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“The source problem"—Transient waves propagating from internal sources in non-stationary media

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Abstract

Direct scattering of propagating transient waves originating from internal sources in non-stationary, inhomogeneous, dispersive, stratified media, is investigated. The starting point is a general, inhomogeneous, linear, first order, 2×2 system of equations. Particular solutions are obtained, as integrals of fundamental waves from point sources distributed throughout the medium. First, resolvent kernels are used to construct time dependent fundamental wave functions at the location of the point source. Wave propagators, closely related to the Green functions, at all times advance these time dependent waves into the surrounding medium. The propagator equations and the propagation of propagator kernel discontinuities along the characteristics of these equations are essential in the distributional proof, which is outlined. As an illustration, three special problems are studied; the inhomogeneous, second order wave equation, and source problems in homogeneous and time invariant media.

1 Introduction

In a previous paper [2], a time domain method for analyzing wave propagation problems in non-stationary media was presented and illustrated with numerical calculations for two model problems. In a subsequent paper [3], the mathematical analysis was further extended and generalized. The suggested technique is based upon the concept of wave splitting, in conjunction with either an invariant imbedding or a Green functions approach. It has its origin from work by Corones, Krueger, et. al. [5, 6, 8, 11]. After these first demonstrations presented in the early 1980's, continued scientific research corroborated the potential of this method. Consequently, a broad variety of direct and inverse scattering problems in time-invariant media was analyzed during the last decade, see Refs 4, 7, 12, 16–19. The same technique applies for the solution of three-dimensional inverse scattering problems, and several interesting results have been reported, see e.g. Refs 13, 30–33. In the present paper, the scattering analysis for non-stationary media is extended to include internal sources in the scattering media.

The non-stationary properties of the scattering media are characterized by rapid changes. The typical time scale of these variations is of the same order as the time needed for the wave to propagate through the slab. Hence, the coefficients of the relevant partial differential equations vary not only in space, but also in time. The literature on the solution of the direct and inverse scattering problems in non-stationary media is sparse. L.P. Nizhnik et. al. [23–25] have suggested a simple model for wave propagation in non-stationary media. An important element is the scattering operator mapping the incident wave on the reflected wave. The scattering operator is factorized into a product of two operators, from which the material parameters can be found. Their work is strictly analytical.

Interesting applications can be found within various fields, e.g. within control theory. Recently, interesting results have appeared connecting the research fields

of elastodynamics and electromagnetics. Thus, H.S. Tzou [29] describes how piezoelectric sensors can be used to control continua. Here, the methods of the present investigations for non-stationary media may prove very useful.

Another relevant field of application is the linearization of non-linear electromagnetic effects. As an example, consider a transient wave propagation problem modeled by the quasi-linear equation

$$c^2(z)\frac{\partial^2 u}{\partial z^2}(z,t) - \frac{\partial^2 u}{\partial t^2}(z,t) - f\left(u(z,t)\,,\frac{\partial u}{\partial z}(z,t)\,,\frac{\partial u}{\partial t}(z,t)\right) = 0 \tag{1.1}$$

Here, u is the dependent field variable, and $c(z)$ is the wave velocity in an inhomogeneous medium. The non-linearity of the problem appears in the last term

$$f\left(u(z,t)\,,\frac{\partial u}{\partial z}(z,t)\,,\frac{\partial u}{\partial t}(z,t)\right) \tag{1.2}$$

The non-linear term (1.2) can be linearized through a two term Taylor expansion centered around a known function $u^i(z,t)$. The result is

$$\begin{aligned} &c^2(z)\frac{\partial^2 u}{\partial z^2}(z,t) - \frac{\partial^2 u}{\partial t^2}(z,t) - f^i(z,t) - \partial_1 f^i(z,t)\,\big(u(z,t)-u^i(z,t)\big) \\ &- \partial_2 f^i(z,t)\,\left(\frac{\partial u}{\partial z}(z,t) - \frac{\partial u^i}{\partial z}(z,t)\right) \\ &- \partial_3 f^i(z,t)\,\left(\frac{\partial u}{\partial t}(z,t) - \frac{\partial u^i}{\partial t}(z,t)\right) = 0 \end{aligned} \tag{1.3}$$

The notation $f^i(z,t)$ simply means the function (1.2) evaluated at the known function $u(z,t) = u^i(z,t)$. Similarly, $\partial_k f^i(z,t),\ k = 1,2,3$, are the partial derivatives of f with respect to u, $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial t}$, respectively, evaluated at $(u^i, \frac{\partial u^i}{\partial z}, \frac{\partial u^i}{\partial t})$. Observe that equation (1.3) is inhomogeneous with source terms related to the known center function $u^i(z,t)$ and with coefficients $\partial_k f^i(z,t),\ k = 1,2,3$, depending on both time and space. Equation (1.3) can be used to find an iterative solution to the original quasi-linear equation (1.1). The procedure is equivalent to the well-known Newton's method, see Ref. [34]. The present paper is of importance in the study of equations like (1.1).

The two previous papers concerning non-stationary media have relevance for both direct and inverse scattering problems [2, 3]. In the direct scattering problem, the material parameters are known, and the goal is to calculate the response of a known incoming field. The inverse problem, on the other hand, assumes knowledge of the incident and the scattered field (data collected exterior to the medium) and the problem is to infer information about the material parameters. The effects of internal sources on the scattered field from an inhomogeneous slab have been studied by J. Corones and Z. Sun [9]. They focus on the inverse problem, and assume source functions to be separable in space and time, in order to solve the inverse problem uniquely. In the present paper, source functions of a more general form are treated. The direct problem is emphasized.

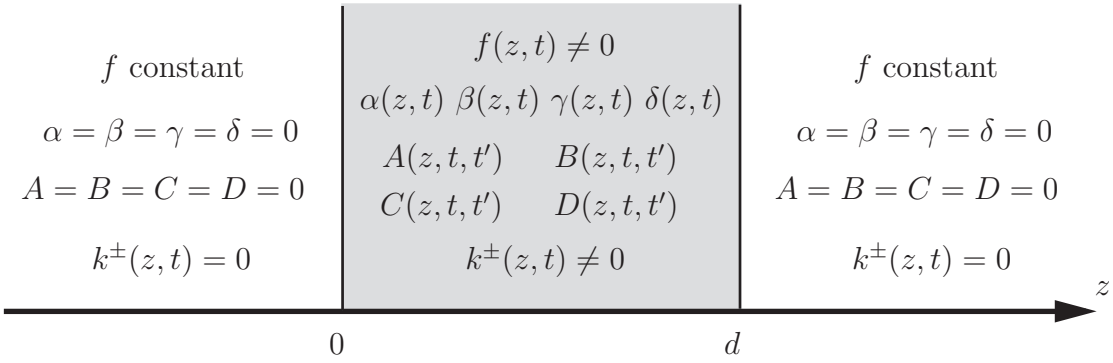

Figure 1: Geometry of the problem.

In Section 2, the geometry, the basic equations, and the boundary conditions of the underlying problem are presented. An outline of how the solutions can be obtained is given in Section 3. The fundamental solutions are treated separately in Section 4, and three special problems which may be solved with the presented technique, are discussed in Section 5. The paper ends with an appendix, containing the propagator equations together with the relevant initial and boundary conditions.

2 Presentation of the problem

The general setting of the wave propagation problems considered here, is similar to the one used in an earlier paper [3]. For convenience it is repeated. The scattering medium which is inhomogeneous and non-stationary, is confined to a slab with thickness d, see Figure 1. The slab is surrounded by a lossless, homogeneous and time-invariant medium. In the previous paper, there are exterior sources to the left of the slab. They generate uniform waves impinging at $z = 0$. The direction of incidence is perpendicular to the slab. The analysis can easily be extended to include exterior sources to the right of the slab. In the present analysis, all exterior sources are omitted and replaced by interior sources, represented by the functions $k^{\pm}(z,t)$. The solution of a general problem, with both exterior and interior sources, can then be obtained after superposition of the waves originating from different kinds of sources.

In the present study, it is assumed that the parameters of the medium vary in one spatial direction, z, and in time t. The analysis starts out from a generalized form of the dynamics of the wave fields. The reason for this approach is to cover a large number of applications, not only those described by second order partial differential equations. Thus, all wave propagation problems which can be cast on the form of (2.1) are considered. Here, the concept of wave splitting plays a significant part. The original, dependent wave variables are transformed into the two wave components $u^{\pm}(z,t)$. How this can be done, is demonstrated in some examples in Ref. 3. For convenience, parts of that information is repeated in Appendix A. Outside the slab, $u^{\pm}(z,t)$ can simply be understood as the right- and leftgoing wave components,

respectively. Inside the slab, however, they are just mathematical quantities. The basic equation is the following 2×2 system of first order linear hyperbolic partial differential equations:

$$\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix} \begin{pmatrix} u^+(z,t) \\ u^-(z,t) \end{pmatrix} = \begin{pmatrix} k^+(z,t) \\ k^-(z,t) \end{pmatrix} \tag{2.1}$$

The operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, and $\mathcal{L}_4$ are

$$\begin{cases} (\mathcal{L}_1 h(z,\cdot))(t) = \dfrac{\partial h}{\partial z}(z,t) + f(z,t)\dfrac{\partial h}{\partial t}(z,t) - \alpha(z,t)h(z,t) \\ \qquad\qquad - \displaystyle\int_{-\infty}^{t} A(z,t,t')h(z,t')\,dt' \\ (\mathcal{L}_2 h(z,\cdot))(t) = -\beta(z,t)h(z,t) - \displaystyle\int_{-\infty}^{t} B(z,t,t')h(z,t')\,dt' \\ (\mathcal{L}_3 h(z,\cdot))(t) = -\gamma(z,t)h(z,t) - \displaystyle\int_{-\infty}^{t} C(z,t,t')h(z,t')\,dt' \\ (\mathcal{L}_4 h(z,\cdot))(t) = \dfrac{\partial h}{\partial z}(z,t) - f(z,t)\dfrac{\partial h}{\partial t}(z,t) - \delta(z,t)h(z,t) \\ \qquad\qquad - \displaystyle\int_{-\infty}^{t} D(z,t,t')h(z,t')\,dt' \end{cases}$$

Due to the lack of exterior sources, the relevant boundary conditions are

$$\begin{cases} u^+(0,t) = 0 \\ u^-(d,t) = 0 \end{cases}$$

Note that the dependent wave quantities are assumed to be quiescent before a fixed time. This property guarantees that all $u^{\pm}(z,t)$ vanish at $t \to -\infty$.

