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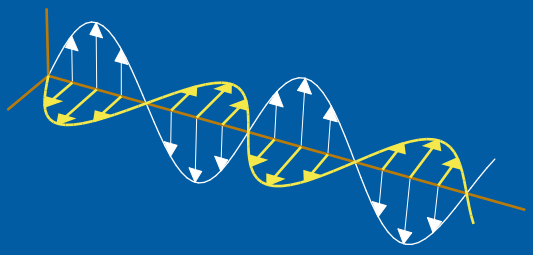

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Abstract

This paper contains an overview of the homogenization of anisotropic materials at fixed frequency using the concept of two-scaled convergence. The homogenized electric and magnetic parameters, the relative permittivity and the relative permeability, respectively, are found by suitable averages of the solution of a local problem in the unit cell. A comparison between the exact homogenization method presented in this paper and the traditional mixture formulae, which are based on physical arguments, is made.

1 Introduction

Many engineering problems contain materials which are heterogeneous on a length scale (microscopic) that is small compared with the typical length scales (macroscopic) of the problem. Nevertheless, the underlying heterogeneous, microscopic structure affects the macroscopic properties of the material. Accurate methods to model these microscopic effects are therefore important to develop.

In electromagnetic problems, when the wavelength is long compared to the periodicity of the microstructure, the homogenized values of the electric or magnetic material parameters are pertinent macroscopic quantities. Several homogenization procedures have been suggested in the literature. Some of these are based upon physical arguments, *e.g.*, the Maxwell Garnett formula, the Böttcher mixture rule or Bruggeman formula, and the coherent potential (CP) formula [15]. They have proven useful in many situations, *e.g.*, low volume fraction of homogeneous spherical or ellipsoidal inclusions in a homogeneous host material, but they fail if the volume fraction is too high or if the inclusions are not spheres or ellipsoids. Then, a more accurate homogenization procedure has to be used, which includes all contributions of the interaction between the inclusions.

In this paper, we review a general mathematical procedure to obtain the homogenized (or effective) material parameters, which includes all interactions effects between the inclusions. Specifically, we employ recent advances in the mathematics of two-scale convergence, which was introduced in 1989 by Nguetseng [11]. Specifically, we review some of the more important results of this convergence concept, but for the mathematical details we refer to the existing literature on this subject.

A typical homogenization situation is depicted in Figure 1, where we, in several steps, shrink the length scale ε, which is the periodicity of the material. The two-scaled convergence predicts the limit of this process, and gives a procedure of how to compute the effective material parameters of the material with microstructure. The results are not restricted to low volume fractions of the inclusions or to a two-phase composition of materials. In fact, the results are quite general and can be applied to a large variety of engineering situations.

The results obtained using two-scale convergence are self-contained in contrast to similar approaches that assume an asymptotic expansion of the solution in terms of the microscopic scale, *e.g.*, see [14]. The theory of two-scale convergence does not rely on such an expansion. The periodic variations in the material parameters

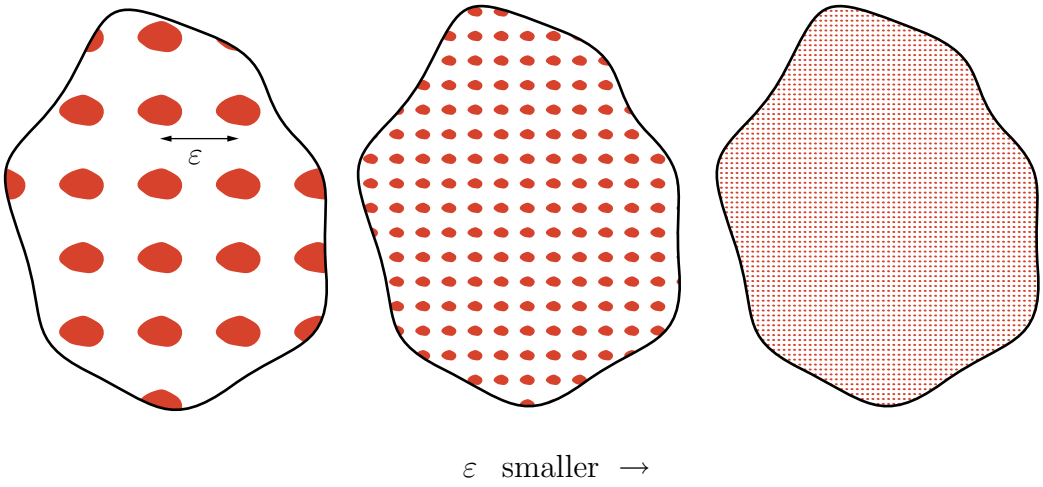

Figure 1: A material with a microstructure with periodicity ε. The size of ε is decreasing from left to right.

generate the same type of variations in the two-scale converged solution, which characterizes the microscopic solution completely.

