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## GLOBAL STABILIZATION FOR A CLASS OF COUPLED NONLINEAR SYSTEMS WITH APPLICATION TO ACTIVE SURGE CONTROL

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**Abstract.** We propose here a new procedure for output feedback design for systems with nonlinearities satisfying quadratic constraints. It provides an alternative for the classical observer-based design and relies on transformation of the closed-loop system with a dynamic controller of particular structure into a special block form. We present two sets of sufficient conditions for stability of the transformed block system and derive matching conditions allowing such a representation for a particular challenging example. The two new tests for global stability proposed for a class of nonlinear systems extend the famous Circle criterion applied for infinite sector quadratic constraints. The study is motivated and illustrated by the problem of output feedback control design for the well-known finite dimensional nonlinear model qualitatively describing surge instabilities in compressors. Assuming that the only available measurement is the pressure rise, we suggest a constructive procedure for synthesis of a family of robustly globally stabilizing feedback controllers. The solution relies on structural properties of the nonlinearity of the model describing a compressor characteristic, which includes earlier known static quadratic constraints and a newly found integral quadratic constraint. Performance of the closed-loop system is discussed and illustrated by simulations.

**Keywords.** Active Surge Control, Output Feedback Control, Quadratic Constraints, Circle Criterion, Nonlinear Systems, Three-state Moore-Greitzer model, Global Stabilization.

**AMS subject classification:** 93D15 (93D21, 93C10).

## 1 Introduction

The study reported in this paper is motivated by the problem of designing globally stabilizing output feedback controllers for the following nonlinear dynamical system

$$\frac{d}{dt}\phi = -\psi + \frac{3}{2}\phi + \frac{1}{2}[1 - (1 + \phi)^3] \quad (1)$$

$$\frac{d}{dt}\psi = \frac{1}{\beta^2}(\phi - u), \quad y = \psi \quad (2)$$

Here  $\psi$  and  $\phi$  are the scalar state variables,  $u$  is the control input,  $\beta$  is a positive constant, and  $y$  is the measured output.

The variables  $\phi$  and  $\psi$  denote deviations of the averaged flow and the total-to-static pressure-rise coefficients from their nominal values, respectively;  $u$  is defined by deviation of the coefficient of the inverse throttle characteristic function from a nominal value; and  $t$  is a scaled time measured in radians of travel of the compressor wheel. For a detailed description we refer the interested reader to [27]: (1)–(2) corresponds to [27, (59)–(60) with  $J \equiv 0$ ]. The parameters of the system (1)–(2), known as the Greitzer model [16, 17], as well as the nonlinearity in (1) can be different from presented. However, it is instructive to consider the system (1)–(2): many of the arguments below can be applied for the system with modified constants and even for a different structure of the nonlinearity.

It should be noted that although sometimes both of the state variables can be assumed as outputs of the system, on-line measurements of the flow require special instrumentation and are typically not feasible. So, here, we consider the case when only the  $\psi$ -variable is available for feedback design.

Difficulties in developing feedback controllers for (1)–(2) are due to the presence of the non-globally Lipschitz (cubic) nonlinearity in the equation (1) that depends on unmeasured  $\phi$ -variable. It is easy to see that the zero dynamics [22, Section 13.2] with respect to the measured output  $y$  is unstable and so designing a globally stabilizing output feedback controller is a challenge. The key for our development is the fact that the nonlinearity satisfies certain quadratic constraints [39], which can be written in the classical form of sector conditions [22, Section 6.1].

Our search for a large family of robust globally stabilizing controllers for (1)–(2) have led us to the following methodology.

- First, the closed-loop system with a dynamic output feedback controller is transformed into a special block form. Possibility of such a transformation results in a set of matching conditions defining the structure of a family of parametrized feedback control laws formed by linear terms and special nonlinear terms defined by the original nonlinearity present in the open-loop system.
- After that, sufficient conditions for the parameters to ensure stability can be derived exploiting the fact that the nonlinearities satisfy certain quadratic constraints.

We continue with some preliminary remarks and a definition of the special instrumental representation for the closed-loop system.

### 1.1 Preliminaries: A class of dynamical systems achieved with proposed output feedback controllers

We consider a particular structured class of output feedback controllers, chosen to ensure that after an appropriate change of coordinates the closed-loop

systems can be rewritten as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} W^{\{x\}}(Cx) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} W^{\{e\}}(x, e), \quad (3)$$

where the combined components of  $x$  and  $e$  define the new state vector,  $A_{11}$ ,  $A_{12}$ ,  $A_{22}$ ,  $B_1$ ,  $B_2$ , and  $C$  are constant matrices of appropriate dimensions,  $W^{\{x\}}$  and  $W^{\{e\}}$  are static nonlinearities that resemble the nonlinearity present in the original dynamics (1)–(2).

The explicit form and the required properties of  $W^{\{x\}}(\cdot)$  and  $W^{\{e\}}(\cdot, \cdot)$  will be discussed below in Section 2 when we establish stability criteria for (3). The partial motivation for the decomposition of the state of the closed-loop system into  $x$  and  $e$  and assuming the structure (3) is the following.

- The vector  $x$  shall be composed from the measured state  $\psi$  and, possibly, some of transformed states of a dynamic feedback controller. The conditions for the stability of the  $x$ -subsystem can sometimes be interpreted as stability conditions under (*dynamic*) *state feedback controller* designed for (1)–(2) assuming that both variables  $\phi$  and  $\psi$  are measured.
- The vector  $e$  shall have a component that can sometimes be interpreted as a state of the error dynamics of a *reduced-order observer* for the unmeasured state  $\phi$ .

We will choose parameters of the controller to stabilize two systems

$$\frac{d}{dt} x = A_{11} x + B_1 W^{\{x\}}(Cx) + A_{12} e(t) \quad \text{with} \quad e(t) \equiv 0 \quad (4)$$

and

$$\frac{d}{dt} e = A_{22} e + B_2 W^{\{e\}}(x(t), e) \quad \text{with} \quad x(t) \quad \text{being an unknown signal} \quad (5)$$

separately and after that look for conditions to ensure stabilization of the whole interconnected system (3).

It is worth noting that since the  $x$ - and  $e$ -subsystems of (3) are both nonlinear and coupled, the standard *separation principle* [33, 37] or one of its extended versions, see e.g. [5, 12, 1], are not applicable. *Cascade* arguments cannot be directly applied for imposing stability of (3) either since (5) is not decoupled from the other subsystem. Note also that even without such a dependence on  $x(t)$  in many cases<sup>1</sup> the stability proofs are not easy<sup>2</sup>.

Despite our motivation in the form of observer-based design, the arguments below will not be focused on establishing new conditions for validity of

<sup>1</sup>Except for special situations like [22, Lemma 4.7].

<sup>2</sup>The difficulty can be seen from the observation that an exponentially vanishing signal  $e(t)$  can make the state of the  $x$ -subsystem unbounded even if the decoupled system is globally exponentially stable [34].

the separation principle, but will be directly aimed at proving global asymptotic stability of the closed-loop system. Nevertheless, some of the derived output feedback controllers can have the interpretation of observer-based design in a broad sense. We will return to this point in Section 6.3.

The rest of the paper is organized as follows: In Section 2 we present two tests for global asymptotic stability of (3) expressed as conditions for stability of the  $x$ - and  $e$ -subsystems and properties of the nonlinearities  $W^{\{x\}}(\cdot)$  and  $W^{\{e\}}(\cdot, \cdot)$ . In Section 3 we discuss a structure of output feedback controllers and matching conditions that allow transforming the closed-loop system into (3). We show in Section 4 how both statements can be used for description of output feedback controllers that are stabilizing for (1)–(2). Results of numerical simulations are presented in Section 5. Finally, we present some discussions in Section 6 and concluding remarks in Section 7. This Section is continued with a description of some key properties of the system (1)–(2).

### 1.2 Preliminaries: Quadratic constraints for the nonlinearity in the surge dynamics (1)–(2)

The nonlinearity of the dynamical system (1)–(2)

$$W^{\{\phi\}}(v) := 1 - (1 + v)^3 \tag{6}$$

has several useful properties presented below.

**Lemma 1** *The static nonlinearity (6) satisfies the incremental quadratic constraint (QC)*

$$e \cdot [W^{\{\phi\}}(v) - W^{\{\phi\}}(v + e)] \geq 0, \quad \forall v, e \in \mathbb{R}^1 \tag{7}$$

and the sector QC<sup>3</sup>.

$$\mathcal{G}_1 [v, W^{\{\phi\}}(v)] = W^{\{\phi\}}(v) \cdot (-v) - \frac{3}{4} v^2 \geq 0, \quad \forall v \in \mathbb{R}^1. \tag{8}$$

*Proof:* Substituting (6) into the left-hand side of (7) we obtain

$$\begin{aligned} e \left[ \{1 - (1 + v)^3\} - \{1 - (1 + v + e)^3\} \right] &= e [(v + e + 1)^3 - (v + 1)^3] \\ &= e^2 \left[ (v + 1)^2 + (v + e + 1)^2 + (v + 1)(v + e + 1) \right] \geq 0 \end{aligned}$$

The second constraint can be derived as follows.

$$\begin{aligned} \mathcal{G}_1 [v, W^{\{\phi\}}(v)] &= W^{\{\phi\}}(v) \cdot (-v) - \frac{3}{4} v^2 = [1 - (1 + v)^3] \cdot (-v) - \frac{3}{4} v^2 \\ &= [3 + 3v + v^2] \cdot (-v)^2 - \frac{3}{4} v^2 = \left(v + \frac{3}{2}\right)^2 \cdot v^2 \geq 0 \end{aligned} \tag{9}$$

---

<sup>3</sup>The QC (8) is stronger than  $W^{\{\phi\}}(v) \cdot (-v) \geq 0$  used in [23] and some other papers. Using the latter we would, in particular, arrive at the more restrictive condition  $\text{Re}\{T(j\omega)\} \leq 0$  instead of (65) in Proposition 2 formulated below.

■

Consider now the general form of a dynamic output feedback control law

$$u = \mathcal{U}(z, y), \quad \dot{z} = \mathcal{F}(z, y) \quad (10)$$

where  $\mathcal{U}(\cdot)$  and  $\mathcal{F}(\cdot)$  are smooth functions of appropriate dimensions.

**Lemma 2** *Suppose  $[0, \tau^{max})$  is the maximal interval of existence of a solution*

$$X(t) = [\phi(t); \psi(t); z(t)]$$

*of the closed-loop system (1), (2), (10), where  $\tau^{max} > 0$  can be finite or not. Let  $F(X)$  be a smooth scalar function. Then, only the following two cases are possible:*

1. *There exists a sequence  $\{t_k\}_{k=1}^{\infty}$  of time instants with  $\lim_{k \rightarrow \infty} t_k = \tau^{max}$  such that the integrals of the quadratic form*

$$\mathcal{G}_1 [v, W^{\{\phi\}}(v)] = W^{\{\phi\}}(v) \cdot (-v) - \frac{3}{4} v^2 \quad (11)$$

*with  $W^{\{\phi\}}$  defined in (6), along this solution with*

$$v(t) = F(X(t))$$

*are strictly positive, i.e.*

$$\int_{t_{k-1}}^{t_k} \mathcal{G}_1 [v(t), W^{\{\phi\}}(v(t))] dt > 0, \quad k = 1, 2, \dots \quad (12)$$

2. *Along this solution,*

$$\text{either } v = F(X(t)) \equiv 0 \quad \text{or} \quad v = F(X(t)) \equiv -3/2$$

*Moreover, the integral in (12) of the quadratic form  $\mathcal{G}_1 [v(t), W^{\{\phi\}}(v(t))]$  identically equals zero for any  $t \in [0, \tau^{max})$ .*

*Proof:* To check (12), observe that integrating the relation (9) along a solution of the closed-loop system over  $[t_k, \tau^{max})$  with  $v(t) = F(X(t))$  results in zero value for any  $t_k \in [0, \tau^{max})$ . Then,  $v(t)$  equals either 0 or  $-\frac{3}{2}$  on  $[t_k, \tau^{max})$  and it is just left to notice that we can shift the time since the system is time-invariant. ■

## 2 Main result: Stability criteria for (3)

Let us postpone the discussion on how to transform the closed-loop system (1), (2), (10) into the form of (3) and search for conditions, under which global stability of (3) follows from properties of the separated  $x$ - and  $e$ -subsystems. It is worth noting that neither  $x$ - nor  $e$ -subsystems are independent, and just assuming asymptotic stability of each of them will not necessary result in asymptotic stability of (3). Stronger properties will be requested and features of the nonlinear functions  $W^{\{x\}}(\cdot)$  and  $W^{\{e\}}(\cdot, \cdot)$  will be used. To this purpose, assumptions of the following properties are made:

**Assumption 1:** *The nonlinearity  $W^{\{x\}}(\cdot)$  satisfies the relations (9) and (12), which are satisfied by  $W^{\{\phi\}}(\cdot)$  defined in (6). ■*

**Assumption 2:** *The nonlinearity  $W^{\{e\}}(\cdot, \cdot)$  satisfies the infinite sector quadratic constraint*

$$\mathcal{G}_2 [e, W^{\{e\}}(x, e)] = e^T \Pi_e W^{\{e\}}(x, e) \geq 0, \quad \forall x, e \quad (13)$$

with  $\Pi_e$  being a constant nonzero matrix, that is the relation similar to (7) defined for the original nonlinearity  $W^{\{\phi\}}(\cdot)$ . ■

The first stability condition will rely on quadratic stabilities of both  $x$ - and  $e$ -subsystems.

