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Hagander, Per; Hansson, Anders

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Per Hagander and Anders Hansson

Department of Automatic Control
Lund Institute of Technology
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Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

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Author(s)
Per Hagander and Anders Hansson

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Abstract
Existence results for the H_2 controller are investigated. Assuming left and right invertibility gives a unique Riccati equation based controller, potentially with closed loop eigenvalues on the unit circle. It is then shown how this controller is optimal if and only if it stabilizes the closed loop system after removal of all its unobservable and uncontrollable modes. This condition is a considerable simplification of a general condition recently derived by Trentelman and Stoorvogel.

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1. Introduction

The LQG or H_2 controller has a long history. Much work was done during the 60:s and 70:s. The topic is treated in many text-books, among others [Kucera, 1991]. The singular cases were, however, not fully understood. This was discovered when people started to work on the optimal H_∞ controller, where similar singular results were needed. In e.g. minimum variance velocity control of a servo motor, where the position is measured, the integrating property from velocity to position makes the problem singular. A good understanding of singular LQG controllers also facilitates the analysis of the so called minimum upcrossing controller, [Hansson and Hagander, 1994].

Two different types of singularities are encountered in H_2 problems. The first type is related to non-uniqueness due to redundant control signals and measurement signals. This type of singularity can be taken care of by certain transformations, see e.g. [Hagander and Hansson, 1994]. The non-uniqueness will in the present work be parameterized by a simple equation related to the Q-parameterization of all stablizing controllers.

The second type of singularities is related to poles on the stability boundary of the closed loop system. In most such cases there exists no optimal controller corresponding to the infimal cost. It will, however, be shown that if and only if all the unstable modes of the closed loop system are in the controller and such that they are canceled by zeros of the controller, then this reduced order controller is indeed an optimal controller. This was actually discussed already in [Kucera, 1980]. To illustrate the result a minimum variance example is given as an introduction.

EXAMPLE 1—Minimum Variance Control
Consider the process model

$$A(q)y(k) = B(q)u(k) + C(q)e(q)$$

where $y(k)$ is the measurement signal, $u(k)$ the control signal, $e(k)$ is a sequence of independent zero mean Gaussian distributed random variables, and $A(q)$, $B(q)$, and $C(q)$ are polynomials in the forward shift operator q. Assume that $C(q)$ has all its zeros inside or on the unit circle, that $\deg A(q) = \deg C(q) = n$, and that $\deg B(q) = n-d$. It is then well-known, see [Åström and Wittenmark, 1990], that the controller that minimizes

$$\mathrm{E}\left\{y^2(k)\right\}$$

is given by $u(k) = -S(q)/R(q)y(k)$, where $S(q)$ and $R(q)$ satisfys the following Diophantine equation

$$A(q)R(q) + B(q)S(q) = P(q)C(q)$$

with $R(0) = S(0) = 0$, and

$$P(q) = q^d \prod_{i=1}^{s}(q - z_i) \prod_{i=s+1}^{n-d} (q - 1/z_i)$$

where z_i are the stable and unstable zeros of $B(q)$ respectively. If in addition $B(q)$ and $C(q)$ have no zeros on the unit circle, then the controller will also be stabilizing. The converse is, however, not always true, as the following example shows. Specialize the process polynomials to

$$\begin{aligned}
A(q) &= q^4 \\
B(q) &= (q-1)(q-2)^2 \\
C(q) &= q(q-1)(q^2 + 8/21q + 4/21)
\end{aligned}$$

There are two closed loop poles at $q = 1$ due to the presence of a factor $(q-1)$ in both $B(q)$ and $C(q)$. The Diophantine equation becomes

$$q^4R(q) + (q-1)(q-2)^2S(q) = q(q-1/2)^2(q-1)q(q-1)(q^2+8/21q+4/21)$$

with solutions $R(q) = q(q-1)^2(q+51/84)$ and $S(q) = -1/84q^2(q-1)^2$. Here the two closed loop poles at $q = 1$ are in the controller, i.e. $(q-1)^2$ are factors of $R(q)$. Furthermore they are canceled by the same factors in $S(q)$, and the reduced order controller

$$u(k) = -\frac{q}{84q+51}y(k) \tag{1}$$

is optimal. □

This example will later on be further investigated using state space descriptions and Riccati-equations instead of polynomial descriptions and Diophantine equations. It will be seen that the cancellation of the factor in $B(q)$ can be interpreted as loss of controllability in the controller, and that the cancellation of the factor in $C(q)$ can be interpreted as loss of observability in the controller.

It is almost trivial to see that the presence of all unstable closed loop poles in the optimal H_2 controller together with cancellation is a sufficient condition for it to be stabilizing. That it is also a necessary condition, is more tricky. The derivation will relay on a very general version of the separation principle. In [Trentelman and Stoorvogel, 1993] a geometric approach is taken to give necessary and sufficient conditions for a stabilizing H_2 controller. It will be seen that the conditions given there are equivalent to the more explicit ones given in this report. In [Trentelman and Stoorvogel, 1993] the more general case when the controller is not unique is also covered.

In [Chen *et al.*, 1993] an algorithm is given for constructing all stabilizing H_2 controllers. There the modes of the controller are not canceled, but instead moved to an arbitrary position inside the unit circle. This then enables the use of the Q-parametrization of all stabilizing controllers in order to give a necessary and sufficient condition for uniqueness of the optimal controller.

The report is organized as follows. In Section 2, first, the problem formulation is given together with the main results, Theorem 1. Then the example of this section is revisited. In Section 3 the proofs are carried out. In Section 4 an example of velocity control of a servo motor is investigated, and in Section 5 some concluding remarks are given. Some useful results on solutions of Riccati equations are given in an appendix.

2. Control Problem and Solution

In this section the H_2 control problem will be formulated. Equations for deriving the solution will be given together with conditions under which there exists an optimal solution. Finally, the results are applied to Example 1.

