Optimal Linear Control for Channels with Signal-to-Noise Ratio Constraints

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Erik Johannesson, Anders Rantzer and Bo Bernhardsson

***Abstract*— We consider the problem of stabilizing and minimizing the disturbance response of a SISO LTI plant, subject to a stochastic disturbance, over an analog communication channel with additive white noise and a signal-to-noise ratio (SNR) constraint. The controller is linear, based on output feedback and has a structure with two degrees of freedom: One part represents sensing and encoding operations and the other part represents decoding and issuing the control signal. It is shown that the problem of simultaneously designing the two optimal controller parts can be solved in two stages: First a functional depending both on the 1- and 2-norms of the Youla parameter is minimized. This minimization can be arbitrarily well approximated by a quasiconvex program. The second stage consists of a spectral factorization.**

I. INTRODUCTION

The trend towards decentralized control systems has in recent years inspired a lot of research on networked control systems (NCS). As control systems are required to operate using non-ideal communication channels between its parts, it becomes important to take into account the impact of these channels on the control performance. Communication constraints, which are a fundamental aspect of NCS, can take various forms depending on the type of communication system used. In digital networks there may be packet drops, bit rate limitations, and time delays. In analog communication systems there may be constraints on the Signal-to-Noise Ratio (SNR).

The NCS considered in this paper uses an analog communication channel and has the architecture seen in Fig. 1. The controller has two degrees of freedom: C can be seen as a sensor/encoder and D as a decoder/controller.

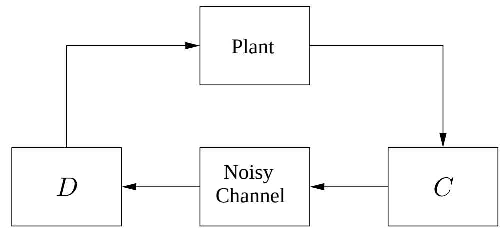

Fig. 1. General feedback system with noisy communication channel. In this paper, C and D are linear and the channel has an SNR constraint.

A. Previous Research

Until a few years ago, the majority of the research on NCS with analog channels was focused on fundamental limitations. Stabilizability of the feedback loop has been characterized for general noisy channels in [12]. For Additive White Noise (AWN) channels, conditions on the SNR for stabilizability were derived, under different assumptions, in [2] and [13]. limitations due to noisy channels have also been characterized in [10] and [6].

More recently, the design problem has gained more attention. Design of an encoder-decoder pair with one degree of freedom, when placing a fixed nominal controller in either part, was studied in [7]. In [4], the decoder was optimized under the assumption of a constant gain encoder, and it was shown that this structure is optimal for first order plants. Another approach was taken in [14], where instead the decoder was fixed to be a unit gain.

The case when the encoder has access to the channel output (feedback channel) has been considered in [1], where it was shown that non-linear strategies can be better than linear if the system is not of order one. Further, linear strategies were studied in [13], [15] and others. The feedback channel makes the problem different, but it is interesting to see that the solution in [15] involves minimizing a functional with a similar structure to the one obtained in this paper.

The problem of optimizing the control performance at a given terminal time was considered in [5] and [3]. The solutions may however yield poor transient performance and therefore be unsuitable for closed-loop control.

B. Main Contribution

The problem of designing the optimal linear output feedback controller with two degrees of freedom is considered. The plant is SISO, LTI and subject to a stochastic disturbance. The objective of the controller is to stabilize the system, satisfy an SNR constraint on the noisy channel and minimize the plant output.

It is shown that the optimal controller can be found by first minimizing a functional which depends on a combination of 1- and 2-norms of the Youla parameter. It is demonstrated that this minimization can be arbitrarily well approximated by a quasiconvex program. The controller is then obtained from a spectral factorization.

The solution technique is based on transfer function representations. It is closely related to the methodology used in [8] for design of an encoder-decoder pair for signal estimation over a channel of the same type as here.

C. Notation

Denote the unit circle by $\mathbb{T}$. For $1 \leq p \leq \infty$, we define the Lebesgue spaces $\mathcal{L}_p$ and the Hardy spaces $\mathcal{H}_p$, over $\mathbb{T}$, in the usual manner. The space of real, rational and proper

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transfer functions is denoted by $\mathcal{R}$. The intersections of $\mathcal{R}$ with $\mathcal{H}_p$ and $\mathcal{L}_p$ are denoted $\mathcal{RH}_p$ and $\mathcal{RL}_p$ respectively. For details, consult standard textbooks such as [11] and [18].

