Fixed-point Proportion of Groups Acting on Rooted Trees

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Bachelor's thesis
2026:K6

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Date: May 04, 2026

Abstract

We review groups acting on rooted trees and discuss their topological and measure-theoretic structure as compact topological groups using the theory of profinite groups and Haar measures. We introduce new notions of fractal-like structures for groups acting on rooted trees: shifted fractality and shifted super strong fractality, as generalisations of fractality and super strong fractality. These new notions generalise measure-theoretic results regarding the pushforward of measures and probabilistic independence. Furthermore, we review the fixed-point process, fixed-point proportion, and application of martingale processes in this context. We apply these theories to show that the fixed-point proportion of a level transitive shifted super-strongly fractal group acting on a bounded tree is zero; this gives a natural extension to the analogous result for super-strongly fractal groups.

Popular Scientific Summary

Group theory is the study of symmetries of objects and their interactions. The subject begins with the work of Galois in studying when a polynomial equation is solvable by radicals by exploiting the symmetries of solutions. Later on, the development in group theory over the centuries brings immense applications in various fields, from the basis of cryptography to the spin structure of particles in quantum mechanics.

A group acting on a rooted tree is a group of symmetries of an infinite tree fixing a special vertex called the root. This class of groups solves many longstanding open problems in group theory, such as Day's, Milnor's, and Burnside's problems. In particular, we will study groups acting on rooted trees as topological groups. Roughly, we consider each symmetry on the infinite tree by stringing the same symmetry but only on the same tree with finitely many levels. This gives a meaningful geometric and probabilistic structure to the group.

The main result of this thesis concerns the fixed-point proportions of groups acting on rooted trees. This object captures the probability of finding a symmetry that fixes at least one infinite path on the whole tree. We exploit the fractal-like structure of a certain class of groups, in which the symmetries in the group repeat on all the subtrees, to show that the fixed-point proportion is always zero when each vertex in the tree does not have too many descendants, and the group is shifted super-strongly fractal and level-transitive. The fixed-point proportion is useful both within group theory and in complex and arithmetic dynamics, which have important applications in modelling dynamical systems that appear in ecology, economics, medicine, statistical mechanics, and so forth.

Acknowledgements

I would like to express my gratitude to my supervisor, Jorge Fariña-Asategui, for his patience, continual support, meticulous feedback and guidance both for the thesis and my future as a mathematician following in his footsteps. It is such an honour to collaborate with him in this and the future works. Even on days filled with sadness, working on this thesis still brings me joy. I would also like to thank Marcus Carlsson for convincing me to take the specialised course in integration theory. Without it, I would not have considered working with measure theory on topological groups. Lastly, I would like to thank Anitha Thillaisundaram for sparking my interest in groups acting on rooted trees and introducing me to Jorge for this thesis.

Contents

Chapter 1

Introduction

A group acting on a rooted tree is simply a collection of symmetries of the infinite tree fixing a special vertex called the root, which forms a group under composition. The group consisting of all the root-fixing symmetries is called the full automorphism group of the tree T, denoted by $\mathrm{Aut}(T)$.

Some of the earliest works on groups acting on rooted trees were by Dark in the form of iterated wreath products; see [6] and [28]. The main novelty of groups acting on rooted trees was the first construction of a group of intermediate growth by Grigorchuk, which is also an example of an amenable but not elementary amenable group. The concept of amenability first arose from the famous Banach-Tarski paradox, where the unit sphere in $\mathbb{R}^3$ can be decomposed into two identical unit spheres. It was due to von Neumann in 1929 [23] that amenability can be rephrased in terms of the existence of translation-invariant finitely additive measures on groups. It was known that finite groups and abelian groups are amenable; moreover, their subgroups, quotients, extensions, and directed unions are all amenable as well. All of which are classified as elementary amenable groups. Hence, the question by Day in 1957 [7]: are all amenable groups elementary amenable? The celebrated result by Kesten in 1959 [19] said that all groups with subexponential growth are amenable; in particular, groups with intermediate growth are amenable. Another result by Chou [4] showed that an elementary amenable group must either have polynomial or exponential growth, so a group of intermediate growth must be amenable but not elementary amenable. Milnor asked in 1968 [21] whether there exists a group with intermediate growth. In 1984, Grigorchuk gave a positive answer to this by showing that the group acting on a rooted tree he constructed in [13] has intermediate growth, and is thus amenable but not elementary amenable; see [14]. This group is now recognised as the first Grigorchuk group, and it solves questions by Day and Milnor; moreover, it also gave a first concrete example of a Burnside group, i.e. an infinite finitely-generated torsion group. Due to these exotic properties, groups acting on rooted trees became a subject of interest in group theory.