This paper is organized in the following way. In Section 2, the basic assumptions of the homogenization problem are stated, and, in Section 3, the fundamental properties of the two-scaled convergence are reviewed. The local problem is stated in Section 4, and a comparison between the classical mixture formulae and the exact homogenization procedure is presented in Section 5. Some conclusions of the theory are discussed in Section 6.

2 Formulation of the problem

Assume Ω is a smooth, bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$, and outward pointing unit normal $\hat{\boldsymbol{\nu}}$. Some requirements on the regularity of the boundary $\partial\Omega$ have to be made to justify the mathematics, but to avoid technical details these are omitted. Moreover, assume that the material in Ω is Y^ε-periodic, *i.e.*, it is a collection of identical cubes with side length ε (Y^ε-cells), see Figure 2. This means that the periodicity of the material is ε, with scaled unit cell εY, where Y is the unit cube in $\mathbb{R}^3$.

In Ω the electromagnetic fields satisfy the Maxwell equations. The field solutions depend on the size of the Y^ε-cells, and, therefore, all fields are indexed by the periodicity ε. The Maxwell equations in the presence of a source term $\boldsymbol{J}^\varepsilon$ are

$$\begin{cases} \nabla \times \boldsymbol{E}^\varepsilon(\boldsymbol{x}) = \mathrm{i}\omega \boldsymbol{B}^\varepsilon(\boldsymbol{x}) \\ \nabla \times \boldsymbol{H}^\varepsilon(\boldsymbol{x}) = \boldsymbol{J}^\varepsilon(\boldsymbol{x}) - \mathrm{i}\omega \boldsymbol{D}^\varepsilon(\boldsymbol{x}) \end{cases} \qquad \boldsymbol{x} \in \Omega \tag{2.1}$$

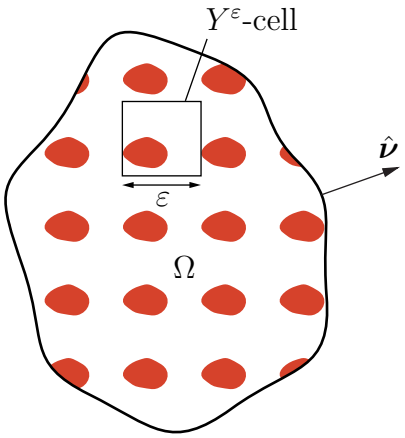

Figure 2: Typical periodic geometry of the material parameters and the definition of the Y^ε-cell.

and

$$\begin{cases} \nabla \cdot \boldsymbol{B}^\varepsilon(\boldsymbol{x}) = \boldsymbol{0} \\ \nabla \cdot \boldsymbol{D}^\varepsilon(\boldsymbol{x}) = 0 \end{cases} \qquad \boldsymbol{x} \in \Omega \tag{2.2}$$

The boundary conditions on $\partial\Omega$ are traditionally taken as

$$\hat{\boldsymbol{\nu}} \times \boldsymbol{E}^\varepsilon(\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \partial\Omega \tag{2.3}$$

which also is adopted in this paper. Other boundary values, such as the penetrable case, are also possible to analyze. In the penetrable case, the excitation is exterior to the domain Ω. This problem is more complex, and the reader is referred to the literature for the solution of this problem [20]. To simplify this review, we omit the discussion of the sources below.

We now state the general assumption made on the relative permittivity and the relative permeability used to describe the material in Ω. The material can be inhomogeneous, and the relative electric permittivity dyadic, $\boldsymbol{\epsilon}(\boldsymbol{x})$, and the relative magnetic permeability dyadic, $\boldsymbol{\mu}(\boldsymbol{x})$, are three-dimensional, complex-valued dyadics, and they satisfy a coercivity and a boundedness condition ($C_{1,2} > 0$)

$$\begin{cases} \mathrm{i}\boldsymbol{\xi} \cdot \left(\boldsymbol{\epsilon}(\boldsymbol{x}) - \boldsymbol{\epsilon}(\boldsymbol{x})^\dagger\right) \cdot \boldsymbol{\xi}^* \geq C_1 |\boldsymbol{\xi}|^2 \\ |\boldsymbol{\epsilon}(\boldsymbol{x}) \cdot \boldsymbol{\xi}| \leq C_2 |\boldsymbol{\xi}| \end{cases} \qquad \boldsymbol{\xi} \in \mathbb{C}^3, \quad \boldsymbol{x} \in \Omega$$

where $\boldsymbol{\epsilon}^\dagger$ denotes the Hermitian of the dyadic $\boldsymbol{\epsilon}$. The first condition states that the medium is passive, *i.e.*, it is dissipative. The second one is a condition that guarantees boundedness of the parameters. Moreover, the quantities are assumed to be Y-periodic, *i.e.*, $\boldsymbol{\epsilon}(\boldsymbol{y} + \hat{\boldsymbol{e}}_k) = \boldsymbol{\epsilon}(\boldsymbol{y})$ for every value of the local variable $\boldsymbol{y} \in \mathbb{R}^3$ and for every $k = 1, 2, 3$. Similar result holds for the relative permeability $\boldsymbol{\mu}$. Due to the dyadic characters of the relative permittivity and the relative permeability, inhomogeneous, anisotropic situations can be analyzed.