For $1 \leq p < \infty$ and scalar transfer functions X and Y, define the p-norms

$$\|X\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|X(e^{i\omega})\right|^p d\omega\right)^{1/p}$$

and the quantity

$$\langle X, Y\rangle = \frac{1}{2\pi}\int_0^{2\pi} X^*(e^{i\omega})Y(e^{i\omega})d\omega.$$

A transfer function $X \in \mathcal{H}_p$ is said to be outer if the set $\{Xq : q \text{ is a polynomial in } z^{-1}\}$ is dense in $\mathcal{H}_p$. If X is rational, then this is equivalent to $X(z) \neq 0$ for $|z| > 1$.

Equalities and inequalities involving functions in $\mathcal{L}_p$ evaluated on $\mathbb{T}$ are to be interpreted as holding almost everywhere on $\mathbb{T}$. That is, the subset of $\mathbb{T}$ in which the (in)equality does not hold is of measure zero. Transfer function arguments will sometimes be omitted when they are clear from context.

II. PROBLEM FORMULATION

Consider the system in Fig. 2. The plant G is assumed to be a finite dimensional, SISO, LTI system whose transfer function $G(z)$ admits a coprime factorization over $\mathcal{RH}_\infty$. The input signals, the disturbance v and the channel noise n are mutually independent white noise sequences with zero mean and identity variance (the signal w is only included in the stability definition and is otherwise assumed to be zero). The system is studied under a stationarity assumption, so the feedback system is required to be internally stable.

The communication channel is assumed to be an AWN channel with a transmission power constraint. Specifically,

$$r(k) = t(k) + n(k), \quad \mathbf{E}(t(k)^2) \leq \sigma^2 \tag{1}$$

where k is the time index, t is the transmitted variable, r is the received variable, n is the channel noise, and $\sigma > 0$ determines the maximum instantaneous transmission power. Since the transmission power constraint in this case is equivalent to an SNR constraint [13], we shall refer to it as the SNR constraint.

The objective is to find the transfer functions of C and D such that $\mathbf{E}(y^2)$, the stationary variance of the plant output, is minimized, while making the system internally stable and satisfying the SNR constraint (1). The search is restricted to linear C, D. However, we make no claim that linear solutions are optimal per se.

Under these assumptions, the relevant variances are given by the closed-loop transfer functions, and we have:

Problem 1:

$$\underset{C,D}{\text{minimize}}\, \|y\|_2^2 = \left\|\frac{G}{1-GDC}\right\|_2^2 + \left\|\frac{DG}{1-GDC}\right\|_2^2$$

subject to

$$\|t\|_2^2 = \left\|\frac{CG}{1-GDC}\right\|_2^2 + \left\|\frac{DCG}{1-GDC}\right\|_2^2 \leq \sigma^2$$

while achieving internal stability of the feedback system. ■

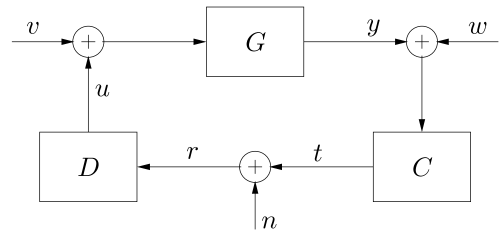

Fig. 2. Model of feedback system with disturbance and a noisy communication channel. The input w is only used for definition of internal stability.

III. SOLUTION

This section is divided into four subsections. In the first, conditions for internal stability of the feedback system are presented. In the second, it is shown how to find optimal C and D if their product DC is given. In the third subsection, a criterion for stabilizability under the SNR constraint is presented, and the factorization result is used to show equivalence between Problem 1 and minimization of a functional in the Youla parameter. Finally, in the fourth subsection, it is shown that this minimization problem can be approximated arbitrarily well by a quasiconvex optimization problem.

A. Internal Stability

In order to analyze internal stability of the closed loop system, we consider the block diagram in Fig. 2. The system can be represented by the closed loop map T,

$$\begin{bmatrix} y \\ t \\ u \end{bmatrix} = T \begin{bmatrix} v \\ w \\ n \end{bmatrix}.$$

The feedback system of Fig. 2 is said to be internally stable if

$$T = \begin{bmatrix} G & DCG & DG \\ CG & C & DCG \\ DCG & DC & D \end{bmatrix} (1 - GDC)^{-1} \in \mathcal{H}_2. \tag{2}$$

This definition implies that, given stochastic input signals with finite variance, all the signals in the system will have bounded variance.

The product DC will play an important role so we introduce the notation $K = DC$. For practical reasons, we will restrict ourselves to considering $K \in \mathcal{RL}_1$. It will be shown in section III-D that this restriction does not change the infimum value of the problem.

Together with (2), $K \in \mathcal{RL}_1$ implies that

$$\begin{bmatrix} 1 & -K \\ -G & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{1-GK} & \frac{K}{1-GK} \\ \frac{G}{1-GK} & \frac{1}{1-GK} \end{bmatrix} \in \mathcal{RH}_\infty, \tag{3}$$

since these transfer functions are rational and have no poles on or outside the unit circle. It is well-known that the set of K satisfying (3) can be parameterized using the Youla parameterization of all stabilizing controllers [18].