The fixed-point proportion is an interesting tool for studying groups acting on rooted trees. Formally, the fixed-point proportion of a group $G \leq \mathrm{Aut}(T)$ is defined as

$$\mathrm{FPP}(G) := \lim_{n\to\infty} \frac{\#\{g \in G_n : g \text{ fixes a vertex at level } n\}}{|G_n|}$$

where G_n is the quotient of G by the n-level stabiliser, isomorphic to the action of G down to the nth layer. As shown in Section 6.2, the fixed-point proportion of G is precisely the probability of finding an element in closure $\overline{G}$ that fixes an infinite rooted path in the tree T.

The fixed-point proportion has its origin in arithmetic dynamics; see [24]. In particular, let a_0 be a rational number and f a polynomial with rational coefficients. We define the sequence $\mathcal{I}_{f,a_0} := \{f^{\circ n}(a_0)\}_{n\in\mathbb{N}_0}$. Many interesting families of numbers can be described in this fashion, for example, the Fermat numbers correspond to $\mathcal{I}_{z^2-2z+2,3}$ and the Mersenne numbers correspond to $I_{2z+1,1}$. One prominent open problem in arithmetic dynamics is to prove which of the terms in $\mathcal{I}_{f,a}$ are primes. An easier related problem is to compute the density of the primes dividing some term in $\mathcal{I}_{f,a_0}$, i.e.

$$\lim_{k\to\infty} \frac{\#\left\{p \text{ prime} : p \leq k \text{ and } p \text{ divides some term in } \mathcal{I}_{f,a_0}\right\}}{\#\{p \text{ prime} : p \leq k\}}.$$

The fixed-point proportion may be used to provide an upper bound for the above density of primes, as shown by Odoni in [24] and later by Jones and Manes in [18]. Moreover, the fixed-point proportion is linked to the computation of the proportion of periodic points of f over residue fields; see [3].

Probabilistic tools of groups acting on rooted trees are key to calculating the fixed-point proportion. Probabilistic methods were proposed by Odoni [24]. Abért and Virág made explicit use of branching processes associated with iterated wreath products in their seminal paper [1] to prove properties of random elements, random subgroups, Hausdorff dimension, and many others.

In 2007, Jones studied martingales on iterated Galois towers; see [16]. Fariña-Asategui and Radi utilised Jones' martingale strategy in [10] to provide sufficient conditions for the fixed-point proportion to be zero. Furthermore, they have shown that super-strongly fractal groups have fixed-point proportion zero by combining the above strategy with a new ergodic theory for fractal groups.

The notion of super-strong fractality belongs to the family of properties describing the fractal-like structure of groups acting on rooted trees. Self-similarity is the weakest notion of fractality on groups acting on rooted trees; it allows an action on the subtree to be described as an action on the whole tree itself. As described by Nekrashevych in his book on self-similar groups, self-similarity was originally defined as automata groups back in the 1980s [22]. It was only after his works on iterated monodromy groups in complex dynamics that self-similarity could be naturally presented in the language of groups acting on rooted trees. The construction of iterated monodromy groups in a discrete analogue of Pink's construction of profinite iterated monodromy groups. An iterated monodromy group is a group acting on the tree of preimages of a transcendental element of a partial self-covering map via its iterated monodromy action, i.e. the action of the first fundamental group on the preimages according to the lifting correspondence. As shown by Nekrashevych, iterated monodromy groups are the prototype of self-similar groups; moreover, they exhibit a stronger fractal-like structure called fractality or recurrence. With fractality, one can obtain the whole group by simply looking at the actions of the vertex stabilisers on any subtrees.

Super-strong fractality is a more recent notion in this family. Roughly speaking, it describes a stronger version of fractality, where the groups can be fully obtained by looking at the actions of level stabilisers on any subtree. Fractal and super-strongly fractal groups exhibit strong links between the group-theoretic notions and probabilistic notions, such as a pushforward property and probabilistic independence, as illustrated by Fariña-Asategui in [8]. There, he also developed an ergodic theory for these groups. We will explore a generalised version of these connections in great detail in Section 5.