**Theorem 1** *Let Assumptions 1 and 2 hold. Suppose that:*

1. *There exist matrices  $P_1 = P_1^T$  and  $Q_1 = Q_1^T > 0$  such that the following inequality*

$$\begin{aligned} & 2x^T P_1 (A_{11}x + B_1 w_1) + \mathcal{G}_1 [Cx, w_1] \\ & = 2x^T P_1 (A_{11}x + B_1 w_1) + (-x^T C^T w_1 - \frac{3}{4} x^T C^T C x) < -x^T Q_1 x, \end{aligned} \quad (14)$$

*holds for all  $x \neq 0$  and  $w_1 \neq 0$ . Moreover, the matrix  $(A_{11} - \frac{3}{4} B_1 C)$  is Hurwitz.*

2. *There exist matrices  $P_2 = P_2^T$  and  $Q_2 = Q_2^T > 0$  such that*

$$e^T (A_{22}^T P_2 + P_2 A_{22}) e < -e^T Q_2 e, \quad e^T (2P_2 B_2 + \Pi_e) w_2 = 0 \quad (15)$$

*for all  $e \neq 0$  and  $w_2 \neq 0$ . Moreover, the matrix  $A_{22}$  is Hurwitz.*

*Then, the nonlinear system (3) is quadratically stable, i.e. there are matrices  $\mathcal{P} = \mathcal{P}^T > 0$  and  $\mathcal{Q} = \mathcal{Q}^T > 0$  such that along any nontrivial solution  $[x(t); e(t)]$  of (3) we have*

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} < - \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}^T \mathcal{Q} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \quad (16)$$

*Proof:* By the standard arguments, see e.g. [39, pp. 93–97], we conclude that the matrices  $P_1$  and  $P_2$  in the inequalities (14) and (15) are positive definite, so that the systems (4) and (5) are quadratically stable.

Let us consider now the positive definite quadratic form

$$\mathcal{W}(x, e) = \begin{bmatrix} x \\ e \end{bmatrix}^T \mathcal{P} \begin{bmatrix} x \\ e \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & \gamma \cdot P_2 \end{bmatrix} \tag{17}$$

with  $\gamma > 0$  and compute the time-derivative of  $\mathcal{W}(\cdot)$  along a solution of (3):

$$\frac{d}{dt} \mathcal{W}(x(t), e(t)) = 2x^T P_1 [A_{11}x + A_{12}e + B_1 W^{\{x\}}(Cx)] + 2\gamma e^T P_2 [A_{22}e + B_2 W^{\{e\}}(x, e)]$$

Since the inequalities (9) and (13) are valid along any solution of (3), we have

$$\begin{aligned} \frac{d}{dt} \mathcal{W} \leq & 2x^T P_1 [A_{11}x + A_{12}e + B_1 W^{\{x\}}(Cx)] + 2\gamma e^T P_2 [A_{22}e + B_2 W^{\{e\}}(x, e)] \\ & + \mathcal{G}_1 [Cx, W^{\{x\}}(Cx)] + 2\gamma \mathcal{G}_2 [e, W^{\{e\}}(x, e)] \end{aligned} \tag{18}$$

Taking into account the relations (14), (15), one can observe that the right-hand side of (18) becomes less or equal to

$$-x^T Q_1 x + 2x^T P_1 A_{12}e - \gamma e^T Q_2 e,$$

which after completing the squares becomes

$$-(R_1 x + r_1 e)^2 - e^T (\gamma Q_2 - r_1^T r_1) e \tag{19}$$

Here  $R_1$  is such that  $Q_1 = R_1^T R_1$ , and  $r_1^T = A_{12}^T P_1 R_1^{-1}$ . If the constant  $\gamma$  is chosen large enough, then the quadratic form (19), which serves as an upper bound for  $\frac{d}{dt} \mathcal{W}(x(t), e(t))$ , is negative definite. ■

The second stability criterion is again based on quadratic stability of the  $e$ -subsystem, but requires weaker properties of the  $x$ -subsystem.

**Theorem 2** *Let Assumptions 1 and 2 hold. Suppose that:*

1. *There exist a matrix  $P_1 = P_1^T$  such that the following inequality*

$$\begin{aligned} & 2x^T P_1 (A_{11}x + B_1 w_1) + \mathcal{G}_1 [Cx, w_1] \\ & = 2x^T P_1 (A_{11}x + B_1 w_1) + (-x^T C^T w_1 - \frac{3}{4} x^T C^T C x) \leq 0 \end{aligned} \tag{20}$$

*holds for all  $x \neq 0$  and  $w_1 \neq 0$ . Moreover, the matrix  $(A_{11} - \frac{3}{4} B_1 C)$  is Hurwitz and the pair  $[C, A_{11}]$  is observable.*

2. *There exist matrices  $P_2 = P_2^T$  and  $Q_2 = Q_2^T > 0$  such that (15) is valid for all  $e \neq 0$  and  $w_2 \neq 0$ . Moreover, the matrix  $A_{22}$  is Hurwitz.*
3. *The system (3) has no nontrivial solution  $[x(t); e(t)]$  along which  $e(t) \equiv 0$  and  $Cx(t) \equiv \text{const} \in \{-\frac{3}{2}; 0\}$ .*



Then, the origin of nonlinear system (3) is locally exponentially stable and globally asymptotically stable. ■

*Proof* is based on the following four claims.

**Claim 1** Under the assumptions of Theorem 2 the system (4) is globally asymptotically stable. ■

With the change of  $w_1$  into  $\tilde{w}_1$  defined by

$$\tilde{w}_1 = w_1 + \frac{3}{4}Cx$$

the inequality (20) has the form

$$2x^T P_1 \left[ \left( A_{11} - \frac{3}{4} B_1 C \right) x + B_1 \tilde{w}_1 \right] + (-x^T C^T \tilde{w}_1) \leq 0 \quad \forall x, \tilde{w}_1$$

By assumption, it has a solution. The facts that  $(A_{11} - \frac{3}{4} B_1 C)$  is Hurwitz and  $[C, A_{11}]$  is observable, imply that  $P_1 > 0$  (see [39, Theorem 2.15, p.97]). Consider the Lyapunov function candidate  $V_1(x) = x^T P_1 x$ . Due to (20) and (9), its time derivative along a solution  $x(t)$  of the system (4) satisfies the non-strict inequality

$$\begin{aligned} & \frac{d}{dt} V_1(x(t)) \\ &= 2x(t)^T P_1 \left( A_{11}x(t) + B_1 W^{\{x\}}(Cx(t)) \right) \\ &\leq 2x(t)^T P_1 \left( A_{11}x(t) + B_1 W^{\{x\}}(Cx(t)) \right) + \mathcal{G}_1 \left[ Cx(t), W^{\{x\}}(Cx(t)) \right] \\ &\leq 0 \end{aligned} \tag{21}$$

It implies an existence of the solutions of the system (4) on an infinite interval of time [22, Theorem 3.3], Lyapunov stability [22, Theorem 4.1], and their boundedness. It is left to verify that  $x(t)$  converges to the origin. Considering the integral form of (21), we have the inequalities

$$\begin{aligned} & V_1(x(t_k)) - V_1(x(t_{k-1})) \\ &= \int_{t_{k-1}}^{t_k} 2x^T P_1 \left( A_{11}x(t) + B_1 W^{\{x\}}(Cx(t)) \right) dt \\ &< \int_{t_{k-1}}^{t_k} 2x(t)^T P_1 \left( A_{11}x(t) + B_1 W^{\{x\}}(Cx(t)) \right) dt + \int_{t_{k-1}}^{t_k} \mathcal{G}_1 \left[ Cx(t), W^{\{x\}}(Cx(t)) \right] dt \\ &\leq 0 \end{aligned} \tag{22}$$

where the sequence  $\{t_k\}_{k=1}^\infty$  is from (12), which by assumption exists for any particular nontrivial solution<sup>4</sup> of (3). Hence,  $V_1(x(T)) \rightarrow c \equiv const$

<sup>4</sup>Note that using these special intervals of time allows us to make one of the inequalities strict. This is the key point of the proof.

as  $T \rightarrow \infty$ . It follows from (21) that the  $\omega$ -limit set of any solution is nonempty, compact, and invariant [22, Lemma 4.1]. Taking a solution  $x_\infty(t)$  of the system (4) from an  $\omega$ -limit set and applying (21) and (22) to it, we conclude that  $c = 0$  and so  $x(T) \rightarrow 0$  as  $T \rightarrow \infty$ . ■

**Claim 2** *There are no solutions of (3) that escape to infinity in finite time.* ■

The time-derivative of the function  $\mathcal{W}(\cdot)$  defined by (17) with  $\gamma = 1$  along any solution of (3) satisfies the inequality

$$\frac{d}{dt} \mathcal{W}(x(t), e(t)) \leq 2x(t)^T P_1 A_{12} e(t) - e(t)^T Q_2 e(t) \leq \varepsilon_1 \mathcal{W}(x(t), e(t)) \quad (23)$$

for some  $\varepsilon_1 > 0$ . Hence, solutions cannot grow faster than exponentially, see e.g. [22, Lemma 3.4], and exist on the infinite interval of time. ■

**Claim 3** *Along any (even unbounded) solution  $[x(t), e(t)]$  of (3),  $e(t)$  exponentially converges to zero.* ■

This fact immediately follows from (15) and the fact that the matrix  $A_{22}$  is Hurwitz.

**Claim 4** *All solutions of (3) are bounded.* ■

The first inequality in (23) can be rewritten as

$$\frac{d}{dt} \mathcal{W}(x(t), e(t)) \leq \varepsilon_2 \cdot \sqrt{\mathcal{W}(x(t), e(t))} \cdot \beta(t) \quad (24)$$

with  $\beta(t) = e(t)$  and  $\varepsilon_2 > 0$ . Integrating (24) results in the following inequality

$$\sqrt{\mathcal{W}(x(T), e(T))} - \sqrt{\mathcal{W}(x(0), e(0))} \leq \varepsilon_3 \cdot \int_0^T \beta(t) dt$$

Exponential convergence of  $\beta(t)$  to zero implies that  $\beta(\cdot) \in L^1[0, +\infty)$ . In turn, integrability of  $\beta(t)$  over the interval  $[0, +\infty)$  implies boundedness of  $\mathcal{W}(\cdot)$  and consequently boundedness of the solution  $[x(t), e(t)]$ . ■

With the four claims at hand, to finish the proof of Theorem 2, one can observe that any solution  $[x(t), e(t)]$  of (3) will have a non-empty  $\omega$ -limit set, while on this set  $e$ -variable should be zero. That is, this set consists of solutions of (4), which are asymptotically stable. This implies that all solutions of (3) converge to the origin. Furthermore, it is readily seen that with the conditions of Theorem 2 the origin is locally exponentially stable by linearization<sup>5</sup>. Hence, it is globally asymptotically stable. ■

With the formulated two criteria for stability of the transformed system (3), it is left to establish the possibility and a constructive procedure for transforming (1), (2), and (10) into such a form. This will be discussed next.