It will be shown in the next subsection how to find the optimal factors C and D, for any given $K \in \mathcal{RL}_1$ such that (3) holds. It turns out that the optimal factors can always be chosen so that $C \in \mathcal{H}_2$ is outer and $D \in \mathcal{L}_2$. For such K, C and D, it is easy to show that $T \in \mathcal{H}_2$. Hence, we make the following characterization of internal stability, which is a slight variation of the Youla parametrization:

Lemma 1: Suppose that $G = NM^{-1}$ is a coprime factorization over $\mathcal{RH}_\infty$ and that $U, V \in \mathcal{RH}_\infty$ satisfy the Bezout identity $VM + UN = 1$. Suppose further that $C \in \mathcal{H}_2$ is outer, that $D \in \mathcal{L}_2$ and that $K = DC \in \mathcal{RL}_1$. Then the closed loop system is internally stable if and only if

$$K = \frac{MQ - U}{NQ + V}, \quad Q \in \mathcal{RH}_\infty. \tag{4}$$

B. Optimal Factorization of K

Suppose for now that the product $K = DC \in \mathcal{RL}_1$ is given, and that (4) holds. Perhaps K is a nominal controller that is designed to have some desired properties and now has to be implemented in the architecture of Fig. 2. Another possibility is that K is optimal in the sense that it is the product of some C and D that is the solution to problem 1.

In either case, a natural question to ask is how to factorize K into C and D such that internal stability is achieved, the SNR constraint is satisfied (if possible) and $\|y\|_2^2$ is minimized (the latter is equivalent to minimizing the impact of the channel noise). If K is "optimal", then finding an optimal factorization should give a solution to Problem 1.

Problem 2 (Optimal Factorization): Given $K \in \mathcal{RL}_1$ such that (4) holds,

$$\underset{C,D}{\text{minimize}} \left\| \frac{G}{1-GK} \right\|_2^2 + \left\| \frac{DG}{1-GK} \right\|_2^2 \tag{5}$$

subject to

$$K = DC, \quad \left\| \frac{CG}{1-GK} \right\|_2^2 + \left\| \frac{KG}{1-GK} \right\|_2^2 \le \sigma^2. \tag{6}$$

while achieving internal stability of the feedback system. ■

Note that the first term in (5) and the second term in (6) are constant. Problem 2 can thus be solved by applying the following lemma. For simplicity, the lemma is written in terms of a transfer function H, which is to be interpreted as $H = G/(1 - GK)$ when considering Problem 2.

Lemma 2: Suppose that $\alpha > 0$, $H \in \mathcal{RH}_\infty$ and that $K \in \mathcal{RL}_1$ satisfies (3). Then the minimum

$$\min_{C \in \mathcal{H}_2, D \in \mathcal{L}_2} \|DH\|_2^2 \tag{7}$$

subject to the constraints

$$K = DC, \qquad \|CH\|_2^2 \le \alpha \tag{8}$$

is attained. The minimum value is

$$\frac{1}{\alpha} \left\|KH^2\right\|_1^2. \tag{9}$$

Moreover, if K is not identically zero, then C and D are optimal if and only if $C \in \mathcal{H}_2$, $D = KC^{-1} \in \mathcal{L}_2$ and

$$|C|^2 = \frac{\alpha}{\|KH^2\|_1} |K| \text{ on } \mathbb{T}. \tag{10}$$

If $K = 0$, then the minimum is achieved by $C = D = 0$.

Proof: The proof is trivial if $K = 0$, so assume that K is not identically zero. Then C is not identically zero and $D = KC^{-1}$. Cauchy-Schwarz's inequality gives that

$$\left\|KC^{-1}H\right\|_2^2 \|CH\|_2^2 \ge \langle \left|KC^{-1}H\right|, |CH| \rangle^2 = \left\|KH^2\right\|_1^2.$$

This shows that (9) is a lower bound on (7). Equality holds if and only if $\left|KC^{-1}H\right| = \lambda |CH|$ for some $\lambda \in \mathbb{R}$ and $\|CH\|_2^2 = \alpha$. It is easily verified that this is equivalent to the optimality condition (10).