Since the medium is Y^ε-periodic, we scale the arguments in $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ with the periodicity ε. In fact, the constitutive relations adopted in this paper are

$$\begin{cases} \boldsymbol{D}^\varepsilon(\boldsymbol{x}) = \epsilon_0 \boldsymbol{\epsilon}(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{E}^\varepsilon(\boldsymbol{x}) \\ \boldsymbol{B}^\varepsilon(\boldsymbol{x}) = \mu_0 \boldsymbol{\mu}(\boldsymbol{x}/\varepsilon) \cdot \boldsymbol{H}^\varepsilon(\boldsymbol{x}) \end{cases} \tag{2.4}$$

This implies that the scaled material parameters used in (2.1) and (2.2) are Y^ε-periodic, *i.e.*,

$$\boldsymbol{\epsilon}((\boldsymbol{x} + \varepsilon \hat{\boldsymbol{e}}_k)/\varepsilon) = \boldsymbol{\epsilon}(\boldsymbol{x}/\varepsilon + \hat{\boldsymbol{e}}_k) = \boldsymbol{\epsilon}(\boldsymbol{x}/\varepsilon), \quad \text{for all } \boldsymbol{x} \in \Omega, \quad k = 1, 2, 3$$

since $\boldsymbol{\epsilon}$ is Y-periodic.

3 The concept of two-scaled convergence

The basic mathematical tool in the analysis of the homogenized or effective material parameters is the concept of two-scaled convergence. In this section, we review some of the properties of this convergence principle. Only the basic properties needed for the results presented in this paper are stated, and the reader who wants to study this convergence concept in more detail is referred to the literature [1–3, 5, 11, 16–19].

The concept of two-scaled convergence was introduced by Nguetseng [11] and has later been analyzed further in the mathematical literature, see *e.g.*, [1–3, 5, 16–19]. A sequence $\{\boldsymbol{u}^\varepsilon\}$ in $L^2(\Omega)^3$ two-scale converges to $\boldsymbol{u}_0 \in L^2(\Omega \times Y)^3$ if [11]

$$\lim_{\varepsilon \to 0} \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) \, \mathrm{d}v_{\boldsymbol{x}} = \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y}) \, \mathrm{d}v_{\boldsymbol{x}} \, \mathrm{d}v_{\boldsymbol{y}}$$

for every smooth vector field $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})$, which is Y-periodic in $\boldsymbol{y}$. The volume measure of $\mathbb{R}^3$ is denoted $\mathrm{d}v_{\boldsymbol{x}}$ or $\mathrm{d}v_{\boldsymbol{y}}$ depending on the integration variable. We denote two-scale convergence by $\boldsymbol{u}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{u}_0$. Allaire [1] proved that it is possible to extend the set of test functions to the set of admissible function, *i.e.*, all functions $\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})$ (Y-periodic in $\boldsymbol{y}$, but not necessarily smooth in $\boldsymbol{x}$ and $\boldsymbol{y}$) such that

$$\lim_{\varepsilon \to 0} \iiint_\Omega |\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}/\varepsilon)|^2 \, \mathrm{d}v_{\boldsymbol{x}} = \iiint_\Omega \iiint_Y |\boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{y})|^2 \, \mathrm{d}v_{\boldsymbol{x}} \, \mathrm{d}v_{\boldsymbol{y}}$$

We easily see that if $\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y})$ then $\boldsymbol{u}^\varepsilon(\boldsymbol{x})$ converges weakly to the mean over the unit cell Y. To see this, take a test function $\boldsymbol{\phi} \in (L^2(\Omega))^3$ that is independent of $\boldsymbol{y}$ and the definition implies

$$\begin{aligned} \lim_{\varepsilon \to 0} \iiint_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) \, \mathrm{d}v_{\boldsymbol{x}} &= \iiint_\Omega \iiint_Y \boldsymbol{u}_0(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) \, \mathrm{d}v_{\boldsymbol{x}} \, \mathrm{d}v_{\boldsymbol{y}} \\ &= \iiint_\Omega \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}) \, \mathrm{d}v_{\boldsymbol{x}} \end{aligned}$$

where the mean over the unit cell Y is denoted

$$\boldsymbol{u}(\boldsymbol{x}) = <\boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})> = \iiint\limits_Y \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})\ \mathrm{d}v_{\boldsymbol{y}}$$

The average over the unit cell (Y-cell) is denoted

$$<f> = \frac{1}{|Y|}\iiint\limits_Y f(\boldsymbol{y})\ \mathrm{d}v_{\boldsymbol{y}}$$

and $|Y|$ is the volume of the unit cell (in our case we have scaled so that $|Y| = 1$). Therefore, $\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{u}_0(\boldsymbol{x},\boldsymbol{y})$ implies that $\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \rightharpoonup \boldsymbol{u}(\boldsymbol{x})$ weakly in $(L^2(\Omega))^3$.