The geometry of the problem is shown in Figure 1. The slowness function $f(z,t)$ is a notation for

$$f(z,t) = \frac{1}{c(z,t)} \geq \epsilon > 0$$

where $c(z,t)$ is the wave front velocity. In order to model also non-stationary memory effects, integral terms have been included in the equation. These memory effects are non-local in time. In the integrals, the variable t describes the current time, whereas the variable t' is an integration variable, related to the starting time of the excitation. The system (2.1) is a strictly hyperbolic system.

The positive function $f(z,t)$ is a continuous, bounded function of the variables z and t everywhere. Furthermore, it is assumed to be constant outside the slab $(0,d)$

$$\begin{cases} f(z,t) = 1/c_0 & , z < 0 \\ f(z,t) = 1/c_d & , z > d \end{cases}$$

and continuously differentiable, with bounded derivatives, in z and t everywhere inside the slab, i.e. $(z,t) \in (0,d) \times (-\infty,\infty)$. This implies that $f(0,t) = 1/c_0$ and $f(d,t) = 1/c_d$ for all times t.

The functions $\alpha(z,t)$, $\beta(z,t)$, $\gamma(z,t)$ and $\delta(z,t)$ are equal to zero outside the slab and they are continuous, bounded functions inside the slab (not necessarily continuous at the edges of the slab).

The functions $A(z,t,t')$, $B(z,t,t')$, $C(z,t,t')$ and $D(z,t,t')$ are always zero outside of the slab $(0,d)$. Due to causality, they vanish identically inside the slab for $t < t'$. For simplicity, the functions $A(z,t,t')$, $B(z,t,t')$, $C(z,t,t')$ and $D(z,t,t')$ are assumed continuous and bounded as functions of the variables z, t and t' in the region $t > t'$, $0 < z < d$.

The source functions $k^{\pm}(z,t)$ are integrable functions in the domain $(z,t) \in (0,d) \times (-\infty,\infty)$. Outside the slab, they vanish identically.

The assumptions described above can, of course, be relaxed and the results presented in this paper then hold for a larger class of parameters. However, the purpose of this paper is not to formulate the results for the weakest set of assumptions, but to exploit the potential of the method for a set of physically reasonable assumptions.

3 Outline of the solution

The main procedure to obtain the solutions of (2.1) is the following:

First, in (2.1) put $k^{-}(z,t) = 0$

$$\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix} \begin{pmatrix} {u_1}^{+}(z,t) \\ {u_1}^{-}(z,t) \end{pmatrix} = \begin{pmatrix} k^{+}(z,t) \\ 0 \end{pmatrix} \tag{3.1}$$

and evaluate the solutions ${u_1}^{\pm}(z,t)$.

Second, in (2.1) put $k^{+}(z,t) = 0$

$$\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix} \begin{pmatrix} {u_2}^{+}(z,t) \\ {u_2}^{-}(z,t) \end{pmatrix} = \begin{pmatrix} 0 \\ k^{-}(z,t) \end{pmatrix} \tag{3.2}$$

and calculate the solutions ${u_2}^{\pm}(z,t)$.

Due to linearity, the solutions of (3.1) and (3.2) can be superposed. Thus, (2.1) is solved by

$$\begin{cases} u^{+}(z,t) = {u_1}^{+}(z,t) + {u_2}^{+}(z,t) \\ u^{-}(z,t) = {u_1}^{-}(z,t) + {u_2}^{-}(z,t) \end{cases} \tag{3.3}$$

The method to find the solutions ${u_1}^{\pm}(z,t)$ of (3.1) is the conventional Green functions technique, e.g. see Refs [22, 27]. In the conventional technique, the Green functions (${E_1}^{\pm}$ and ${E_2}^{\pm}$ below) are solutions of fundamental equations. In order to avoid confusion, it should immediately be pointed out that the Green functions approach mentioned in Section 1 is a different technique. It is a method to solve direct and inverse scattering problems in the time domain. It was first used by Krueger and Ochs in 1988 [20], and generalized to non-stationary media in Ref. 3. In this paper the Krueger and Ochs method is even further extended, and the corresponding Green functions are the propagator kernels $G^{\pm}$ and $Q^{\pm}$, see Appendix B. The name "Green functions" was chosen, because the Green functions of the Krueger and Ochs method map the excitation at the boundary to the interior of the medium, which

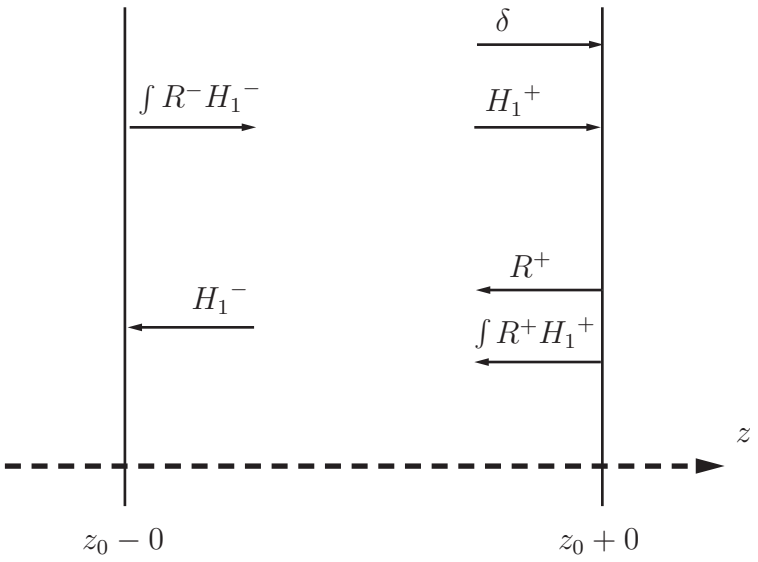

Figure 2: Initiation of fundamental waves.

is also a property of the conventional Green functions. Thus, let the set of Green functions $E_1{}^{\pm}(z,t;z_0,t_0)$ be the fundamental solutions of

$$\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix} \begin{pmatrix} E_1{}^+(z,t;z_0,t_0) \\ E_1{}^-(z,t;z_0,t_0) \end{pmatrix} = \begin{pmatrix} \delta(z-z_0)\delta(t-t_0) \\ 0 \end{pmatrix} \tag{3.4}$$

where $0 < z < d$, with the boundary conditions

$$\begin{cases} E_1{}^+(0,t;z_0,t_0) = 0 \\ E_1{}^-(d,t;z_0,t_0) = 0 \end{cases}$$

The fundamental solutions $E_1{}^{\pm}(z,t;z_0,t_0)$ can now be understood as the waves arising in the slab after it has been exposed to a rightgoing impulse introduced at the space-time point (z_0,t_0). Figure 2 shows how these waves are initiated. The medium to the right of $z = z_0$, i.e. in $z > z_0$, immediately responds to the impulse δ by sending out a leftgoing reflected wave (labeled R^+) from $z = z_0$. Now, the medium to the left of $z = z_0$ is excited. A rightgoing reflected wave is sent out from $z = z_0$, and again, the medium to the right of $z = z_0$ responds and sends out a leftgoing wave from $z = z_0$, which is added to the leftgoing immediate response of the δ-function mentioned above. Thus, with exception of the δ-pulse, all waves at $z = z_0$ have their origin in the reaction of the medium to the impulse. In Section 4, it will be seen that all responding waves can be represented by regular functions. For the moment, let $H_1{}^-$ be the notation of the sum of the leftgoing regular waves at $z = z_0$, and let $\int R^- H_1{}^-$ be a symbol of the reaction from the medium in $z < z_0$ to $H_1{}^-$. Similarly, $H_1{}^+$ is the sum of the rightgoing regular waves, and $\int R^+ H_1{}^+$ is a symbol of the response from the medium in $z > z_0$ to $H_1{}^+$.

The wave fronts of the two resulting wave solutions $E_1{}^+(z_0+0,t;z_0,t_0)$ and $E_1{}^-(z_0-0,t;z_0,t_0)$ are bound to move along the characteristics $\tau^+(z;z_0,t)$ and

$\tau^-(z;z_0,t)$, respectively. Note that $\tau^+(z;z_0,t)$ is a characteristic trace of the upper u^+-equation in (2.1). In Ref. 3, a special section is reserved for discussion of these characteristics. The function $\tau^+ = \tau^+(z;z_0,t)$ is a solution to the differential equation

$$\frac{\mathrm{d}\tau^+}{\mathrm{d}z}(z;z_0,t) = f(z,\tau^+(z;z_0,t)) \tag{3.5}$$

with the initial condition (the curve passes through the point (z_0,t))

$$\tau^+(z_0;z_0,t) = t$$

Another expression needed in the derivation of the results in this paper is

$$\frac{\partial\tau^+}{\partial z_0}(z;z_0,t) + f(z_0,t)\frac{\partial\tau^+}{\partial t}(z;z_0,t) = 0 \tag{3.6}$$

This expression has been proved in Ref. 3. Likewise, the function $\tau^- = \tau^-(z;z_0,t)$ which is a characteristic trace of the lower u^--equation in (2.1), is a solution to the differential equation

$$\frac{\mathrm{d}\tau^-}{\mathrm{d}z}(z;z_0,t) = -f(z,\tau^-(z;z_0,t)) \tag{3.7}$$

and the initial condition

$$\tau^-(z_0;z_0,t) = t$$

In analogy to (3.6) the expression

$$\frac{\partial\tau^-}{\partial z_0}(z;z_0,t) - f(z_0,t)\frac{\partial\tau^-}{\partial t}(z;z_0,t) = 0$$

holds for $\tau^-(z;z_0,t)$. Figure 3 illustrates some typical characteristic traces $\tau^\pm(z;z_0,t)$ for different t. They are the paths along which the wave fronts of the fundamental solutions ${E_1}^\pm(z_0 \pm 0,t;z_0,t_0)$ propagate through the slab.

The two propagator expressions below have been derived from Duhamel's integral [10]. They are equivalent to the expressions defining the Green functions, see Ref. 3, Section 5.