Control Problem

Consider the following state space description

$$\begin{pmatrix} x(k+1) \\ z(k) \\ y(k) \end{pmatrix} = \begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \\ u(k) \end{pmatrix} \tag{2}$$

where $w(k) \in R^l$ is a sequence of independent zero mean Gaussian random variables with covariance I, $u(k) \in R^m$ is the control signal, $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measurement signal, and $z(k) \in R^q$ is the signal to be controlled. It will be assumed that $D_{yu} = 0$. Denote by $\mathcal{D}$ the set of linear, proper, and time-invariant controllers, and by $\mathcal{D}_s$ the subset of $\mathcal{D}$ which stabilizes (2), i.e. the set of controllers which are such that the eigenvalues of the closed loop system have absolute values strictly less than one. Let the control signal be given by

$$u(k) = -H(q)y(k)$$

where $H \in \mathcal{D}$. Further introduce the following performance index:

$$J(H) = \lim_{k\to\infty} \mathrm{E}\left\{z^T(k)z(k)\right\}, \quad H \in \mathcal{D}_s \tag{3}$$

Since $H \in \mathcal{D}_s$, it is no loss in generality to assume that $x(0) = 0$ when evaluating J. Consider the following optimal control problem

$$\min_{H\in\mathcal{D}_s} J(H) \tag{4}$$

which is known as the H_2 problem. This is a convex problem, and hence the infimum always exists. However, the set $\mathcal{D}_s$ is open, and thus the infimum will not always be a minimum, i.e. the smallest value of the performance index J may be attained by a controller which does not internally stabilize the closed loop system. This is one of the questions that will be dealt with in the present report. The other question is uniqueness of the solution, and how to parameterize all candidate solutions in case of non-uniqueness.

Main Results

Introduce the following matrices:

$$P_c(z) = \begin{pmatrix} zI - A & -B_u \\ C_z & D_{zu} \end{pmatrix}$$
$$P_o(z) = \begin{pmatrix} zI - A & -B_w \\ C_y & D_{yw} \end{pmatrix}$$

and the following standing assumptions:

(A1) : (A, B_u) stabilizable

(A2) : (C_y, A) detectable

If (A1) and (A2) do not hold, then $\mathcal{D}_s$ will be empty, and hence there will be no optimal controller.

It will be shown that the optimal controllers, whenever they exist, can be obtained by solving the Riccati equations

$$\begin{aligned} S &= (A - B_uL)^TS(A - B_uL) + (C_z - D_{zu}L)^T(C_z - D_{zu}L) \\ G &= B_u^TSB_u + D_{zu}^TD_{zu} \\ G\begin{pmatrix} L & L_v & L_w \end{pmatrix} &= \begin{pmatrix} B_u^TSA + D_{zu}^TC_z & B_u^TS & D_{zu}^T \end{pmatrix} \end{aligned} \tag{5}$$

and

$$\begin{aligned} P &= (A - KC_y)P(A - KC_y)^T + (B_w - KD_{yw})(B_w - KD_{yw})^T \\ H &= C_yPC_y^T + D_{yw}D_{yw}^T \\ H\begin{pmatrix} K \\ K_x \\ K_v \\ K_w \end{pmatrix}^T &= \begin{pmatrix} APC_y^T + B_wD_{yw}^T \\ PC_y^T \\ B_wD_{yw}^T \\ D_{zw}D_{yw}^T \end{pmatrix}^T \end{aligned} \tag{6}$$

using for instance the Kleinman iteration procedure described in e.g. [Hagander and Hansson, 1994]. These are the unique real symmetric matrices $S \geq 0$ and

$P \geq 0$ such that there exist L and K such that $A - B_u L$ and $A - KC_y$ have all their eigenvalues inside or on the unit circle. They are also the maximal solutions. Let

$$\begin{aligned} A_{co} &= A - B_u L - KC_y + B_u D_c C_y \\ B_c &= K - B_u D_c \\ C_c &= L - D_c C_y \\ D_c &= LK_x + L_v K_v + L_w K_w \end{aligned} \tag{7}$$

and define the controller

$$H_{nom}(q) := C_{nom}(qI - A_{nom})^{-1}B_{nom} + D_{nom} = C_c(qI - A_{co})^{-1}B_c + D_c \tag{8}$$

where all uncontrollable and unobservable modes in $H_{nom}(q)$ should be canceled. How to cancel these modes will be described in Theorem 3 and the proof of Lemma 5. Some of the results will be derived under the assumptions

(A3) : $\max_z \text{rank} P_c(z) = n + m$

(A4) : $\max_z \text{rank} P_o(z) = n + p$

which are actually equivalent to the uniqueness of the optimal controller, i.e. $G > 0$ and $H > 0$ see e.g. [Hagander and Hansson, 1994], and usually referred to as left and right invertibility of (A, B_u, C_z, D_{zu}) and (A, B_w, C_y, D_{yw}) respectively, e.g. [Silverman, 1976].

THEOREM 1
Assume that the uniqueness conditions (A3) and (A4) hold. Then there exists a solution to (4) if and only if $H_{nom}(q)$ as defined in (8) is in $\mathcal{D}_s$, i.e. is stabilizing. Further, this solution is unique. □

Remark. The proof of this theorem will be carried out in Theorem 3 below, where the structure of the cancellations is further investigated.

The existence of an H_2 controller is thus easily investigated when the controller is unique. The parametrization of all H_2 controllers is slightly more tricky, and it will be postponed to the end of next section. Further there are no simple existence conditions available for the general case, see [Trentelman and Stoorvogel, 1993] for a non-explicit one.

Example Revisited

The process model in Example 1 can be casted as in (2) with

$$\begin{pmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{pmatrix} = \left(\begin{array}{cccc|c|c} 0 & 1 & 0 & 0 & -13/21 & 1 \\ 0 & 0 & 1 & 0 & -4/21 & -5 \\ 0 & 0 & 0 & 1 & -4/21 & 8 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Notice that assumptions (A3) and (A4) are fulfilled. The solutions of the Riccati equations are given by

$$S = \begin{pmatrix} * \end{pmatrix}, \quad \begin{pmatrix} L \\ L_v \end{pmatrix} = \frac{1}{64}\begin{pmatrix} 0 & 4 & -12 & 3 \\ 4 & -12 & 3 & * \end{pmatrix}, \quad L_w = 0$$

$$P = 0, \quad K_x = 0, \quad K = K_v = \frac{1}{21}\begin{pmatrix} -13 & -4 & -4 & 0 \end{pmatrix}^T, \quad K_w = 1$$

Thus

$$A_{co} = \begin{pmatrix} 0.6071 & 0.9375 & 0.1875 & -0.0469 \\ 0.2500 & 0.3125 & 0.0625 & 0.2344 \\ 0.0952 & -0.5000 & 1.5000 & 0.6250 \\ 0.0476 & 0.2500 & -0.7500 & 0.1875 \end{pmatrix}, \qquad B_c = \begin{pmatrix} -0.6071 \\ -0.2500 \\ -0.0952 \\ -0.0476 \end{pmatrix}$$

$$C_c = \begin{pmatrix} 0.0119 & 0.0625 & -0.1875 & 0.0469 \end{pmatrix}, \qquad D_c = -0.0119$$

where the controllable and observable part is only of first order, and it is easily verified that this gives the same controller as in (1). Notice that

$$\begin{aligned} P(q) &= \det\left[qI - (A - B_u L)\right] \\ C(q) &= \det\left[qI - (A - KC_y)\right] \end{aligned}$$