It is also not hard to prove that if $\boldsymbol{u}^\varepsilon(\boldsymbol{x}) \to \boldsymbol{u}(\boldsymbol{x})$ strongly in $(L^2(\Omega))^3$ (norm convergence in $(L^2(\Omega))^3$), then it also two-scale converges to the same limit. The converse is, however, not true. Moreover, if $a(\boldsymbol{x},\boldsymbol{y})$ is a smooth Y-periodic function in $\boldsymbol{y}$, then $a^\varepsilon(\boldsymbol{x}) = a(\boldsymbol{x},\boldsymbol{x}/\varepsilon)$ two-scale converges to $a(\boldsymbol{x},\boldsymbol{y})$. In fact, every $a(\boldsymbol{x},\boldsymbol{y}) \in L^2(\Omega \times Y)$ is obtained as a two-scale limit of a function $a^\varepsilon(\boldsymbol{x})$ in $L^2(\Omega)$ [1]. We can conclude that the convergence in the two-scaled sense is more general than both the usual strong and weak convergence concepts.

The two usual concepts of convergence — strong (norm convergence) and weak convergence — are too coarse instruments to preserve the information of the field at the microscale. The concept of two-scale convergence, however, preserves some of the relevant information of field at the microscale. This is proved to be an ideal tool in the homogenization procedure of materials with microscopic scale, see *e.g.*, [1, 2, 5, 11, 16–20].

The existence of a two-scale limit of a family of fields $\boldsymbol{E}^\varepsilon$ and $\boldsymbol{H}^\varepsilon$, such as the solutions to the Maxwell equations in (2.1) and (2.2), as $\varepsilon \to 0$, is guaranteed provided a uniform bound can be established, that is, we need uniform bounds on the fields of the following form (*a priori* estimates):

$$\|\boldsymbol{E}^\varepsilon\|_2^2 \le C_1, \quad \|\nabla\times\boldsymbol{E}^\varepsilon\|_2^2 \le C_2, \quad \|\boldsymbol{H}^\varepsilon\|_2^2 \le C_3, \quad \|\nabla\times\boldsymbol{H}^\varepsilon\|_2^2 \le C_4, \text{ for all } \varepsilon \quad (3.1)$$

where

$$\|\boldsymbol{u}\|_2^2 = \iiint\limits_\Omega |\boldsymbol{u}(\boldsymbol{x})|^2\ \mathrm{d}v_{\boldsymbol{x}}$$

The constants C_i, $i = 1, 2, 3, 4$, do not depend on the parameter ε, but only on the domain Ω, and the material parameters in Ω. These *a priori* estimates, are fundamental in the development of the homogenization procedure [1, 11]. For the boundary conditions in (2.3) such an *a priori* estimate can be established [17]. The *a priori* estimates for other appropriate boundary conditions, such as the penetrable case, can also be obtained, see [20], but the analysis is more complex.

4 The local problem

The local problem is reviewed in this section. Most of the mathematical details, however, are omitted. These are easily found in the literature [1, 3, 11, 16, 17].

The *a priori* estimates in (3.1) suffice to guarantee that there are subsequences of $\{\boldsymbol{E}^\varepsilon\}$ and $\{\boldsymbol{H}^\varepsilon\}$, respectively, that two-scale converge [11, 16, 17], *i.e.*,

$$\begin{cases} \boldsymbol{E}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{E}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y}) \\ \boldsymbol{H}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{x}) + \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y}) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \nabla\times\boldsymbol{E}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}}\times\boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{y}) \\ \nabla\times\boldsymbol{H}^\varepsilon \overset{\text{2-s}}{\rightharpoonup} \nabla_{\boldsymbol{x}}\times\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}}\times\boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{y}) \end{cases}$$

where the fields, $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$, are the averaged fields over the unit cell Y, *i.e.*,

$$\begin{cases} \boldsymbol{E}(\boldsymbol{x}) = <\boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y})> = \displaystyle\iiint_Y \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{y})\ \mathrm{d}v_{\boldsymbol{y}} \\ \boldsymbol{H}(\boldsymbol{x}) = <\boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y})> = \displaystyle\iiint_Y \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{y})\ \mathrm{d}v_{\boldsymbol{y}} \end{cases}$$

and the scalar fields Φ_1 and Ψ_1, and the vector fields $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, $\boldsymbol{E}_1$ and $\boldsymbol{H}_1$ denote the limit functions. Notice that the first terms in the two-scaled convergence in (4.1), $\boldsymbol{E}$ and $\boldsymbol{H}$, are independent of the local variable $\boldsymbol{y}$, and the second terms are gradients w.r.t. the local variable $\boldsymbol{y}$. The two-scaled limits contain information about the field on the microscopic level, and, moreover, the fields $\boldsymbol{E}^\varepsilon$ and $\boldsymbol{H}^\varepsilon$ do not converge strongly to $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively, *i.e.*, $\lim_{\varepsilon\to 0}\|\boldsymbol{E}^\varepsilon(\boldsymbol{x}) - \boldsymbol{E}(\boldsymbol{x})\| \neq 0$ [1]. However, we have [20]

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\|\boldsymbol{E}^\varepsilon(\boldsymbol{x}) - \boldsymbol{E}_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) - \varepsilon\boldsymbol{E}_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\| = 0 \\ \lim\limits_{\varepsilon\to 0}\|\boldsymbol{H}^\varepsilon(\boldsymbol{x}) - \boldsymbol{H}_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) - \varepsilon\boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon)\| = 0 \end{cases}$$

This is the so called corrector result.