For waves propagated to the right, i.e. $z > z_0$, they are

$$\begin{cases} E^+(z,t;z_0,t_0) = p^+(z_0,z,\tau^+(z_0;z,t))E^+(z_0,\tau^+(z_0;z,t);z_0,t_0) \\ \qquad + \displaystyle\int_{-\infty}^{\tau^+(z_0;z,t)} G^+(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t')E^+(z_0,t';z_0,t_0)\,dt' \\ E^-(z,t;z_0,t_0) = \\ \qquad \displaystyle\int_{-\infty}^{\tau^+(z_0;z,t)} G^-(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t')E^+(z_0,t';z_0,t_0)\,dt' \end{cases} \tag{3.8}$$

with the wave front factor defined as

$$p^+(z_0,z,t) = \exp\left\{\int_{z_0}^{z} \alpha(\zeta,\tau^+(\zeta;z_0,t))\,d\zeta\right\}$$

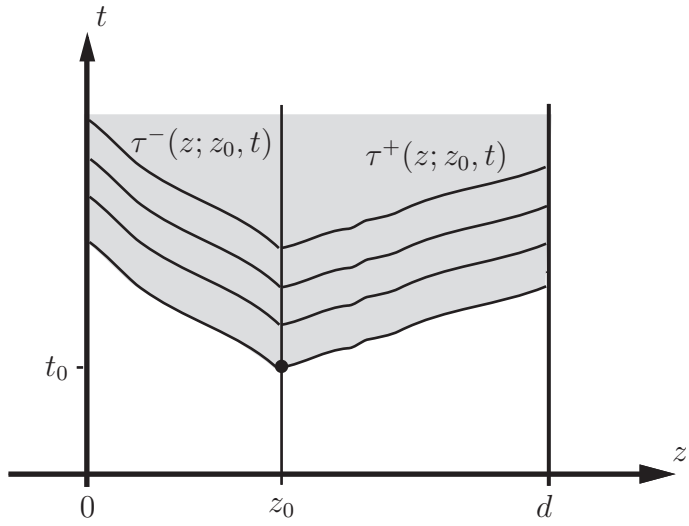

Figure 3: Geometry for calculation of fundamental solutions.

and for waves propagated to the left, i.e. $z < z_0$, they are

$$\begin{cases} E^-(z,t;z_0,t_0) = p^-(z_0,z,\tau^-(z_0;z,t))E^-(z_0,\tau^-(z_0;z,t);z_0,t_0) \\ \qquad + \displaystyle\int_{-\infty}^{\tau^-(z_0;z,t)} Q^-(z_0,z,\tau^-(z_0;z,t),t')p^-(z_0,z,t')E^-(z_0,t';z_0,t_0)\,dt' \\ E^+(z,t;z_0,t_0) = \\ \qquad \displaystyle\int_{-\infty}^{\tau^-(z_0;z,t)} Q^+(z_0,z,\tau^-(z_0;z,t),t')p^-(z_0,z,t')E^-(z_0,t';z_0,t_0)\,dt' \end{cases} \tag{3.9}$$

with the wave front factor defined as

$$p^-(z_0,z,t) = \exp\left\{\int_{z_0}^{z} \delta(\zeta,\tau^-(\zeta;z_0,t))\,d\zeta\right\}$$

The propagator kernels $G^\pm(z_0,z,t,t')$ and $Q^\pm(z_0,z,t,t')$ are generalizations of the Green functions $G^\pm(z,t,t')$ in Ref. 3. They are solutions to initial-boundary value problems equivalent to those of the Green functions. Wave propagators have also been used in other contexts; recent results can be found in Ref. 14, 15. The propagator equations, relevant in this paper, can be found in Appendix B.

The two propagator expressions (3.8) and (3.9) can now be used to propagate ${E_1}^+(z_0+0,t;z_0,t_0)$ into the region $z > z_0$, and ${E_1}^-(z_0-0,t;z_0,t_0)$ into the region $z < z_0$, respectively. Note that subscript r is used below to denote fundamental waves propagating to the right into $z > z_0$, and subscript l is the index of fundamental waves propagating to the left into $z < z_0$.

The solutions $u_1^\pm(z,t)$ of system (3.1) can now be obtained from the fundamental

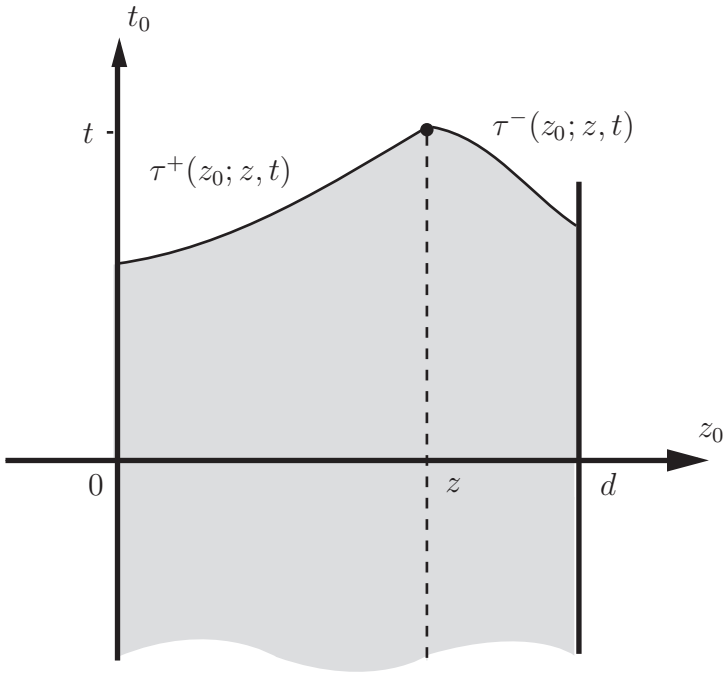

Figure 4: Calculation of ${u_1}^{\pm}(z,t)$.

waves. They are

$$\left\{\begin{aligned} {u_1}^{+}(z,t) &= \int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} E_{1r}^{+}(z,t;z_0,t_0) k^+(z_0,t_0)\, dt_0 \right\} dz_0 \\ &\quad + \int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} E_{1l}^{+}(z,t;z_0,t_0) k^+(z_0,t_0)\, dt_0 \right\} dz_0 \\ {u_1}^{-}(z,t) &= \int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} E_{1r}^{-}(z,t;z_0,t_0) k^+(z_0,t_0)\, dt_0 \right\} dz_0 \\ &\quad + \int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} E_{1l}^{-}(z,t;z_0,t_0) k^+(z_0,t_0)\, dt_0 \right\} dz_0 \end{aligned}\right. \tag{3.10}$$

The area of integration is shown in Figure 4. Due to causality, only source points (z_0, t_0) located within the shaded area (domain of dependence), contribute to the wave field in (z,t). Observe that the characteristic curves $t_0 = \tau^{\pm}(z_0; z, t)$ actually belong to the area of integration. This is essential, since, the δ-functions appearing in the final expressions (4.16) and (4.19) otherwise get lost. The proof that the expressions (3.10) are solutions of the set (3.1) is straightforward, provided it has already been proved that ${E_1}^{\pm}$ are weak solutions of the set (3.4), see Section 4. The upper limits of the integrals in (3.10) may be extended to infinity, since ${E_1}^{\pm}$ vanish beyond the wave front.

Equivalent methods can be used to calculate the solutions ${u_2}^{\pm}(z,t)$ of the system

(3.2). The fundamental solutions $E_2{}^{\pm}(z,t;z_0,t_0)$ of the system

$$\begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix} \begin{pmatrix} E_2{}^{+}(z,t;z_0,t_0) \\ E_2{}^{-}(z,t;z_0,t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ \delta(z-z_0)\delta(t-t_0) \end{pmatrix} \tag{3.11}$$

where $0 < z < d$, with the boundary conditions

$$\begin{cases} E_2{}^{+}(0,t;z_0,t_0) = 0 \\ E_2{}^{-}(d,t;z_0,t_0) = 0 \end{cases}$$

can easily be understood as the response of the medium to a leftgoing impulse introduced at $z = z_0$, $t = t_0$. The solutions $u_2{}^{\pm}(z,t)$ are now given by

$$\begin{cases} u_2{}^{+}(z,t) = \displaystyle\int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} E_{2r}^{+}(z,t;z_0,t_0) k^{-}(z_0,t_0)\, dt_0 \right\} dz_0 \\ \qquad\qquad + \displaystyle\int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} E_{2l}^{+}(z,t;z_0,t_0) k^{-}(z_0,t_0)\, dt_0 \right\} dz_0 \\ u_2{}^{-}(z,t) = \displaystyle\int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} E_{2r}^{-}(z,t;z_0,t_0) k^{-}(z_0,t_0)\, dt_0 \right\} dz_0 \\ \qquad\qquad + \displaystyle\int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} E_{2l}^{-}(z,t;z_0,t_0) k^{-}(z_0,t_0)\, dt_0 \right\} dz_0 \end{cases} \tag{3.12}$$

4 Fundamental solutions

A very important step in our efforts to find the solutions of the equations (2.1) is to derive the solutions $E_1{}^{\pm}(z,t;z_0,t_0)$ and $E_2{}^{\pm}(z,t;z_0,t_0)$ of the fundamental systems (3.4) and (3.11). As already mentioned, this investigation is a continuation of the work already published in the Refs 2,3. Together, they aim at determining the total wave response in a slab exposed to external and internal sources. It is advantageous from both the analytical and the numerical points of view, if the two parts have some common features. The propagator expressions (3.8) and (3.9) are generalizations of the Green functions operators of Ref. 3. The same expressions can therefore be used to propagate waves from both external and internal sources to all parts of the slab. It is not claimed that the method suggested here is the only practical one, but it works well in this larger context.

It should also be pointed out that the single aim of the first three parts of this section, is to find plausible expressions, so the discussion does not claim to be mathematically complete. The fact, that these expressions really are weak solutions of (3.4) and (3.11), then has to be verified in a proof which is correct also from a distributional point of view. Since this proof is quite extensive, it cannot be completely accounted for here. However, an outline of the proof and a discussion of some pertinent features are carried through in the last subsection of this section.

4.1 Resolvent equations

Prior to getting involved in the main discussion, some details should be sorted out. They all concern a method which will be resorted to frequently, namely the use of resolvent kernels and equations.

Therefore, consider a time-dependent function $g(t)$, square integrable in $t \in [t_0, T]$. Assume that $g(t)$ is related to a function $f(t)$ in a way which in $t \in [t_0, T]$ can be described by a Volterra equation of the second kind

$$g(t) = f(t) + \int_{t_0}^{t} K(t,t')f(t')\,dt' \tag{4.1}$$

The kernel $K(t,t')$ is assumed to be square integrable in $(t,t') \in [t_0,T] \times [t_0,T]$. By causality $K(t,t') = 0$, $t < t'$, which explains the upper limit of the integral.

The Volterra equation (4.1) has a unique solution given by

$$f(t) = g(t) + \int_{t_0}^{t} P(t,t')g(t')\,dt' \tag{4.2}$$

The resolvent kernel $P(t,t')$ is square integrable in $(t,t') \in [t_0,T] \times [t_0,T]$ and uniquely related to $K(t,t')$ through the integral equation

$$P(t,t') + K(t,t') + \int_{t'}^{t} K(t,t'')P(t'',t')\,dt'' = 0 \tag{4.3}$$

Furthermore, the kernels K and P satisfy the relation of symmetry

$$\int_{t'}^{t} K(t,t'')P(t'',t')\,dt'' = \int_{t'}^{t} P(t,t'')K(t'',t')\,dt'' \tag{4.4}$$

Conversely, if (4.2) is the Volterra equation, and $f(t)$ is a square integrable function in $t \in [t_0,T]$, then (4.2) has one and only one solution which has the form of (4.1).