3. Derivation of the Results

In this section Theorem 1 will be proved. In the first subsection the separation principle will be shown to hold under very weak conditions. This will enable the stochastic approach to solve the H_2 problem presented in this report. Then in the second subsection a well-known sufficient condition for the existence of a solution, stating "no zeros on the unit circle", will be given. This will be utilized to derive a both necessary and sufficient condition in case of uniqueness of the controller. In the third subsection the conditions derived in this report will be related to the ones presented in [Trentelman and Stoorvogel, 1993]. The assumptions on uniqueness will be relaxed in the fourth subsection, where the Q-parametrization will be utilized to give a parametrization of all H_2 candidate controllers in case of non-uniqueness. The results are more explicit than the ones in [Chen *et al.*, 1993]. However, no simple existence results are known when assumptions (A3) and (A4) are not fulfilled.

The Separation Principle

The approach taken in this report is the classical stochastic approach for solving LQG or H_2 problems utilizing separation. To this end introduce the following observer

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_u u(k) + K\tilde{y}(k), \quad \hat{x}(0) = 0 \\ \tilde{y}(k) &= y(k) - C_y \hat{x}(k) \end{aligned} \tag{9}$$

usually called a stationary Kalman filter, where K is a solution of (6) such that $\operatorname{eig}(A - KC_y) \leq 1$. Define $\tilde{x}(k) = x(k) - \hat{x}(k)$. It then holds that

$$\begin{aligned} \tilde{x}(k+1) &= A_o \tilde{x}(k) + B_N w(k), \quad \tilde{x}(0) = 0 \\ \tilde{y}(k) &= C_y \tilde{x}(k) + D_{yw} w(k) \end{aligned} \tag{10}$$

where $A_o = A - KC_y$, and $B_N = B_w - KD_{yw}$. Since there is no guarantee for A_o being stable, some care has to be taken in order to get orthogonality between $\hat{x}(k)$ and $\tilde{x}(k)$. The following lemma will clarify.

LEMMA 1
It holds that $\hat{x}(k)$ and $\tilde{x}(k)$ as defined in (9) and (10) are orthogonal in stationarity, i.e. $\mathrm{E}\left\{\hat{x}(k)\tilde{x}^T(k)\right\} \to 0, \quad k \to \infty$. Further the stationary covariance of $\tilde{x}(k)$ is given by the solution P of (6).

Proof: It is well-known that there exists a time-varying Kalman filter, i.e. $K = K(k)$, which computes an estimate $\hat{x}(k)$ of $x(k)$ such that $\hat{x}(k)$ and $\tilde{x}(k)$ are orthogonal, provided that the initial value $\hat{x}(0)$ is correctly chosen. Further if A_o is stable then there also exists a stationary Kalman-filter with the properties given above. However, if $\tilde{x}(0) = 0$, then it can be shown that there still exists a stationary Kalman-filter, even if A_o is not stable. This follows from the fact that the unstable modes of A_o are not controllable from B_N, and hence these modes will be identically zero, provided that $\tilde{x}(0) = 0$, which make them uncorrelated with the corresponding $\hat{x}(k)$-modes. That the unstable modes of A_o are not controllable from B_N is the dual of Lemma 7 in the appendix. □

Remark. It can be shown that $\hat{x}(k) = \mathrm{E}\{x(k)|\mathcal{Y}(k-1)\}$, where $\mathcal{Y}(k-1) = \begin{pmatrix} y(k-1) & y(k-2) & \cdots \end{pmatrix}$.

In order to allow for not only strictly proper controllers but also for proper controllers, an estimate of $x(k)$ based on $\mathcal{Y}(k)$ is needed as well as estimates of $e_x(k) = B_w w(k)$ and $e_z(k) = D_{zw} w(k)$.

LEMMA 2
It holds that

$$\mathrm{E}\left\{ \begin{pmatrix} x(k) \\ e_x(k) \\ e_z(k) \end{pmatrix} \middle| \mathcal{Y}(k) \right\} - \left\{ \begin{pmatrix} K_x \\ K_v \\ K_w \end{pmatrix} \tilde{y}(k) + \begin{pmatrix} \hat{x}(k) \\ 0 \\ 0 \end{pmatrix} \right\} \to 0, \quad k \to \infty$$

Proof: The result follows by Lemma 1 and [Åström, 1970, Theorem 3.2. and Theorem 3.3]. □

THEOREM 2—Separation Principle
For any $H(q) \in \mathcal{D}_s$ it holds that

$$J(H(q)) = \lim_{k\to\infty} \mathrm{E}\left\{ [u(k) + L\hat{x}(k) + D_c\tilde{y}(k)]^T G [u(k) + L\hat{x}(k) + D_c\tilde{y}(k)] \right\} + J^*$$

where J^* is independent of $H(q)$.

Proof: The first step of the proof is a tedious completion of squares utilizing (5) and the fact that $\lim_{k\to\infty} \mathrm{E}\left\{x^T(k+1)Sx(k+1) - x^T(k)Sx(k)\right\} = 0$ for any stabilizing controller yielding

$$\begin{aligned} J(H(q)) = \mathrm{E}\Big\{ & [u(k) + Lx(k) + L_v e_x(k) + L_w e_z(k)]^T \\ & \cdot G[u(k) + Lx(k) + L_v e_x(k) + L_w e_z(k)] \Big\} + \bar{J} \end{aligned}$$

where $\bar{J}$ is independent of $H(q)$. Then by (6), (7), Lemma 2, and the orthogonality between the estimates and the estimation errors the result follows. □

Sufficient and Necessary Conditions for Existence

Let $A_c = A - B_u L$, and introduce the following assumptions as an alternative to (A3) and (A4)

(A5) : $\mathrm{rank}_{|z|=1} P_c(z) = \max_z \mathrm{rank} P_c(z)$

(A6) : $\mathrm{rank}_{|z|=1} P_o(z) = \max_z \mathrm{rank} P_o(z)$

which are equivalent to no zeros on the unit circle. Notice that these conditions were not fulfilled in Example 1. Then the following sufficient condition holds:

LEMMA 3
Under assumptions (A5) and (A6) the controller $H_{nom}(q)$ as defined in (8) is a solution to (4).