The unknown functions Φ_1 and Ψ_1, which contain the information of the behavior of the fields on the microscale, are found by employing the Maxwell equations (2.1) and (2.2), and the constitutive relations (2.4). A separation of variables arguments implies that these functions can be obtained as the solution of a local problem. The result is [1, 11, 17]

$$\begin{cases} \nabla_{\boldsymbol{y}}\Phi_1(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_e(\boldsymbol{y})\cdot\boldsymbol{E}(\boldsymbol{x}) = -\nabla_{\boldsymbol{y}}\displaystyle\sum_{i=1}^{3}\chi_e^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i\cdot\boldsymbol{E}(\boldsymbol{x}) \\ \nabla_{\boldsymbol{y}}\Psi_1(\boldsymbol{x},\boldsymbol{y}) = -\nabla_{\boldsymbol{y}}\boldsymbol{\chi}_h(\boldsymbol{y})\cdot\boldsymbol{H}(\boldsymbol{x}) = -\nabla_{\boldsymbol{y}}\displaystyle\sum_{i=1}^{3}\chi_h^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i\cdot\boldsymbol{H}(\boldsymbol{x}) \end{cases}$$

where

$$\boldsymbol{\chi}_e(\boldsymbol{y}) = \sum_{i=1}^{3}\chi_e^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i, \qquad \boldsymbol{\chi}_h(\boldsymbol{y}) = \sum_{i=1}^{3}\chi_h^i(\boldsymbol{y})\hat{\boldsymbol{e}}_i$$

The functions $\chi_e^i(\boldsymbol{y})$ and $\chi_h^i(\boldsymbol{y})$ are determined by the solution of the local problem in the unit cell Y $(i = 1, 2, 3)$ [1, 11]

$$\begin{cases} \nabla_{\boldsymbol{y}} \cdot \big(\boldsymbol{\epsilon}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \chi_e^i(\boldsymbol{y})\big) = \nabla_{\boldsymbol{y}} \cdot (\boldsymbol{\epsilon}(\boldsymbol{y}) \cdot \hat{\boldsymbol{e}}_i) \\ \nabla_{\boldsymbol{y}} \cdot \big(\boldsymbol{\mu}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \chi_h^i(\boldsymbol{y})\big) = \nabla_{\boldsymbol{y}} \cdot (\boldsymbol{\mu}(\boldsymbol{y}) \cdot \hat{\boldsymbol{e}}_i) \end{cases} \tag{4.2}$$

or in a weak formulation

$$\begin{cases} \iiint\limits_Y \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \boldsymbol{\epsilon}(\boldsymbol{y}) \cdot \big(\hat{\boldsymbol{e}}_i - \nabla_{\boldsymbol{y}} \chi_e^i(\boldsymbol{y})\big)\, \mathrm{d}v_{\boldsymbol{y}} = 0 \\ \iiint\limits_Y \nabla_{\boldsymbol{y}} w(\boldsymbol{y}) \cdot \boldsymbol{\mu}(\boldsymbol{y}) \cdot \big(\hat{\boldsymbol{e}}_i - \nabla_{\boldsymbol{y}} \chi_h^i(\boldsymbol{y})\big)\, \mathrm{d}v_{\boldsymbol{y}} = 0 \end{cases}$$

for all Y-periodic test function w.

The local problem in (4.2) is an electrostatic problem in the unit cell Y. These equations are solved for the unknowns $\chi_e^i(\boldsymbol{y})$ and $\chi_h^i(\boldsymbol{y})$ in the unit cell Y in each unit direction $\hat{\boldsymbol{e}}_i$, $i = 1, 2, 3$, with periodic boundary conditions on ∂Y. From these solutions we then find the effective relative permittivity and permeability dyadics in terms of the following averages [1, 11]:

$$\begin{cases} \boldsymbol{\epsilon}^h = <\boldsymbol{\epsilon}(\boldsymbol{y}) \cdot (\mathbf{I}_3 - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y}))> = \iiint\limits_Y \{\boldsymbol{\epsilon}(\boldsymbol{y}) - \boldsymbol{\epsilon}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_e(\boldsymbol{y})\}\, \mathrm{d}v_{\boldsymbol{y}} \\ \boldsymbol{\mu}^h = <\boldsymbol{\mu}(\boldsymbol{y}) \cdot (\mathbf{I}_3 - \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_h(\boldsymbol{y}))> = \iiint\limits_Y \{\boldsymbol{\mu}(\boldsymbol{y}) - \boldsymbol{\mu}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \boldsymbol{\chi}_h(\boldsymbol{y})\}\, \mathrm{d}v_{\boldsymbol{y}} \end{cases} \tag{4.3}$$