<sup>5</sup>Note that the quadratic constraints in the form of sector conditions imply exponential stability for  $x$ - and  $e$ -subsystems with all the linear functions  $W^{\{x\}}(\cdot)$  and  $W^{\{e\}}(\cdot, \cdot)$  satisfying the quadratic constraints, i.e. the ones appearing after linearization.

### 3 Feedback controllers for (1)–(2) and sufficient conditions to meet the structure of (3)

Here we introduce a class of output feedback controllers (10) for (1)–(2) and discuss matching conditions for the closed-loop system to be equivalent to the dynamical system (3). We will separate the linear and nonlinear parts in the control law, restrict ourselves to linear changes of coordinates and look for sufficient conditions to perform the transformation.

#### 3.1 Parametric set of controllers

Consider the family of output feedback controllers (10) with the following structure

$$u = \Lambda^{\{u\}} \begin{bmatrix} \psi \\ z \end{bmatrix} + W^{\{u\}}(\psi, z), \quad \frac{d}{dt} z = \Lambda^{\{z\}} \begin{bmatrix} \psi \\ z \end{bmatrix} + W^{\{z\}}(\psi, z), \quad z \in \mathbb{R}^m, \tag{25}$$

where  $\Lambda^{\{u\}} = [\Lambda_{\psi}^{\{u\}}, \Lambda_z^{\{u\}}]$ ,  $\Lambda^{\{z\}} = [\Lambda_{\psi}^{\{z\}}, \Lambda_z^{\{z\}}]$  are constant matrices of appropriate dimensions;  $W^{\{u\}}(\cdot)$ ,  $W^{\{z\}}(\cdot)$  are static nonlinearities. With such a feedback controller the dynamics of the system (1)–(2) becomes

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_{\psi}^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_{\psi}^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix}}_{= \mathcal{A}_{cl}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{\beta^2} & 0 \\ 0 & 0 & \mathbf{1}_m \end{bmatrix}}_{= \mathcal{B}_{cl}} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W^{\{u\}}(\psi, z) \\ W^{\{z\}}(\psi, z) \end{bmatrix} \tag{26}$$

Below we present the conditions that allow rewriting the system (26) in the the form (3) with  $x$ - and  $e$ -variables being defined by linear combinations of the components of the state vector  $[\phi; \psi; z]$ , i.e.

$$x = T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix} = T_{\psi}^{\{x\}} \psi + T_z^{\{x\}} z, \quad e = T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} = T_{\phi}^{\{e\}} \phi + T_{\psi}^{\{e\}} \psi + T_z^{\{e\}} z \tag{27}$$

where  $T^{\{x\}}$  and  $T^{\{e\}}$  are constant matrices of appropriate dimensions. As seen, the  $x$ -variable is constructed from the measured quantity and internal states of the controller, while the  $e$ -variable depends on all components of the state vector of (26).

#### 3.2 The first matching condition: the $e$ -dynamics

Differentiating the  $e$ -variable defined by (27), (26) and equating the result with the  $e$ -dynamics in (3), we obtain the first matching condition written

as

$$\begin{aligned}
 T^{\{e\}} \left( \mathcal{A}_{cl} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \mathcal{B}_{cl} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W^{\{u\}}(\psi, z) \\ W^{\{z\}}(\psi, z) \end{bmatrix} \right) &= \underbrace{T^{\{e\}} \begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix}}_{=\dot{e}} \\
 &= \underbrace{A_{22} T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}}_{=e} + B_2 W^{\{e\}} \left( \underbrace{T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}}_{=x}, \underbrace{T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}}_{=e} \right)
 \end{aligned}$$

With some loss of generality, let us consider separately the relations between the linear and nonlinear terms. We obtain the identity between matrices

$$T^{\{e\}} \mathcal{A}_{cl} \equiv A_{22} T^{\{e\}} \tag{28}$$

and the identity for the nonlinearities

$$T^{\{e\}} \mathcal{B}_{cl} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W^{\{u\}}(\psi, z) \\ W^{\{z\}}(\psi, z) \end{bmatrix} \equiv B_2 W^{\{e\}} \left( T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}, T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \right) \quad \forall \phi, \psi, z \tag{29}$$

The quadratic constraint (13) originally expressed in the  $x$ - and  $e$ -variables becomes

$$[\phi, \psi, z^T] \left( T^{\{e\}} \right)^T \Pi_e W^{\{e\}} \left( T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}, T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \right) \geq 0, \quad \forall \phi, \psi, z \tag{30}$$

### 3.3 The second matching condition: the $x$ -dynamics

Differentiating the  $x$ -variable defined by (27), (26) and equating the result with the  $x$ -dynamics in (3), we obtain the second matching condition written as

$$\begin{aligned}
 [\mathbf{0}, T^{\{x\}}] \left( \mathcal{A}_{cl} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \mathcal{B}_{cl} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W^{\{u\}}(\psi, z) \\ W^{\{z\}}(\psi, z) \end{bmatrix} \right) &= [\mathbf{0}, T^{\{x\}}] \begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \frac{d}{dt} x \\
 &= \underbrace{A_{11} T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}}_{=x} + \underbrace{A_{12} T^{\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}}_{=e} + B_1 W^{\{x\}} \left( \underbrace{C T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}}_{=x} \right)
 \end{aligned}$$

This identity will be satisfied if we equate the linear and nonlinear terms. Namely, if the relations between the matrices

$$[\mathbf{0}, T^{\{x\}}] \mathcal{A}_{cl} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \equiv A_{12} T^{\{e\}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

$$[\mathbf{0}, T^{\{x\}}] \mathcal{A}_{cl} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \mathbf{1}_m \end{bmatrix} \equiv A_{11} T^{\{x\}} + A_{12} T^{\{e\}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \mathbf{1}_m \end{bmatrix} \tag{32}$$

hold, and if the nonlinearities are equal

$$[\mathbf{0}, T^{\{x\}}] \mathcal{B}_{cl} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W^{\{u\}}(\psi, z) \\ W^{\{z\}}(\psi, z) \end{bmatrix} \equiv B_1 W^{\{x\}} \left( CT^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix} \right) \quad \forall \phi, \psi, z \tag{33}$$

The first column of  $\mathcal{A}_{cl}$  in (26) cannot be changed by the controller, hence the relation (31) is

$$\frac{1}{\beta^2} T_\psi^{\{x\}} = A_{12} T_\phi^{\{e\}} \quad \text{with} \quad T^{\{x\}} = [T_\psi^{\{x\}}, T_z^{\{x\}}], \quad T^{\{e\}} = [T_\phi^{\{e\}}, T_\psi^{\{e\}}, T_z^{\{e\}}] \tag{34}$$

The constraint (9) with the quadratic form  $\mathcal{G}_1[\cdot, \cdot]$  becomes rewritten as

$$\mathcal{G}_1 \left[ CT^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix}, W^{\{x\}} \left( T^{\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix} \right) \right] \geq 0, \quad \forall \psi, z \tag{35}$$

### 3.4 Example

To illustrate some of the above conditions for matching, for instance, imposed on the  $e$ -dynamics in Section 3.2, let us consider the simplest case of the controller (25) with no dynamics. For this case several of the parameters will have predefined zero values and the matching relations become simpler: Indeed, the variables  $x$  and  $e$  are scalars and the linear transforms (27) take the form

$$x = [T_\psi^{\{x\}}, 0] \begin{bmatrix} \psi \\ z \end{bmatrix}, \quad e = [T_\phi^{\{e\}}, T_\psi^{\{e\}}, 0] \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \quad x \in \mathbb{R}^1, \quad e \in \mathbb{R}^1$$

where  $T_\psi^{\{x\}}$ ,  $T_\phi^{\{e\}}$  and  $T_\psi^{\{e\}}$  are scalar parameters. The equation (28) that matches the linear parts of the  $e$ -dynamics is reduced now to

$$[T_\phi^{\{e\}}, T_\psi^{\{e\}}] \begin{bmatrix} \frac{3}{2} & -1 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_\psi^{\{u\}} \end{bmatrix} = A_{22} [T_\phi^{\{e\}}, T_\psi^{\{e\}}] \tag{36}$$

The next statement describes the necessary and sufficient conditions for this (Sylvester-like) equation to have non-trivial solutions

**Lemma 3** *The set of parameters  $\{T_\phi^{\{e\}}, T_\psi^{\{e\}}, \Lambda_\psi^{\{u\}}, A_{22}\}$ , which satisfy (36), is non-trivial if and only if  $\{\Lambda_\psi^{\{u\}}, A_{22}\}$  satisfy the equation*

$$(A_{22})^2 + \left[ \frac{1}{\beta^2} \Lambda_\psi^{\{u\}} - \frac{3}{2} \right] \cdot A_{22} + \frac{1}{\beta^2} - \frac{3}{2\beta^2} \Lambda_\psi^{\{u\}} = 0 \quad (37)$$

*Proof* is based on the observation that (36) has nontrivial solution if and only if the determinant of the matrix

$$\begin{bmatrix} \left( \frac{3}{2} - A_{22} \right) & -1 \\ \frac{1}{\beta^2} & \left( -\frac{1}{\beta^2} \Lambda_\psi^{\{u\}} - A_{22} \right) \end{bmatrix}$$

is zero. This results in the equation (37). The pair  $\left[ T_\phi^{\{e\}}, T_\psi^{\{e\}} \right]$  can be found then computing a left eigenvector corresponding to the zero eigenvalue. ■

If the parameters  $\{T_\phi^{\{e\}}, T_\psi^{\{e\}}, \Lambda_\psi^{\{u\}}, A_{22}\}$  are chosen as described in Lemma 3 and if the nonlinearity  $W^{\{u\}}(\cdot)$  of the controller (25) is taken as

$$W^{\{u\}}(\psi) = w_u \cdot (1 - (1 + t_u \cdot \psi)^3) \quad (38)$$

where  $w_u, t_u$  are constants, then the relation (29) defines  $B_2$  and  $W^{\{e\}}(\cdot, \cdot)$ . In turn, the quadratic constraint (30) can be met with an appropriate choice of  $w_u$  and  $t_u$ . Indeed, the right-hand side of the inequality (30) is then equal, up to a constant factor, to

$$\left[ \frac{1}{2} T_\phi^{\{e\}} W^{\{\phi\}}(\phi) - \frac{1}{\beta^2} T_\psi^{\{e\}} W^{\{\psi\}}(\psi) \right] \cdot \left( T_\phi^{\{e\}} \phi + T_\psi^{\{e\}} \psi \right)$$

Taking the factor  $\frac{1}{2} \left( T_\phi^{\{e\}} \right)^2$  outside the brackets, the last expression has the form

$$\frac{1}{2} \left( T_\phi^{\{e\}} \right)^2 \left[ W^{\{\phi\}}(\phi) - \frac{2}{\beta^2} \frac{T_\psi^{\{e\}}}{T_\phi^{\{e\}}} \cdot w_u \cdot (1 - (1 + t_u \cdot \psi)^3) \right] \cdot \left( \phi + \frac{T_\psi^{\{e\}}}{T_\phi^{\{e\}}} \psi \right)$$

and it is reduced to

$$\begin{aligned} & \frac{1}{2} \left( T_\phi^{\{e\}} \right)^2 \left[ W^{\{\phi\}}(\phi) - (1 - (1 + t_u \cdot \psi)^3) \right] \cdot (\phi - t_u \cdot \psi) \\ & = \frac{1}{2} \left( T_\phi^{\{e\}} \right)^2 \left[ (1 + t_u \cdot \psi)^3 - (1 + \phi)^3 \right] \cdot (\phi - t_u \cdot \psi) \end{aligned} \quad (39)$$

provided that  $w_u$  and  $t_u$  are taken as

$$w_u = \frac{\beta^2}{2} \frac{T_\phi^{\{e\}}}{T_\psi^{\{e\}}}, \quad t_u = -\frac{T_\psi^{\{e\}}}{T_\phi^{\{e\}}}$$

The relation (39) is already sign-definite for any  $\phi$  and  $\psi$  as requested in (30).