Proof: By Theorem 2 and since $G \geq 0$ it holds that $u(k) = -L\hat{x}(k) - D_c\tilde{y}(k)$ minimizes the performance index J. This is the same control signal as the one defined by H_{nom}. Further this controller is stabilizing, since the closed loop system is governed by

$$\begin{pmatrix} x(k+1) \\ \tilde{x}(k+1) \end{pmatrix} = \begin{pmatrix} A_c & B_u C_c \\ 0 & A_o \end{pmatrix} \begin{pmatrix} x(k) \\ \tilde{x}(k) \end{pmatrix} + \begin{pmatrix} B_N + B_c D_{yw} \\ B_N \end{pmatrix} w(k) \tag{11}$$

where A_c and A_o are stable under conditions (A5) and (A6) by e.g. [Hagander and Hansson, 1994, Theorem 1] and its dual version. □

It is possible to relax assumptions (A5) and (A6), i.e. to treat the case with zeros on the unit circle. This will, however, only be done under the uniqueness assumptions here. Hence assume that (A3) and (A4) hold for the rest of this subsection. Let U_c and U_o be transformations that bring A_c and A_o, respectively, to block diagonal form

$$\begin{aligned} A_c U_c &= U_c \begin{pmatrix} J_{cs} & 0 \\ 0 & J_{cu} \end{pmatrix} \\ A_o U_o &= U_o \begin{pmatrix} J_{os} & 0 \\ 0 & J_{ou} \end{pmatrix} \end{aligned} \tag{12}$$

where J_{cs} and J_{os} contain the stable parts. The theorem that follows characterizes the solutions of H_2 problems and proves the results of Theorem 1.

THEOREM 3

Assume that (A3) and (A4) hold. Then there exists a solution to (4) if and only if there exists a state transformation T such that

$$\begin{aligned} A_{co} T &= T \begin{pmatrix} J_{co} & 0 & * \\ * & J_{ou} & * \\ 0 & 0 & J_{cu} \end{pmatrix} \\ T^{-1} B_c &= \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix} \\ C_c T &= \begin{pmatrix} C_1 & 0 & C_3 \end{pmatrix} \end{aligned} \tag{13}$$

where J_{cu} and J_{ou} are given by (12). Further the controller is unique and given by (8), i.e.

$$H_{nom}(q) = C_1(qI - J_{co})^{-1} B_1 + D_c$$

Remark 1. In words this means that there exists an optimal controller if and only if all unstable closed loop modes are in the controller (7) and such that the ones from A_o are unobservable in C_c and the ones from A_c are uncontrollable from B_c. It will be seen that in the case of uncontrollability in the controller there are modes in the closed loop system which are not influenced by the noise, and hence do not influence the other modes. In the case of unobservability in the controller there are modes in the closed loop system which are not influencing the rest of the dynamics, although they are influenced by the noise. As well as providing a proof of Theorem 1 this contains a state space description of the properties of the controller (7).

Proof: The first part of the proof will be done along ideas from [Trentelman and Stoorvogel, 1994] by considering a sequence of controllers $H_i \in \mathcal{D}_s$, $i \geq 0$, which are such that for all $\varepsilon > 0$, there exists N such that $J(H_i) \leq J^* + \varepsilon$, for all $i \geq N$, Consider the perturbation of (2) obtained by the following replacements

$$
\begin{aligned}
B_w &\leftrightarrow \begin{pmatrix} B_w & \varepsilon I \end{pmatrix} \\
D_{yw} &\leftrightarrow \begin{pmatrix} D_{yw} & 0 \end{pmatrix} \\
C_z &\leftrightarrow \begin{pmatrix} C_z \\ \varepsilon I \end{pmatrix} \\
D_{zu} &\leftrightarrow \begin{pmatrix} D_{zu} \\ 0 \end{pmatrix}
\end{aligned}
$$

Denote the value of the performance index for the perturbed system and any $H \in \mathcal{D}_s$ by $J_\varepsilon(H)$. Notice that the perturbation does not influence the feedback loop, so $\mathcal{D}_s$ is also the set of controllers that stabilize the perturbed system. This system satisfies (A5) and (A6) and has an optimal controller $H_\varepsilon \in \mathcal{D}_s$ for all $\varepsilon > 0$ by Lemma 3. Denote the corresponding optimal value of the performance index by J_ε^*. Since the perturbation is linear it holds that $J^* \le J(H) \le J_\varepsilon(H)$ for all $H \in \mathcal{D}_s$. Especially this holds for $H = H_\varepsilon$, i.e.

$$J^* \le J(H_\varepsilon) \le J_\varepsilon^*$$

By Lemma 9 in the appendix it holds that S_ε and P_ε converge to S and P, which are solutions of (5) and (6), respectively. Hence $J_\varepsilon^* \to J^*$, $\varepsilon \to 0$, which by Theorem 2 implies that one control signal minimizing J is given by

$$u(k) = -L\hat{x}(k) - D_c\tilde{y}(k)$$

Since $G > 0$ by assumption (A3), this control signal is unique. It can also be expressed as

$$
\begin{aligned}
\hat{x}(k+1) &= A_{co}\hat{x}(k) + B_c y(k) \\
-u(k) &= C_c\hat{x}(k) + D_c y(k)
\end{aligned}
$$

where A_{co}, B_c, C_c, and D_c are unique by assumptions (A3) and (A4), see e.g. Lemma 8 in the appendix and [Hagander and Hansson, 1994]. Further the closed loop system is governed by (11). Hence there exists an optimal controller if and only if all unstable closed loop modes are in the controller and such that they do not have to be implemented, i.e. are either unobservable in C_c or uncontrollable from B_c. It only remains to show that uncontrollable modes in the controller are modes of A_c and that unobservable modes of the controller are modes of A_o. To this end let

$$\begin{pmatrix} u & U \end{pmatrix}$$

be a state transformation that brings A_{co} to Jordan form, i.e.

$$A_{co}\begin{pmatrix} u & U \end{pmatrix} = \begin{pmatrix} u & U \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ * & J \end{pmatrix}$$

and let

$$\begin{pmatrix} v^T \\ V^T \end{pmatrix}$$

be the inverse of $\begin{pmatrix} u & U \end{pmatrix}$. Further assume that λ is uncontrollable, i.e.

$$\begin{pmatrix} v^T \\ V^T \end{pmatrix} B_c = \begin{pmatrix} 0 \\ \bar{B} \end{pmatrix}$$

Introduce the new state ξ via

$$\xi = \begin{pmatrix} \xi_\lambda \\ \xi_J \end{pmatrix} = \begin{pmatrix} v^T \\ V^T \end{pmatrix}\hat{x}$$

