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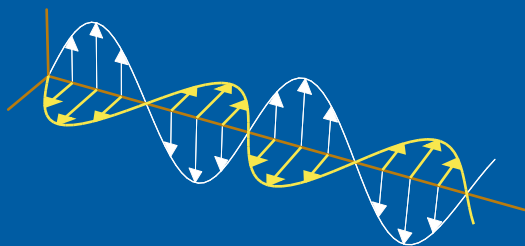
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# Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

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## Abstract

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.

## 1 Introduction

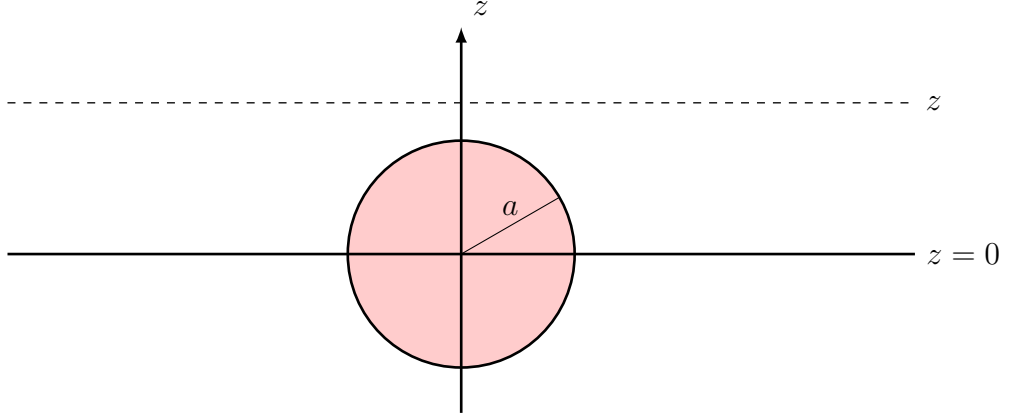
In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in *e.g.*, [3–8, 10, 11, 13, 16–19, 21–25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see *e.g.*, [12, 14, 20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9, 18, 20]

$$I_l(z) = \frac{k^2}{2\pi} \iint_{\mathbb{R}^2} H(r-a) h_l^{(1)}(kr) P_l(\cos \theta) \, dx \, dy, \quad z \in \mathbb{R} \quad (1.1)$$

where  $H(x)$  denotes the Heaviside function,  $h_l^{(1)}(kr)$  the spherical Hankel function, and  $P_l(x)$  the Legendre polynomial of order  $l$ , respectively. We have also adopted the spherical coordinates,  $r = \sqrt{x^2 + y^2 + z^2}$  and  $\theta$  ( $\cos \theta = z/r$ ), and the wave number  $k$ . The domain of integration is the plane  $z = \text{constant}$ , excluding the sphere of radius  $a > 0$  at the center, see Figure 1. For a given value of  $|z| \leq a$ , the radius of the excluded circle is  $\sqrt{a^2 - z^2}$ . For  $|z| \geq a$  the integration is the entire  $x$ - $y$  plane. This integral, for a given  $a > 0$ , is a non-trivial function of  $z \in \mathbb{R}$ . To ensure convergence of the integral at infinity, we assume the wave number  $k$  has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of  $z$  and the index  $l = 0, 1, 2, \dots$ , is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral  $I_l(z)$  is developed in Sections 2 and 3. The indefinite Fourier transform of  $I_l(z)$  is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.



**Figure 1:** The geometry of the integration domain — the plane  $z = \text{constant}$  (dotted line), and the exclusion volume — the sphere of radius  $a$  located at the origin (in gray).

## 2 The integral $I_l(z)$

Rewrite the integral  $I_l(z)$  in (1.1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1.1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)} \left( k \sqrt{\rho^2 + z^2} \right) P_l \left( z / \sqrt{\rho^2 + z^2} \right) \rho \, d\rho, \quad z \in \mathbb{R} \quad (2.1)$$

where

$$h(z) = \begin{cases} \sqrt{a^2 - z^2}, & -a \leq z \leq a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials,  $P_l(-x) = (-1)^l P_l(x)$ , we see that also  $I_l(-z) = (-1)^l I_l(z)$ . Thus, it suffices to evaluate the integral for  $z > 0$ . In particular,  $I_l(0) = 0$ , if  $l$  is an odd integer. From (2.1) we also easily compute the integral for  $l = 0$ , *viz.*,

$$I_0(z) = \begin{cases} e^{-ikz}, & z \leq -a \\ ikah_0^{(1)}(ka) = e^{ika}, & -a \leq z \leq a \\ e^{ikz}, & z \geq a \end{cases}$$