It only remains to verify the existence of $C \in \mathcal{H}_2$ and $D = KC^{-1} \in \mathcal{L}_2$ such that (10) holds. Since K satisfies (3), it can be written as in (4) with $M, N, Q, U, V \in \mathcal{RH}_\infty$ and thus

$$\log|K| = \log|MQ - U| - \log|NQ + V|.$$

By Lemma 6 (in appendix), $\log|MQ - U| \in \mathcal{L}_1$ and $\log|NQ + V| \in \mathcal{L}_1$. Thus $\log|K| \in \mathcal{L}_1$, so K satisfies

$$\int_0^{2\pi} \log\left|K\left(e^{i\omega}\right)\right| d\omega > -\infty. \tag{11}$$

It follows from Theorem 4 (in appendix) that there exists an outer function $C \in \mathcal{H}_2$ such that (10) holds. Finally, $D = KC^{-1} \in \mathcal{L}_2$ since

$$\left\|KC^{-1}\right\|_2^2 = \frac{1}{\alpha} \left\|KH^2\right\|_1 \|K\|_1 < \infty.$$

■

Note that thanks to Szegő's theorem, C can always be chosen to be $\mathcal{H}_2$ and outer instead of, for example, $\mathcal{L}_2$, without any increase in the optimal value. However, changing this restriction could give more possible solutions. For example, we could choose D to be $\mathcal{H}_2$ and outer. In this paper, we settle on restricting C in this way since it simplifies the characterization of internal stability.

Analyzing the structure of the solution to the factorization problem, it is interesting to note that the magnitudes of C and D are directly proportional to the square root of the magnitude of K, on the unit circle. In other words, the dynamics of K is "evenly" distributed on both sides of the communication channel. The static gain of C is such that the SNR constraint is active.

C. Minimum Variance Control

We will now present a condition for stabilizability of a plant under the SNR constraint. First, we need to define the set of admissible pairs (C, D),

$$\Theta_{C,D} = \Big\{(C,D) : C \in \mathcal{H}_2 \text{ is outer}, D \in \mathcal{L}_2, DC \in \mathcal{RL}_1, \\ T \in \mathcal{H}_2, \left\| \frac{CG}{1-GDC} \right\|_2^2 + \left\| \frac{DCG}{1-GDC} \right\|_2^2 \le \sigma^2 \Big\},$$

and the set of admissible Q,

$$\Theta_Q = \Big\{ Q : Q \in \mathcal{RH}_\infty, K = \frac{MQ - U}{NQ + V} \in \mathcal{RL}_1, \\ \|MNQ - NU\|_2^2 < \sigma^2 \Big\}. \tag{12}$$

The next lemma says that the smallest SNR compatible with stabilization by linear filters can be found by considering Θ_Q. That is, by minimizing $\|MNQ - NU\|_2^2$. This result was previously presented as Theorem III.2 in [2], where an analytical formula for the smallest SNR was also given, showing that the SNR requirement depends not only on the unstable plant poles but also on the non-minimum phase zeros and the relative degree. The condition is included here in the present form to simplify the main theorem.

Lemma 3: Suppose that $\sigma > 0$, $G = NM^{-1}$ is a coprime factorization over $\mathcal{RH}_\infty$ and that $U, V \in \mathcal{RH}_\infty$ satisfy the Bezout identity $VM + UN = 1$. Then there exists $(C, D) \in \Theta_{C,D}$ if and only if there exists $Q \in \Theta_Q$.

Proof: Suppose that $(C, D) \in \Theta_{C,D}$ and

$$\left\| \frac{DCG}{1 - GDC} \right\|_2^2 = \sigma^2.$$

Then

$$\left\| \frac{CG}{1 - GDC} \right\|_2 = 0 \quad \Rightarrow \quad \left\| \frac{DCG}{1 - GDC} \right\|_2 = 0,$$

which is a contradiction. Hence,

$$\left\| \frac{DCG}{1 - GDC} \right\|_2^2 < \sigma^2 \quad \forall (C, D) \in \Theta_{C,D}. \tag{13}$$

By Lemma 1 there exists $Q \in \mathcal{RH}_\infty$ such that (4) holds for $K = DC \in \mathcal{RL}_1$. Moreover,

$$\sigma^2 > \left\| \frac{KG}{1 - GK} \right\|_2^2 = \|MNQ - NU\|_2^2, \tag{14}$$

so $Q \in \Theta_Q$.

Suppose conversely that $Q \in \Theta_Q$. Then (3) and (14) hold. Define

$$\alpha = \sigma^2 - \left\| \frac{KG}{1 - GK} \right\|_2^2, \qquad H = \frac{G}{1 - GK}. \tag{15}$$

Then Lemma 2 shows existence of an outer $C \in \mathcal{H}_2$ and $D \in \mathcal{L}_2$ such that $DC = K$ and $\|CH\|_2^2 \le \alpha$. It then follows from Lemma 1 that $T \in \mathcal{H}_2$, so $(C, D) \in \Theta_{C,D}$. ■

The following theorem is one of the main results of this paper. It shows that the infimum of Problem 1 is equal to the infimum of an optimization problem in the Youla parameter.