where $\mathbf{I}_3$ is the identity dyadic in $\mathbb{R}^3$. This procedure provides a constructive algorithm to find the homogenized parameters — solve (4.2) and compute the averages in (4.3). We also see that in general the homogenized parameters $\boldsymbol{\epsilon}^h$ and $\boldsymbol{\mu}^h$ are anisotropic dyadics, even though the material parameters $\boldsymbol{\epsilon}(\boldsymbol{y})$ and $\boldsymbol{\mu}(\boldsymbol{y})$ inside the unit cell are isotropic, *i.e.*, proportional to the identity dyadic $\mathbf{I}_3$, $\boldsymbol{\epsilon}(\boldsymbol{y}) = \epsilon(\boldsymbol{y})\mathbf{I}_3$ and $\boldsymbol{\mu}(\boldsymbol{y}) = \mu(\boldsymbol{y})\mathbf{I}_3$. Notice that the homogenized parameters in (4.3) are constant dyadics in Ω. It can be proven that if $\boldsymbol{\epsilon}$ is a symmetric dyadic (reciprocal material), then $\boldsymbol{\epsilon}^h$ is also a symmetric dyadic [20]. Similar results hold for the homogenized permeability dyadic $\boldsymbol{\mu}^h$.

Moreover, it is possible to prove [20] that the averaged fields, $\boldsymbol{E}(\boldsymbol{x})$ and $\boldsymbol{H}(\boldsymbol{x})$, satisfy the Maxwell equations with the homogenized relative permittivity and permeability $\boldsymbol{\epsilon}^h$ and $\boldsymbol{\mu}^h$ given by (4.3), *i.e.*,

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{x}) = \mathrm{i}\omega\mu_0 \boldsymbol{\mu}^h \cdot \boldsymbol{H}(\boldsymbol{x}) \\ \nabla \times \boldsymbol{H}(\boldsymbol{x}) = -\mathrm{i}\omega\epsilon_0 \boldsymbol{\epsilon}^h \cdot \boldsymbol{E}(\boldsymbol{x}) \end{cases}$$

and

$$\begin{cases} \nabla \cdot \big(\boldsymbol{\mu}^h \cdot \boldsymbol{H}(\boldsymbol{x})\big) = \mathbf{0} \\ \nabla \cdot \big(\boldsymbol{\epsilon}^h \cdot \boldsymbol{E}(\boldsymbol{x})\big) = \mathbf{0} \end{cases}$$