### 2.1 Solution outside the interval $[-a, a]$

In the interval  $z > a$ , the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], *i.e.*, for a

general value of  $z \neq 0$

$$\begin{aligned} & h_l^{(1)} \left( k \sqrt{\rho^2 + z^2} \right) P_l \left( \pm |z| / \sqrt{\rho^2 + z^2} \right) \\ &= \frac{i^{-l}}{2\pi} \iint_{\mathbb{R}^2} P_l (\pm k_z / k) e^{i\mathbf{k}_t \cdot \boldsymbol{\rho} + i k_z |z|} \frac{k}{k_z} \frac{dk_x dk_y}{k^2}, \quad z \gtrless 0 \end{aligned}$$

where  $\boldsymbol{\rho} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ ,  $\mathbf{k}_t = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$ ,  $k_t = |\mathbf{k}_t|$ , and  $k_z$  is defined by

$$k_z = (k^2 - k_t^2)^{1/2} = \begin{cases} \sqrt{k^2 - k_t^2} & \text{for } k_t < k \\ i\sqrt{k_t^2 - k^2} & \text{for } k_t > k \end{cases}$$

For  $z > a$ , we get from (1.1)

$$\begin{aligned} I_l(z) &= \frac{k^2}{2\pi} \iint_{\mathbb{R}^2} \frac{i^{-l}}{2\pi} \left( \iint_{\mathbb{R}^2} P_l (k_z / k) e^{i\mathbf{k}_t \cdot \boldsymbol{\rho} + i k_z |z|} \frac{k}{k_z} \frac{dk_x dk_y}{k^2} \right) dx dy \\ &= i^{-l} \iint_{\mathbb{R}^2} P_l (k_z / k) e^{i k_z |z|} \delta(\mathbf{k}_t) \frac{k}{k_z} dk_x dk_y = i^{-l} e^{i k_z z} \end{aligned}$$

by orthogonality or completeness of the planar waves.<sup>1</sup> This implies that the integral for  $z > a$  is

$$I_l(z) = i^{-l} e^{i k_z z}, \quad z > a$$

and consequently, by parity, or analogous calculations

$$I_l(z) = i^l e^{-i k_z z}, \quad z < -a$$

We observe that the integral outside the interval  $[-a, a]$  is not singular as  $a \rightarrow 0$ . In fact, the module is constant 1.

### 3 Solution of the integral $I_l(\eta)$ , $-a \leq z \leq a$

We have already obtained a solution of the integral in the interval  $|z| > a$ , and we now concentrate on finding a solution of the integral in the interval  $-a \leq z \leq a$ .

The Erdélyi operators  $\mathcal{Y}_n^m$  in Ref. 12 are instrumental in finding a closed formula for the integral  $I_l(z)$ . From [12, Th. 3.13], we have the following very useful result:

$$D \left( h_l^{(1)}(kr) P_l(\cos \theta) \right) = \frac{l+1}{2l+1} h_{l+1}^{(1)}(kr) P_{l+1}(\cos \theta) - \frac{l}{2l+1} h_{l-1}^{(1)}(kr) P_{l-1}(\cos \theta)$$

where  $D = -k^{-1}(\partial/\partial z)$ . The  $D$  operator and the Erdélyi operators are related by  $\mathcal{Y}_1^0 = \sqrt{\frac{3}{4\pi}} D_1^0 = \sqrt{\frac{3}{4\pi}} D$ .

---

<sup>1</sup>To ensure convergence of the integral at infinity, assume the wave number  $k$  has an arbitrary small, positive imaginary part.

Apply the differential operator  $D$  to the integral  $I_l(z)$  in (2.1), and use the relation above. We obtain, since  $h'(z)h(z) = -z$ , the following recursion relation:<sup>2</sup>

$$DI_l(z) = -kzh_l^{(1)}(ka)P_l(z/a) + \frac{l+1}{2l+1}I_{l+1}(z) - \frac{l}{2l+1}I_{l-1}(z), \quad -a \leq z \leq a$$

with initial condition  $I_0(z) = ika h_0^{(1)}(ka)$ .

In the dimensionless variables  $\eta = kz$  and  $\xi = ka > 0$ , this leads to the recursion relation,  $l = 0, 1, 2, \dots$  (note the mild change in notation)

$$I_{l+1}(\eta) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)\frac{\eta}{\xi}P_l(\eta/\xi) - \frac{2l+1}{l+1}\frac{d}{d\eta}I_l(\eta) + \frac{l}{l+1}I_{l-1}(\eta), \quad -\xi \leq \eta \leq \xi$$

The recursion relation is conveniently put in a more generic form by introducing the variable  $x = \eta/\xi \in [-1, 1]$ . The dependent variable is now  $x$ , and  $\xi$  is a parameter. Retaining the same notation for the integral, but with a change of the independent variable, we get

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I'_l(x) + \frac{l}{l+1}I_{l-1}(x), \quad -1 \leq x \leq 1$$

The following proposition states the surprisingly simple and elegant solution of this recursion relation.