Returning to the problem of zero fixed-point proportion, many interesting subgroups are far from being fractal or super-strongly fractal, so one cannot directly apply [10, Theorem 1] to calculate the fixed-point proportion of these groups. This motivates us to find a suitable generalisation of super-strong fractality such that the theorem in [10] holds for a larger class of groups. Indeed, we defined these new properties in Chapter 5 and set out to establish them, along with some other natural assumptions, as a sufficient condition for zero fixed-point proportion. Note that, related to our result, Fariña-Asategui and Radi also proved a generalisation amongst fractal groups to the class of mixing groups; see [11].

The work in this thesis will be used in an upcoming paper [27] to show the independence of positive or zero fixed-point proportion, and the Hausdorff dimension.

Overview. In Chapter 2, we will look at the basic definitions related to groups acting on rooted trees, such as iterated wreath products and several examples.

Chapter 3 introduces basic notions of profinite groups and profinite topologies. This topological perspective is crucial for developing a measure-theoretic framework, thus enabling the use of probabilistic tools.

We dedicate Chapter 4 to realising groups acting on rooted trees as topological groups with the congruence topology and discussing preliminary results and definitions from measure theory.

In Chapter 5, we generalise the notions of fractality and super-strong fractality and show some relation with measure-theoretic properties. In fact, we will show in Theorem 5.6 (shifted probabilistic independence theorem) that generalised super-strong fractality is equivalent to probabilistic independence. We note that the focus of this chapter is also quite fascinating on its own as a link between algebra and analysis. For the interested reader, we suggest a paper by Fariña-Asategui and Radi [12], which discusses an even stronger form of probabilistic independence for branch groups.

Lastly, we will use the shifted notions of fractality to develop the tools needed to generalise a sufficient condition for the fixed-point proportion to be zero in Chapter 6. We first discuss fixed-point processes and martingales, and then define the fixed-point proportions and prove the aforementioned result.

Notation. We denote by $\mathcal{P}(S)$ the power set of a set S. We denote by $\#S$ the cardinality of a finite set S. We use the convention that the set of natural numbers $\mathbb{N}$ does not include zero, and we write $\mathbb{N}_0$ for the set of non-negative integers.

Chapter 2

Groups acting on rooted trees

In this chapter, we introduce the notions of rooted trees and their automorphism groups. We shall fix some notation to aid us in the coming parts. Lastly, we give a few concrete examples of groups acting on rooted trees arising in group theory and related fields.

2.1. Rooted trees

A *tree* T is a connected graph that contains no cycle, and it is *rooted* if there is a marked vertex; the marked vertex is called the *root* of T and is commonly denoted by $\emptyset_T$ or $\emptyset$ for simplicity.

For every vertex $v \in T \setminus \{\emptyset\}$ except for the root, there is a unique vertex adjacent to v that lies in the path between the root $\emptyset$ and v; we call this vertex the *predecessor* of v. The remaining vertices adjacent to v are called the *descendants* of v. All the vertices adjacent to the root $\emptyset$ are also said to be the *descendants* of $\emptyset$. We shall assume throughout that each vertex has only finitely many descendants. A vertex $w \in T$ is said to be *below* v, if the unique path between the root $\emptyset$ and w contains v.

A rooted tree admits a natural metric space structure given by the distance function

$$d : T \times T \to [0, \infty)$$

with $d(v, w) := \#\{\text{edges in the unique path between } v \text{ and } w\}$. We denote by $|v|$ the *length* or the *level* of the vertex $v \in T$, i.e. $|v| := d(\emptyset, v)$. For $n \in \mathbb{N}_0$, the *nth layer of* T is the set

$$\mathcal{L}_n := \{v \in T : |v| = n\}$$

of vertices at distance n from the root.

For each vertex $v \in T$, we denote by T_v the *subtree of* T *rooted at* v, i.e. the rooted tree containing the root v and all the vertices below v. We note that $T_\emptyset = T$. Furthermore, for each $n \in \mathbb{N} \cup \{\infty\}$, we define the *$n$-truncated tree* T^n as the finite connected subgraph of T containing the root and all the vertices at level no greater than n. Note that we regard T^n as a rooted tree with the same root $\emptyset$, and $T^\infty = T$.