## 4 Sufficient conditions for robust global stabilization of (1), (2) via output feedback

Here we present two families of dynamic output feedback control laws (25) that meet the full set of matching conditions of the previous Section to fit the structure of (3). With additional constraints on coefficients the first family of controllers satisfies the conditions of Theorem 1 and the second satisfies the conditions of Theorem 2 ensuring in each case global asymptotic stability of the respective closed-loop system.

### 4.1 Output feedback controller design: Example 1

Consider a subset of output feedback controllers (25) with  $z \in \mathbb{R}^1$  and the transformations (27) defined by

$$x = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{= T\{x\}} \begin{bmatrix} \psi \\ z \end{bmatrix} \in \mathbb{R}^2, \quad e = \underbrace{\begin{bmatrix} 1, & -t_\psi, & -t_z \end{bmatrix}}_{= T\{e\}} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \in \mathbb{R}^1 \quad (40)$$

Suppose that the nonlinearities  $W^{\{u\}}(\cdot)$ ,  $W^{\{z\}}(\cdot)$  in the controller (25) are similar and defined as

$$W^{\{u\}}(\psi, z) = w_u \cdot W(\psi, z), \quad W^{\{z\}}(\psi, z) = w_z \cdot W(\psi, z), \quad (41)$$

where  $w_u$ ,  $w_z$  are constants and

$$W(\psi, z) = 1 - (1 + t_\psi \psi + t_z z)^3 \quad (42)$$

With such a choice, the controller (25) is

$$u = \Lambda_\psi^{\{u\}} \psi + \Lambda_z^{\{u\}} z + w_u \cdot W(\psi, z), \quad \frac{d}{dt} z = \Lambda_\psi^{\{z\}} \psi + \Lambda_z^{\{z\}} z + w_z \cdot W(\psi, z), \quad (43)$$

with  $z \in \mathbb{R}^1$ , and the closed-loop system (26) takes the form

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_\psi^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_\psi^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{\beta^2} w_u & 0 \\ 0 & 0 & w_z \end{bmatrix} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi, z) \\ W(\psi, z) \end{bmatrix} \quad (44)$$

Here the nonlinearities  $W^{\{\phi\}}(\cdot)$  and  $W(\cdot)$  are defined by (6) and (42) respectively;  $\Lambda_\psi^{\{u\}}$ ,  $\Lambda_z^{\{u\}}$ ,  $\Lambda_\psi^{\{z\}}$ ,  $\Lambda_z^{\{z\}}$  are scalar constant parameters.

The most demanding part in rewriting (44) in the form of (3) is the relation (28). For this case it requires finding a solution of the linear matrix

equation

$$[\tau_\phi, \tau_\psi, \tau_z] \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_\psi^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_\psi^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} = A_{22} [\tau_\phi, \tau_\psi, \tau_z], \quad (45)$$

from which the parameters  $t_\psi$  and  $t_z$  are computed as

$$t_\psi = -\frac{\tau_\psi}{\tau_\phi}, \quad t_z = -\frac{\tau_z}{\tau_\phi}, \quad \text{if } \tau_\phi \neq 0. \quad (46)$$

The conditions for (45) to have a nontrivial solution are given next<sup>6</sup>.

**Lemma 4** *There exist non-trivial solutions  $[\tau_\phi, \tau_\psi, \tau_z]$  for (45) if and only if the parameters  $\Lambda_\psi^{\{u\}}$ ,  $\Lambda_z^{\{u\}}$ ,  $\Lambda_\psi^{\{z\}}$ ,  $\Lambda_z^{\{z\}}$ , and  $A_{22}$  satisfy the equation*

$$-A_{22}^3 + \left[ \frac{3}{2} - \frac{\Lambda_\psi^{\{u\}}}{\beta^2} + \Lambda_z^{\{z\}} \right] A_{22}^2 + \left[ \frac{3\Lambda_\psi^{\{u\}}}{2\beta^2} - \frac{1}{\beta^2} + \frac{\Lambda_\psi^{\{u\}}\Lambda_z^{\{z\}}}{\beta^2} - \frac{3\Lambda_z^{\{z\}}}{2} - \frac{\Lambda_z^{\{u\}}\Lambda_\psi^{\{z\}}}{\beta^2} \right] A_{22} + \frac{\Lambda_z^{\{z\}}}{\beta^2} - \frac{3\Lambda_\psi^{\{u\}}\Lambda_z^{\{z\}}}{2\beta^2} + \frac{3\Lambda_z^{\{u\}}\Lambda_\psi^{\{z\}}}{2\beta^2} = 0 \quad (47)$$

*Proof* is based on the observation that (45) has a solution if and only if the determinant of the matrix

$$\begin{bmatrix} \left( \frac{3}{2} - A_{22} \right) & -1 & 0 \\ \frac{1}{\beta^2} & \left( -\frac{1}{\beta^2} \Lambda_\psi^{\{u\}} - A_{22} \right) & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_\psi^{\{z\}} & \left( \Lambda_z^{\{z\}} - A_{22} \right) \end{bmatrix}$$

is zero. This results in the equation (47). The row  $[\tau_\phi, \tau_\psi, \tau_z]$  can be found then as the left eigenvector corresponding to the zero eigenvalue. ■

The relation (29) takes for this case the form

$$\frac{1}{2} W^{\{\phi\}}(\phi) + \left[ \frac{1}{\beta^2} t_\psi w_u - t_z w_z \right] W(\psi, z) = B_2 W^{\{e\}}(x, e)$$

If the parameters satisfy the relation

$$\frac{1}{\beta^2} t_\psi w_u - t_z w_z = -\frac{1}{2}, \quad (48)$$

then  $B_2$  is equal to  $\frac{1}{2}$  and  $W^{\{e\}}(x, e) = W^{\{\phi\}}(\phi) - W(\psi, z)$ . With this choice, the inequality (30) is satisfied with  $\Pi_e = -1$  since

$$e \cdot (-1) \cdot W^{\{e\}}(x, e) = -(\phi - t_\psi \psi - t_z z) \cdot \left[ (1 + t_\psi \psi + t_z z)^3 - (1 + \phi)^3 \right] \geq 0 \quad (49)$$

<sup>6</sup>However, not all solutions of (45) might be of interest. Indeed, if for some parameters the equation (45) has a solution with  $\tau_\phi = 0$ , then this solution cannot be used for transformations of coordinates  $[\phi; \psi; e]$  into  $[x; e]$  as done in (40).



As seen, the quadratic form is non-negative for any  $\phi, \psi, z$ .

The matching conditions for the  $x$ -dynamics are straightforward for the case and allow defining coefficients of matrices  $A_{11}, A_{12}, B_1$  as well as the nonlinearity  $W^{\{x\}}(\cdot)$  without imposing additional constraints on parameters of the controller. Namely, with  $T^{\{x\}} = \mathbf{1}_2$  the relation (31) uniquely defines the column  $A_{12} \in \mathbb{R}^{2 \times 1}$  as follows

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_{\psi}^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_{\psi}^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{\beta^2} \\ 0 \end{bmatrix} = A_{12} [1, -t_{\psi}, -t_z] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A_{12} \end{aligned} \tag{50}$$

In turn, the relation (32) that takes for the case the form

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ \frac{1}{\beta^2} & -\frac{1}{\beta^2} \Lambda_{\psi}^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ 0 & \Lambda_{\psi}^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\beta^2} \Lambda_{\psi}^{\{u\}} & -\frac{1}{\beta^2} \Lambda_z^{\{u\}} \\ \Lambda_{\psi}^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} \\ = A_{11} + A_{12} [1, -t_{\psi}, -t_z] \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = A_{11} + \begin{bmatrix} -\frac{1}{\beta^2} t_{\psi} & -\frac{1}{\beta^2} t_z \\ 0 & 0 \end{bmatrix} \end{aligned}$$

uniquely defines coefficients of  $A_{11} \in \mathbb{R}^{2 \times 2}$  as follows

$$A_{11} = \begin{bmatrix} -\frac{1}{\beta^2} (\Lambda_{\psi}^{\{u\}} - t_{\psi}) & -\frac{1}{\beta^2} (\Lambda_z^{\{u\}} - t_z) \\ \Lambda_{\psi}^{\{z\}} & \Lambda_z^{\{z\}} \end{bmatrix} \tag{51}$$

The relation (33) is now

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{\beta^2} w_u & 0 \\ 0 & 0 & w_z \end{bmatrix} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi, z) \\ W(\psi, z) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\beta^2} w_u \\ w_z \end{bmatrix} W(\psi, z) = B_1 W^{\{x\}}(Cx)$$

Hence

$$B_1 = \begin{bmatrix} -\frac{1}{\beta^2} w_u \\ w_z \end{bmatrix}, \quad W^{\{x\}}(Cx) = W(\psi, z), \quad Cx = t_{\psi}\psi + t_z z = [t_{\psi}, t_z] \begin{bmatrix} \psi \\ z \end{bmatrix} \tag{52}$$

One can readily check that with such choices of  $W^{\{x\}}(Cx)$  and of the row  $C$  the nonlinearity satisfies the quadratic constraint (35). Summing up the

manipulations made with the closed-loop system (44), we obtain the following.

**Lemma 5** *Suppose the coefficients of the controller (43) with  $W(\cdot)$  defined by (42) satisfy the relations (45)–(48). Then, the following is true.*

1. *The closed-loop system (44) can be equivalently written as (3), where  $x$  and  $e$  are given by (40) and*
  - *The scalar  $A_{22}$  is defined from the equation (45), the scalar  $B_2$  equals  $\frac{1}{2}$ .*
  - *The matrices  $A_{11}$ ,  $A_{12}$ , and  $B_1$  are defined by (50)–(52).*
2. *The nonlinear function  $W^{\{e\}}(\cdot, \cdot)$  is defined by  $W^{\{e\}}(x, e) := W^{\{\phi\}}(\phi) - W(\psi, z)$  and satisfies the infinite sector condition (13) with  $\Pi_e = -1$ .*
3. *The nonlinearity  $W^{\{x\}}(\cdot)$  is defined by  $W^{\{x\}} := W(\psi, z)$  and the linear output*

$$v_x = t_\psi \psi + t_z z = Cx \quad (53)$$

*of the  $x$ -dynamics satisfies the relation (9) defined by the quadratic form*

$$\mathcal{G}_1 [v_x, W^{\{x\}}(v_x)] = W^{\{x\}}(v_x) \cdot (-v_x) - \frac{3}{4} v_x^2 \geq 0, \quad \forall \psi, z$$

*with the redefined state and the linear output. ■*

Once the coefficients of the controller (43), for which after a change of coordinates the closed-loop system (44) can be rewritten as (3), are found, we can apply Theorem 1 and search for a set of parameters corresponding to stabilizing controllers. The result based on applying the Frequency Theorem [39] to verify (14) is formulated next.