**Proposition 3.1.** *The recursion relation*

$$I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I'_l(x) + \frac{l}{l+1}I_{l-1}(x), \quad l = 0, 1, 2, \dots \quad (3.1)$$

with initial condition

$$I_0(z) = i\xi h_0^{(1)}(\xi)$$

has the solution

$$I_l(x) = -\xi h_{l+1}^{(1)}(\xi)P_l(x) + \sum_{k=0}^{[l/2]} (-1)^k \left( \xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}(x), \quad l = 0, 1, 2, \dots \quad (3.2)$$

---

<sup>2</sup>Outside the interval  $z \in [-a, a]$  the recursion relation reads

$$I_{l+1}(z) = \frac{2l+1}{l+1}DI_l(z) + \frac{l}{l+1}I_{l-1}(z), \quad I_0(z) = e^{ikz} \quad z \geq a$$

which is easily solved by induction over the integer  $l$ . The result is

$$I_l(z) = i^{-l}e^{ikz}, \quad z \geq a$$

in agreement with the result above.



**Proof.** We prove the proposition by induction over the integer  $l$ . The recursion relation (3.2) is true for  $l = 0$ , due to the properties of the spherical Hankel functions [15, 10.16.1]. We have from (3.2)

$$I_0(x) = \xi h_{-1}^{(1)}(\xi) = \xi \left( \frac{\pi}{2\xi} \right)^{1/2} H_{-1/2}^{(1)}(\xi) = i\xi \left( \frac{\pi}{2\xi} \right)^{1/2} H_{1/2}^{(1)}(\xi) = i\xi h_0^{(1)}(\xi)$$

Now assume the solution (3.2) holds for all integers less than or equal to  $l$ , and we want to prove that it holds for  $l + 1$ . We have from (3.1) and the induction assumption

$$\begin{aligned} I_{l+1}(x) &= \frac{2l+1}{l+1} \xi h_l^{(1)}(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I_l'(x) + \frac{l}{l+1} I_{l-1}(x) \\ &= \xi h_l^{(1)}(\xi) P_{l+1}(x) + \frac{2l+1}{\xi(l+1)} \xi h_{l+1}^{(1)}(\xi) P_l'(x) \\ &\quad - \frac{2l+1}{\xi(l+1)} \sum_{k=0}^{[l/2]} (-1)^k \left( \xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}'(x) \\ &\quad + \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left( \xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi) \right) P_{l-1-2k}(x) \end{aligned}$$

where we used the following recursion relation for the Legendre polynomials:

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x)$$

We conclude that  $I_{l+1}(x)$  is a polynomial in  $x$  of the order  $l+1$ , and therefore can be expanded in a series of Legendre polynomials. The form is

$$I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)$$

where  $a_n$  depends on  $l$  and  $\xi$ . The coefficients  $a_n$  are determined by orthogonality of the Legendre polynomials.

$$a_n = \frac{2l+3-4n}{2} \int_{-1}^1 I_{l+1}(x) P_{l+1-2n}(x) dx$$

The first coefficient is special.

$$a_0 = \xi h_l^{(1)}(\xi) = -\xi h_{l+2}^{(1)}(\xi) + \left( \xi h_{l+2}^{(1)}(\xi) + \xi h_l^{(1)}(\xi) \right)$$

Proceed in the same way with the remaining coefficients,  $n = 1, 2, \dots, [(l+1)/2]$ .

$$\begin{aligned} a_n &= \frac{2l+1}{l+1} \frac{2l+3-4n}{2} h_{l+1}^{(1)}(\xi) I_{l,l+1-2n} \\ &\quad - \frac{2l+1}{l+1} \frac{2l+3-4n}{2} \sum_{k=0}^{[l/2]} (-1)^k \left( h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right) I_{l-2k,l+1-2n} \\ &\quad + \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left( \xi h_{l-2k}^{(1)}(\xi) + \xi h_{l-2-2k}^{(1)}(\xi) \right) \delta_{k,n-1} \end{aligned}$$

where we used the notion

$$I_{k,n} = \int_{-1}^1 P'_k(x) P_n(x) \, dx = \begin{cases} 0, & 0 \leq k \leq n \\ 1 - (-1)^{k+n}, & 0 \leq n < k \end{cases}$$

Use this result, and the following recursion relation for the spherical Hankel functions:

$$(2l+1)h_l^{(1)}(\xi) = \xi h_{l+1}^{(1)}(\xi) + \xi h_{l-1}^{(1)}(\xi) \quad (3.3)$$