Lastly, a rooted tree T is *infinite* if it contains infinitely many vertices, and it is called *spherically homogeneous*, if every vertex at the same distance from the root $\emptyset$ has the same number of descendants. Assuming spherical homogeneity, for $n \in \mathbb{N}$, we define d_n to be the common number of descendants of the vertices at level $n - 1$, i.e. $d_n := \#\mathcal{L}_n / \#\mathcal{L}_{n-1}$. In what follows, we shall assume that $2 \le d_n < \infty$.

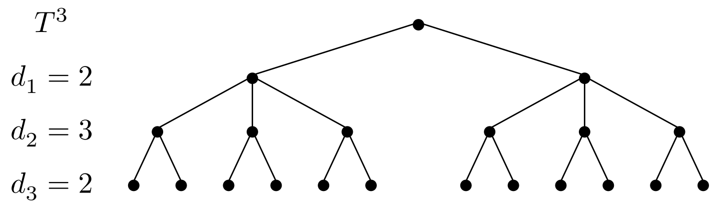

Figure 1: The 3rd level truncation of a spherically homogeneous rooted tree.

A certain kind of rooted trees called regular rooted trees are especially well behaved and often encountered in complex and arithmetic dynamics. A rooted tree T is *regular* if every vertex has the same number of descendants. In particular, we say that T is a *d-adic tree* or a *d-ary rooted tree* if d is the common number of descendants.

2.2. Groups acting on rooted trees

Let T be a rooted tree, finite or infinite. Recall that a *graph automorphism* of T is a permutation of the vertices of T that preserves adjacency. The set $\mathrm{Aut}(T)$ of graph automorphisms on T fixing the root forms a group under composition. The group $\mathrm{Aut}(T)$ acts on T via a right action, which we denote via exponential notation, with composition given by

$$v^{gh} = (v^g)^h.$$

It is easy to see that the action of $\mathrm{Aut}(T)$ on T preserves the levels, i.e. $|v^g| = |v|$ for every $v \in T$ and $g \in \mathrm{Aut}(T)$.

Let G be a subgroup of $\mathrm{Aut}(T)$. As $\mathrm{Aut}(T)$ acts on the rooted tree T, every vertex $v \in T$ gives a *vertex stabiliser* subgroup in G given by

$$\mathrm{st}_G(v) := \{g \in G : v^g = v\}.$$

When $G = \mathrm{Aut}(T)$, we write $\mathrm{st}(v)$ instead. For each $n \in \mathbb{N}_0$ no larger than the maximum level of the tree, we define the *nth level stabiliser* subgroup, denoted by $\mathrm{St}_G(n)$, as the intersection of all the vertex stabilisers in G of vertices at level n, i.e.

$$\mathrm{St}_G(n) := \bigcap_{v \in \mathcal{L}_n} \mathrm{st}_G(v).$$

When $G = \mathrm{Aut}(T)$, we write $\mathrm{St}(n)$ instead. Clearly, $\mathrm{St}_G(n)$ is normal in G and of finite index. We write $G_n := G/\mathrm{St}_G(n)$ for the *nth congruence quotient* of G. Furthermore, we can identify the group G_n with a subgroup of $\mathrm{Aut}(T^n)$ by considering the induced action of G on the n-truncated tree T^n. Note that $\mathrm{st}_G(v) = \mathrm{st}(v) \cap G$ and $\mathrm{St}_G(n) = \mathrm{St}(n) \cap G$.

From this point onwards, let us assume further that T is spherically homogeneous. We may treat the vertices of T as words in the following manner:

1. let $L_n = \{1, 2, \ldots, d_n\}$ for $n \in \mathbb{N}$;
2. for each $v \in T$, make a correspondence between the descendants of v and the letters in $L_{|v|+1}$;
3. identify each vertex v in T by a word obtained from the unique (directed) path from the root to v, where, for every $n \in \mathbb{N}$, the nth letter of the word is given by the corresponding letter in L_n.

Let v and u be words in the above alphabets where $v = x_1 x_2 \cdots x_r$ with $x_i \in L_i$ and $u = y_1 y_2 \cdots y_l$ with $y_i \in L_{r+i}$. We can see that v corresponds to a vertex in T and u corresponds to a vertex in T_v. Observe that the concatenation $vu = x_1 \cdots x_r y_1 \cdots y_l$ is a word obtain from the above procedure, and thus corresponds to a vertex on T, we denote the corresponding vertex also by vu. When this is the case, we say that v and u are *compatible* (with respect to the tree T).