Theorem 1: Make the same assumptions and definitions as in Lemma 3. Further, suppose that $\Theta_{C,D}$ is non-empty and introduce $A = N^2$, $B = NV$, $E = MN$ and $F = -NU$ and the functionals

$$\varphi(C, D) = \left\| \frac{G}{1 - GDC} \right\|_2^2 + \left\| \frac{DG}{1 - GDC} \right\|_2^2$$

$$\psi(Q) = \|AQ + B\|_2^2 + \frac{\|(AQ + B)(EQ + F)\|_1^2}{\sigma^2 - \|EQ + F\|_2^2}. \tag{16}$$

Then

$$\inf_{(C,D) \in \Theta_{C,D}} \varphi(C, D) = \inf_{Q \in \Theta_Q} \psi(Q).$$

Moreover, suppose $Q \in \Theta_Q$ minimizes $\psi(Q)$. Then let

$$K = (MQ - U)(NQ + V)^{-1}.$$

If K is not identically zero, then $(C, D) \in \Theta_{C,D}$ minimize $\varphi(C, D)$ if and only if $K = DC$ and

$$|C|^2 = \frac{\sigma^2 - \left\| \frac{KG}{1 - GK} \right\|_2^2}{\left\| \frac{KG^2}{(1 - GK)^2} \right\|_1} |K| \ \text{ on } \mathbb{T}. \tag{17}$$

If $K = 0$, then $\varphi(C, D)$ is minimized by $C = D = 0$.

Proof: Define the sets

$$\Theta_{C,D}(K) = \{(C, D) : (C, D) \in \Theta_{C,D}, DC = K\}$$

$$\Theta_K = \left\{ K : K \in \mathcal{RL}_1 \text{ satisfies (3)}, \left\| \frac{KG}{1 - GK} \right\|_2^2 < \sigma^2 \right\}.$$

Note that due to (13),

$$(C, D) \in \Theta_{C,D} \Leftrightarrow (C, D) \in \Theta_{C,D}(K) \text{ for some } K \in \Theta_K.$$

since the additional inequality in Θ_K imposes no restriction.

Now the minimization problem will be rewritten through a series of equalities, followed by an explanation of each step.

$$\begin{aligned}
\inf_{(C,D) \in \Theta_{C,D}} \varphi(C, D) &= \inf_{K \in \Theta_K} \left(\inf_{(C,D) \in \Theta_{C,D}(K)} \varphi(C, D) \right) \\
&= \inf_{K \in \Theta_K} \left(\left\| \frac{G}{1 - GK} \right\|_2^2 + \inf_{(C,D) \in \Theta_{C,D}(K)} \left\| \frac{DG}{1 - GK} \right\|_2^2 \right) \\
&= \inf_{K \in \Theta_K} \left(\left\| \frac{G}{1 - GK} \right\|_2^2 + \frac{\left\| \frac{KG^2}{(1 - GK)^2} \right\|_1^2}{\sigma^2 - \left\| \frac{KG}{1 - GK} \right\|_2^2} \right) \\
&= \inf_{Q \in \Theta_Q} \|AQ + B\|_2^2 + \frac{\|(AQ + B)(EQ + F)\|_1^2}{\sigma^2 - \|EQ + F\|_2^2}
\end{aligned}$$

In the first equality, the minimization over $\Theta_{C,D}$ is parameterized by the product $K = DC$. In the second equality, the first term of $\varphi(C, D)$ is moved out from the inner minimization since it is constant for fixed K. In the third equality, Lemma 2 is applied with α and H given by (15). In the fourth equality, the parameterization (4) is used, since (3) holds. The Bezout identity gives that

$$\frac{G}{1 - GK} = AQ + B, \quad \frac{KG}{1 - GK} = EQ + F.$$

The optimality conditions for C and D follow from the application of Lemma 2. ■

D. Quasiconvex Approximation

It will now be shown that the minimization of $\psi(Q)$ over Θ_Q can be arbitrarily well approximated by a quasiconvex optimization problem. To this end, the constraint $K \in \mathcal{RL}_1$, which is non-convex in Q, is removed. The minimization is thus done over the convex set

$$\check{\Theta}_Q = \left\{ Q : Q \in \mathcal{RH}_\infty, \|EQ + F\|_2^2 < \sigma^2 \right\}. \tag{18}$$

instead of over Θ_Q. Clearly, $\Theta_Q \subseteq \check{\Theta}_Q$. This change of optimization domain is motivated by the following theorem:

Theorem 2: With definitions given by (12), (16) and (18),

$$\inf_{Q \in \Theta_Q} \psi(Q) = \inf_{Q \in \check{\Theta}_Q} \psi(Q). \tag{19}$$

