{n}}$ be the exterior normal to Γ. For a wide class of boundaries, metallic for example, the perfect electric conductor (PEC) boundary condition for the electric field, $\hat{\boldsymbol{n}} \times \boldsymbol{E} = \boldsymbol{0}$ on Γ, applies.

Let now $\boldsymbol{A} = (A_x, A_y, A_z)^T$ be a vector field in Ω; it can be represented as $\boldsymbol{A} = (\boldsymbol{A}_\perp, A_z)^{\mathrm{T}}$ where $\boldsymbol{A}_\perp := A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$ is the transverse and A_z the longitudinal part. It is easily seen that the *curl* operator reads

$$\nabla \times \boldsymbol{A} = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \partial_z \begin{pmatrix} \boldsymbol{A}_\perp \\ A_z \end{pmatrix} - \begin{bmatrix} 0 & W\nabla_\perp \\ \nabla_\perp \cdot W & 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{A}_\perp \\ A_z \end{pmatrix} \tag{0.3}$$

where

$$W := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\mathbf{z}} \times I_3$$

and $\nabla_\perp := \partial_x\hat{\mathbf{x}} + \partial_y\hat{\mathbf{y}}$ is the formal transverse gradient.

An infinite waveguide is a cylinder

$$\Omega = \Omega_\perp \times \mathbb{R}$$

where $\Omega_\perp \subset \mathbb{R}^2$ is a domain with $\Gamma_\perp$. Observe that the wall of the waveguide is $\Gamma = \Gamma_\perp \times \mathbb{R}$ and $\hat{\boldsymbol{n}}$ coincides with its transverse part and is the exterior normal

to $\Gamma_\perp$, whereas $\hat{\boldsymbol{\tau}} := W\hat{\boldsymbol{\nu}}$ is the tangent vector. The PEC boundary condition now reads

$$\hat{\boldsymbol{\tau}} \cdot \boldsymbol{E}_\perp = 0 \quad , \quad E_z = 0 \text{ on } \Gamma \tag{0.4}$$

The fact that the longitudinal variable z runs $\mathbb{R}$ allows us to formulate the Maxwell system as an evolution equation with respect to this variable. Indeed, letting

$$C := \begin{bmatrix} 0 & W\nabla_\perp \\ \nabla_\perp \cdot W & 0 \end{bmatrix}$$

the Maxwell system is written

$$\partial_z \mathsf{V}\mathsf{e} = (\mathsf{A}_0 + \mathrm{i}\omega \mathsf{M})\mathsf{e}$$

where $\mathsf{V} := \hat{\mathbf{z}} \times \mathsf{J}$ and $\mathsf{A}_0 := C\mathsf{J}$. Define now a Hilbert space $\mathcal{X}$ of functions of the transverse variables and consider the E/M field e as vector–valued a function

$$\mathsf{e} \;:\; \mathbb{R} \ni z \mapsto \mathsf{e}(\cdot, \cdot, z) \in \mathcal{X}$$

Then A_0 can be realized as an unbounded operator in $\mathcal{X}$ and the PEC conditions are incorporated in the domain of A_0. Actually, if we separate $\mathsf{u} \in \mathcal{X}$ into "electric" and "magnetic" part

$$\mathsf{u} =: \begin{pmatrix} \mathsf{u}^e \\ \mathsf{u}^h \end{pmatrix},$$

then A_0 is given explicitly by

$$\mathsf{A}_0 \mathsf{u} = \begin{pmatrix} -W\nabla_\perp \mathsf{u}^h_z \\ -\nabla_\perp \cdot W\mathsf{u}^h_\perp \\ W\nabla_\perp \mathsf{u}^e_z \\ \nabla_\perp \cdot W\mathsf{u}^e_\perp \end{pmatrix} \tag{0.5}$$

The first step is to prove that A_0 is the generator of a strongly continuous group in $\mathcal{X}$. The second is to realize Maxwell system as a perturbed abstract degenerate evolution problem

$$\mathsf{V}\mathsf{e}'(z) = (\mathsf{A}_0 + \mathrm{i}\omega\mathsf{M}(\omega))\mathsf{e}(z) \tag{0.6}$$

and apply relevant perturbation arguments in order to establish well-posedness. The research presented here implements exactly this program.

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