**Proposition 1** *Consider the closed-loop system (44). Suppose that the parameters are such that the relations (45)–(48) are satisfied, the matrices  $A_{11}$ ,  $A_{12}$ ,  $A_{22}$ ,  $B_1$ ,  $B_2$ ,  $C$  are defined as in Lemma 5 and the following conditions hold:*

1. *The inequality*

$$\operatorname{Re} \{T(j\omega)\} - \frac{3}{4} |T(j\omega)|^2 < 0 \quad (54)$$

*is valid for all  $\omega \geq 0$ , where*

$$T(s) = -C (sI_2 - A_{11})^{-1} B_1 = \frac{-\frac{1}{2}s + p_0}{s^2 + l_1s + l_0} \quad (55)$$

2. *The  $2 \times 2$  matrix  $(A_{11} - \frac{3}{4} B_1 C)$  is Hurwitz.*

3. The scalar  $A_{22}$  is negative.

Then with any sets of these parameters the closed-loop system (44), i.e., the system (1), (2) with the dynamic output feedback controller (43) defined by these parameters, is quadratically stable. ■

*Proof:* Since the matrix  $(A_{11} - \frac{3}{4}B_1C)$  is stable, the pair  $[A_{11}, B_1]$  is stabilizable. Then the strict inequality (54) implies the solvability of (14) with respect to matrix  $P_1$  (see [39, Theorem 2.13, p. 92]). The inequality (15) is scalar with  $B_2 = \frac{1}{2}$  and  $\Pi_e = -1$ , then  $P_2$  can be chosen as 1. So all the conditions of Theorem 1 are met and the quadratic stability of (44) follows. ■

### 4.2 Output feedback controller design: Example 2

Consider the modification of the family of output feedback controllers (43) by adding new internal state as a dynamic extension bringing an integral action. Namely  $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2$ , the linear transforms (27) are given by

$$x = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{= T\{x\}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^3, \quad e = \underbrace{[1, -t_\psi, -t_{z_1}, 0]}_{= T\{e\}} \begin{bmatrix} \phi \\ \psi \\ z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^4 \quad (56)$$

and the output feedback controller is

$$\left\{ \begin{array}{l} u = \underbrace{[\Lambda_\psi^{\{u\}}, \Lambda_{z_1}^{\{u\}}, \Lambda_{z_2}^{\{u\}}]}_{= \Lambda^{\{u\}}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} + \underbrace{w_u \cdot W(\psi, z_1)}_{= W^{\{u\}}(\cdot)} \\ \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \Lambda_\psi^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ -t_\psi & -t_{z_1} & 0 \end{bmatrix}}_{= \Lambda^{\{z\}}} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} w_{z_1} \\ 0 \end{bmatrix}}_{= W^{\{z\}}(\cdot)} W(\psi, z_1) \end{array} \right. \quad (57)$$

where the nonlinearity  $W(\psi, z_1)$  is similar to (42) and defined as

$$W(\psi, z_1) = 1 - (1 + t_\psi \psi + t_{z_1} z_1)^3. \quad (58)$$

The closed-loop system (1)-(2) with the controller (57) takes the form

$$\begin{bmatrix} \dot{\phi} \\ \dot{\psi} \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_{\psi}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_1}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ 0 & \Lambda_{\psi}^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ 0 & -t_{\psi} & -t_{z_1} & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{w_u}{\beta^2} & 0 \\ 0 & 0 & w_{z_1} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} W^{\{\phi\}}(\phi) \\ W(\psi, z_1) \\ W(\psi, z_1) \end{bmatrix} \tag{59}$$

Among matching conditions of Section 3 to rewrite (44) in the form (3)—as in the previous example—the relation (28) is the most challenging to satisfy. For this case it requires searching a solution of the linear matrix equation

$$[\tau_{\phi}, \tau_{\psi}, \tau_{z_1}, 0] \begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 \\ \frac{1}{\beta^2} & -\frac{\Lambda_{\psi}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_1}^{\{u\}}}{\beta^2} & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ 0 & \Lambda_{\psi}^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ 0 & -t_{\psi} & -t_{z_1} & 0 \end{bmatrix} = A_{22} [\tau_{\phi}, \tau_{\psi}, \tau_{z_1}, 0], \tag{60}$$

from which the parameters  $t_{\psi}$ ,  $t_z$  and  $\tau_{\phi}$ ,  $\tau_{\psi}$ ,  $\tau_{z_1}$  are related as

$$t_{\psi} = -\frac{\tau_{\psi}}{\tau_{\phi}}, \quad t_z = -\frac{\tau_{z_1}}{\tau_{\phi}}, \quad \text{if } \tau_{\phi} \neq 0. \tag{61}$$

However, it is straightforward to check that the equation (60) has a solution if and only if the conditions of Lemma 4 are valid and, in addition, the following identity holds:

$$\Lambda_{z_2}^{\{z_1\}} \tau_{z_1} = \frac{1}{\beta^2} \Lambda_{z_2}^{\{u\}} \tau_{\psi} \tag{62}$$

Arguments similar to the ones presented above in Example 1 allow verification that the other matching conditions are met provided the relation (48) is valid. Hence, we obtain the following.

**Lemma 6** *Suppose the coefficients of the controller (57) with  $W(\psi, z_1)$  defined by (58) satisfy the relations (45)–(48) and (62). Then, the following is true.*

1. *The closed-loop system (59) can be equivalently written as (3), where  $x$  and  $e$  are given by (56) and*

- *The scalar  $A_{22}$  is defined from the equation (60) and the scalar  $B_2$  is equal to  $\frac{1}{2}$ .*
- *The matrices  $A_{11}$ ,  $A_{12}$ , and  $B_1$  are defined as follows:*

$$A_{11} = \begin{bmatrix} -\frac{\Lambda_{\psi}^{\{u\}} - t_{\psi}}{\beta^2} & -\frac{\Lambda_{z_1}^{\{u\}} - t_{z_1}}{\beta^2} & -\frac{\Lambda_{z_2}^{\{u\}}}{\beta^2} \\ \Lambda_{\psi}^{\{z_1\}} & \Lambda_{z_1}^{\{z_1\}} & \Lambda_{z_2}^{\{z_1\}} \\ -t_{\psi} & -t_{z_1} & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \frac{1}{\beta^2} \\ 0 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\frac{w_u}{\beta^2} \\ w_{z_1} \\ 0 \end{bmatrix} \tag{63}$$

2. The nonlinear function  $W^{\{e\}}$  is defined by  $W^{\{e\}}(x, e) := W^{\{\phi\}}(\phi) - W(\psi, z_1)$  and satisfies the infinite sector condition

$$e \cdot \Pi_e \cdot W^{\{e\}}(x, e) \geq 0 \quad \forall x, \forall e$$

similar to (7), (30), and (49) with  $\Pi_e = -1$ .

3. The nonlinearity  $W^{\{x\}}(\cdot)$  is defined by  $W^{\{x\}} := W(\psi, z_1)$  and the linear output

$$v_x = t_\psi \psi + t_{z_1} z_1 = \underbrace{[t_\psi, t_{z_1}, 0]}_{= C} \begin{bmatrix} \psi \\ z_1 \\ z_2 \end{bmatrix} \quad (64)$$

of the  $x$ -dynamics satisfies the relation (9) defined by the quadratic form

$$\mathcal{G}_1 [v_x, W^{\{x\}}(v_x)] = W^{\{x\}}(v_x) \cdot (-v_x) - \frac{3}{4} v_x^2 \geq 0, \quad \forall \psi, z$$

with the redefined state and the linear output. ■

With coefficients of controllers (57) as in Lemma 6, we can apply Theorem 2 using the Frequency Theorem [39] to verify (20) and to identify stabilizing controllers in the family.

**Proposition 2** Consider the closed-loop system (59). Suppose that the parameters are such that the relations (45)–(48) and (62) are satisfied, the matrices  $A_{11}$ ,  $A_{12}$ ,  $A_{22}$ ,  $B_1$ ,  $B_2$ ,  $C$  are defined as in Lemma 6 and the following conditions hold:

1. The inequality

$$\operatorname{Re}\{T(j\omega)\} - \frac{3}{4}|T(j\omega)|^2 \leq 0 \quad (65)$$

is valid for any  $\omega \geq 0$ , where

$$T(s) = -C(sI_3 - A_{11})^{-1} B_1 = \frac{-\frac{1}{2}s^2 + p_1 s}{s^3 + l_2 s^2 + l_1 s + l_0} \quad (66)$$

2. The matrix  $(A_{11} - \frac{3}{4} B_1 C)$  is Hurwitz.  
 3. The pair  $[C, A_{11}]$  is observable and the pair  $[A_{11}, B_1]$  is controllable.  
 4. The scalar  $A_{22}$  is negative.

Then, with any sets of these parameters the closed-loop system (59), i.e., the system (1), (2) with the dynamic output feedback controller (57) defined by these parameters, is globally asymptotically and locally exponentially stable. ■

*Proof:* The inequality (65) together with conditions 2 and 3 of Proposition 2 imply that the inequality (20) has a solution  $P_1$  and it is positive definite. The scalar  $P_2$  in (15) can be chosen as 1. To check that the  $x$ -subsystem of (59), i.e. (4), has no nontrivial solutions consistent with the constraints

$$v_x(t) = Cx(t) \equiv 0, \forall t \quad \text{or} \quad v_x(t) = Cx(t) \equiv -\frac{3}{2}, \forall t$$

let us consider the two cases separately.

**Case 1:** If there is a solutions  $x(t)$  with  $v(t) \equiv 0$ , then it is the solution of the system

$$\dot{x} = A_{11}x + B_1W^{\{x\}}(0) = A_{11}x$$

Differentiating  $v(t)$  twice with respect to time, we obtain

$$\dot{v}(t) = CA_{11}x(t) \equiv 0, \quad \ddot{v}(t) = CA_{11}^2x(t) \equiv 0.$$

Hence the solution  $x(t) \neq 0$ , if exists, should satisfy the relation

$$\begin{bmatrix} C \\ CA_{11} \\ CA_{11}^2 \end{bmatrix} x(t) \equiv 0_{3 \times 1}$$

But this contradicts the observability of  $[C, A_{11}]$ .

**Case 2:** If there is a solutions  $x(t)$  with  $v(t) \equiv -\frac{3}{2}$ , then it is a solution of the system

$$\dot{x} = A_{11}x + B_1W^{\{x\}}(-\frac{3}{2}) = A_{11}x + B_1(1 - (1 - \frac{3}{2})^3) = A_{11}x + B_1\frac{9}{8}$$

Differentiating  $v(t)$  twice with respect to time, we obtain

$$\dot{v}(t) = CA_{11}x(t) + CB_1\frac{9}{8} \equiv 0, \quad \ddot{v}(t) = C^2A_{11}^2x(t) + CA_{11}B_1\frac{9}{8} \equiv 0.$$

The last two relations together with  $v(t) \equiv -\frac{3}{2}$  can be rewritten as

$$\begin{bmatrix} C \\ CA_{11} \\ CA_{11}^2 \end{bmatrix} x(t) \equiv \begin{bmatrix} -\frac{3}{2} \\ -CB_1\frac{9}{8} \\ -CA_{11}B_1\frac{9}{8} \end{bmatrix}$$

The pair  $[C, A_{11}]$  is observable, then the  $3 \times 3$ -matrix in the left-hand side of the last equation is of full rank. Hence  $x(t)$  is an equilibrium. The  $x$ -dynamics has the particular structure and the last differential equation (see (59)) with respect to  $z_2$ -variable is

$$\frac{d}{dt} z_2(t) = -v(t)$$

If  $x(t) = [\psi(t); z_1(t); z_2(t)]$  is an equilibrium, then  $v(t)$  should be zero. This contradicts the assumption that  $v(t) \equiv -\frac{3}{2}$ . Summarizing, all the conditions of Theorem 2 are checked and so global asymptotic stability of the closed-loop system (59) follows. ■

## 5 Choosing coefficients of the designed output feedback controllers for (1)–(2)

To illustrate the procedure for solving the matching equations and finding coefficients for the control laws suggested in Proposition 1 and 2 of Section 4, we present here two numerical examples of successful designs.

### 5.1 Example for Proposition 1

Suppose that  $\beta = 1$  and the scalars  $\Lambda_\psi^{\{u\}}$ ,  $\Lambda_z^{\{u\}}$ ,  $\Lambda_\psi^{\{z\}}$ ,  $\Lambda_z^{\{z\}}$  are chosen as

$$\Lambda_\psi^{\{u\}} = -17, \quad \Lambda_z^{\{u\}} = -4, \quad \Lambda_\psi^{\{z\}} = -\frac{207}{2}, \quad \Lambda_z^{\{z\}} = -\frac{47}{2}. \quad (67)$$

Then, the equation (46) with the matrix  $\mathcal{A}_{cl}$  takes the form

$$[\tau_\phi, \tau_\psi, \tau_z] \begin{bmatrix} \frac{3}{2} & -1 & 0 \\ 1 & 17 & 4 \\ 0 & -\frac{207}{2} & -\frac{47}{2} \end{bmatrix} = A_{22} [\tau_\phi, \tau_\psi, \tau_z]$$

The values of  $A_{22}$ , for which this equation has a non-trivial solution, are determined by the equation (37). They are

$$A_{22} = \{-0.5, -1, -3.5\}$$

and all are negative. The corresponding left eigenvectors define the constants  $t_\psi$ ,  $t_z$  by (46). For example, if  $A_{22} = -3.5$ , then

$$t_\psi = 5, \quad t_z = 1. \quad (68)$$

The last matching condition from Section 3 defines the constraint on values of  $w_u$ ,  $w_z$ . This is the linear relation (48), which is now

$$-\frac{1}{2} = \frac{1}{\beta^2} t_\psi w_u - t_z w_z = 5w_u - w_z$$

The values

$$w_u = -1, \quad w_z = -\frac{9}{2} \quad (69)$$

as well as many others, meet this condition. With such assignments all coefficients of the controller (43) are fixed and the matrices  $A_{11}$ ,  $B_1$  and  $C$  can be found from the relations (50)–(52) as follows

$$A_{11} = \begin{bmatrix} 22 & 5 \\ -\frac{207}{2} & -\frac{47}{2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -\frac{9}{2} \end{bmatrix}, \quad C = [5, 1] \quad (70)$$

To verify that this controller is stabilizing, we check one by one the conditions of Proposition 1:

1. The transfer functions (55) is now

$$T(s) = \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + \frac{3}{2}s + \frac{1}{2}}. \tag{71}$$

Since  $-T(s)$  is positive real [22, Sec. 6.3], it satisfies (54).