Proof: Only a sketch of the proof is provided here due to limited space. Clearly, $\inf_{Q\in\Theta_Q}\psi(Q) \geq \inf_{Q\in\check{\Theta}_Q}\psi(Q)$ since $\Theta_Q \subseteq \check{\Theta}_Q$. Conversely, for any $\check{Q} \in \check{\Theta}_Q \backslash \Theta_Q$, define Q as a small perturbation of $\check{Q}$, such that $NQ+V$ has no zeros on $\mathbb{T}$. Existence of such perturbations can be shown using the implicit function theorem. For sufficiently small perturbations, $|\psi(Q) - \psi(\check{Q})|$ and $\left|\|EQ+F\|_2 - \|E\check{Q}+F\|_2\right|$ will be small by continuity. Hence, $Q \in \Theta_Q$ and the stated equality follows from $\inf_{Q\in\Theta_Q}\psi(Q) \leq \inf_{Q\in\check{\Theta}_Q}\psi(Q)$. ■

Theorem 2 shows that the approximation can be made sufficiently accurate. If the obtained K has poles on the unit circles, a small perturbation must be done to make factorization possible. It will now be shown that the approximation is quasiconvex. To this end, define the functional

$$\rho(a,e) = \frac{1}{2\pi}\int_0^{2\pi} a(\omega)^2 d\omega + \frac{\left(\frac{1}{2\pi}\int_0^{2\pi} a(\omega)e(\omega)d\omega\right)^2}{\sigma^2 - \frac{1}{2\pi}\int_0^{2\pi} e(\omega)^2 d\omega},$$

with domain

$$\left\{(a,e) : a(\omega), e(\omega) \geq 0 \ \forall\omega, \frac{1}{2\pi}\int_0^{2\pi} e(\omega)^2 d\omega < \sigma^2\right\}.$$

Lemma 4: Suppose $Q \in \check{\Theta}_Q$. Then $\psi(Q) \leq \gamma$ if and only if there exists (a,e) such that $\rho(a,e) \leq \gamma$ and

$$a \geq |AQ+B|, \quad e \geq |EQ+F| \quad \forall\omega. \tag{20}$$

Proof: Suppose $\psi(Q) \leq \gamma$. Then it is enough to define $a = |AQ+B|$ and $e = |EQ+F|$. Conversely, suppose that a and e satisfies the stated conditions. It then follows from inspection of the functionals that $\psi(a,e) \leq \rho(a,e) \leq \gamma$. ■

Lemma 5: The functional $\rho(a,e)$ is quasiconvex.

Proof: $\rho(a,e) - \gamma$ is the Schur complement of

$$\mathcal{M} = \begin{bmatrix} \frac{1}{2\pi}\int_0^{2\pi} e(\omega)^2 d\omega - \sigma^2 & \frac{1}{2\pi}\int_0^{2\pi} a(\omega)e(\omega)d\omega \\ \frac{1}{2\pi}\int_0^{2\pi} a(\omega)e(\omega)d\omega & \frac{1}{2\pi}\int_0^{2\pi} a(\omega)^2 d\omega - \gamma \end{bmatrix}.$$

So $\rho(a,e) \leq \gamma$ if and only if $\mathcal{M} \preceq 0$. Equivalently,

$$\frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} e(\omega) \\ a(\omega) \end{bmatrix} \begin{bmatrix} e(\omega) \\ a(\omega) \end{bmatrix}^T d\omega \preceq \begin{bmatrix} \sigma^2 & 0 \\ 0 & \gamma \end{bmatrix}. \tag{21}$$

Pre- and postmultiplication with $z \in \mathbb{R}^2$ gives the equivalent condition

$$\left\|\begin{bmatrix} e & a \end{bmatrix} z\right\|_{\mathcal{L}_2}^2 \leq z^T \begin{bmatrix} \sigma^2 & 0 \\ 0 & \gamma \end{bmatrix} z \quad \forall z,$$

which is convex in (e,a). ■

Theorem 3: The problem of minimizing $\psi(Q)$ over $\check{\Theta}_Q$ is quasiconvex.

Proof: Suppose that $Q_1, Q_2 \in \check{\Theta}_Q$, $\psi(Q_1) \leq \gamma$, $\psi(Q_2) \leq \gamma$ and $0 \leq \theta \leq 1$. Then by Lemma 4 $\exists a_1, a_2, e_1, e_2$ such that $\rho(a_1,e_1) \leq \gamma$ and $\rho(a_2,e_2) \leq \gamma$. By Lemma 5, $\rho(\theta a_1 + (1-\theta)a_2, \theta e_1 + (1-\theta)e_2) \leq \gamma$. Moreover, the constraints (20) are convex in (a,e,Q). Hence, it follows from Lemma 4 that $\psi(\theta Q_1 + (1-\theta)Q_2) \leq \gamma$. ■

IV. Procedure for Numerical Solution

A. Optimization Program

By Lemma 4, the problem can be solved by minimizing $\rho(a,e)$. In other words, minimize γ subject to (20), $\frac{1}{2\pi}\int_0^{2\pi} e(\omega)^2 d\omega < \sigma^2$ and $\mathcal{M} \preceq 0$. However, this problem is infinite-dimensional, so the integrals must be discretized and Q must be given a finite basis representation.