It holds that

$$\xi(k+1) = \begin{pmatrix} \lambda & 0 \\ * & J \end{pmatrix} \xi(k) + \begin{pmatrix} 0 \\ \bar{B} \end{pmatrix} y(k)$$

Since $\hat{x}(0) = 0$, it follows that $\xi_\lambda(k) = 0$ for all k. Hence

$$\hat{x}(k+1) = UJV^T\hat{x}(k) + U\bar{B}\,y(k) = \bar{A}_c\hat{x}(k) + B_c\left[C_y\tilde{x}(k) + D_{yw}w(k)\right]$$

where

$$\bar{A}_c = UJV^T + U\bar{B}\,C_y$$

The estimation error is governed by

$$\tilde{x}(k+1) = A_o\tilde{x}(k) + \tilde{A}_c\hat{x}(k) + B_N w(k)$$

where

$$\tilde{A}_c = \begin{pmatrix} u & U \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ * & 0 \end{pmatrix} \begin{pmatrix} v^T \\ V^T \end{pmatrix}$$

Hence the closed loop system has the system matrix

$$\begin{pmatrix} \bar{A}_c & B_cC_y \\ \tilde{A}_c & A_o \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ B_2C_yu & J + B_2C_yU & B_2C_yu & B_2C_yU \\ \lambda & 0 & v^TA_ou & v^TA_oU \\ * & 0 & V^TA_ou & V^TA_oU \end{pmatrix}$$

which has the eigenvalues 0, $\mathrm{eig}(J + B_2C_yU)$ and $\mathrm{eig}(A_o)$. Since the eigenvalues of A_c are given by λ and $\mathrm{eig}(J + B_2C_yU)$, it follows that all uncontrollable eigenvalues of the controller are also eigenvalues of the closed loop system. Further they are eigenvalues of A_c, and they can be positioned arbitrarily, e.g. at zero as with the realization related to $\bar{A}_c$.

Now follows a dual version to prove that the unobservable modes of the controller are modes of A_o. To this end let

$$\begin{pmatrix} u & U \end{pmatrix}$$

be a state transformation that brings A_{co} to Jordan form, i.e.

$$A_{co}\begin{pmatrix} u & U \end{pmatrix} = \begin{pmatrix} u & U \end{pmatrix}\begin{pmatrix} \lambda & * \\ 0 & J \end{pmatrix}$$

and let

$$\begin{pmatrix} v^T \\ V^T \end{pmatrix}$$

be the inverse of $\begin{pmatrix} u & U \end{pmatrix}$. Further assume that λ is unobservable, i.e.

$$C_c\begin{pmatrix} u & U \end{pmatrix} = \begin{pmatrix} 0 & \bar{C} \end{pmatrix}$$

Introduce the new state ξ via

$$\xi = \begin{pmatrix} \xi_\lambda \\ \xi_J \end{pmatrix} = \begin{pmatrix} v^T \\ V^T \end{pmatrix} \hat{x}$$

It holds that

$$\begin{aligned} \xi(k+1) &= \begin{pmatrix} \lambda & * \\ 0 & J \end{pmatrix} \xi(k) + \begin{pmatrix} v^T \\ V^T \end{pmatrix} B_c y(k) \\ u(k) &= -C_2\xi_J(k) - D_c y(k) \end{aligned}$$

Hence any $\xi_\lambda(k)$ will yield the same control signal. Let $\xi_\lambda(k+1) = v^T B_c y(k)$, which implies

$$\hat{x}(k+1) = UJV^T\hat{x}(k) + B_c y(k) = \bar{A}\hat{x}(k) + B_u u(k) + K\tilde{y}(k)$$

where

$$\bar{A} = A + UJV^T - A_{co}$$

It is easily verified that the estimation error is governed by

$$\tilde{x}(k+1) = \bar{A}_o\tilde{x}(k) + \tilde{A}x(k) + B_N w(k)$$

where

$$\begin{aligned}\bar{A}_o &= \bar{A} - KC_y \\ \tilde{A} &= A - \bar{A}\end{aligned}$$

Further the state x is governed by

$$x(k+1) = A_c x(k) + B_u C_c \tilde{x}(k) + (B_c D_{yw} + B_N)w(k)$$

Hence the closed loop system has the system matrix

$$\begin{pmatrix} A_c & B_u C_c \\ \tilde{A} & \bar{A}_o \end{pmatrix} \sim \begin{pmatrix} v^T A_c u & v^T A_c U & 0 & v^T B_u C_2 \\ V^T A_c u & V^T A_c U & 0 & V^T B_u C_2 \\ \lambda & * & 0 & v^T B_u C_2 \\ 0 & 0 & 0 & J + V^T B_u C_2 \end{pmatrix}$$

which has the eigenvalues 0, $\mathrm{eig}(J + V^T B_u C_2)$ and $\mathrm{eig}(A_c)$. Since the eigenvalues of A_o are given by λ and $\mathrm{eig}(J + V^T B_u C_2)$, it follows that all unobservable eigenvalues of the controller also are eigenvalues of the closed loop system. Further they are eigenvalues of A_o, and they can be positioned arbitrarily, e.g. at zero as with the realization related to $\bar{A}_o$. This concludes the proof. □

Remark 2. It is obvious from the transformations in the proof above that the modes corresponding to the change of eigenvalue from λ to 0 are the ones which are removed from the closed loop system when implementing the controller as a reduced order controller. It should be stressed that what is novel in the approach in this report as compared to the approach in [Chen *et al.*, 1993] is that the unstable modes of the closed loop system are not only moved but that they are actually removed from the closed loop system by cancellations in the controller.

Remark 3. How to compute the state transformation T of Theorem 3 is described in Lemma 5 below.

Relation to Trentelman and Stoorvogel

Throughout this subsection it will be assumed that (A3) and (A4) hold. Let $\mathcal{V}_g$ be the invariant subspace associated with the stable eigenvalues of A_c, and let $\mathcal{S}_g$ be the invariant subspace associated with the unstable eigenvalues of A_o.