2. The eigenvalues of the matrix  $A_{11} - \frac{3}{4}B_1C_1$  are  $\{-1, -0.875\}$ ; hence, it is Hurwitz.

3. By our choice,  $A_{22} = -3.5$ ; and it is negative.

So, the assumptions of Proposition 1 are all valid; therefore, the controller (43) with the coefficients (67)–(69) quadratically stabilizes the system (1)–(2). Fig. 1 depicts the evolution of the variables  $\phi$  and  $\psi$  for the solution of the closed-loop system with initial conditions at  $\phi(0) = -2.07$ ,  $\psi(0) = 0.5$ ,  $z(0) = 0$ .

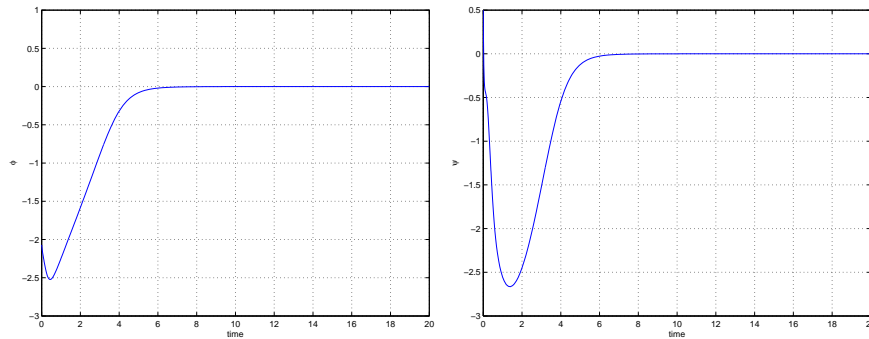


Figure 1: The behavior of the  $\phi$ -and  $\psi$ -variables for the solution of the closed-loop system (44) with the dynamical controller (43) when coefficients are chosen as in (67)–(69) and the initial conditions are  $\phi(0) = -2.07$ ,  $\psi(0) = 0.5$ ,  $z(0) = 0$ .

### 5.2 Example for Proposition 2

Suppose  $\beta = 1$  and the scalars  $\Lambda_{\psi}^{\{u\}}$ ,  $\Lambda_{z_1}^{\{u\}}$ ,  $\Lambda_{\psi}^{\{z_1\}}$ ,  $\Lambda_{z_1}^{\{z_1\}}$  are chosen as in (67), i.e.,

$$\Lambda_{\psi}^{\{u\}} = -17, \quad \Lambda_{z_1}^{\{u\}} = -4, \quad \Lambda_{\psi}^{\{z_1\}} = -\frac{207}{2}, \quad \Lambda_{z_1}^{\{z_1\}} = -\frac{47}{2}, \tag{72}$$

the gain  $A_{22}$  is  $-3.5$  and the coefficients  $t_{\psi}$ ,  $t_{z_1}$ ,  $w_u$ ,  $w_{z_1}$  are as in (68), (69), i.e.,

$$t_{\psi} = 5, \quad t_{z_1} = 1, \quad w_u = -1, \quad w_{z_1} = -\frac{9}{2} \tag{73}$$

These parameters for the controller (57) meet all the matching conditions of Lemma 6 except the relation (62), which defines a constraint on coefficients



$\Lambda_{z_2}^{\{z_1\}}$  and  $\Lambda_{z_1}^{\{u\}}$ . With (73) this constraint has the form

$$\Lambda_{z_2}^{\{z_1\}} = 5 \cdot \Lambda_{z_1}^{\{u\}}$$

If, for instance,

$$\Lambda_{z_2}^{\{z_1\}} = 4, \quad \Lambda_{z_1}^{\{u\}} = \frac{4}{5} \tag{74}$$

then it is satisfied. With such assignments all coefficients of the controller (57) are fixed and the matrices  $A_{11}$ ,  $B_1$  and  $C$  can be found from the relations similar to (50)–(52) as follows

$$A_{11} = \begin{bmatrix} 22 & 5 & -\frac{4}{5} \\ -\frac{207}{2} & -\frac{47}{2} & 4 \\ -5 & -1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -\frac{9}{2} \\ 0 \end{bmatrix}, \quad C = [5, 1, 0]$$

To verify that the controller is stabilizing, we check one by one the conditions of Proposition 2:

1. The transfer function (66) is now

$$T(s) = \frac{-\frac{1}{2}s^2 - \frac{1}{2}s}{s^3 + \frac{3}{2}s^2 + \frac{1}{2}s + \frac{4}{5}}$$

The transfer function  $-T(s)$  is not strictly positive real (SPR)<sup>7</sup>, but it satisfies the inequality (65).

2. The eigenvalues of the matrix  $(A_{11} - \frac{3}{4}B_1C)$  are  $\{-0.118 \pm 0.688j, -1.639\}$ ; hence, it is Hurwitz.
3. The controllability and observability matrices  $W_c$  and  $W_o$  for this case are

$$W_c = [B_1, A_{11}B_1, A_{11}^2B_1] = \begin{bmatrix} 1 & -0.5 & 0.65 \\ -4.5 & 2.25 & -3.125 \\ 0 & -0.5 & 0.25 \end{bmatrix}$$

$$W_o = \begin{bmatrix} C \\ CA_{11} \\ CA_{11}^2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 0 \\ 6.5 & 1.5 & 0 \\ -12.25 & -2.75 & 0.8 \end{bmatrix}$$

and both are of full rank. Hence, the pair  $[A_{11}, B_1]$  is controllable and the pair  $[C, A_{11}]$  is observable.

4. By construction,  $A_{22} = -3.5$ ; and it is negative.

All the conditions are valid; then, the controller (57) with the coefficients (72)–(74) stabilizes the system (1)–(2). Fig. 2 depicts the evolution of the variables  $\phi$  and  $\psi$  for the closed-loop system with the initial conditions  $\phi(0) = -2.07$ ,  $\psi(0) = 0.5$ ,  $z_1(0) = z_2(0) = 0$ .

<sup>7</sup>Going back to footnote 3, we note that it is important here to use the QC (8) instead of the weaker one since the latter leads to the SPR condition, which is not satisfied for this transfer function.

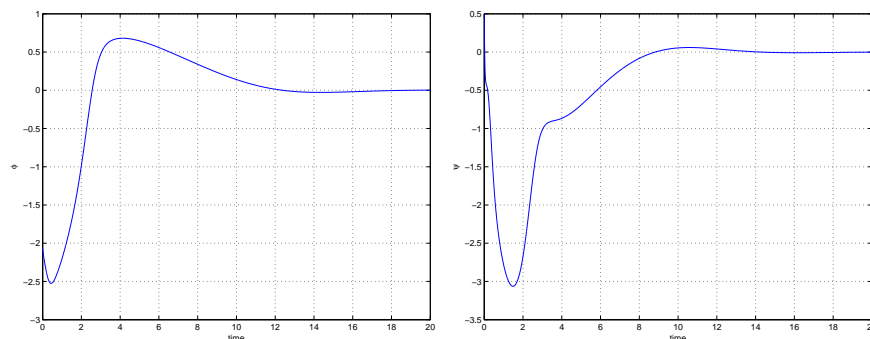


Figure 2: The solution of the closed-loop system with the dynamical controller (59) when coefficients are chosen as in (72)–(74).

To conclude, we have designed two globally stabilizing output feedback controllers for the nominal system (1)–(2) and have verified that they work in simulations. Let us now discuss the issue of robustness and give some comments on the contribution and the proposed approach.

## 6 Discussion and Comments

Our design is based on a structural transformation and on the fact that for each nonlinearity in the transformed system from a specified class there exists a quadratic Lyapunov function. Intuitively, certain degree of robustness is expected with respect to parametric uncertainties and measurement noise, see e.g., [22, Lemma 9.4]. A few related comments are given below.

### 6.1 Robustness of the closed-loop system (26) with respect to parametric uncertainty

We have developed a constructive procedure for synthesis of stabilizing controllers for (1)–(2), application of which leads to verification of two separate sets of conditions:

- The first one is one of the stability criteria for the dynamical system (3) presented in Theorems 1 and 2.
- The second one consists of the matching conditions for the closed-loop system to be equivalent to (3) for controllers with a certain structure.

The stability criteria explicitly rely on structural properties of nonlinearities of the system written as two quadratic constraints (7) and (12). These relations are valid for solutions of the closed-loop system (26) irrespective of

whether the matching conditions of Section 3 are in place or not<sup>8</sup>.

To illustrate this point, let us re-use the found quadratic constraints and apply the *Circle criterion* [22, 39] directly to re-prove the stability of the closed-loop system in Example 1 (Section 5.1), which has the form

$$\frac{d}{dt} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 & 0 \\ 1 & 17 & 4 \\ 0 & -\frac{207}{2} & -\frac{47}{2} \end{bmatrix}}_{= A} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \\ 0 & -\frac{9}{2} \end{bmatrix}}_{= [B_1, B_2]} \underbrace{\begin{bmatrix} 1 - (1 + \phi)^3 \\ 1 - (1 + 5\psi + z)^3 \end{bmatrix}}_{= \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}} \quad (75)$$

The outputs of the static nonlinearities

$$W_1 = W^{\{\phi\}}(v_1) = 1 - \left( 1 + \underbrace{(\phi)}_{= v_1} \right)^3 \quad \text{and} \quad W_2 = W^{\{\phi\}}(v_2) = 1 - \left( 1 + \underbrace{(5\psi + z)}_{= v_2} \right)^3$$

and the outputs of the linear parts of the system (75)

$$v_1 = \phi = \underbrace{[1, 0, 0]}_{= C_1} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \quad \text{and} \quad v_2 = 5\psi + z = \underbrace{[0, 5, 1]}_{= C_2} \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}$$

satisfy at least the next three quadratic constraints

$$-W_1 \cdot v_1 - \frac{3}{4} |v_1|^2 \geq 0, \quad -W_2 \cdot v_2 - \frac{3}{4} |v_2|^2 \geq 0, \quad -(W_1 - W_2) \cdot (v_1 - v_2) \geq 0 \quad (76)$$

valid for  $\forall \phi, \psi, z$ . Following the *Circle criterion*, to validate stability of the system (75) it is enough to check the next two conditions [39]

1. There are constants  $\tau_1 \geq 0, \tau_2 \geq 0, \tau_3 \geq 0$  such that  $\tau_1 + \tau_2 + \tau_3 > 0$  and the inequality

$$-\tau_1 \text{Re} \{ \tilde{w}_1^* \tilde{v}_1 + \frac{3}{4} |\tilde{v}_1|^2 \} - \tau_2 \text{Re} \{ \tilde{w}_2^* \tilde{v}_2 + \frac{3}{4} |\tilde{v}_2|^2 \} - \tau_3 \text{Re} \{ (\tilde{w}_1 - \tilde{w}_2)^* (\tilde{v}_1 - \tilde{v}_2) \} < 0 \quad (77)$$

with

$$\begin{aligned} \tilde{v}_1 &= \bar{T}_{11}(j\omega) \tilde{w}_1 + \bar{T}_{12}(j\omega) \tilde{w}_2 = C_1 (j\omega I_3 - A)^{-1} [B_1 \tilde{w}_1 + B_2 \tilde{w}_2] \\ \tilde{v}_2 &= \bar{T}_{21}(j\omega) \tilde{w}_1 + \bar{T}_{22}(j\omega) \tilde{w}_2 = C_2 (j\omega I_3 - A)^{-1} [B_1 \tilde{w}_1 + B_2 \tilde{w}_2] \end{aligned} \quad (78)$$

holds for all  $\tilde{w}_1 \in \mathbb{C}^1, \tilde{w}_2 \in \mathbb{C}^1, \omega \in \mathbb{R}^1$ . If the inequality (77) is met with  $\tau_i > 0$ , then the  $i$ -th quadratic constraint is called active.