We get

$$\begin{aligned} a_n &= \frac{2l+1}{l+1} (2l+3-4n) \left( h_{l+1}^{(1)}(\xi) - \underbrace{\sum_{k=0}^{n-1} (-1)^k \left( h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right)}_{=h_{l+1}^{(1)}(\xi) + (-1)^{n-1} h_{l+1-2n}^{(1)}(\xi)} \right) \\ &\quad + \frac{l}{l+1} (-1)^{n-1} \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \\ &= \frac{2l+1}{l+1} (-1)^n (2l+3-4n) h_{l+1-2n}^{(1)}(\xi) - \frac{l}{l+1} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \\ &= (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) \end{aligned}$$

Collecting the results gives

$$\begin{aligned} I_{l+1}(x) &= \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x) \\ &= -\xi h_{l+2}^{(1)}(\xi) P_{l+1}(x) + \sum_{n=0}^{[(l+1)/2]} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) P_{l+1-2n}(x) \end{aligned}$$

which is the statement (3.2) for  $l+1$ , and the proposition is proved.  $\square$

Alternative expressions of the integral  $I(z)$  in the interval  $z \in [-a, a]$  can be found. The following corollary shows some.

**Corollary 3.1.** *The integral  $I(z)$  in Proposition 3.1 has the following alternative expressions:*

$$I_l(x) = -\xi h_{l+1}^{(1)}(\xi) P_l(x) + \sum_{k=0}^{[l/2]} (-1)^k (2l-4k+1) h_{l-2k}^{(1)}(\xi) P_{l-2k}(x), \quad l = 0, 1, 2, \dots \quad (3.4)$$

and

$$\begin{aligned} I_l(x) &= i^{1-l} \xi h_0^{(1)}(\xi) P_{l-2[l/2]}(x) \\ &\quad + \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k-1}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2k-2}(x)), \quad l = 0, 1, 2, \dots \end{aligned} \quad (3.5)$$

and

$$I_l(x) = i^{1-l} \xi h_0^{(1)}(\xi) P_{l-2[l/2]}(x) - \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k-1}^{(1)}(\xi) \frac{2l-4k-1}{(l-2k-1)(l-2k)} P'_{l-2k-1}(x), \quad l = 0, 1, 2, \dots \quad (3.6)$$

where the two last sums are zero for  $l = 0, 1$ .

**Proof.** The solution in (3.4) is equivalent to (3.2), which is easily seen since the spherical Hankel functions  $h_l^{(1)}(\xi)$  satisfy the recursion relation (3.3). The representation in (3.5) is simply a rearrangement of the sum in (3.2). We obtain from (3.2) ( $l = 0, 1, 2, \dots$ )

$$\begin{aligned} I_l(x) &= \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-1-2k}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2-2k}(x)) \\ &\quad + (-1)^{[l/2]} \xi h_{l-1-2[l/2]}^{(1)}(\xi) P_{l-2[l/2]}(x) \\ &= \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-1-2k}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2-2k}(x)) \\ &\quad + i^{1-l} \xi h_0^{(1)}(\xi) P_{l-2[l/2]}(x) \end{aligned}$$

where we used [15, 10.16.1]

$$h_{-1}^{(1)}(\xi) = i h_0^{(1)}(\xi)$$

Finally, the relation (3.6) from (3.5) with the use of the recursion relation

$$l(l+1)(P_{l+1}(x) - P_{l-1}(x)) = -(2l+1)(1-x^2)P'_l(x)$$

□

In the original variables  $z$  and  $a$ , we have

$$I_l(z) = i^{1-l} k a h_0^{(1)}(ka) P_{l-2[l/2]}(z/a) + \sum_{n=0}^{[l/2]-1} (-1)^n k a h_{l-2n-1}^{(1)}(ka) (P_{l-2n}(z/a) - P_{l-2n-2}(z/a)), \quad l = 0, 1, 2, \dots$$

or

$$I_l(z) = -k a h_{l+1}^{(1)}(ka) P_l(z/a) + \sum_{k=0}^{[l/2]} (-1)^k (2l-4k+1) h_{l-2k}^{(1)}(ka) P_{l-2k}(z/a), \quad l = 0, 1, 2, \dots$$

and we see that the integral  $I_l(z)$  can be written as a finite sum of spherical waves (except the first term). The most singular term in powers of  $ka$  is of the order  $(ka)^{1-l}$  (order  $O(1)$  if  $l = 0$ ), which is most easily seen from the representation in (3.5).

## 4 Fourier transform of $I_l(z)$

The indefinite Fourier transform of the function  $I_l(z)$  has also importance in the analysis of [9]. More specifically, our goal in this section is to compute

$$\widehat{I}_l^\pm(z) = k \int_{z_0}^z I_l(t) e^{\pm ikt} dt, \quad z \geq z_0, \quad l = 0, 1, 2, \dots \quad (4.1)$$

where  $z_0$  is a fixed number such that  $z_0 < -a$ .