For $N \geq 2$, define the grid points $\{\omega_k\}_{k=1}^N$, where $\omega_1 = 0$ and $\omega_{k+1} - \omega_k = 2\pi/N$. Then $\mathcal{M} \preceq 0$ is approximated by

$$\begin{bmatrix} \frac{1}{N}\sum_{k=1}^N e(\omega_k)^2 - \sigma^2 & \frac{1}{N}\sum_{k=1}^N a(\omega_k)e(\omega_k) \\ \frac{1}{N}\sum_{k=1}^N a(\omega_k)e(\omega_k) & \frac{1}{N}\sum_{k=1}^N a(\omega_k)^2 - \gamma \end{bmatrix} \preceq 0,$$

or

$$\begin{bmatrix} N\sigma^2 & 0 \\ 0 & N\gamma \end{bmatrix} - \begin{bmatrix} e(\omega_1) & a(\omega_1) \\ e(\omega_2) & a(\omega_2) \\ \vdots & \vdots \\ e(\omega_N) & a(\omega_N) \end{bmatrix}^T I_N \begin{bmatrix} e(\omega_1) & a(\omega_1) \\ e(\omega_2) & a(\omega_2) \\ \vdots & \vdots \\ e(\omega_N) & a(\omega_N) \end{bmatrix} \succeq 0.$$

Using Schur complement again, this is equivalent to

$$\begin{bmatrix} 1 & & & & e(\omega_1) & a(\omega_1) \\ & 1 & & & e(\omega_2) & a(\omega_2) \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & e(\omega_N) & a(\omega_N) \\ e(\omega_1) & e(\omega_2) & \dots & e(\omega_N) & N\sigma^2 & 0 \\ a(\omega_1) & a(\omega_2) & \dots & a(\omega_N) & 0 & N\gamma \end{bmatrix} \succeq 0. \tag{22}$$

The remaining constraints can be approximated by

$$a(\omega_k) \geq \left|A(e^{i\omega_k})Q(e^{i\omega_k}) + B(e^{i\omega_k})\right|, \ k = 1 \dots N \tag{23}$$

$$e(\omega_k) \geq \left|E(e^{i\omega_k})Q(e^{i\omega_k}) + F(e^{i\omega_k})\right|, \ k = 1 \dots N \tag{24}$$

$$\frac{1}{N}\sum_{k=1}^N e(\omega_k)^2 < \sigma^2 \tag{25}$$

Minimizing γ subject to (22)–(25) is a semidefinite program with second-order cone constraints. By definition of the integral, the approximations converge as $N \to \infty$, so the value of the approximated problem is arbitrarily close to (19) for sufficiently large N.

B. Algorithm for solving Problem 1

1) Determine a coprime factorization of G and calculate A, B, E, F as defined in Theorem 1.
2) Parameterize Q by a finite basis representation, for example as an FIR filter.
3) Choose N large and calculate the grid points.
4) Minimize γ subject to (22)–(25). If the problem is infeasible it could mean that a larger σ is needed to stabilize the plant. This can be checked analytically using the condition in [2]. If σ is sufficiently large, the problem could still become infeasible if N is too small or Q is too coarsely parameterized.
5) Calculate K using (4) and determine a stable and outer spectral factor of $|K|$. Most likely, this has to be done approximately.
6) Using this spectral factor, calculate C according to (17). Finally, let $D = KC^{-1}$.

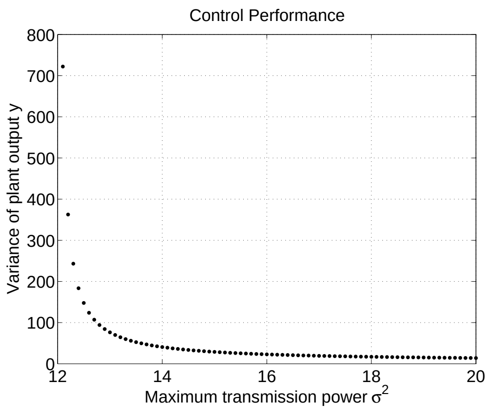

Fig. 3. Variance of the plant output y as a function of the maximum allowed transmission power σ^2, for the plant $G = 1/(z(z-2))$. The variance grows unbounded as σ^2 approaches the lower limit for stabilization.

V. NUMERICAL EXAMPLE

Consider the plant $G = 1/(z(z-2))$. It has one unstable pole and a one-sample time delay. Using Theorem III.2 in [2], we determine that stabilization is possible for $\sigma^2 > 12$.

Problem 1 was solved for various values of σ^2, using the method in section IV with Matlab and the toolboxes [9] and [16]. In the optimization program, $N = 629$ grid points were used and Q was parametrized as an FIR filter with length 20. The performance is plotted in Fig. 3 as a function of σ^2. It can be seen that the variance of the plant output grows unbounded as σ^2 approaches 12 and the feedback system comes closer to instability.