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<sup>8</sup>These conditions are obviously needed if one would like to use one of the Theorems presented above to prove stability. However, their value is to give a hint on how to choose the structure of a feedback controller such that its parameters can be tuned to ensure stability using quadratic constraints.

2. There are row matrices  $K_1$  and  $K_2$  such that the linear relations

$$W_1 = K_1 \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \quad \text{and} \quad W_2 = K_2 \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix}$$

satisfy all the active quadratic constraints and the matrix

$$(A + B_1 K_1 + B_2 K_2)$$

is Hurwitz.

To use these statements for the system (75) let  $\tau_1 = 0$ ,  $\tau_2 = 1$ , and  $\tau_3 = \tau > 0$ ; so that the first quadratic constraint is not active, while the second and the third ones are taken into account. To meet the second condition we suggest to choose the following gains

$$K_1 = -\frac{3}{4}C_2 \quad \text{and} \quad K_2 = -\frac{3}{4}C_2$$

It is readily seen that with this choice the active constraints are satisfied:

$$-W_2 \cdot v_2 - \frac{3}{4}|v_2|^2 = \frac{3}{4}|v_2|^2 - \frac{3}{4}|v_2|^2 = 0, \quad -(W_1 - W_2) \cdot (v_1 - v_2) = -0 \cdot (v_1 - v_2) = 0$$

The eigenvalues of  $(A + B_1 K_1 + B_2 K_2)$  are

$$\lambda = \{-0.875, -1, -3.5\}$$

The inequality (77) is reduced to

$$-\operatorname{Re} \left\{ \tilde{w}_2^* \tilde{v}_2 + \frac{3}{4} |\tilde{v}_2|^2 \right\} - \tau \operatorname{Re} \left\{ (\tilde{w}_1 - \tilde{w}_2)^* (\tilde{v}_1 - \tilde{v}_2) \right\} < 0 \quad (79)$$

By straightforward computations, it is equivalent to the inequality

$$\begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix}^* \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} \operatorname{Re} \{T_{11}(j\omega)\} & T_{12}(j\omega) \\ T_{12}(j\omega)^* & \tau \cdot \operatorname{Re} \{T_{22}(j\omega)\} \end{bmatrix}}_{\Pi(j\omega)} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{bmatrix} > 0$$

that should be valid for some  $\tau > 0$  and  $\tilde{w}_1 \in \mathbb{C}^1$ ,  $\tilde{w}_2 \in \mathbb{C}^1$ ,  $\omega \in \mathbb{R}^1$ . Here

$$\begin{aligned} T_{11}(s) &= \frac{0.5}{s + 0.875} \\ T_{12}(s) &= \frac{2.5s + 7}{(s + 3.5)(s + 1)(s + 0.875)} \\ T_{22}(s) &= \frac{0.5}{s + 3.5} \end{aligned}$$

The diagonal elements of  $\Pi(j\omega)$  are positive definite, therefore this  $2 \times 2$  matrix is positive definite for some  $\tau > 0$  if its determinant is positive. It is

not hard to show that

$$\begin{aligned} \det \Pi(j\omega) &= \tau \cdot \operatorname{Re} \{T_{11}(j\omega)\} \cdot \operatorname{Re} \{T_{22}(j\omega)\} - |T_{12}(j\omega)|^2 = \\ &= \frac{\omega^2 \left(\tau \frac{49}{64} - \frac{25}{4}\right) + \left(\tau \frac{49}{64} - 49\right)}{|j\omega + 3.5|^2 |-\omega^2 + 1.5j\omega + 0.5|^2} \end{aligned}$$

The last expression—and hence (79)—is positive if  $\tau > 64$ . The alternative proof of stability of the closed-loop system (75) is completed.

The above arguments and inequality (77) are more general stability conditions than proposed in Theorem 1. They allow using all three quadratic constraints and showing how to introduce a new quadratic constraint in the analysis if it is found. Verifying the inequality (77) for a system with perturbed values of parameters of the closed-loop system allows obtaining bounds on the perturbations for which quadratic stability is preserved.

## 6.2 Robustness of the closed-loop system (26) with respect to hidden dynamics: Simulation of the 3-State Moore-Greitzer model with non-trivial stall

The system (1)-(2) describes only a part of dynamics for the so-called three-state Moore-Greitzer model [26, 18, 27, 15], see in particular [27, (59),(60), and (61)]

$$\begin{aligned} \frac{d}{dt} \phi &= \frac{3}{2} \phi - \psi + \frac{1}{2} [1 - (1 + \phi)^3] - 3R(1 + \phi) \\ \frac{d}{dt} \psi &= \frac{1}{\beta^2} (\phi - u) \\ \frac{d}{dt} R &= -\sigma R^2 - \sigma R (2\phi + \phi^2), \quad R(0) \geq 0 \end{aligned} \tag{80}$$

where  $\sigma > 0$  is another parameter. The dynamics of the additional state variable  $R$ , known as stall, is hidden and is often considered as a dynamic perturbation to (1)–(2). The variable  $R$  cannot be measured and used for feedback design.

It can be shown that for any (even unbounded) solution of (80) the positive variable  $R(t)$  quickly enters the strip  $0 \leq R < 1$  and never leaves it [30, 31]. It is of interest to check whether the controllers designed above for the surge subsystem will work for the extended one (80). The typical trajectories of the two closed-loop systems with the design of the controllers of Example 1 and of Example 2 above are shown below. In these simulations, we have some noise added to the measured value of  $\psi$ .

In Figure 3 we show a trajectory with the design based on Proposition 1. As seen, this robustly stabilizing controller for the surge subsystem does not stabilize the origin of the extended system obtained by adding the stall dynamics. Note that it is not hard to verify that addition of the stall dynamics results in creation of a new equilibrium that is locally asymptotically stable.

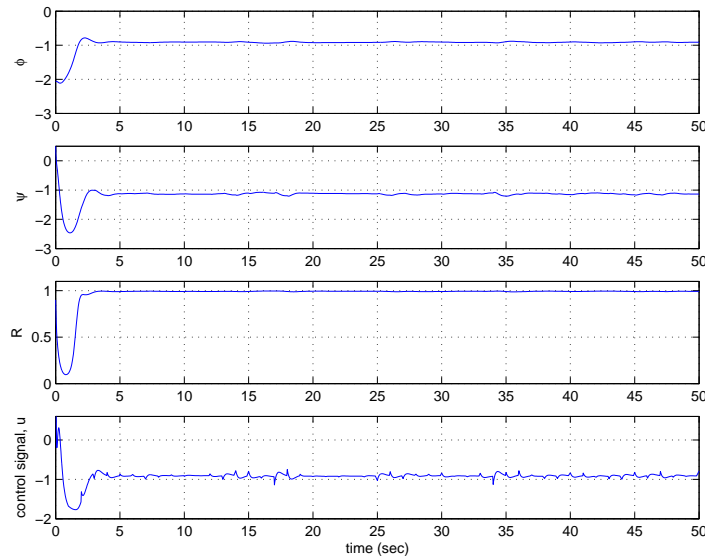


Figure 3: The solution of the closed-loop system with the dynamical controller from Proposition 1 and measurement noise.

In Figure 4 we show a trajectory with the design based on Proposition 2. As seen, this robustly stabilizing controller for the surge subsystem does stabilize the whole system when the stall dynamics is added. Note that the presence of the integral action in the controller prevents the appearance of other equilibria. In fact, it can be easily proved that there is a unique equilibrium of the closed-loop system with stall dynamics. Simulations with various initial conditions suggest that the system is still globally asymptotically stable. However, a rigorous proof of this assessment is not finished yet.

Meanwhile, let us come back to our result for the surge subsystem (1)–(2) and compare the proposed here approach with a more classical observer-based feedback control design.

### 6.3 Theorems 1-2 versus the Separation Principle

The presented decomposition of the state vector of the system (3) into vectors  $x$  and  $e$ , the structure of assumptions of Theorems 1 and 2, as well as arguments of their proofs may look related to the ideas behind the *separation principle* [22, bottom of p. 611]: “...that allows us to separate the [controller] design into two tasks. First we design a state feedback controller that stabilizes the system ... Then, an output feedback controller is obtained by replacing the state  $x$  by its estimate  $\hat{x}$  provided by the [high-gain] observer.”

In order to relate our design with a possible one directly based on a *Separation Principle*, we elaborate further on the first example from Section 5

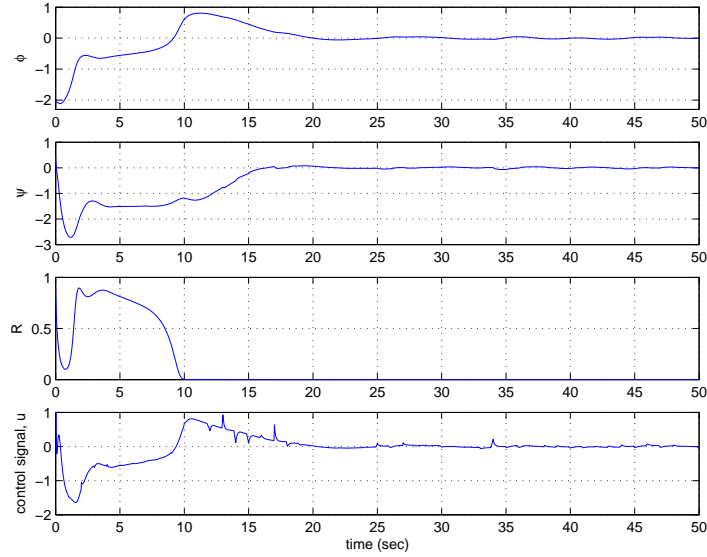


Figure 4: The solution of the closed-loop system with the dynamical controller from Proposition 2 and measurement noise.

and approach the task of static state feedback design for the system (1)–(2). Consider a family of state feedback controllers of the form

$$u = -\phi - \beta^2 \left[ \lambda_\phi \cdot \phi + \lambda_\psi \cdot \psi + w_u \cdot W^{\{\phi\}}(\phi) \right] \quad (81)$$

where  $\lambda_\phi$ ,  $\lambda_\psi$ ,  $w_u$  are constants and the nonlinearity  $W^{\{\phi\}}(\cdot)$  is from (6). The system (1)–(2) with the controller (6) is

$$\frac{d}{dt} \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{2} & -1 \\ \lambda_\phi & \lambda_\psi \end{bmatrix}}_{A_{st}} \begin{bmatrix} \phi \\ \psi \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ w_u \end{bmatrix}}_{B_{st}} W^{\{\phi\}}(v), \quad v = \underbrace{\begin{bmatrix} 1, 0 \end{bmatrix}}_{= C_{st}} \begin{bmatrix} \phi \\ \psi \end{bmatrix} \quad (82)$$

The nonlinearity  $W^{\{\phi\}}(\cdot)$  satisfies the QC (9). Hence, the conditions similar to assumptions 1 and 2 of Theorem 1 can be used for description of stabilizing controllers (81). Namely, if the transfer function

$$T(s) = -C_{st} (sI_2 - A_{st})^{-1} B_{st} = \frac{-\frac{1}{2}s + \tilde{p}_0}{s^2 + \tilde{l}_1 s + \tilde{l}_0} \quad (83)$$

satisfies the inequality

$$\operatorname{Re} \{T(j\omega)\} - \frac{3}{4} |T(j\omega)|^2 < 0, \quad \forall \omega \geq 0 \quad (84)$$

and the matrix  $(A_{st} - \frac{3}{4}B_{st}C_{st})$  is Hurwitz, then the closed-loop system (82) is quadratically stable and the corresponding static state feedback control law (81) is stabilizing. For instance, this is the case for the parameters

$$\lambda_\phi = 5, \quad \lambda_\psi = -3, \quad w_u = 1$$

Then the matrices of (82) are

$$A_{st} = \begin{bmatrix} \frac{3}{2} & -1 \\ \frac{1}{5} & -3 \end{bmatrix}, \quad B_{st} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, \quad C_{st} = [1, 0] \quad (85)$$

and the transfer function

$$T(s) = -C_{st}(sI_2 - A_{st})^{-1}B_{st} = \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2 + \frac{3}{2}s + \frac{1}{2}},$$

while the eigenvalues of  $(A_{st} - \frac{3}{4}B_{st}C_{st})$  are  $\{-1, -0.875\}$ . As seen, the conditions are the same as we found for the  $x$ -subsystem in Example 1, when the matrices  $A_{11}$ ,  $B_1$ ,  $C$  were computed as (70) to meet all the matching conditions.