THEOREM 4—Trentelman and Stoorvogel
Assume that (A3) and (A4) hold. Then there exists a solution to (4) if and only if

(C1) : $\mathrm{Im} B_c \subset \mathcal{V}_g$

(C2) : $\mathcal{S}_g \subset \mathrm{Ker}\, C_c$

(C3) : $(A - B_u D_c C_y)\mathcal{S}_g \subset \mathcal{V}_g$

(C4) : $\mathcal{S}_g \subset \mathcal{V}_g$

Proof: This follows immediately from the conditions given in [Trentelman and Stoorvogel, 1993]. □

It will now be shown that these conditions are equivalent to the ones given in Theorem 3. To this end let

$$U_c = \begin{pmatrix} U_{cs} & U_{cu} \end{pmatrix}$$
$$U_o = \begin{pmatrix} U_{os} & U_{ou} \end{pmatrix}$$

and denote the inverses of these transformations by

$$V_c^T = \begin{pmatrix} V_{cs}^T \\ V_{cu}^T \end{pmatrix}$$
$$V_o^T = \begin{pmatrix} V_{os}^T \\ V_{ou}^T \end{pmatrix}$$

LEMMA 4

Assume that (A3) and (A4) hold. Then the conditions (C1)–(C4) of Theorem 4 are equivalent to :

(I) : All unstable modes of A_c are uncontrollable from B_c, i.e. $V_{cu}^T B_c = 0$.

(II) : All unstable modes of A_o are unobservable from C_c, i.e. $C_c U_{ou} = 0$.

(III) : $\mathrm{Im} U_{ou} \subset \mathrm{Im} U_{cs}$

Proof: It is trivial that (C2) is equivalent to (II), and that (C4) is equivalent to (III). That (C1) is equivalent to (I) follows from the fact that (C1) is equivalent to $\mathrm{Im} B_c \subset \mathrm{Im} U_{cs}$, which is equivalent to $\mathrm{Im} V_{cu} \subset \mathrm{Ker} B_c^T$, which is equivalent to (I). Further it holds that $(A - B_u D_c C_y) U_{ou} = (A_c + B_u C_c) U_{ou} = A_c U_{ou}$ by (II) or equivalently by (C2). By (III) or equivalently by (C4) there exist α such that $U_{ou} = U_{cs}\alpha$. Hence $(A - B_u D_c C_y) U_{ou} = A_c U_{cs}\alpha = U_{cs} J_{cs}\alpha$, where the second equality follows by the definition of U_{cs}. Hence conditions (C2) and (C4) imply condition (C3). □

LEMMA 5

Assume that (A3) and (A4) hold. Then the existence of a transformation T satisfying (13) of Theorem 3 is equivalent to conditions (I)–(III) of Lemma 4.

Proof: Assume that the condition for existence in Theorem 3 holds. Then multiply the first equation of (13) by $\begin{pmatrix} 0 & I & 0 \end{pmatrix}^T$ from the right. This implies that $U_{ou} = T \begin{pmatrix} 0 & I & 0 \end{pmatrix}^T$. Multiplying the first equation of (13) by $\begin{pmatrix} 0 & 0 & I \end{pmatrix} T^{-1}$ from the left implies that $V_{cu}^T = \begin{pmatrix} 0 & 0 & I \end{pmatrix} T^{-1}$. Hence $C_c U_{ou} = 0$, $V_{cu}^T B_c = 0$ and $V_{cu}^T U_{ou} = 0$, which are equivalent to conditions (I)–(III). Now assume the converse, i.e. that the conditions of Lemma 4 hold. Condition (III) implies that there exists α such that $U_{ou} = U_{cs}\alpha$. Notice that the columns of α are linearly independent, and that

$$V_c^T U_o = \begin{pmatrix} V_{cs}^T U_{os} & \alpha \\ V_{cu}^T U_{os} & 0 \end{pmatrix}$$

Consider

$$\begin{aligned}
V_c^T A_{co} U_o &= V_c^T U_o \left\{ \begin{pmatrix} J_{os} & 0 \\ 0 & J_{ou} \end{pmatrix} - V_o^T B_u \begin{pmatrix} C_c U_{os} & 0 \end{pmatrix} \right\} \\
&= \begin{pmatrix} * & \alpha J_{ou} \\ * & 0 \end{pmatrix} \\
&= \left\{ \begin{pmatrix} J_{cs} & 0 \\ 0 & J_{cu} \end{pmatrix} - \begin{pmatrix} V_{cs}^T B_c \\ 0 \end{pmatrix} C_y U_c \right\} V_c^T U_o \\
&= \begin{pmatrix} * & (J_{cs} - V_{cs}^T B_c C_y U_{cs})\alpha \\ * & 0 \end{pmatrix}
\end{aligned}$$

This implies that

$$(J_{cs} - V_{cs}^T B_c C_y U_{cs})\alpha = \alpha J_{ou}$$

Let β be such that $\begin{pmatrix} \beta & \alpha \end{pmatrix}$ is a basis, and such that

$$(J_{cs} - V_{cs}^T B_c C_y U_{cs}) \begin{pmatrix} \beta & \alpha \end{pmatrix} = \begin{pmatrix} \beta & \alpha \end{pmatrix} \begin{pmatrix} J_{co} & 0 \\ * & J_{ou} \end{pmatrix}$$

Now, consider

$$U_c^{-1} A_{co} U_c \begin{pmatrix} \beta & \alpha & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} J_{cs} - V_{cs}^T B_c C_y U_{cs} & * \\ 0 & J_{cu} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} J_{co} & 0 \\ * & J_{ou} \end{pmatrix} & * \\ 0 & J_{cu} \end{pmatrix}$$

Hence with

$$T = U_c \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U_{cs}\beta & U_{ou} & U_{cu} \end{pmatrix}$$

it holds that

$$A_{co} T = T \begin{pmatrix} J_{co} & 0 & * \\ * & J_{cu} & * \\ 0 & 0 & J_{ou} \end{pmatrix}$$

and by condition (II) it holds that

$$C_c T = \begin{pmatrix} C_1 & 0 & C_3 \end{pmatrix}$$

for some C_1 and C_3. Further

$$T^{-1} = \begin{pmatrix} \begin{pmatrix} \beta & \alpha \end{pmatrix}^{-1} & 0 \\ 0 & I \end{pmatrix} V_c$$

and hence it follows by condition (I) that

$$T^{-1} B_c = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}$$

for some B_1 and B_2. This concludes the proof. □

This shows that the conditions of theorems 3 and 4 for existence of a solution to (4) are equivalent. Notice that the conditions of Theorem 3 are formulated in terms of A_{co}, while Theorem 4 relates to A_c and A_o.

Parameterization of all Solutions

In this subsection assumptions (A3) and (A4) will be relaxed. The discussion will be intuitive.

When assumptions (A3) and (A4) are not fulfilled, there are redundant control signal or measurement signals, respectively. Further there is no unique H_2-controller. The conditions for existence get more involved, see [Trentelman and Stoorvogel, 1993], and the main concern of this section is to give a parametrization of all controllers yielding the infimum of the performance index. The question of when there exists an optimal controller, and which controllers are the optimal ones will not be dealt with here.