A stringent proof of these conclusions is given in Ref. 28, for functions belonging to the class L_2. From the proof building on iterated kernels, it also follows that the solutions $f(t)$ and $g(t)$ of the Volterra equations (4.1) and (4.2) are square integrable functions in $t \in [t_0,T]$. Additional information of Volterra equations can be found in Ref. 21.

In the sequel, the kernel functions of interest, and their corresponding resolvent kernels are

$$K^+(z_0,t,t') = -\int_{t'}^{t} R^+(z_0,t,t'')R^-(z_0,t'',t')\,dt''$$

with $P^+(z_0,t,t')$, and

$$K^-(z_0,t,t') = -\int_{t'}^{t} R^-(z_0,t,t'')R^+(z_0,t'',t')\,dt''$$

with $P^-(z_0,t,t')$.

Here, $R^-(z_0,t,t')$ and $R^+(z_0,t,t')$ are the physical reflection kernels of the two subslabs $(0,z_0)$ and (z_0,d). Note that the boundary conditions for the Green functions imply that $R^+(z_0,t,t')$ and $R^-(z_0,t,t')$ may be exchanged for $G^-(z_0,z_0,t,t')$ and $Q^+(z_0,z_0,t,t')$, respectively, see Appendix B.

If the relations (4.1) and (4.2) are applied to each pair of connected kernel and resolvent kernel functions above, the following set of relations is obtained

$$\begin{cases} f(t) + \displaystyle\int_{t_0}^{t} K^{\pm}(z_0,t,t')f(t')\,dt' = g(t) \\ g(t) + \displaystyle\int_{t_0}^{t} P^{\pm}(z_0,t,t')g(t')\,dt' = f(t) \end{cases} \tag{4.5}$$

The resolvent equations are

$$P^{\pm}(z_0,t,t') + K^{\pm}(z_0,t,t') + \int_{t'}^{t} K^{\pm}(z_0,t,t'')P^{\pm}(z_0,t'',t')\,dt'' = 0 \tag{4.6}$$

The relations of symmetry are

$$\int_{t'}^{t} K^{\pm}(z_0,t,t'')P^{\pm}(z_0,t'',t')\,dt'' = \int_{t'}^{t} P^{\pm}(z_0,t,t'')K^{\pm}(z_0,t'',t')\,dt'' \tag{4.7}$$

4.2 Fundamental solutions for $z = z_0$

Now return to the fundamental set of equations (3.4) and integrate from $z = z_0 - 0$ to $z = z_0 + 0$. This yields two equations in the variable t. These equations can be combined with the second equation of each of the sets (3.8) and (3.9), evaluated at $z = z_0$. Therefore, the first step taken in order to solve (3.4) is to find the solutions to the system of equations

$$\begin{cases} {E_1}^{+}(z_0+0,t;z_0,t_0) - {E_1}^{+}(z_0-0,t;z_0,t_0) = \delta(t-t_0) \\ {E_1}^{-}(z_0+0,t;z_0,t_0) - {E_1}^{-}(z_0-0,t;z_0,t_0) = 0 \\ {E_1}^{-}(z_0+0,t;z_0,t_0) = \displaystyle\int_{t_0}^{t} R^{+}(z_0,t,t'){E_1}^{+}(z_0+0,t';z_0,t_0)\,dt' \\ {E_1}^{+}(z_0-0,t;z_0,t_0) = \displaystyle\int_{t_0}^{t} R^{-}(z_0,t,t'){E_1}^{-}(z_0-0,t';z_0,t_0)\,dt' \end{cases} \tag{4.8}$$

Here ${E_1}^{\pm}(z_0 \mp 0,t;z_0,t_0)$ can be eliminated and the resulting equations may be reorganized into

$$\begin{cases} {E_1}^{+}(z_0+0,t;z_0,t_0) - \displaystyle\int_{t_0}^{t} \left\{ {E_1}^{+}(z_0+0,t';z_0,t_0) \int_{t'}^{t} R^{-}(z_0,t,t'')R^{+}(z_0,t'',t')\,dt'' \right\} dt' \\ \qquad\qquad = \delta(t-t_0) \\ {E_1}^{-}(z_0-0,t;z_0,t_0) - \displaystyle\int_{t_0}^{t} R^{+}(z_0,t,t'){E_1}^{+}(z_0+0,t';z_0,t_0)\,dt' = 0 \end{cases} \tag{4.9}$$

The kernel function $K^-(z_0,t,t')$ is recognized in the first equation, and the solutions of (4.9) are obtained after comparison with (4.5). For $t \geq t_0$, they are

$$\begin{cases} {E_1}^+(z_0+0,t;z_0,t_0) = \delta(t-t_0) + P^-(z_0,t,t_0) \\ {E_1}^-(z_0-0,t;z_0,t_0) = R^+(z_0,t,t_0) + \displaystyle\int_{t_0}^{t} R^+(z_0,t,t')P^-(z_0,t',t_0)\,dt' \end{cases} \tag{4.10}$$

Note that ${E_1}^\pm(z_0 \pm 0,t;z_0,t_0)$ vanish for $t < t_0$.

For completeness, it should be mentioned that there is an alternative way to formulate the solutions of (4.9). The equations (4.9) can be rearranged into the equivalent set

$$\begin{cases} {E_1}^-(z_0-0,t;z_0,t_0) - \displaystyle\int_{t_0}^{t} \left\{ {E_1}^-(z_0-0,t';z_0,t_0) \int_{t'}^{t} R^+(z_0,t,t'')R^-(z_0,t'',t')\,dt'' \right\} dt' \\ \qquad\qquad = R^+(z_0,t,t_0) \\ {E_1}^+(z_0+0,t;z_0,t_0) - \displaystyle\int_{t_0}^{t} R^-(z_0,t,t'){E_1}^-(z_0-0,t';z_0,t_0)\,dt' = \delta(t-t_0) \end{cases} \tag{4.11}$$

The procedure above can be repeated. Hereby, the kernel superscripts $\pm$ are interchanged, implying that the relevant choice of equations in the relation (4.5) are the upper ones. For $t \geq t_0$, the alternative expressions are

$$\begin{cases} {E_1}^+(z_0+0,t;z_0,t_0) = \delta(t-t_0) + \displaystyle\int_{t_0}^{t} R^-(z_0,t,t')R^+(z_0,t',t_0)\,dt' \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} \left\{ R^-(z_0,t,t') \int_{t_0}^{t'} P^+(z_0,t',t'')R^+(z_0,t'',t_0)\,dt'' \right\} dt' \\ {E_1}^-(z_0-0,t;z_0,t_0) = R^+(z_0,t,t_0) + \displaystyle\int_{t_0}^{t} P^+(z_0,t,t')R^+(z_0,t',t_0)\,dt' \end{cases} \tag{4.12}$$

A final remark is that the two sets of solutions, (4.10) and (4.12), are equivalent. The proof, which is very straightforward, needs not to be carried out in full detail here. The argument which is strictly classical, involves the relations (4.5)– (4.7), together with the unique solvability of Volterra equations of the second kind. For later use, note the conclusions

$$\begin{cases} P^-(z_0,t,t_0) - \displaystyle\int_{t_0}^{t} R^-(z_0,t,t')R^+(z_0,t',t_0)\,dt' \\ \qquad - \displaystyle\int_{t_0}^{t} \left\{ R^-(z_0,t,t') \int_{t_0}^{t'} P^+(z_0,t',t'')R^+(z_0,t'',t_0)\,dt'' \right\} dt' = 0 \\ \displaystyle\int_{t_0}^{t} R^+(z_0,t,t')P^-(z_0,t',t_0)\,dt' - \int_{t_0}^{t} P^+(z_0,t,t')R^+(z_0,t',t_0)\,dt' = 0 \end{cases} \tag{4.13}$$

The solutions of the fundamental system (3.11) can be derived in an analogous way. The equations obtained by this procedure are similar to the ones already

presented. Some minor, but important, differences deserve attention. First of all, in the sets (4.8) and (4.9), ${E_1}^{\pm}$ is everywhere replaced by ${E_2}^{\pm}$. In (4.8), the right sides of the two first equations are interchanged. This means that the right side of the first equation equals 0, while the δ-function appears on the right side of the second equation. In (4.9), the right side of the first equation should be replaced by $-R^-(z_0,t,t_0)$, whereas the right side of the second equation turns into $-\delta(t-t_0)$. For $t \geq t_0$, the solutions are

$$\begin{cases} {E_2}^+(z_0+0,t;z_0,t_0) = -R^-(z_0,t,t_0) - \displaystyle\int_{t_0}^{t} P^-(z_0,t,t')R^-(z_0,t',t_0)\,dt' \\ {E_2}^-(z_0-0,t;z_0,t_0) = -\delta(t-t_0) - \displaystyle\int_{t_0}^{t} R^+(z_0,t,t')R^-(z_0,t',t_0)\,dt' \\ \qquad - \displaystyle\int_{t_0}^{t} \left\{ R^+(z_0,t,t') \int_{t_0}^{t'} P^-(z_0,t',t'')R^-(z_0,t'',t_0)\,dt'' \right\} dt' \end{cases} \tag{4.14}$$

Due to causality ${E_2}^{\pm}(z_0 \pm 0,t;z_0,t_0)$ vanish for $t < t_0$.

Alternative expressions for these solutions can be obtained from a system of equations similar to (4.11). In (4.11), the right side of the first equation should be replaced by $-\delta(t-t_0)$, while the δ-function in the right side of the second equation turns into 0. The solutions for $t \geq t_0$ are

$$\begin{cases} {E_2}^+(z_0+0,t;z_0,t_0) = -R^-(z_0,t,t_0) - \displaystyle\int_{t_0}^{t} R^-(z_0,t,t')P^+(z_0,t',t_0)\,dt' \\ {E_2}^-(z_0-0,t;z_0,t_0) = -\delta(t-t_0) - P^+(z_0,t,t_0) \end{cases} \tag{4.15}$$

The two sets of solutions (4.14) and (4.15) are equal, since

$$\begin{cases} \displaystyle\int_{t_0}^{t} P^-(z_0,t,t')R^-(z_0,t',t_0)\,dt' - \int_{t_0}^{t} R^-(z_0,t,t')P^+(z_0,t',t_0)\,dt' = 0 \\ P^+(z_0,t,t_0) - \displaystyle\int_{t_0}^{t} R^+(z_0,t,t')R^-(z_0,t',t_0)\,dt' \\ \qquad - \displaystyle\int_{t_0}^{t} \left\{ R^+(z_0,t,t') \int_{t_0}^{t'} P^-(z_0,t',t'')R^-(z_0,t'',t_0)\,dt'' \right\} dt' = 0 \end{cases}$$

The proof is similar to that of (4.13).