This observation shows that, even though it might be difficult (or impossible) to meet literally all 8 parameters of  $A_{st}$ ,  $B_{st}$ ,  $C_{st}$  with some choice of controller coefficients of (43), one can interpret the  $x$ -subsystem of Example 1 as closed-loop system (1)-(2) under the full state feedback (81): Indeed, we do not need to compare and match the coefficients of (85) and (70) (even after a similarity transformation), but rather focus on properties of the associated transfer functions  $T(s)$ . If there is a choice of  $T(s)$  that meets the conditions of quadratic stability with the QC (9), then reaching this transfer function by the full-state feedback control as well as by appropriate choice of coefficients using output feedback control would be the tasks. However, it is worth to note that matching the structure of a particular  $T(s)$  for full state and output feedback designs might not be feasible for both cases simultaneously.

Another comment that relates the presented output feedback design to the *separation principle* is the observation that the  $e$ -variable for Example 1 written as

$$e = \phi - t_\psi\psi - t_z z = \phi - \hat{\phi}$$

and its dynamics can be seen as the error variable and the reduced-order observer for  $\phi$ . Assumption 2 of Theorem 1 implies that such error dynamics is exponentially stable, see also [24, 23, 2]. So the analogy of Example 1 with the observer-based feedback design and the *separation principle* is clearly in place. However, there are decompositions of the state of the closed-loop system (3) into  $x$  and  $e$ , which might not easily admit such interpretations and arguments. For instance, one can consider the case when  $\dim e > \dim \phi = 1$ .



## 6.4 Bibliographical remarks

Based on a PDE model of compression systems, Moore and Greitzer applied Galerkin approximation to provide a simplified ODE model of the pressure and flow behavior of a compressor system [16, 26, 27], which is in the form of (80). Using the Moore-Greitzer model, considerable research was carried out on the analysis and control of the stall and surge, revealing instabilities and difficult challenges in nonlinear control [6, 7, 8, 9, 20, 21, 23, 28, 29, 30, 31, 32, 35, 36]. In addition, several approaches to stabilization were made based on models described by partial differential equations underlying the Moore-Greitzer model [6, 19]. Parametrized nonlinear state feedback stabilizing the Moore-Greitzer system to the right of the compressor characteristic was proposed in [38]. Various approaches to observer designs relevant for output-feedback stabilization of Moore-Greitzer systems are to be found in [7, 10, 28, 13, 2, 3, 4]. Among these observer design circle criterion formulations with attention to quadratic constraints are provided. Whereas stabilization of the Moore-Greitzer model were not always accomplished, several valuable partial results were provided. Sometimes, these results have provided important theoretical progress on nonlinear output feedback. However, there is still no controllers that are shown to provide acceptable performance in an experiment even for the practical systems where the stall dynamics can be disregarded. Proposing such a design would make an enormous impact on many industries.

## 7 Conclusion

We have suggested two families of robust dynamic output feedback controllers that globally stabilize a well-known two dimensional model for surge instability of compressor systems assuming only availability of measurements for one state associated with the drop in pressure. Theoretical results are rigorously proved using quadratic constraints; while the feedback controllers are systematically designed using the Frequency Theorem and a new constructive procedure using a feedback transformation of the dynamics into a special block form. Controllers from the first family ensure global exponential stabilization. The ones from the second family provide integral action but only ensure local exponential and global asymptotic stability. Performance and robustness with respect to measurement noise and presence of hidden stall dynamics are verified by simulations. Some relations with direct application of the circle criterion and observer-based designs are drawn.

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## References

- [1] M. Arcak, “A global separation theorem for a new class of nonlinear observers,” *Proc. 41st IEEE Conf. Decision and Control*, pp. 676–681, 2002.
- [2] M. Arcak and P.V. Kokotović, “Nonlinear observers: a circle criterion design and robustness analysis,” *Automatica*, vol. 37 (12), pp. 1923–1930, 2001.
- [3] M. Arcak and P.V. Kokotović, “Observer-based control of systems with slope-restricted nonlinearities,” *IEEE Trans. Automatic Control*, vol. 46 (7), pp. 1146–1150, 2001.
- [4] M. Arcak and P.V. Kokotović, “Feasibility conditions for circle criterion designs,” *Systems and Control Letters*, vol. 42 (5), pp. 405–412, 2001.
- [5] A.N. Atassi and H.K. Khalil, “A separation principle for the stabilization of a class of nonlinear systems,” *IEEE Trans. Automatic Control*, vol. 44, pp. 1672–1687, 1999.
- [6] A. Banaszuk, H.A. Hauksson, and I. Mezić, “A backstepping controller for a nonlinear partial differential equation model of compression system instabilities,” *Proc. 36th Conf. Decision and control*, pp. 1050–1055, 1997.
- [7] B. Bøhagen and J.T. Gravdahl, “Circle criterion observer for a compression system,” *Proc. 2007 American Control Conf.*, pp. 3553–3558, 2007.
- [8] B. Bøhagen and J.T. Gravdahl, “Active surge control of compression system using drive torque,” *Automatica*, vol. 44 (4), pp. 1135–1140, 2008.
- [9] K.K. Botros, “Development in centrifugal compressor surge control—A technology assessment,” *ASME J. Turbomachinery*, vol. 116, pp. 133–142, 1994.
- [10] N.A. Chaturvedi and S.P. Bhat, “Output-feedback semiglobal stabilization of stall dynamics for preventing hysteresis and surge in axial-flow compressors,” *IEEE Trans. Control Systems Technology*, vol. 14 (2), pp. 301–307, 2006.
- [11] B. de Jager, “Rotating stall and surge control: A survey,” *Proc. of 34th Conf. Decision and Control*, pp. 1857–1862, 1995.
- [12] R. Johansson and A. Robertsson, “Observer-based strict positive real (SPR) feedback control system design,” *Automatica*, vol. 38, pp. 1557–1564, 2002.
- [13] X. Fan and M. Arcak, “Observer design for systems with multivariable monotone nonlinearities,” *Systems and Control Letters*, vol. 50, pp. 319–330, 2003.
- [14] D. Fontaine, S. Liao, J.D. Paduano, and P.V. Kokotović, “Nonlinear control experiments on axial flow compressor,” *IEEE Trans. Automatic Control*, vol. 12, pp. 683–693, 2004.
- [15] J.T. Gravdahl and O. Egeland, *Compressor surge and rotating stall: Modeling and control*, London: Springer-Verlag, 1999.
- [16] E.M. Greitzer, “Surge and rotating stall in axial flow compressors. Part I, II,” *ASME J. Eng. for Power*, vol. 98 (2), pp. 190–217, 1976.
- [17] E.M. Greitzer, “The stability of pumping systems—the 1980 Freeman scholar lecture,” *ASME J. Fluids Eng.*, vol. 103, pp. 193–243, 1981.
- [18] E.M. Greitzer and F.K. Moore, “A theory of post-stall transients in axial compressor systems: Part II—Application,” *J. Eng. Gas Turbines and Power*, 108, pp. 231–239, 1986.
- [19] G. Hagen, I. Mezić, and B. Bamieh, “Distributed control design for parabolic evolution equations: Application to compressor stall control,” *IEEE Trans. Automatic Control*, vol. 89 (8), pp. 1247–1258, 2004.

- [20] J. van Helvoirt, *Centrifugal compressor surge, modeling and identification for control*, Ph.D. thesis (NUR 978), Technische Universiteit Eindhoven, Netherlands, 2007.
- [21] J. van Helvoirt and B. de Jager, "Dynamic model including piping acoustics of a centrifugal compression system," *J. Sound and Vibration*, vol. 302 (1-2), pp. 361–378, 2007.
- [22] H. Khalil, *Nonlinear systems* (3rd ed.), Prentice Hall, Upper Saddle River, 2002.
- [23] M. Krstić, D. Fontaine, P.V. Kokotović, and J.D. Paduano, "Useful nonlinearities and global stabilization of bifurcations in a model of jet engine surge and stall", *IEEE Trans. Automatic Control*, vol. 43, pp. 1739–1745, 1998.
- [24] M. Krstić, I. Kanellakopoulos, and P.V. Kokotović, *Nonlinear and adaptive control design*, New York: Wiley, 1995.
- [25] M. Maggiore and K.M. Passino, "A separation principle for a class of non-UCO systems", *IEEE Trans. Automatic Control*, vol. 48(7), pp. 1122–1133, 2003.
- [26] F.K. Moore, "A theory of rotating stall of multistage axial compressors: Parts I-III", *J. Eng. Gas Turbines and Power*, 106, pp. 313–336, 1984.
- [27] F.K. Moore and E.M. Greitzer, "A theory of post-stall transients in axial compressor systems: Part I-Development of equations," *J. Eng. Gas Turbines and Power*, vol. 108, pp. 68–76, 1986.
- [28] J.D. Paduano, E.M. Greitzer, and A. Epstein, "Compression system stability and active control", *Annual Reviews Fluid Mechanics*, vol. 33, pp. 491–517, 2001.
- [29] J.D. Paduano, L. Valavani, A. Epstein, E.M. Greitzer, and G.R. Guenette, "Modeling for control of rotating stall", *Automatica*, vol. 30(9), pp. 1357–1373, 1994.
- [30] A.S. Shiriaev, R. Johansson, and A. Robertsson, "Some comments on output feedback stabilization of Moore-Greitzer compressor model," Proc. of *43rd Conf. Decision and Control*, pp. 4465–4466, 2004.
- [31] A.S. Shiriaev, R. Johansson, A. Robertsson, and L.B. Freidovich, "Output feedback stabilization of the Moore-Greitzer compressor model," Proc. of *44th IEEE Conf. Decision and Control / the European Control Conf.*, pp. 1102–1107, 2005.
- [32] A.S. Shiriaev, R. Johansson, A. Robertsson, and L.B. Freidovich, "Criteria for global stability of coupled systems with application to robust output feedback design for active surge control," Proc. of *IEEE Multi-conf. on Systems and Control*, pp. 1021–1026, 2009.
- [33] H.A. Simon, "Dynamical programming under uncertainty with quadratic criterion function," *Econometrica*, vol. 24 (1), pp. 74–81, 1956.
- [34] A.R. Teel and J. Hespanha "Examples of GES systems that can be driven to infinity by arbitrarily small additive decaying exponentials," *IEEE Trans. on Automatic Control*, vol. 49 (8), pp. 1407–1410, 2004.
- [35] F. Willems and B. de Jager, "Modeling and control of compressor flow instabilities," *IEEE Control Systems Magazine*, vol. 19 (5), pp. 8–18, 1999.
- [36] F. Willems, W.P.M.H. Heemels, B. de Jager, and A.A. Stoorvogel, "Positive feedback stabilization of centrifugal compressor surge," *Automatica*, vol. 38 (2), pp. 311–318, 2002.
- [37] W.M. Wonham, "On separation theorem of stochastic control," *SIAM J. Control*, vol. 6 (2), pp. 312–326, 1968.
- [38] M.Q. Xiao and T. Basar, "Analysis and Control of multi-mode axial flow compression system models," *J. Dynamic Systems, Measurement and Control*, vol. 122, pp. 393–401, 2000.
- [39] V.A. Yakubovich, G.A. Leonov, and A.Kh. Gel'g, *Stability of stationary sets in control systems with discontinuous nonlinearities*, Singapore: World Scientific, 2004.

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