For $n \in \mathbb{N} \cup \{\infty\}$ and a vertex $v \in T$, we define the *section of g at v of depth n* as the unique automorphism $g|_v^n \in \mathrm{Aut}(T_v^n)$ such that

$$(vu)^g = v^g u^{g|_v^n}$$

for every compatible word u of length at most n. Set $g|_v := g|_v^\infty$. One can derive that

$$(gh)|_v^n = (g|_v^n)(h|_{v^g}^n).$$

For every $v \in T$, define the functions $\varphi_v^n : G \to \mathrm{Aut}(T_v^n)$ as $\varphi_v^n(g) := g|_v^n$ and $\varphi_v := \varphi_v^\infty$. These functions, upon restriction to the vertex stabiliser of v in G, are group homomorphisms.

We call a subset $A \subseteq \langle \varphi_v^n(G) \rangle \leq \mathrm{Aut}(T_v^n)$ a set of *n-patterns*. For any set A of n-patterns in a group $\langle \varphi_v^n(G) \rangle \leq \mathrm{Aut}(T_v^n)$ acting on T_v^n, the set

$$C_A^v := \left\{ g \in G : g|_{\emptyset_{T_v}}^n \in A \right\} \subseteq G \leq \mathrm{Aut}(T_v)$$

is called the *cone over A in G*, where $\emptyset_{T_v}$ is the root of T_v and $g|_{\emptyset_{T_v}}^n$ is the section of g at $\emptyset_{T_v}$ in T_v. When v is the root of T, we write C_A instead of $C_A^\emptyset$ for simplicity. Note also that $C_{G_n} = G$ and $C_{1_n} = \mathrm{St}_G(n)$ for all $n \in \mathbb{N}$.

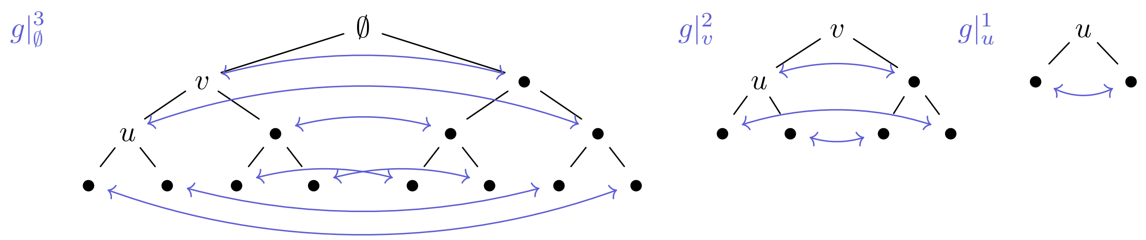

Figure 2: Sections of g at different vertices and of different depths.

The *wreath product* of a group A by a group H acting on a set S by permutations is denoted $A \wr_S H$ and given by

$$A \wr_S H := A^S \rtimes H,$$

where the action of H is extended to an action on $A^S = \prod_{s \in S} A$ given by

$$\left((a_s)_{s \in S} \right)^h = (a_{h^{-1} \cdot s})_{s \in S}.$$

We shall drop S from the notation for the wreath product when it is clear from context.

The group of symmetries of the square D_8 is an example of a wreath product, as it is isomorphic to $C_2 \wr_{\{1,2\}} C_2 = (C_2 \times C_2) \rtimes C_2$ where $C_2 = \langle (1\,2) \rangle$ acts on $\{1, 2\}$ by permutation. To see this, we consider a square in the $\mathbb{R}^2$ plane with the centre at the origin. Denote

$(C_2 \times C_2) \rtimes C_2 = (A \times B) \rtimes C$ where $A = B = C = C_2$ are the corresponding groups in the wreath product. By letting A and B act on the square by reflections along the horizontal and vertical axes, respectively and observing that reflections along the two axes commute, we obtain that $A \times B$ is the group of reflections of the square. Now, C acts on the set $\{1, 2\} = \{\text{x-axis}, \text{y-axis}\}$ by switching the two axes, and hence rotating the square. Therefore, we have $D_8 = C_2 \wr C_2$.