4.3 Propagation of fundamental solutions

The fundamental solution ${E_1}^+(z_0+0,t;z_0,t_0)$ in (4.10), or alternatively (4.12), may now be propagated into $z > z_0$ by means of (3.8) to get ${E_{1r}}^{\pm}(z,t;z_0,t_0)$, and the fundamental solution ${E_1}^-(z_0-0,t;z_0,t_0)$ may be propagated into $z < z_0$ using (3.9) to find ${E_{1l}}^{\pm}(z,t;z_0,t_0)$. In parallel, ${E_{2r}}^{\pm}(z,t;z_0,t_0)$ and ${E_{2l}}^{\pm}(z,t;z_0,t_0)$ can be obtained from ${E_2}^{\pm}(z_0 \pm 0,t;z_0,t_0)$ in (4.14) or alternatively (4.15). Note that ${E_1}^+(z_0+0,t;z_0,t_0)$ and ${E_2}^-(z_0-0,t;z_0,t_0)$ are sums of distributions ($\delta(t-t_0)$

and $-\delta(t-t_0)$, respectively) and regular functions (${H_1}^+(z_0,t,t_0)$ and ${H_2}^-(z_0,t,t_0)$, respectively), whereas ${E_1}^-(z_0-0,t;z_0,t_0)$ and ${E_2}^+(z_0+0,t;z_0,t_0)$ are solely regular, (${H_1}^-(z_0,t,t_0)$ and ${H_2}^+(z_0,t,t_0)$).

The propagated solutions of the fundamental system (3.4) are for $t \geq \tau^+(z;z_0,t_0)$ and $z_0 \leq z \leq d$

$$\begin{cases} {E_{1r}}^+(z,t;z_0,t_0) = p^+(z_0,z,\tau^+(z_0;z,t))\,\delta(\tau^+(z_0;z,t)-t_0) \\ \qquad\qquad + p^+(z_0,z,\tau^+(z_0;z,t)){H_1}^+(z_0,\tau^+(z_0;z,t),t_0) \\ \qquad\qquad + p^+(z_0,z,t_0)G^+(z_0,z,\tau^+(z_0;z,t),t_0) \\ \qquad\qquad + \displaystyle\int_{t_0}^{\tau^+(z_0;z,t)} G^+(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t'){H_1}^+(z_0,t',t_0)\,dt' \\ {E_{1r}}^-(z,t;z_0,t_0) = p^+(z_0,z,t_0)G^-(z_0,z,\tau^+(z_0;z,t),t_0) \\ \qquad\qquad + \displaystyle\int_{t_0}^{\tau^+(z_0;z,t)} G^-(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t'){H_1}^+(z_0,t',t_0)\,dt' \end{cases} \tag{4.16}$$

and for $t \geq \tau^-(z;z_0,t_0)$ and $0 \leq z < z_0$

$$\begin{cases} {E_{1l}}^-(z,t;z_0,t_0) = p^-(z_0,z,\tau^-(z_0;z,t)){H_1}^-(z_0,\tau^-(z_0;z,t),t_0) \\ \qquad\qquad + \displaystyle\int_{t_0}^{\tau^-(z_0;z,t)} Q^-(z_0,z,\tau^-(z_0;z,t),t')p^-(z_0,z,t'){H_1}^-(z_0,t',t_0)\,dt' \\ {E_{1l}}^+(z,t;z_0,t_0) = \\ \qquad\qquad \displaystyle\int_{t_0}^{\tau^-(z_0;z,t)} Q^+(z_0,z,\tau^-(z_0;z,t),t')p^-(z_0,z,t'){H_1}^-(z_0,t',t_0)\,dt' \end{cases} \tag{4.17}$$

Due to causality, ${E_{1r}}^\pm(z,t;z_0,t_0) = 0$, $t < \tau^+(z;z_0,t_0)$, and ${E_{1l}}^\pm(z,t;z_0,t_0) = 0$, $t < \tau^-(z;z_0,t_0)$.

The propagated solutions of the fundamental system (3.11) are for $t \geq \tau^+(z;z_0,t_0)$ and $z_0 < z \leq d$

$$\begin{cases} {E_{2r}}^+(z,t;z_0,t_0) = p^+(z_0,z,\tau^+(z_0;z,t)){H_2}^+(z_0,\tau^+(z_0;z,t),t_0) \\ \qquad\qquad + \displaystyle\int_{t_0}^{\tau^+(z_0;z,t)} G^+(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t'){H_2}^+(z_0,t',t_0)\,dt' \\ {E_{2r}}^-(z,t;z_0,t_0) = \\ \qquad\qquad \displaystyle\int_{t_0}^{\tau^+(z_0;z,t)} G^-(z_0,z,\tau^+(z_0;z,t),t')p^+(z_0,z,t'){H_2}^+(z_0,t',t_0)\,dt' \end{cases} \tag{4.18}$$

and for $t \geq \tau^-(z; z_0, t_0)$ and $0 \leq z \leq z_0$

$$\begin{cases} E_{2l}{}^-(z,t; z_0, t_0) = -p^-(z_0, z, \tau^-(z_0; z, t))\, \delta(\tau^-(z_0; z, t) - t_0) \\ \qquad + p^-(z_0, z, \tau^-(z_0; z, t)) H_2{}^-(z_0, \tau^-(z_0; z, t), t_0) \\ \qquad - p^-(z_0, z, t_0) Q^-(z_0, z, \tau^-(z_0; z, t), t_0) \\ \qquad + \displaystyle\int_{t_0}^{\tau^-(z_0;z,t)} Q^-(z_0, z, \tau^-(z_0; z, t), t') p^-(z_0, z, t') H_2{}^-(z_0, t', t_0)\, dt' \\ E_{2l}{}^+(z,t; z_0, t_0) = -p^-(z_0, z, t_0) Q^+(z_0, z, \tau^-(z_0; z, t), t_0) \\ \qquad + \displaystyle\int_{t_0}^{\tau^-(z_0;z,t)} Q^+(z_0, z, \tau^-(z_0; z, t), t') p^-(z_0, z, t') H_2{}^-(z_0, t', t_0)\, dt' \end{cases} \tag{4.19}$$

Due to causality, $E_{2r}{}^\pm(z, t; z_0, t_0) = 0$, $t < \tau^+(z; z_0, t_0)$, and $E_{2l}{}^\pm(z, t; z_0, t_0) = 0$, $t < \tau^-(z; z_0, t_0)$.

4.4 The proof—Some guidelines

It now remains to show that the suggested expressions (4.16) and (4.17) actually are solutions of (3.4) in a distributional sense, and that similarly (4.18) and (4.19) are weak solutions of (3.11). Since both of the proofs follow the same principles, it is sufficient to discuss one of them.

The notation

$$\mathbf{L}^* = \begin{pmatrix} \mathcal{L}_1{}^* & \mathcal{L}_3{}^* \\ \mathcal{L}_2{}^* & \mathcal{L}_4{}^* \end{pmatrix}$$

with

$$\begin{cases} (\mathcal{L}_1{}^* h(z, \cdot))(t) = -\dfrac{\partial h}{\partial z}(z,t) - \dfrac{\partial (f\, h)}{\partial t}(z,t) - \alpha(z,t) h(z,t) \\ \qquad\qquad - \displaystyle\int_t^\infty A(z, t', t)\, h(z, t')\, dt' \\ (\mathcal{L}_2{}^* h(z, \cdot))(t) = -\beta(z,t) h(z,t) - \displaystyle\int_t^\infty B(z, t', t)\, h(z, t')\, dt' \\ (\mathcal{L}_3{}^* h(z, \cdot))(t) = -\gamma(z,t) h(z,t) - \displaystyle\int_t^\infty C(z, t', t)\, h(z, t')\, dt' \\ (\mathcal{L}_4{}^* h(z, \cdot))(t) = -\dfrac{\partial h}{\partial z}(z,t) + \dfrac{\partial (f\, h)}{\partial t}(z,t) - \delta(z,t) h(z,t) \\ \qquad\qquad - \displaystyle\int_t^\infty D(z, t', t)\, h(z, t')\, dt' \end{cases}$$

is introduced for the formal adjoint to the matrix operator

$$\mathbf{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 \\ \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix}$$

Let

$$\mathbf{\Phi} = \begin{pmatrix} \phi^+(z,t) \\ \phi^-(z,t) \end{pmatrix}$$

where $\phi^{\pm}(z,t)$ are infinitely differentiable test functions with compact support in $(z,t) \in (0,d) \times (-\infty,\infty)$. Also, denote

$$\mathbf{E}_1 = \begin{pmatrix} {E_1}^+(z,t;z_0,t_0) \\ {E_1}^-(z,t;z_0,t_0) \end{pmatrix}$$

and

$$\boldsymbol{\delta} = \begin{pmatrix} \delta(z-z_0)\delta(t-t_0) \\ 0 \end{pmatrix}$$

Now, if the notation $< \cdot \, , \cdot >$ is used for the inner product, then $\mathbf{E}_1$ is a distributional solution of (3.4) if

$$< \mathbf{E}_1, \mathbf{L}^*\mathbf{\Phi} > = < \boldsymbol{\delta}, \mathbf{\Phi} >, \text{ for all } \mathbf{\Phi}$$

If this is written on integral form, one gets

$$\begin{aligned} &\int_{-\infty}^{\infty} \left\{ \int_0^{z_0} \left[{E_{1l}}^+ {\mathcal{L}_1}^* \phi^+ + {E_{1l}}^+ {\mathcal{L}_3}^* \phi^- + {E_{1l}}^- {\mathcal{L}_2}^* \phi^+ + {E_{1l}}^- {\mathcal{L}_4}^* \phi^- \right] dz \right\} dt \\ &+ \int_{-\infty}^{\infty} \left\{ \int_{z_0}^{d} \left[{E_{1r}}^+ {\mathcal{L}_1}^* \phi^+ + {E_{1r}}^+ {\mathcal{L}_3}^* \phi^- + {E_{1r}}^- {\mathcal{L}_2}^* \phi^+ + {E_{1r}}^- {\mathcal{L}_4}^* \phi^- \right] dz \right\} dt \\ &= \phi^+(z_0,t_0) \end{aligned} \tag{4.20}$$

The procedure is now to calculate the left side of equation (4.20) separately, therebye manipulating the terms to fit into groups of similar terms. Within each group subgroups of canceling terms can be identified. After all cancelations have been performed, one single term should remain. For equation (4.20) to be satisfied, that term must be equal to the right side of equation (4.20).