Also in case of non-uniqueness, controllers defined by (8) yields the infimal performance index, see the proof of Theorem 3, but they are not necessarily stabilizing. S and P are still unique, see appendix, but L and K are not unique. Further the closed loop system is governed by (11), and L and K can be chosen such that all modes of the closed loop system are stable except the ones corresponding to loss of rank in conditions (A5) and (A6). Hence the closed loop has all its eigenvalues inside or on the unit circle for any such choice of the controllers defined by (8). Now a version of the Q-parametrization with Q being a transfer function matrix with all its poles inside or on the unit circle will yield all closed loop systems with eigenvalues inside or on the unit circle. This follows from the fact that the Q-parametrization holds for any "stability" region. Hence all controllers yielding a "stable" closed loop system are given by

$$u(k) = -L\hat{x}(k) - D_c\tilde{y}(k) + Q(q)\tilde{y}(k)$$

which results in the following value of the performance index:

$$J(H(q)) - J^* = \lim_{k\to\infty} \mathrm{E}\left\{[Q(q)\tilde{y}(k)]^T G\,[Q(q)\tilde{y}(k)]\right\} = \frac{1}{2\pi}\mathrm{tr}\oint H_Q(z)H_Q^*(z)\frac{dz}{z}$$

where $H_Q(z) = \sqrt{G}Q(z)\sqrt{H}$. Hence all controllers corresponding to the infimum of the performance index can be obtained as the ones solving the equation

$$GQ(z)H = 0 \tag{14}$$

It is obvious that $Q(z) = 0$ is the unique solution of this equation if and only if $G > 0$ and $H > 0$, which is equivalent to conditions (A3) and (A4). Further the controller defined by (8) is unique under the same conditions. This shows that the solution to the H_2-problem is unique, whenever it exists, if and only if conditions (A3) and (A4) hold. Notice that this was proven already in Theorem 3 without considering the "Q"-parameterization above. In the general case it may happen that $Q(q) = 0$ gives closed loop poles on the unit circle, that are canceled in the controller for some $Q(q) \neq 0$ satisfying (14).

4. Velocity Control of a Servo Motor

In this section is given an example of minimum variance velocity control of a servo motor based on a position sensor. For this control problem it is inherent that a closed loop pole is positioned at plus one due to the integrating property from velocity to position. It will be seen that it is possible to remove this unstable mode from the closed loop system in case of a special noise process. Notice that this rather artificial noise demonstrates the cancellation of the unstable mode.

EXAMPLE 2—Velocity Control
Let the process be a sampled version, sample interval $h = -\ln 0.75 = 0.2875$, of the generic servo with transfer function

$$Y(s) = \frac{1}{s(s+1)}U(s)$$

which corresponds to the following discrete time state space description:

$$\begin{pmatrix} A & B_u \\ C_z & D_{zu} \\ C_y & D_{zy} \end{pmatrix} = \left(\begin{array}{cc|c} 0.75 & 0 & 0.25 \\ 0.25 & 1 & 0.0377 \\ \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right)$$

Further introduce the following artificial noise description

$$B_w = B_u = \begin{pmatrix} 0.125 & 0.0188 \end{pmatrix}^T, \quad D_{yw} = 1, \quad D_{zw} = 0$$

The solutions of the Riccati equations are given by $S = C_z^T C_z$, $P = 0$, and

$$L = \begin{pmatrix} 3 & 0 \end{pmatrix}, \quad L_v = \begin{pmatrix} 4 & 0 \end{pmatrix}, \quad L_w = 0$$
$$K = K_v = B_w, \quad K_x = K_w = 0$$

The closed loop system eigenvalues are then the eigenvalues of

$$A_c = \begin{pmatrix} 0 & 0 \\ 0.1370 & 1 \end{pmatrix}, \quad A_o = \begin{pmatrix} 0.75 & -0.25 \\ 0.25 & 0.9623 \end{pmatrix}$$

Three of them are inside the unit circle, while there is one at plus one. Further the controller realization is

$$A_{co} = \begin{pmatrix} 0 & 0 \\ 0.1370 & 1 \end{pmatrix}, \qquad B_c = 0$$
$$C_c = \begin{pmatrix} 3 & -1 \end{pmatrix}, \qquad D_c = 1$$

which corresponds to the transfer operator description $S(q)/R(q)$, where

$$R(q) = \det(qI - A_{co}) = q^2 - q, \quad S(q) = q^2 - q$$

and it is obvious that the closed loop pole at one is in the controller and uncontrollable. Hence the optimal controller is proportional and given by

$$u(k) = -y(k)$$

□

5. Conclusions

In this report the existence of H_2 controllers has been investigated. Special attention has been given to the case of "zeros on the unit circle", assumptions (A5) and (A6). Intuition about cancellation in the controller from the polynomial SISO minimum variance case has been shown to carry over to the more general H_2 case.

For the case of uniqueness of the controller it has been shown that there exists an H_2 controller if and only if all unstable modes of the closed loop system, when applying the controller obtained by solving the Riccati equations, are modes also of the controller and such that they are unobservable or uncontrollable. This condition is a striking interpretation of the conditions (C1)–(C4) in [Trentelman and Stoorvogel, 1993], and it is easier to check. Further it shows that the optimal controller given in [Chen *et al.*, 1993] is nonminimal. There unobservable or uncontrollable modes of the controller are not removed, just moved inside the unit circle.

When the controller is not unique, i.e. when assumptions (A3) and (A4) are not fulfilled, the approach taken in this report has not been as fruitful as for the case of uniqueness of the controller. However, it has been possible to give a parameterization of all candidates for optimal controllers, i.e. a parameterization of all controllers yielding the infimum of the performance index by means of a slight extension of the so called Q-parameterization. It is believed that this parameterization could be utilized to give a sufficient and necessary condition for the existence of a solution also in the case of non-uniqueness.

Further it is believed that the insight gained in this report may be fruitfully utilized when analyzing other singular control problems, such as e.g. the optimal H_∞ controller.