The function  $I_l(t)$  has explicit forms in the three intervals  $[z_0, -a]$ ,  $(-a, a)$ , and  $[a, \infty)$ . The explicit forms are:

$$I_l(t) = i^l e^{-ikt}, \quad t \leq -a$$

and in the interval  $t \in (-a, a)$  as a finite sum of spherical waves

$$\begin{aligned} I_l(t) &= i^{1-l} k a h_0^{(1)}(ka) P_{l-2[l/2]}(t/a) \\ &+ \sum_{n=0}^{[l/2]-1} (-1)^n k a h_{l-2n-1}^{(1)}(ka) (P_{l-2n}(t/a) - P_{l-2n-2}(t/a)) \end{aligned}$$

In the interval  $t \geq a$

$$I_l(t) = i^{-l} e^{ikt}$$

To compute the indefinite Fourier transform we need to calculate the function

$$h_l^\pm(z) = k \int_{-a}^z P_l(t/a) e^{\pm ikt} dt = k a \int_{-1}^{z/a} P_l(t) e^{\pm ika t} dt, \quad |z| \leq a \quad (4.2)$$

For  $z = a$  the integral is a spherical Bessel function, *viz.*,

$$h_l^\pm(a) = k \int_{-a}^a P_l(t/a) e^{\pm ikt} dt = k a \int_{-1}^1 P_l(t) e^{\pm ika t} dt = 2ka(\pm i)^l j_l(ka)$$

We divide the interval  $[z_0, z]$  in three parts. In the interval  $z_0 \leq z < -a$ , we have

$$\widehat{I}_l^\pm(z) = i^l k \int_{z_0}^z e^{i(\pm 1 - 1)kt} dt = i^l \left\{ \begin{aligned} &k(z - z_0) \\ &\frac{1}{2i} (e^{-2ikz_0} - e^{-2ikz}) \end{aligned} \right.$$

and in the interval  $-a < z < a$ , we have

$$\begin{aligned} \widehat{I}_l^\pm(z) &= i^l \left\{ \begin{aligned} &k(-a - z_0) \\ &\frac{1}{2i} (e^{-2ikz_0} - e^{2ika}) \end{aligned} \right. + i^{1-l} k a h_0^{(1)}(ka) h_{l-2[l/2]}^\pm(z) \\ &+ \sum_{n=0}^{[l/2]-1} (-1)^n k a h_{l-2n-1}^{(1)}(ka) (h_{l-2n}^\pm(z) - h_{l-2n-2}^\pm(z)) \end{aligned}$$

and in the interval  $a < z$ , we have

$$\begin{aligned}\widehat{I}_l^\pm(z) = & \mathbf{i}^l \left\{ \frac{k(-a - z_0)}{2\mathbf{i}} (e^{-2\mathbf{i}kz_0} - e^{2\mathbf{i}ka}) \right. \\ & + \mathbf{i}^{1-l} 2(ka)^2 h_0^{(1)}(ka) (\pm \mathbf{i})^{l-2[l/2]} j_{l-2[l/2]}(ka) \\ & + 2(ka)^2 (\pm \mathbf{i})^l \sum_{n=0}^{[l/2]-1} h_{l-2n-1}^{(1)}(ka) (j_{l-2n}(ka) + j_{l-2n-2}(ka)) \\ & \left. + \mathbf{i}^{-l} \left\{ \frac{1}{2\mathbf{i}} (e^{2\mathbf{i}kz} - e^{2\mathbf{i}ka}) \right. \right. \\ & \left. \left. k(z - a) \right\} \right\}\end{aligned}$$

## 5 Indefinite integral of Legendre polynomials

It remains to find an effective method to compute the functions  $h_l^\pm(z)$  in (4.2). To this end, define

$$h_l(\eta, \zeta) = \int_{-1}^{\eta} P_l(t) e^{\mathbf{i}\zeta t} dt, \quad |\eta| \leq 1 \quad (5.1)$$

We see that  $h_l(1, \zeta) = 2\mathbf{i}^l j_l(\zeta)$ . In terms of the functions  $h_l(\eta, \zeta)$ , the functions  $h_l^\pm(z)$  are

$$h_l^\pm(z) = ka h_l(z/a, \pm ka)$$

Our ambition in this section is to find an efficient method to compute the integrals in (5.1). We express the function  $h_l(\eta, \zeta)$  as a recursion relation.