The expressions for ${E_{1r}}^{\pm}$ and ${E_{1l}}^{\pm}$ given by (4.16) and (4.17), respectively, are inserted into the left side of equation (4.20). The regular parts ${H_1}^{\pm}$ of ${E_1}^{\pm}$ are fetched from the relation (4.10). Explicitely, they are

$$\begin{cases} {H_1}^+(z_0,t,t_0) = P^-(z_0,t,t_0) \\ {H_1}^-(z_0,t,t_0) = R^+(z_0,t,t_0) + \displaystyle\int_{t_0}^{t} R^+(z_0,t,t')P^-(z_0,t',t_0)\, dt' \end{cases}$$

The operators ${\mathcal{L}_1}^*$, ${\mathcal{L}_2}^*$, ${\mathcal{L}_3}^*$ and ${\mathcal{L}_4}^*$ are given by the expressions above. The terms are manipulated to fit into one of the four double integral groups

$$\int_{-\infty}^{+\infty} \left\{ \int_0^{z_0} \phi^{\pm}(z,t) \bullet dz \right\} dt \quad \text{or} \quad \int_{-\infty}^{+\infty} \left\{ \int_{z_0}^{d} \phi^{\pm}(z,t) \bullet dz \right\} dt$$

for left or right propagated waves, respectively. Note the two possible signs in $\phi^{\pm}$. The symbol $\bullet$ is a place holder for a function in the variables z and t.

Terms which from the start involve partial derivatives of $\phi^{\pm}$ are integrated by parts. For t-derivatives this can be done only after a change of the order of integration. Together with the double integrals, the operation also yields some single integrals in t. Special care must be taken upon derivation of the propagator kernels

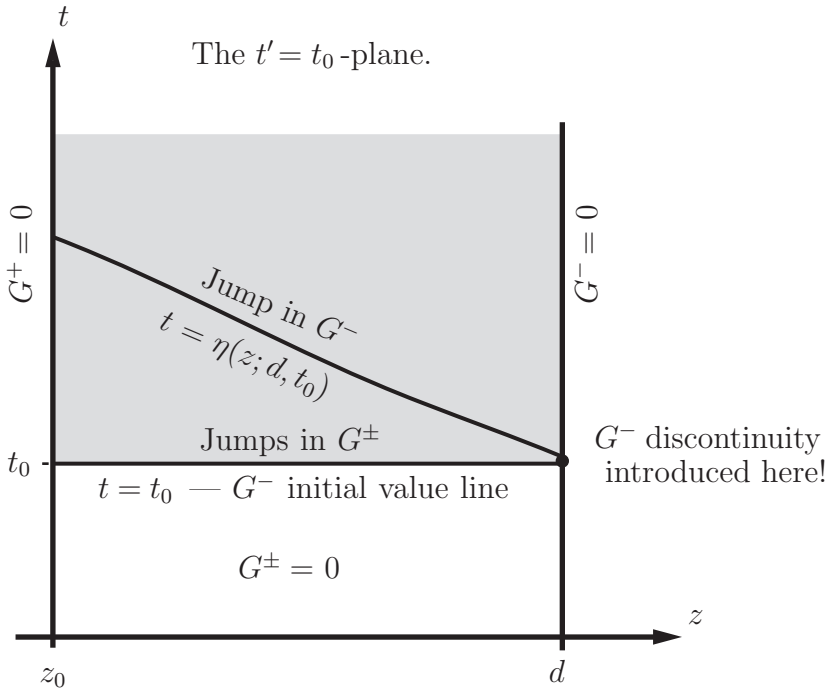

Figure 5: Discontinuities in G^+ and G^-.

$G^\pm$ and $Q^\pm$. Jumps appear in $G^\pm$ across the characteristic $t = t_0$ of equation (B.1), and in G^- across the characteristic $t = \eta(z; d, t_0)$ of (B.2), see Figure 5. The latter jump is introduced by a possible discontinuity in the values of $G^-(z_0, d, t_0, t_0)$, which below the characteristic is given by the initial condition, and above by the boundary condition $G^-(z_0, d, t_0, t_0) = 0$. Similar jumps appear in $Q^\pm$. The derivatives of the propagator kernels therefore exist only in a distributional sense. They are given by the expressions involving δ-functions.

In terms containing the integral parts of the $\mathcal{L}^*$-operators, the order of integration can be exchanged according to the pattern

$$\int_{-\infty}^{+\infty} \left\{ \int_0^{z_0} \left[f(z,t) \int_t^{+\infty} A(z,t',t)\phi^+(z,t')\, dt' \right] dz \right\} dt$$
$$= \int_{-\infty}^{+\infty} \left\{ \int_0^{z_0} \left[\phi^+(z,t) \int_{-\infty}^{t} A(z,t,t') f(z,t')\, dt' \right] dz \right\} dt$$

After these operations, the main groups of terms can be identified. Three different types of terms can be distinguished: several double integral terms, a single integral term in z originating from the δ-function term in (4.16), and some single integral terms in t. Among the double integral terms, many cancel, either directly by use of the initial condition for G^-, see Appendix B, or due to (3.6), (B.1) or (B.2). Most of the terms resulting from discontinuities cancel two by two. However, two discontinuity terms, caused by jumps in G^- across $t = t_0$, add. The resulting term cancels the only single z-integral term mentioned above. Among the t-integral terms, several vanish due to the boundary conditions $G^+(z_0, z_0, t, t_0) = 0$

and $Q^-(z_0, z_0, t, t_0) = 0$.

After left and right propagated waves have been treated independently, only a few terms remain, and it is found that t-integral terms originating from left propagated terms can be paired with terms originating from right propagated terms to cancel. At last, only one term originating from the δ-function term in (4.16), remains. After integration it yields $\phi^+(z_0, t_0)$, which is equal to the right side of equation (4.20)! Therefore ${E_1}^{\pm}(z, t; z_0, t_0)$, as given by (4.16), (4.17) and (4.10), are clearly fundamental solutions of (3.4)!

The relations (4.13) can be obtained directly from the relations (4.5) - (4.7). If the K^+-equation in (4.5) is applied directly to the identity

$$R^+(z_0, t, t_0) + \int_{t_0}^{t} P^+(z_0, t, t')R^+(z_0, t', t_0)\, dt' \\ \equiv R^+(z_0, t, t_0) + \int_{t_0}^{t} P^+(z_0, t, t')R^+(z_0, t', t_0)\, dt'$$

then P^+ can be eliminated from one side of the identity. The two sides of the resulting equality can be multiplied from the front by R^- and integrated. The aim is to obtain an equation of the same form as (4.6), with the lower choice of sign. The kernel K^+ can be substituted with its R^+R^--equivalent and after changes in the order of integration, the unique solvability of Volterra equations can be used to obtain the relations (4.13). At last, if (4.13) is substituted into (4.10), then relation (4.12) is verified.

5 Three special cases

Naturally, the general method presented in the preceding sections, is applicable to a great variety of direct scattering problems. Here, the solutions of some selected, special problems of general interest are given. The focus is on the simplifications encountered.

5.1 The inhomogeneous wave equation

Consider the generalized, inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial z^2}(z, t) - \frac{\partial^2 (f^2 u)}{\partial t^2}(z, t) + J[u] = g(z, t) \tag{5.1}$$

The linear functional $J[u]$ may consist of terms proportional to u, $\frac{\partial u}{\partial z}$ and $\frac{\partial u}{\partial t}$, but also of integral terms with the integrand linear in u. One example is the generalized wave equation in Ref. 3, with the linear functional given by

$$J[u] = \mathcal{A}(z, t)\frac{\partial u}{\partial z}(z, t) + \mathcal{B}(z, t)\frac{\partial u}{\partial t}(z, t) + \mathcal{C}(z, t)u(z, t) \tag{5.2}$$

Now, the following wave splitting, which is a transformation of the dependent variable, is applied to (5.1):

$$\begin{pmatrix} u^+(z,t) \\ u^-(z,t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -\frac{1}{f(z,t)}\partial_t^{-1} \\ 1 & \frac{1}{f(z,t)}\partial_t^{-1} \end{pmatrix} \begin{pmatrix} u(z,t) \\ \frac{\partial u}{\partial z}(z,t)) \end{pmatrix}$$

with the anti-derivative ∂_t^{-1} defined as

$$\partial_t^{-1} g(t) = \int_{-\infty}^{t} g(t')\, dt'$$

The result is a first order hyperbolic system which has the form of (2.1). For the wave equation defined by (5.2), the functions $\alpha(z,t)$, $\beta(z,t)$, $\gamma(z,t)$, $\delta(z,t)$, and $A(z,t,t')$, $B(z,t,t')$, $C(z,t,t')$, $D(z,t,t')$ can be found in Ref. 3. The simplifications, which can be observed in the solutions of (5.1), have their origin in the similarity of the source functions $k^{\pm}(z,t)$ in the right side of (2.1); they differ only in sign. They are

$$k^{\pm}(z,t) = \mp k(z,t) \tag{5.3}$$

with

$$k(z,t) = \frac{1}{2f(z,t)} \int_{-\infty}^{t} g(z,t')\, dt'$$

The solution $u(z,t)$ can be deduced from the wave splitting

$$u(z,t) = u^+(z,t) + u^-(z,t)$$

The final expressions may now be obtained from (3.3), (3.10), and (3.12), after insertion of the source functions (5.3) and the fundamental solutions given by (4.16) – (4.19). The observation

$${H_1}^{\pm}(z_0,t;z_0,t_0) - {H_2}^{\pm}(z_0,t;z_0,t_0) = D^{\mp}(z_0,t,t_0)$$

for the regular parts ${H_1}^{\pm}$ and ${H_2}^{\pm}$ of the fundamental solutions ${H_1}^{\pm}$ and ${H_2}^{\pm}$, see subsection 4.3, reduces the number of terms. The solution $u(z,t)$ of (5.1) for $z \in [0,d]$, $t \in \mathbb{R}$ may be written as a sum of right and left propagated terms