We define an isomorphism $\psi : \mathrm{Aut}(T) \to \prod_{v \in L_1} \mathrm{Aut}(T_v) \rtimes S_{d_1}$ via $\psi : g \mapsto (g_v)_{v \in L_1} \cdot g|_{\emptyset}^{1}$. Hence

$$\mathrm{Aut}(T) \cong \prod_{v \in L_1} \mathrm{Aut}(T_v) \rtimes S_{d_1} \cong \mathrm{Aut}(T_v)^{d_1} \rtimes S_{d_1} = \mathrm{Aut}(T_v) \wr S_{d_1}.$$

If T is infinite, we obtain by induction an isomorphism between $\mathrm{Aut}(T)$ and the infinite iterated wreath product

$$\mathrm{Aut}(T) \cong \cdots \wr S_{d_n} \wr \cdots \wr S_{d_2} \wr S_{d_1}.$$

This leads to the fact that we can identify every automorphism of a rooted tree by labelling each vertex v of the tree with the permutation $g|_v^1$ that acts on its descendants. We call $g|_v^1$ the *label of g at v*. The set of all labels of an element $g \in \mathrm{Aut}(T)$ is called the *portrait of g*, and it uniquely determines the element g.

2.3. Examples

To conclude the chapter, we give a few examples of group acting on rooted trees.

Example 2.1 (Dihedral Groups). The dihedral group

$$D_{2n} := \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle \cong C_n \rtimes C_2$$

can be realised as a group acting on a finite rooted tree T of 2 levels with $d_1 = 2$, $d_2 = n$. We define via ψ the elements $a, b \in \mathrm{Aut}(T)$ by

$$\psi(a) = \tau \quad \text{and} \quad \psi(b) = (\sigma, \sigma^{-1})$$

where $\tau = (1\,2)$ and $\sigma = (1\,2\,\cdots\,n)$. One can check that a corresponds to the reflection s, and b corresponds to the rotation r.

Example 2.2 (Infinite Dihedral Group). The infinite dihedral group

$$D_\infty := \langle x, y \mid x^2 = y^2 = 1 \rangle$$

admits a faithful action on the binary rooted tree T where the actions x and y correspond to the respective element a and b in $\mathrm{Aut}(T)$ given by

$$\psi(a) = (1, 1)\sigma \quad \text{and} \quad \psi(b) = (a, b),$$

where b is recursively defined. Observe that a and b both have order 2, and one can check that $G := \langle a, b \rangle$ is indeed isomorphic to D_∞.

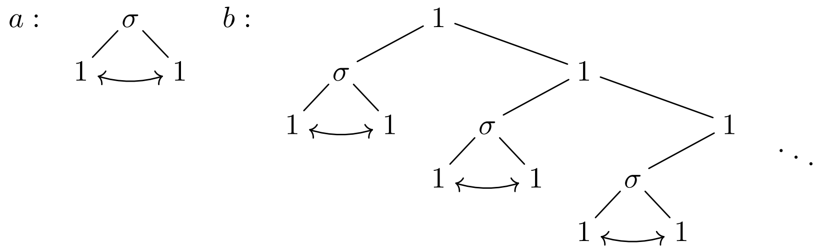

Figure 3: The portraits of a and b on the binary rooted tree.

The group G is linked to the dynamics of the Chebyshev polynomial $2z^2 - 1$. It is worth noting that the infinite dihedral group can also be embedded into the group of automorphisms of the d-adic tree, for any integer $d \geq 2$, by considering the dth degree Chebyshev polynomial instead. For a reference, we refer the reader to [22, Section 6.11.2].

Example 2.3 (Basilica Group)**.** The Basilica group B is a subgroup of $\mathrm{Aut}(T)$ with T the binary rooted tree T. It was first introduced by Grigorchuk and Żuk in [15]. The group B is generated by a and b given by

$$\psi(a) = (1, b)\sigma \quad \text{and} \quad \psi(b) = (1, a).$$

The group B is connected to the dynamics of the polynomial $z^2 - 1$, and it takes its name from the shape of the Julia set of $z^2 - 1$, which resembles the basilica of San Marco mirrored in the water; see [22, Section 6.11.1].

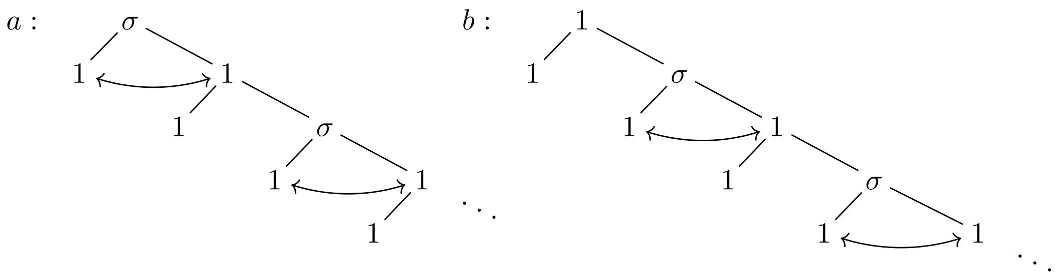

Figure 4: The portraits of a and b on the binary rooted tree.