### 5.1 Solution by recursion

The following recursion relation of Legendre polynomials is useful:

$$P_l(t) = \frac{1}{2l+1} (P'_{l+1}(t) - P'_{l-1}(t))$$

Integration by parts then implies  $(P_l(-1) = (-1)^l)$

$$\begin{aligned}h_l(\eta, \zeta) &= \int_{-1}^{\eta} P_l(t) e^{\mathbf{i}\zeta t} dt = \frac{1}{2l+1} \int_{-1}^{\eta} (P'_{l+1}(t) - P'_{l-1}(t)) e^{\mathbf{i}\zeta t} dt \\ &= \frac{1}{2l+1} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{\mathbf{i}\zeta \eta} - \frac{\mathbf{i}\zeta}{2l+1} (h_{l+1}(\eta, \zeta) - h_{l-1}(\eta, \zeta))\end{aligned}$$

or solving for  $h_{l+1}(\eta, \zeta)$

$$h_{l+1}(\eta, \zeta) = \frac{1}{\mathbf{i}\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{\mathbf{i}\zeta \eta} - \frac{2l+1}{\mathbf{i}\zeta} h_l(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \dots$$

The functions  $h_l(\eta, \zeta)$  can therefore be found by iteration with starting values

$$h_0(\eta, \zeta) = \frac{1}{\mathbf{i}\zeta} (e^{\mathbf{i}\zeta \eta} - e^{-\mathbf{i}\zeta}) = \frac{1}{\mathbf{i}\zeta} P_0(\eta) e^{\mathbf{i}\zeta \eta} + h_0^{(2)}(\zeta) = \eta h_0^{(1)}(\zeta \eta) + h_0^{(2)}(\zeta)$$

and

$$\begin{aligned} h_1(\eta, \zeta) &= \frac{1}{i\zeta} (\eta e^{i\zeta\eta} + e^{-i\zeta}) + \frac{1}{\zeta^2} (e^{i\zeta\eta} - e^{-i\zeta}) \\ &= \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) e^{i\zeta\eta} + i h_1^{(2)}(\zeta) = i \eta^2 h_1^{(1)}(\zeta\eta) + i h_1^{(2)}(\zeta) \end{aligned}$$

To find the general solution to this recursion scheme, we start by solving the homogeneous difference equation.

**Lemma 5.1.** *The solution to the homogeneous difference equation*

$$a_{l+1} + \frac{2l+1}{i\zeta} a_l - a_{l-1} = 0, \quad l = 1, 2, 3, \dots$$

given the initial values  $a_0$  and  $a_1$  is

$$\begin{aligned} a_l &= -\frac{\zeta^2}{2i} \left( a_0 h_0^{(2)'}(\zeta) - i a_1 h_0^{(2)}(\zeta) \right) i^l h_l^{(1)}(\zeta) \\ &\quad + \frac{\zeta^2}{2i} \left( a_0 h_0^{(1)'}(\zeta) - i a_1 h_0^{(1)}(\zeta) \right) i^l h_l^{(2)}(\zeta), \quad l = 2, 3, 4, \dots \end{aligned}$$

**Proof.** Two linearly independent solutions to the homogeneous difference equation in the lemma are  $i^l h_l^{(1)}(\zeta)$  and  $i^l h_l^{(2)}(\zeta)$ , which is easily proved by the recursion relation  $f_{l+1}(z) - (2l+1)f_l(z)/z + f_{l-1}(z) = 0$ , where  $f_l(z)$  is any spherical Bessel or Hankel function. The general solution therefore is

$$a_l = c_1 i^l h_l^{(1)}(\zeta) + c_2 i^l h_l^{(2)}(\zeta), \quad l = 2, 3, 4, \dots$$

where  $c_1$  and  $c_2$  are constants determined by the starting values  $a_0$  and  $a_1$ . Explicitly, we get

$$\begin{cases} c_1 h_0^{(1)}(\zeta) + c_2 h_0^{(2)}(\zeta) = a_0 \\ c_1 i h_1^{(1)}(\zeta) + c_2 i h_1^{(2)}(\zeta) = a_1 \end{cases}$$

with solution

$$\begin{cases} c_1 = -\frac{\zeta^2}{2i} \left( a_0 h_0^{(2)'}(\zeta) - i a_1 h_0^{(2)}(\zeta) \right) \\ c_2 = \frac{\zeta^2}{2i} \left( a_0 h_0^{(1)'}(\zeta) - i a_1 h_0^{(1)}(\zeta) \right) \end{cases}$$

where we used the Wronskian of the spherical Hankel functions.

$$h_n^{(2)}(z) h_n^{(1)'}(z) - h_n^{(2)'}(z) h_n^{(1)}(z) = \frac{2i}{z^2}$$

and  $h_0^{(1,2)'}(z) = -h_1^{(1,2)}(z)$ . This completes the proof of the lemma.  $\square$

We are now ready for the solution to the inhomogeneous difference equation in  $h_l(\eta, \zeta)$  above. We formulate this as a lemma.