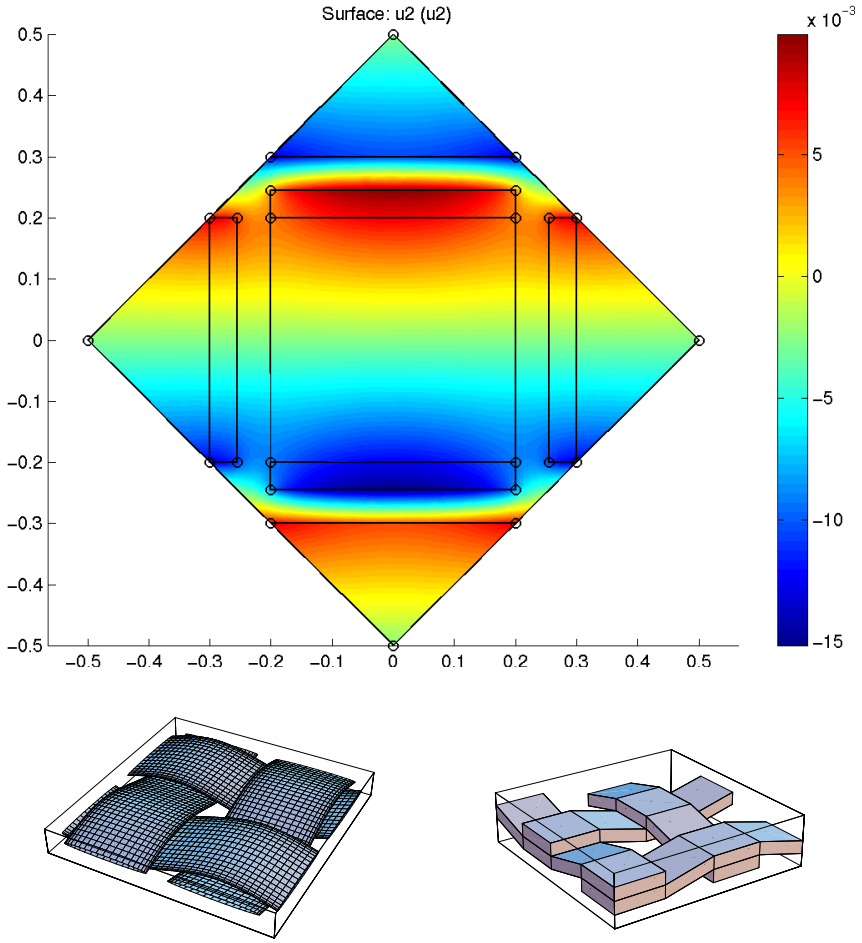

Figure 3: A numerical simulation of the local field in the unit cell (two-dimensional version) of a two-shaft glass fiber weave. The lower left figure shows the glass fiber weave and the lower right figure the computational model of this weave that is used in this numerical simulation. The upper figure shows the local field in the unit cell in a horizontal cross section of the right lower figure.

The local problem, (4.2), is solved by a suitable numerical approach, *e.g.*, FEM. We illustrate the homogenization procedure by an example that occurs in radome applications. In radomes the matrix material is usually reinforced by glass fiber to give the radome mechanical structure. The periodicity of the glass fiber is in general small compared to the wavelength of the application. Therefore, a homogenization of the complex material gives material parameters that accurately model the interaction between the material and the electromagnetic fields. The electromagnetic parameters of this homogenization are uniaxial. A numerical simulation in two dimensions of the local field in the unit cell Y of a two-shaft weave is illustrated in Figure 3. The glass fiber weave is depicted in the bottom left figure, and the model of this glass fiber is illustrated in the lower right figure. The upper figure illustrates the solution of the local problem in the unit cell. The intensity of the local field is

shown in a horizontal cross section of the unit cell.

5 Comparison with classical mixture formulae

The classical mixture formulae are well-known to the electrical engineers. They are derived under the assumption that all scattering effects can be ignored [15]. This implies that they give an approximate value of the material parameters of the homogenized material, which is most accurate at low volume fractions. This is in contrast to the exact method described in this paper, where all scattering effects are included, and which gives an accurate value of the homogenized material parameters for all volume fractions. Moreover, the classical mixture formulae apply only to certain generic form of the inclusions, while the exact homogenization procedure has very small limitations in its range of applications — both to geometry and to material components. To illustrate the differences and the similarities between these two ways of calculations, we make a comparison between the procedures in this section. All the details of the analysis can be found in [6].

The classical mixture formulae apply to homogeneous, spherical or ellipsoidal inclusions in a homogeneous, isotropic or anisotropic background material [15]. Therefore, let the material consist of two homogeneous materials; an isotropic background material with relative permittivity $\epsilon_{\rm b}$, and periodically arranged spherical inclusions with relative permittivity $\epsilon_{\rm i}$. We denote the radius of the sphere by a, and all materials are non-magnetic, $\mu = 1$. The volume fraction of the inclusions is then $f = 4\pi a^3/3$.

Several of the effective relative permittivity expressions of a mixture of homogeneous spherical inclusions are represented in the formula [15]

$$\frac{\epsilon^h - \epsilon_{\rm b}}{\epsilon^h + 2\epsilon_{\rm b} + \nu(\epsilon^h - \epsilon_{\rm b})} = f\frac{\epsilon_{\rm i} - \epsilon_{\rm b}}{\epsilon_{\rm i} + 2\epsilon_{\rm b} + \nu(\epsilon^h - \epsilon_{\rm b})}$$

where ϵ_h is the effective relative permittivity of the mixture. The integer ν represents different mixture formulas, *e.g.*, $\nu = 0$ gives the Maxwell Garnett formula, $\nu = 2$ gives the Böttcher mixture rule or Bruggeman formula, and $\nu = 3$ gives the coherent potential (CP) formula. The Maxwell Garnett formula is explicitly given by

$$\epsilon^h = \epsilon_{\rm b} + \frac{3f\epsilon_{\rm b}(\epsilon_{\rm i} - \epsilon_{\rm b})}{\epsilon_{\rm i} + 2\epsilon_{\rm b} - f(\epsilon_{\rm i} - \epsilon_{\rm b})} \tag{5.1}$$

Another mixture formula was derived by Lord Rayleigh and is given by [4, 7–10, 12, 13, 15]

$$\epsilon^h = \epsilon_{\rm b} + \frac{3f\epsilon_{\rm b}}{(\epsilon_{\rm i} + 2\epsilon_{\rm b})/(\epsilon_{\rm i} - \epsilon_{\rm b}) - f - 1.305f^{10/3}(\epsilon_{\rm i} - \epsilon_{\rm b})/(\epsilon_{\rm i} + 4\epsilon_{\rm b}/3)}$$

which for small volume fraction f can be expanded as

$$\epsilon^h = \epsilon_{\rm b} + \frac{3f\epsilon_{\rm b}(\epsilon_{\rm i} - \epsilon_{\rm b})}{\epsilon_{\rm i} + 2\epsilon_{\rm b} - f(\epsilon_{\rm i} - \epsilon_{\rm b})} + 9f\epsilon_{\rm b}\frac{155a^{10}(\epsilon_{\rm i} - \epsilon_{\rm b})^3}{(3\epsilon_{\rm i} + 4\epsilon_{\rm b})\left[\epsilon_{\rm i} + 2\epsilon_{\rm b} - f(\epsilon_{\rm i} - \epsilon_{\rm b})\right]^2} + \dots$$