$$u(z,t) = u_r(z,t) + u_l(z,t)$$

with

$$\begin{aligned} u_r(z,t) = &- \int_0^z p^+(z_0,z,\tau^+(z_0;z,t))\, k(z_0,\tau^+(z_0;z,t))\, dz_0 \\ &- \int_0^z \left\{ p^+(z_0,z,\tau^+(z_0;z,t)) \int_{-\infty}^{\tau^+(z_0;z,t)} k(z_0,t_0)\, D^-(z_0,\tau^+(z_0;z,t),t_0)\, dt_0 \right\} dz_0 \\ &- \int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} p^+(z_0,z,t_0)\, k(z_0,t_0) \left[G^+(z_0,z,\tau^+(z_0;z,t),t_0) \right.\right. \\ &\qquad \left.\left. + G^-(z_0,z,\tau^+(z_0;z,t),t_0) \right] dt_0 \right\} dz_0 \\ &- \int_0^z \left\{ \int_{-\infty}^{\tau^+(z_0;z,t)} k(z_0,t_0) \left[\int_{t_0}^{\tau^+(z_0;z,t)} p^+(z_0,z,t') D^-(z_0,t',t_0) \right.\right. \\ &\qquad \left.\left. \left[G^+(z_0,z,\tau^+(z_0;z,t),t') + G^-(z_0,z,\tau^+(z_0;z,t),t') \right] dt' \right] dt_0 \right\} dz_0 \end{aligned}$$

and

$$
\begin{aligned}
u_l(z,t) = &- \int_z^d p^-(z_0,z,\tau^-(z_0;z,t))\, k(z_0,\tau^-(z_0;z,t))\, dz_0 \\
&- \int_z^d \left\{ p^-(z_0,z,\tau^-(z_0;z,t)) \int_{-\infty}^{\tau^-(z_0;z,t)} k(z_0,t_0)\, D^+(z_0,\tau^-(z_0;z,t),t_0)\, dt_0 \right\} dz_0 \\
&- \int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} p^-(z_0,z,t_0)\, k(z_0,t_0) \left[Q^+(z_0,z,\tau^-(z_0;z,t),t_0) \right.\right. \\
&\qquad \left.\left. + Q^-(z_0,z,\tau^-(z_0;z,t),t_0) \right] dt_0 \right\} dz_0 \\
&- \int_z^d \left\{ \int_{-\infty}^{\tau^-(z_0;z,t)} k(z_0,t_0) \left[\int_{t_0}^{\tau^-(z_0;z,t)} p^-(z_0,z,t') D^+(z_0,t',t_0) \right.\right. \\
&\qquad \left.\left. \left[Q^+(z_0,z,\tau^-(z_0;z,t),t') + Q^-(z_0,z,\tau^-(z_0;z,t),t') \right] dt' \right] dt_0 \right\} dz_0
\end{aligned}
$$

5.2 The homogeneous medium

In the homogeneous medium, all medium dependent quantities are constant with respect to the spatial variable z. Therefore, the functions f, α, β, γ and δ exclusively depend on the observation time t, and the functions A, B, C and D solely depend on the observation time t and the excitation time t'.

For the characteristic traces $\tau^\pm$, this means that the differential equations (3.5) and (3.7) turn into

$$\frac{\mathrm{d}\tau^\pm}{\mathrm{d}z} = \pm f(\tau^\pm)$$

These equations are separable, and direct integration shows that the three parameters z, z_0 and t_0 appearing in $\tau^\pm$ reduce into the two $z - z_0$ and t_0, so for the homogeneous medium, in stead of $\tau^\pm(z;z_0,t_0)$, a proper choice of notation is $\tau^\pm(z-z_0,t_0)$. As a consequence, it follows that the wave front factors $p^\pm(z_0,z,t)$ in the equations (3.8) and (3.9) simplify into $p^\pm(z-z_0,t)$, as well.

The propagator equations (B.1)–(B.4) play an important part in the analysis. In an infinite, homogeneous medium, they, too, demonstrate a $z - z_0$ -dependence, so there, the relevant notation for the propagator kernels is $G^\pm(z-z_0,t,t')$ and $Q^\pm(z-z_0,t,t')$. The advantages primarily show in the computational part. Therefore, compared to the general case, where, in principle, z_0 has to be chosen to cover all of the medium, which is quite time consuming, it now suffices to calculate $G^\pm(z_0,z,t,t')$ and $Q^\pm(z_0,z,t,t')$ for a single choice of z_0.

Further reductions in computer time and memory consumption may be expected from the simplifications occurring in Section 4. One aspect is that the coding will be more straightforward. In the general case, the data flow has to be planned carefully in order to avoid too spacious data fields. In the infinite, homogeneous medium it follows that the kernel functions $K^\pm$ along with the resolvent kernels $P^\pm$ depend on t and t'. Furthermore, the fundamental solutions $E_1^\pm(z_0 \pm 0,t;z_0,t_0)$ and $E_2^\pm(z_0 \pm 0,t;z_0,t_0)$ together with their regular parts $H_1^\pm$ and $H_2^\pm$ also depend on t and t_0 only, and, therefore, the propagated solutions $E_{1r}^\pm$, $E_{1l}^\pm$, $E_{2r}^\pm$ and $E_{2l}^\pm$ in

(4.16)–(4.19), which in the general case are 4-parameter functions (z, t, z_0 and t_0), now turn into 3-parameter functions ($z - z_0$, t and t_0). As a conclusion, note that $E_{1r}^{+}(z, t; z_0, t_0)$ in (4.16) simplifies to

$$\begin{aligned} {E_{1r}}^{+}(z - z_0, t, t_0) = {} & p^{+}(z - z_0, \tau^{+}(z - z_0, t))\, \delta(\tau^{+}(z - z_0, t) - t_0) \\ & + p^{+}(z - z_0, \tau^{+}(z - z_0, t)) {H_1}^{+}(\tau^{+}(z - z_0, t), t_0) \\ & + p^{+}(z - z_0, t_0) G^{+}(z - z_0, \tau^{+}(z - z_0, t), t_0) \\ & + \int_{t_0}^{\tau^{+}(z - z_0, t)} G^{+}(z - z_0, \tau^{+}(z - z_0, t), t') p^{+}(z - z_0, t') {H_1}^{+}(t', t_0)\, dt' \end{aligned}$$

Even for a homogeneous slab, some simplifications may occur. In Ref. 26 it is shown that in a time-invariant, homogeneous slab, the infinite medium propagators contain sufficient information to recover the propagated waves for all times. It is conjectured that this method can be generalized to include the non-stationary, homogeneous slab.

In the homogeneous medium, the characteristic traces may easily be straightened out, since the slowness function f only depends on t. For more details, see Ref. 3, Appendix A.

5.3 The time invariant medium

If a medium is invariant under time translations, the computational simplifications are of the same size as those obtained for the homogeneous medium. In the time invariant medium, however, the three significant parameters are z, z_0 and $t - t_0$.

First, consider the simplifications occurring in the basic set of equations (2.1). In the stationary medium, the multiplicative coefficient functions f, α, β, γ and δ depend on z only. The integral kernels A, B, C and D depend on z and on the time variable $t - t'$, which can be interpreted as the period of time that has passed since the medium was excited at $t = t'$. For the characteristic traces $\tau^{\pm}$, the governing differential equations

$$\frac{\mathrm{d}\tau^{\pm}}{\mathrm{d}z} = \pm f(z)$$

are easy to integrate.

As another consequence of the invariance under time translations, it can be noted that the wave front factors $p^{\pm}(z_0, z, t)$ become $p^{\pm}(z_0, z)$. This is so, because of the lack of time dependence in the functions α and δ.

The changes in the coefficient functions and integral kernels of the propagator equations (B.1)–(B.4) agree with those already described for the basic equations (2.1). This means that again the two time variables t and t' are replaced by the single variable $t - t'$, so the simplified notations of the propagator kernels are $G^{\pm}(z_0, z, t - t')$ and $Q^{\pm}(z_0, z, t - t')$. Therefore, compared to the general case, where a wide range of t'-values have to be chosen, it is sufficient to calculate $G^{\pm}(z_0, z, t, t')$ and $Q^{\pm}(z_0, z, t, t')$ for the single value $t' = 0$. The savings in computer time and memory are considerable.

The expressions of Section 4 also simplify. The proper notation of the integral kernels $K^{\pm}$ and the resolvent kernels $P^{\pm}$ is $K^{\pm}(z_0, t-t')$ and $P^{\pm}(z_0, t-t')$. Similarly, the fundamental solutions ${E_1}^{\pm}(z_0 \pm 0, t; z_0, t_0)$ and ${E_2}^{\pm}(z_0 \pm 0, t; z_0, t_0)$ and their regular parts $H_1^{\pm}$ and $H_2^{\pm}$ are described by the two parameters z_0 and $t - t_0$ only.

Finally, the propagated solutions $E_{1r}^{\pm}$, $E_{1l}^{\pm}$, $E_{2r}^{\pm}$ and $E_{2l}^{\pm}$ in (4.16)–(4.19) simplify in similar manners. As an example, note that ${E_{1r}}^{+}(z, t; z_0, t_0)$ in (4.16) becomes

$$\begin{aligned}
{E_{1r}}^{+}(z_0, z, t - t_0) = {} & p^{+}(z_0, z)\, \delta(\tau^{+}(z_0; z, t) - t_0) \\
& + p^{+}(z_0, z) {H_1}^{+}(z_0, \tau^{+}(z_0; z, t) - t_0) \\
& + p^{+}(z_0, z) G^{+}(z_0, z, \tau^{+}(z_0; z, t) - t_0) \\
& + p^{+}(z_0, z) \int_{t_0}^{\tau^{+}(z_0; z, t)} G^{+}(z_0, z, \tau^{+}(z_0; z, t) - t') {H_1}^{+}(z_0, t' - t_0)\, dt'
\end{aligned}$$

In the time invariant medium, further computational advantages are attained, if the characteristic traces are straightened out by a travel time coordinate transformation. The details are given in e.g. Ref. 2, Section 2.

6 Conclusions

This paper contains a continuation of the analysis of transient waves in non-stationary media. Source terms are added to the general, linear, first order 2×2 system of hyperbolic equations. Integral solutions for the direct scattering problem are obtained from fundamental waves, originating from point sources. Propagator equations are introduced. They are similar to the Green functions equations, leading to wave solutions from external sources. The propagator kernels are used to propagate fundamental wave solutions throughout the medium. A proof, showing that the suggested expressions are weak solutions of the fundamental equations, is discussed in some detail. Finally, it is shown that the presented general theory can be used to analyze source problems in homogeneous media, as well as in media invariant to time translations. The starting point must not necessarily be from a system of first order equations, second order inhomogeneous wave equations may also be analyzed.

Computer programs for numerical calculations of non-stationary media have been developed. Through careful planning of the data flow, it has been possible to restrict the storage of data to 3-dimensional data fields. From considerations of space, the discussion of the numerical implementation and the numerical results have been collected in a separate paper, see Ref. 1.