**Lemma 5.2.** Define an iteration scheme by

$$h_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{i\zeta\eta} - \frac{2l+1}{i\zeta} h_l(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \dots$$

with starting values

$$h_0(\eta, \zeta) = \eta h_0^{(1)}(\zeta\eta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta, \zeta) = i \left( \eta^2 h_1^{(1)}(\zeta\eta) + h_1^{(2)}(\zeta) \right)$$

The solution is

$$h_l(\eta, \zeta) = f_l(\eta, \zeta) e^{i\zeta\eta} + i^l h_l^{(2)}(\zeta), \quad l = 0, 1, 2, 3, \dots$$

where

$$f_l(\eta, \zeta) = i^l h_l^{(1)}(\zeta) \left\{ \sum_{k=1}^l \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=0}^k i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \right. \right. \\ \left. \left. + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \right) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\}, \quad l = 0, 1, 2, \dots$$

**Proof.** We first subtract the part of the solution that contains the spherical Hankel function of the second kind  $h_l^{(2)}(\zeta)$  and the exponential function  $e^{i\zeta\eta}$ . To this end, let  $h_l(\eta, \zeta) = f_l(\eta, \zeta) e^{i\zeta\eta} + i^l h_l^{(2)}(\zeta)$ . The recursion relation for  $f_l(\eta, \zeta)$  is easily found by the use of the recursion relation  $h_{l+1}^{(2)}(z) = (2l+1)h_l^{(2)}(z)/z - h_{l-1}^{(2)}(z)$ . We get the new difference equation

$$f_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) - \frac{2l+1}{i\zeta} f_l(\eta, \zeta) + f_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \dots$$

with starting values

$$f_0(\eta, \zeta) = \frac{1}{i\zeta} P_0(\eta)$$

and

$$f_1(\eta, \zeta) = \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right)$$

To simplify the notation, we put the difference equation in a standard form [1].

$$a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \quad n = 1, 2, \dots$$

where

$$\begin{cases} a_n = f_{n-1}(\eta, \zeta) \\ p_1(n) = \frac{2n+1}{i\zeta} \\ p_0(n) = -1 \\ q(n) = \frac{1}{i\zeta} (P_{n+1}(\eta) - P_{n-1}(\eta)) \end{cases}$$

with initial values

$$\begin{cases} a_1 = \frac{1}{i\zeta} P_0(\eta) \\ a_2 = \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) \end{cases}$$

A solution to the homogeneous difference equation is (see Lemma 5.1)

$$y_l = i^{l-1} h_{l-1}^{(1)}(\zeta)$$

The final solution then is [1], ( $n = 3, 4, \dots$ )

$$a_n = \left( \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{p_0(j)y_j}{y_{j+2}} \left( \sum_{l=1}^{k-1} \frac{q(l)}{y_{l+2}} \left[ \prod_{m=1}^l \frac{p_0(m)y_m}{y_{m+2}} \right]^{-1} + \frac{a_2}{y_2} - \frac{a_1}{y_1} \right) + \frac{a_1}{y_1} \right) y_n$$

Insert the explicit values, and we obtain

$$\begin{aligned} f_l(\eta, \zeta) = & \left\{ \sum_{k=1}^l \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=1}^{k-1} h_n^{(1)}(\zeta) \frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{i^n} + i h_1^{(1)}(\zeta) P_0(\eta) \right. \right. \\ & \left. \left. - h_0^{(1)}(\zeta) \left( P_1(\eta) + i \frac{1}{\zeta} P_0(\eta) \right) \right) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i^l h_l^{(1)}(\zeta), \quad l = 2, 3, 4, \dots \end{aligned}$$

This relation holds also for  $l = 0, 1$ , provided the sums with upper limit smaller than the lower limit are interpreted as zero.

We now simplify the sum in this expression.

$$\begin{aligned} S = & - \sum_{n=1}^{k-1} i^{-n} h_n^{(1)}(\zeta) (P_{n+1}(\eta) - P_{n-1}(\eta)) + i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \\ = & i h_1^{(1)}(\zeta) (P_2(\eta) - P_0(\eta)) + h_2^{(1)}(\zeta) (P_3(\eta) - P_1(\eta)) - i h_3^{(1)}(\zeta) (P_4(\eta) - P_2(\eta)) \\ & + \dots - i^{-k+2} h_{k-2}^{(1)}(\zeta) (P_{k-1}(\eta) - P_{k-3}(\eta)) - i^{-k+1} h_{k-1}^{(1)}(\zeta) (P_k(\eta) - P_{k-2}(\eta)) \\ & + i h_1^{(1)}(\zeta) P_0(\eta) - h_0^{(1)}(\zeta) P_1(\eta) \\ = & - \left( h_0^{(1)}(\zeta) + h_2^{(1)}(\zeta) \right) P_1(\eta) + i \left( h_1^{(1)}(\zeta) + h_3^{(1)}(\zeta) \right) P_2(\eta) \\ & + \left( h_2^{(1)}(\zeta) + h_4^{(1)}(\zeta) \right) P_3(\eta) - i \left( h_3^{(1)}(\zeta) + h_5^{(1)}(\zeta) \right) P_4(\eta) + \dots \\ & - i^{-k+2} \left( h_{k-2}^{(1)}(\zeta) + h_k^{(1)}(\zeta) \right) P_{k-1}(\eta) - i^{-k+1} \left( h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta) \\ & + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \end{aligned}$$