ν	β
0	$\alpha^2/(3\epsilon_\mathrm{b})$
2	$\alpha^2\epsilon_\mathrm{i}/(\epsilon_\mathrm{b}(\epsilon_\mathrm{i}+2\epsilon_\mathrm{b}))$
3	$\alpha^2(4\epsilon_\mathrm{i}-\epsilon_\mathrm{b})/(3\epsilon_\mathrm{b}(\epsilon_\mathrm{i}+2\epsilon_\mathrm{b}))$
Rayleigh	$\alpha^2/(3\epsilon_\mathrm{b})$

Table 2: The different coefficients in an expansion of $\epsilon^h = \epsilon_\mathrm{b} + \alpha f + \beta f^2$ for small volume fractions f and different mixture formulae. The constant $\nu = 0$ for the Maxwell Garnett formula, $\nu = 2$ for the Böttcher mixture rule or Bruggeman formula, and $\nu = 3$ for the coherent potential (CP) formula. The coefficient $\alpha = 3\epsilon_\mathrm{b}(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})/(\epsilon_\mathrm{i} + 2\epsilon_\mathrm{b})$.

Notice that the first term in this expansion is the Maxwell Garnett formula, (5.1).

Some of these classical mixture formulae are derived for a random distribution of spheres, but they are nevertheless often applied to a regular lattice problem.

All classical mixture formulae have their domain of validity for small volume fractions f. The differences between the mixture formulae are best seen from the power series expansion in f, *i.e.*,

$$\epsilon^h = \epsilon_\mathrm{b} + \alpha f + \beta f^2 + O(f^3)$$

The coefficient $\alpha = 3\epsilon_\mathrm{b}(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})/(\epsilon_\mathrm{i} + 2\epsilon_\mathrm{b})$ are present in all these formulae, and this contribution represents the dipole contribution. The β coefficients for the mixture formulas are all different and they are given in Table 2 [15, p. 164].

The heterogenous material with spherical inclusions has been solved with the complete homogenization procedure that is reviewed in this paper. The results and the details of the analysis are given in [6]. From this complete solution, it is possible to extract a small volume fraction expansion. The analysis is rather complex, and we refer the interested reader to this reference for details. It is possible to explicitly write down the complete solution of the local problem, but for comparative reasons the low volume fraction result is most illustrative. The result is for small radii a of the sphere is [6]

$$\epsilon^h = \epsilon_\mathrm{b} + \frac{3f\epsilon_\mathrm{b}(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})}{\epsilon_\mathrm{i} + 2\epsilon_\mathrm{b} - f(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})} + \frac{9f\epsilon_\mathrm{b}(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})^3 S}{(3\epsilon_\mathrm{i} + 4\epsilon_\mathrm{b})\left[\epsilon_\mathrm{i} + 2\epsilon_\mathrm{b} - f(\epsilon_\mathrm{i} - \epsilon_\mathrm{b})\right]^2} + O(a^{15})$$

where

$$S = 196a^{10}\left(4S_1 + 2S_2 - S_3\right)^2 \approx 155a^{10}$$

where the sums S_1, S_2, and S_3 explicitly are

$$S_1 = \sum_{ijk=1}^{\infty} \frac{3i^2j^2 - i^4}{(i^2+j^2+k^2)^{9/2}}, \qquad S_2 = \sum_{ij=1}^{\infty} \frac{3i^2j^2 - 2i^4}{(i^2+j^2)^{9/2}}, \qquad S_3 = \sum_{i=1}^{\infty} \frac{1}{i^5}$$

Note that the first term in this exact homogenization procedure is the Maxwell Garnett formula in (5.1), and, moreover, that the Rayleigh formulae is correct if

powers of order fifteen and higher in the radius are ignored. All other mixture formulae do not give the correct behavior beyond the dipole term. Therefore, for this configuration, we see that the Maxwell Garnett (and Rayleigh) formula is the most reliable one.

6 Conclusions

In this paper, a review of the homogenization of a heterogeneous, anisotropic material is presented. The solution of the problem relies on the concept of two-scale convergence. The solution of the local problem is determined by the properties of the material on the microscopic scale, from which, by proper averages, the homogenized values of the electric and the magnetic properties of the material are found. The method provides an exact method to find the homogenized parameters, and the restrictions on geometry and materials are very small. A comparison between the classical mixture formulae and the exact homogenization shows that the classical mixture formulae are correct to dipole order, but also that they all give the incorrect higher order terms. The most accurate ones are the Maxwell Garnett and the Rayleigh formulae.

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