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7. Appendix—Some Results on Riccati Equations

Some maybe novel results on solutions of Riccati equations are collected in this appendix. Consider the Riccati equation (5) which for ease of reference is given below:

$$\begin{aligned} S &= (A - B_u L)^T S (A - B_u L) + (C_z - D_{zu} L)^T (C_z - D_{zu} L) \\ G &= B_u^T S B_u + D_{zu}^T D_{zu} \\ GL &= B_u^T S A + D_{zu}^T C_z \end{aligned} \tag{15}$$

Let the stabilizability (A1) be a standing assumption, and remember that $A_c = A - B_u L$. Introduce the notation $C_N = D_{zu} L - C_z$. Then (15) can be written as

$$S = A_c^T S A_c + C_N^T C_N$$

LEMMA 6
There always exists a solution (S, L) to (15) such that S is real, symmetric and $S \geq 0$ and such that the eigenvalues of A_c are inside or on the unit circle.

Proof: See [Hagander and Hansson, 1994] □

For any real symmetric solution $S \geq 0$ to (15) and any corresponding A_c introduce the state transformation $T = \begin{pmatrix} T_- & T_0 & T_+ \end{pmatrix}$ such that

$$A_c^T T = T \operatorname{diag}(J_-, J_0, J_+)$$

where J_-, J_0, and J_+ are blocks with eigenvalues outside the unit circle, on the unit circle, and inside the unit circle, respectively.

LEMMA 7
For any solution $S \geq 0$ to (15) it holds that

$$C_N \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$
$$S \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$

Proof: For any real symmetric $S \geq 0$ solving (15) it holds by

$$S = (A_c^T)^k S A_c^k + \sum_{i=0}^{k-1} (A_c^T)^i C_N^T C_N A_c^i, \quad k \geq 1$$

that $C_N \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$ and $ST_- = 0$. The equation for L in (15) implies that $B_u^T S A_c = D_{zu}^T C_N$. Since $C_N T_0 = 0$, it follows that $B_u^T S T_0 = 0$. Further

$$T_0^T S = J_0^T T_0^T S A_c + T_0^T C_N^T C_N = J_0^T T_0^T S A_c$$

Now by the stabilizability of (A, B_u) there exist L_0 such that $A - B_u L_0$ is stable. Summing up gives

$$T_0^T S = J_0^T \left[T_0^T S (A - B_u L_0) + T_0^T S B_u (L_0 - L) \right] = J_0^T T_0^T S (A - B_u L_0)$$

which is equivalent to $X(A - B_u L_0) = JX$, where $X = T_0^T S$ and $J = J_0^{-1}$. Since the eigenvalues of J are on the unit circle, and since $A - B_u L_0$ is stable, it follows that $X = 0$. Hence it has been shown that

$$C_N \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$
$$S \begin{pmatrix} T_- & T_0 \end{pmatrix} = 0$$

for any real symmetric solution $S \geq 0$ of the Riccati-equation. □

LEMMA 8
Let (S, L) be a solution to (15) such that S is real, symmetric and $S \geq 0$ and such that the eigenvalues of A_c are inside or on the unit circle. Then S is unique.

Proof: Such an S certainly exists by Lemma 6. Further there exists a state transformation $T = \begin{pmatrix} T_0 & T_+ \end{pmatrix}$ such that

$$A_c^T T = T \operatorname{diag}(J_0, J_+)$$

where J_0, and J_+ are Jordan blocks with eigenvalues on the unit circle, and inside the unit circle, respectively. Further it is straight forward to show that $T_+^T S T_+$ is unique, [Hagander and Hansson, 1994, Lemma 1]. Since $ST_0 = 0$ by Lemma 7, and since T is invertible, it follows that S is unique. □

Remark. It can be shown that this solution is also the maximal solution.

Finally the continuity of the solution of the Riccati equation with respect to the input data will be proven.

LEMMA 9
Assume that (A5) holds, and that A, B_u, C_z, and D_{zu} are continuous functions of a parameter ρ. Then the solution S of the Riccati-equation (15) that stabilizes A_c will also be a continuous function of ρ.

Proof: The algebraic Riccati-equations can be written

$$
\begin{aligned}
S &= (A - B_u L)^T S (A - B_u L) + \begin{pmatrix} I & -L^T \end{pmatrix} Q \begin{pmatrix} I & -L^T \end{pmatrix}^T \\
G &= Q_2 + B_u^T S B_u \\
GL &= Q_{12}^T + B_u^T S A
\end{aligned}
$$

where

$$
Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} = \begin{pmatrix} C_z^T \\ D_{zu}^T \end{pmatrix} \begin{pmatrix} C_z & D_{zu} \end{pmatrix}
$$

Consider the Lyapunov-equations

$$
\begin{aligned}
S_1(\rho) &= A_c(\rho_1)^T S_1(\rho) A_c(\rho_1) + \bar{C}(\rho_1) Q(\rho) \bar{C}(\rho_1)^T \\
S_2(\rho) &= A_c(\rho_2)^T S_2(\rho) A_c(\rho_2) + \bar{C}(\rho_2) Q(\rho) \bar{C}(\rho_2)^T
\end{aligned}
$$

where $\bar{C}(\rho) = \begin{pmatrix} I & -L^T(\rho) \end{pmatrix}$. It is easily verified that $\Delta_1(\rho_1,\rho_2) = S_1(\rho_2) - S_1(\rho_1)$ and $\Delta_2(\rho_1,\rho_2) = S_2(\rho_1) - S_2(\rho_2)$ satisfies the following Lyapunov-equations

$$
\begin{aligned}
\Delta_1(\rho_1,\rho_2) &= A_c(\rho_1)^T \Delta_1(\rho_1,\rho_2) A_c(\rho_1) + \bar{C}(\rho_1)[Q(\rho_2) - Q(\rho_1)]\bar{C}(\rho_1)^T \\
\Delta_2(\rho_1,\rho_2) &= A_c(\rho_2)^T \Delta_2(\rho_1,\rho_2) A_c(\rho_2) + \bar{C}(\rho_2)[Q(\rho_1) - Q(\rho_2)]\bar{C}(\rho_2)^T
\end{aligned}
$$

Since $A_c(\rho)$ is stable it follows that the solutions $\Delta_1(\rho_1,\rho_2)$ and $\Delta_2(\rho_1,\rho_2)$ are continuous functions of ρ_2 and ρ_1 respectively. Further they approach zero as $|\rho_1 - \rho_2| \to 0$. By the optimality it follows that $S(\rho_1) = S_1(\rho_1) \le S_2(\rho_1)$ and $S(\rho_2) = S_2(\rho_2) \le S_1(\rho_2)$, which implies

$$
-\Delta_1(\rho_1,\rho_2) \le S(\rho_1) - S(\rho_2) \le \Delta_2(\rho_1,\rho_2)
$$

Hence $S(\rho_1) - S(\rho_2) \to 0$ as $|\rho_1 - \rho_2| \to 0$, from which the continuity follows. □