Acknowledgment

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Appendix A The coefficient functions

Consider propagation of electromagnetic waves in inhomogeneous and time-varying media. The relevant constitutive relations in this example are

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0\epsilon(z,t)\boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) \end{cases}$$

Here μ_0 is the permeability of vacuum. Observe that the permittivity $\epsilon_0\epsilon(z,t) > 0$ varies in time, as well as in space. This is a model of an inhomogeneous, non-dispersive, non-stationary medium. For the sake of simplicity, dispersive memory terms are omitted. However, the more complex model, where these integral terms are included, is as straightforward to analyze.

With the usual assumption of an electric field $\boldsymbol{E}$ that is transverse to the z-axis and that depends on z and t only, the wave equation is

$$\frac{\partial^2 E}{\partial z^2}(z,t) - \frac{\partial^2(f^2E)}{\partial t^2}(z,t) = 0 \tag{A.1}$$

where

$$f(z,t) = \sqrt{\mu_0\epsilon_0\epsilon(z,t)}$$

This equation is a special case of a more generalized wave equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial z^2}(z,t) - \frac{\partial^2(f^2u)}{\partial t^2}(z,t) \\ &\qquad + \mathcal{A}(z,t)\frac{\partial u}{\partial z}(z,t) + \mathcal{B}(z,t)\frac{\partial u}{\partial t}(z,t) + \mathcal{C}(z,t)u(z,t) = 0 \end{aligned} \tag{A.2}$$

which also has applications in, e.g. linear acoustics in media where the propagation conditions change rapidly with time.

In order to see how equation (A.2) is related to the general hyperbolic wave equation (2.1), the concept of wave splitting is introduced. The wave splitting can be defined in several different ways. The definition adopted here renders a very simple $u^{\pm}$-dynamics for the wave equation in (A.1). Thus, proceeding formally, the wave splitting is defined by the following transformation of the dependent variables:

$$\begin{pmatrix} u^+(z,t) \\ u^-(z,t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\frac{1}{f(z,t)}\partial_t^{-1} \\ 1 & \frac{1}{f(z,t)}\partial_t^{-1} \end{pmatrix}\begin{pmatrix} u(z,t) \\ \frac{\partial u}{\partial z}(z,t)) \end{pmatrix}$$

which generalizes the wave splitting introduced in Ref [6]. The new fields $u^{\pm}(z,t)$ satisfy a first order 2×2 system of hyperbolic partial differential equations, which is identical to the generalized $u^{\pm}$-dynamics in (2.1). The explicit expressions of the coefficients are

$$\begin{cases} \alpha(z,t) = -\frac{1}{2}\frac{\partial}{\partial z}\ln f(z,t) - \frac{3}{2}\frac{\partial f}{\partial t}(z,t) - \frac{1}{2}\mathcal{A}(z,t) + \frac{1}{2}\frac{\mathcal{B}(z,t)}{f(z,t)} \\ \beta(z,t) = +\frac{1}{2}\frac{\partial}{\partial z}\ln f(z,t) - \frac{1}{2}\frac{\partial f}{\partial t}(z,t) + \frac{1}{2}\mathcal{A}(z,t) + \frac{1}{2}\frac{\mathcal{B}(z,t)}{f(z,t)} \\ \gamma(z,t) = +\frac{1}{2}\frac{\partial}{\partial z}\ln f(z,t) + \frac{1}{2}\frac{\partial f}{\partial t}(z,t) + \frac{1}{2}\mathcal{A}(z,t) - \frac{1}{2}\frac{\mathcal{B}(z,t)}{f(z,t)} \\ \delta(z,t) = -\frac{1}{2}\frac{\partial}{\partial z}\ln f(z,t) + \frac{3}{2}\frac{\partial f}{\partial t}(z,t) - \frac{1}{2}\mathcal{A}(z,t) - \frac{1}{2}\frac{\mathcal{B}(z,t)}{f(z,t)} \end{cases}$$

and

$$\begin{cases} A(z,t,t') = \frac{1}{2}\frac{1}{f(z,t)}\left[f(z,t')\frac{\partial \mathcal{A}}{\partial t'}(z,t') - \frac{\partial \mathcal{B}}{\partial t'}(z,t') + \mathcal{C}(z,t')\right] \\ B(z,t,t') = \frac{1}{2}\frac{1}{f(z,t)}\left[-f(z,t')\frac{\partial \mathcal{A}}{\partial t'}(z,t') - \frac{\partial \mathcal{B}}{\partial t'}(z,t') + \mathcal{C}(z,t')\right] \\ C(z,t,t') = -A(z,t,t') \\ D(z,t,t') = -B(z,t,t') \end{cases}$$

Appendix B Propagator equations

This Appendix contains a repetition and an extension of the results given in Section 5 of Ref. 3. The propagator kernels $G^{\pm}(z_0,z,t,t')$ of the expressions in (3.8) satisfy the following Green functions equations valid in $z_0 < z < d$, $t > t'$:

$$\begin{aligned} &\frac{\partial G^+}{\partial z}(z_0,z,t,t') - \alpha(z,\tau^+(z;z_0,t))G^+(z_0,z,t,t') + G^+(z_0,z,t,t')\alpha(z,\tau^+(z;z_0,t')) \\ &\quad - \beta(z,\tau^+(z;z_0,t))G^-(z_0,z,t,t') - A(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t'))\,\frac{\partial \tau^+}{\partial t'}(z;z_0,t') \\ &\quad - \int_{t'}^{t} A(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t''))\,\frac{\partial \tau^+}{\partial t''}(z;z_0,t'')G^+(z_0,z,t'',t')\,dt'' \\ &\quad - \int_{t'}^{t} B(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t''))\,\frac{\partial \tau^+}{\partial t''}(z;z_0,t'')G^-(z_0,z,t'',t')\,dt'' = 0 \end{aligned} \tag{B.1}$$

and

$$\begin{aligned} &\frac{\partial G^-}{\partial z}(z_0,z,t,t') - 2f(z,\tau^+(z;z_0,t))\left(\frac{\partial \tau^+}{\partial t}(z;z_0,t)\right)^{-1}\frac{\partial G^-}{\partial t}(z_0,z,t,t') \\ &\quad - \delta(z,\tau^+(z;z_0,t))G^-(z_0,z,t,t') + G^-(z_0,z,t,t')\alpha(z,\tau^+(z;z_0,t')) \\ &\quad - \gamma(z,\tau^+(z;z_0,t))G^+(z_0,z,t,t') - C(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t'))\,\frac{\partial \tau^+}{\partial t'}(z;z_0,t') \\ &\quad - \int_{t'}^{t} C(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t''))\,\frac{\partial \tau^+}{\partial t''}(z;z_0,t'')G^+(z_0,z,t'',t')\,dt'' \\ &\quad - \int_{t'}^{t} D(z,\tau^+(z;z_0,t),\tau^+(z;z_0,t''))\,\frac{\partial \tau^+}{\partial t''}(z;z_0,t'')G^-(z_0,z,t'',t')\,dt'' = 0 \end{aligned} \tag{B.2}$$

with the initial condition

$$G^-(z_0,z,t,t')\big|_{t'=t} = -\frac{1}{2}\frac{\gamma(z,\tau^+(z;z_0,t))}{f(z,\tau^+(z;z_0,t))}\,\frac{\partial\tau^+}{\partial t}(z;z_0,t)$$

and the boundary values

$$\begin{cases} G^+(z_0,z_0,t,t') = 0 \\ G^-(z_0,d,t,t') = 0 \\ G^-(z_0,z_0,t,t') = R^+(z_0,t,t') \end{cases}$$

Note that $R^+(z_0,t,t')$ is the physical reflection kernel of a slab in position (z_0,d), see Ref. 3. In the limit $z_0=0$, the propagator kernels $G^\pm(0,z,t,t')$ are identical to the Green functions $G^\pm(z,t,t')$ in Ref. 3, and the equations (B.1) and (B.2) turn into the corresponding Green functions equations of Ref. 3.

The propagator kernels $Q^\pm(z_0,z,t,t')$ of expressions (3.8) satisfy the following Green functions equations valid in $0<z<z_0$, $t>t'$:

$$\begin{aligned} &\frac{\partial Q^-}{\partial z}(z_0,z,t,t') - \delta(z,\tau^-(z;z_0,t))Q^-(z_0,z,t,t') + Q^-(z_0,z,t,t')\delta(z,\tau^-(z;z_0,t')) \\ &\quad - \gamma(z,\tau^-(z;z_0,t))Q^+(z_0,z,t,t') - D(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t'))\,\frac{\partial\tau^-}{\partial t'}(z;z_0,t') \\ &\quad - \int_{t'}^{t} D(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t''))\,\frac{\partial\tau^-}{\partial t''}(z;z_0,t'')Q^-(z_0,z,t'',t')\,dt'' \\ &\quad - \int_{t'}^{t} C(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t''))\,\frac{\partial\tau^-}{\partial t''}(z;z_0,t'')Q^+(z_0,z,t'',t')\,dt'' = 0 \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} &\frac{\partial Q^+}{\partial z}(z_0,z,t,t') + 2f(z,\tau^-(z;z_0,t))\left(\frac{\partial\tau^-}{\partial t}(z;z_0,t)\right)^{-1}\frac{\partial Q^+}{\partial t}(z_0,z,t,t') \\ &\quad - \alpha(z,\tau^-(z;z_0,t))Q^+(z_0,z,t,t') + Q^+(z_0,z,t,t')\delta(z,\tau^-(z;z_0,t')) \\ &\quad - \beta(z,\tau^-(z;z_0,t))Q^-(z_0,z,t,t') - B(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t'))\,\frac{\partial\tau^-}{\partial t'}(z;z_0,t') \\ &\quad - \int_{t'}^{t} B(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t''))\,\frac{\partial\tau^-}{\partial t''}(z;z_0,t'')Q^-(z_0,z,t'',t')\,dt'' \\ &\quad - \int_{t'}^{t} A(z,\tau^-(z;z_0,t),\tau^-(z;z_0,t''))\,\frac{\partial\tau^-}{\partial t''}(z;z_0,t'')Q^+(z_0,z,t'',t')\,dt'' = 0 \end{aligned} \tag{B.4}$$

with the initial condition

$$Q^+(z_0,z,t,t')\big|_{t'=t} = \frac{1}{2}\frac{\beta(z,\tau^-(z;z_0,t))}{f(z,\tau^-(z;z_0,t))}\,\frac{\partial\tau^-}{\partial t}(z;z_0,t)$$

and the boundary values

$$\begin{cases} Q^-(z_0,z_0,t,t') = 0 \\ Q^+(z_0,0,t,t') = 0 \\ Q^+(z_0,z_0,t,t') = R^-(z_0,t,t') \end{cases}$$

Here, $R^-(z_0, t, t')$ is the physical reflection kernel of a slab in position $(0, z_0)$.

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