VI. CONCLUSIONS

This paper has three important contributions. First of all, it shows that the considered design problem is in some sense equivalent to minimizing a functional of the Youla parameter. The functional has an interesting structure as it depends both on 1- and 2-norms of the parameter. Second, it gives a numerical algorithm for finding an arbitrarily good approximation of the solution to the optimization problem. Third, conditions are given for the optimal factorization of a nominal controller, so that it can be implemented in the system architecture of Fig. 1.

This work suggests many areas for future research. For example, the extension to MIMO plants, or an investigation of the (sub)optimality of linear solutions.

APPENDIX

The following lemma consists of one of the results stated in Theorem 17.17 in [11].

Lemma 6: Suppose $0 < p \le \infty$, $X \in \mathbf{H_p}$, and X is not identically zero. Define

$$\bar{X}(e^{i\omega}) = \lim_{r \to 1^+} X(re^{i\omega}).$$

Then $\log|\bar{X}| \in \mathbf{L_1}$.

The following theorem is a generalization of the Fejér-Riesz Theorem and can be found in [17].

Theorem 4 (Szegő): Suppose that $f(\omega)$ is a non-negative function on $\omega \in [-\pi, \pi]$, that is Lebesgue integrable and that $\int_{-\pi}^{\pi} \log f(\omega)\, d\omega > -\infty$. Then there exists an outer function $X \in \mathbf{H_2}$ such that for almost all $\omega \in [-\pi, \pi]$ it holds that $X(e^{i\omega}) = \lim_{r\to 1^+} X(re^{i\omega})$ and $f(\omega) = |X(e^{i\omega})|^2$.

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REFERENCES

[1] R. Bansal and T. Basar, "Simultaneous design of measurement and control strategies for stochastic systems with feedback," *Automatica*, vol. 25, no. 5, pp. 679–694, 1989.

[2] J. Braslavsky, R. Middleton, and J. Freudenberg, "Feedback stabilization over signal-to-noise ratio constrained channels," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1391–1403, Aug. 2007.

[3] J. S. Freudenberg and R. H. Middleton, "Stabilization and performance over a gaussian communication channel for a plant with time delay," *2009 American Control Conference*, pp. 2148–2153, 2009.

[4] J. Freudenberg, R. Middleton, and J. Braslavsky, "Stabilization with disturbance attenuation over a gaussian channel," in *Decision and Control, 2007 46th IEEE Conference on*, dec. 2007, pp. 3958 –3963.

[5] ——, "Minimum variance control over a gaussian communication channel," *2008 American Control Conference*, pp. 2625–2630, 2008.

[6] J. Freudenberg, R. Middleton, and V. Solo, "Stabilization and disturbance attenuation over a gaussian communication channel," *Automatic Control, IEEE Transactions on*, vol. 55, no. 3, pp. 795 –799, mar. 2010.

[7] G. C. Goodwin, D. E. Quevedo, and E. I. Silva, "Architectures and coder design for networked control systems," *Automatica*, vol. 44, no. 1, pp. 248 – 257, 2008.

[8] E. Johannesson, A. Rantzer, B. Bernhardsson, and A. Ghulchak, "Encoder and decoder design for signal estimation," in *Proc. American Control Conference*, Baltimore, USA, June 2010.

[9] J. Löfberg, "Yalmip : A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004. [Online]. Available: http://control.ee.ethz.ch/ joloef/yalmip.php

[10] N. Martins and M. Dahleh, "Feedback control in the presence of noisy channels: "Bode-like" fundamental limitations of performance," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1604–1615, Aug. 2008.

[11] W. Rudin, *Real and Complex Analysis*, 3rd ed. McGraw-Hill Science/Engineering/Math, May 1986.

[12] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link — part I: Scalar systems," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3369–3395, Aug. 2006.

[13] E. I. Silva, G. C. Goodwin, and D. E. Quevedo, "Control system design subject to SNR constraints," *Automatica*, vol. 46, no. 2, pp. 428 – 436, 2010.

[14] E. I. Silva, D. E. Quevedo, and G. C. Goodwin, "Optimal controller design for networked control systems," in *Proceedings of the 17th World Congress, The International Federation of Automatic Control*, Seoul, Korea, July 2008.

[15] E. Silva, M. Derpich, and J. Østergaard, "On the minimal average data-rate that guarantees a given closed loop performance level," in *Proceedings of the 2nd IFAC Workshop on Distributed Estimation and Control in Networked Systems (NecSys)*, Annecy, France, 2010.

[16] J. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999, version 1.3 available from `http://sedumi.ie.lehigh.edu/`.

[17] G. Szegő, *Orthogonal polynomials*, 4th ed. American Mathematical Society, Providence, RI, 1975.

[18] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.