The recursion relation  $h_{l+1}^{(1)}(z) + h_{l-1}^{(1)}(z) = (2l+1)h_l^{(1)}(z)/z$  implies

$$\begin{aligned} S = & -3 \frac{h_1^{(1)}(\zeta)}{\zeta} P_1(\eta) + 5i \frac{h_2^{(1)}(\zeta)}{\zeta} P_2(\eta) + 7 \frac{h_3^{(1)}(\zeta)}{\zeta} P_3(\eta) - 9i \frac{h_4^{(1)}(\zeta)}{\zeta} P_4(\eta) + \dots \\ & - i^{-k+1} (2k+1) \frac{h_k^{(1)}(\zeta)}{\zeta} P_k(\eta) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \\ = & - \sum_{n=1}^k i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \end{aligned}$$



which gives

$$f_l(\eta, \zeta) = \left\{ \sum_{k=1}^l \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=1}^k i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \right. \right. \\ \left. \left. + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) - i \frac{h_0^{(1)}(\zeta)}{\zeta} P_0(\eta) \right) \right. \\ \left. - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i^l h_l^{(1)}(\zeta)$$

or

$$f_l(\eta, \zeta) = \left\{ \sum_{k=1}^l \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=0}^k i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) \right. \right. \\ \left. \left. + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \right) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i^l h_l^{(1)}(\zeta)$$

This completes the lemma.  $\square$

In conclusion, the functions  $h_l^\pm(z)$  defined in (4.2) can be expressed in the function  $h(\eta, \zeta)$  in (5.1). Specifically, we have

$$h_l^\pm(z) = ka h_l(z/a, \pm ka)$$

## 6 Summary and explicit terms

This paper contains an evaluation of a non-trivial integral that occurs in the formulation of scattering by randomly distributed obstacles.

To summarize, the integral  $I_l(z)$  in (1.1) has been solved and the solution outside the interval  $[-a, a]$  is a simple exponential function in  $kz$ , while inside the interval  $[-a, a]$ , the solution can be found in a finite series of spherical waves. The finite sum of spherical waves depends on the two parameters  $kz$  and  $ka$ , or, more precisely, the parameter  $ka$  and a polynomial of the order  $l$  in the parameter  $z/a$ . Several equivalent solutions are presented in the paper, one of them is ( $l = 0, 1, 2, \dots$ )

$$I_l(z) = \begin{cases} i^l e^{-ikz}, & z \leq -a \\ i^{1-l} ka h_0^{(1)}(ka) P_{l-2[l/2]}(z/a) \\ \quad + \sum_{n=0}^{[l/2]-1} (-1)^n ka h_{l-2n-1}^{(1)}(ka) (P_{l-2n}(z/a) - P_{l-2n-2}(z/a)), & z \in [-a, a] \\ i^{-l} e^{ikz}, & z \geq a \end{cases}$$

The first integrals,  $l = 0, 1, 2$ , are of interest for low-frequency expansions. For  $l = 0$  the integral is

$$I_0(z) = \begin{cases} e^{-ikz}, & z \leq -a \\ e^{ika}, & z \in [-a, a] \\ e^{ikz}, & z \geq a \end{cases}$$

and for  $l = 1$  the result is

$$I_1(z) = \begin{cases} ie^{-ikz}, & z \leq -a \\ -ie^{ika}\frac{z}{a}, & z \in [-a, a] \\ -ie^{ikz}, & z \geq a \end{cases}$$

For  $l = 2$  the result is

$$I_2(z) = \begin{cases} -e^{-ikz}, & z \leq -a \\ e^{ika} \frac{(ka)^2(3i + ka) - 3(i + ka)(kz)^2}{2(ka)^3}, & z \in [-a, a] \\ -e^{ikz}, & z \geq a \end{cases}$$

and we notice that the integral contains a polynomial in  $z/a$  of order  $l$ .

Moreover, the indefinite Fourier transform of  $I_l(z)$  has also been investigated. More precisely, the integral, see (4.1)

$$\widehat{I}_l^\pm(z) = k \int_{z_0}^z I_l(t) e^{\pm ikt} dt, \quad z \geq z_0, \quad l = 0, 1, 2, \dots$$

is shown to have a solution expressed in spherical waves.

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