Chapter 3

Profinite groups

The goal of this section is to study the structure of certain groups acting on rooted trees by realising them as profinite groups. The advantage is that profinite groups have rich algebraic and topological structures, which allow us to study subgroups and finite quotients comprehensively. We will go through the general construction of profinite groups and profinite topologies, and a few results on them. We will apply these results to groups acting on rooted trees in Section 4.1. We refer the reader to [5, Chapter 9] and [26] for in-depth expositions on topological groups and profinite groups, respectively.

3.1. Preliminaries on topological groups

A group G equipped with a topology is called a *topological group* if the following maps are continuous:

(a) the map $G \times G \to G$ given by $(g, h) \mapsto gh$, where $G \times G$ is equipped with the product topology;

(b) the map $G \to G$ given by $g \mapsto g^{-1}$.

Essentially, a topological group is a set equipped with compatible group and topological space structures. We call an isomorphism between topological groups that is also a homeomorphism a *topological isomorphism*, i.e. an isomorphism preserving both the group structure and the topology.

Lemma 3.1 (see [5, Theorem 9.12]). *Let G be a topological group and $h \in G$. Then, the following holds:*

(a) *the maps $g \mapsto gh$, $g \mapsto hg$, and $g \mapsto g^{-1}$ are homeomorphisms from $G \to G$;*

(b) *if $\mathcal{U}$ is a base of open neighbourhoods of the identity in G, then*

$$\{hU : U \in \mathcal{U}\} \quad \text{and} \quad \{Uh : U \in \mathcal{U}\}$$

are bases of open neighbourhoods of h.

The proof of the above result is standard, and it follows directly from the definition of topological groups.

A collection of sets $\mathcal{S}$ is said to be *filtered from below* if for $S_1, S_2 \in \mathcal{S}$, there exists $S \in \mathcal{S}$ such that $S \subseteq S_1 \cap S_2$. The following lemma is a standard result; see [2, Chapter 3, Proposition 1] We provide a proof here for the convenience of the reader.

Lemma 3.2. *Let $\mathcal{N}$ be a collection of normal subgroups of G filtered from below. Then the topology on G generated by the right (or left) cosets of normal subgroups in $\mathcal{N}$ endows G with a topological group structure.*

Proof. Let $g, h \in G$ be arbitrary and N be a normal subgroup in the collection $\mathcal{N}$. First, the preimage of Ng under inversion is $g^{-1}N = Ng^{-1}$, which is open. Second, we will show that the preimage of the open set Ngh under the multiplication map is open with respect to the product topology on $G \times G$. If x and y are elements of G such that their product is in Ngh, we observe that the open set $Nx \times Ny$ is contained in the preimage of Ngh because

$$NxNy = Nxy = Ngh.$$

Thus, the preimage of Ngh under multiplication is simply the union of $Nx \times Ny$ for every pair (x, y) such that $xy \in Ngh$, which is open. ∎

Due to Lemma 3.2, it suffices to define a base of neighbourhoods of the identity consisting of normal subgroups filtered from below to endow a group with a topological group structure. The most intuitive and relevant example is when we have a (descending) filtration $\{N_i\}_{i \in \mathbb{N}}$,

$$N_1 \geq N_2 \geq N_3 \geq \cdots \geq \bigcap_{i \in \mathbb{N}} N_i = 1.$$

It is plain that the intersection $N_i \cap N_j$ is precisely the smaller normal subgroup amongst N_i and N_j.

The following lemma allows one to connect topological information about the "shape", openness and closedness, of a subgroup to the group-theoretic information about the "size", particularly the index, of it. This lemma is a modification of [26, Lemma 2.1.2]. We add the proof for the reader's convenience.

Lemma 3.3. *Let G be a topological group. Then*
(a) *every open subgroup is closed;*
(b) *every finite-index closed subgroup is open;*
(c) *if G is compact, then a subgroup is open if and only if it is closed and of finite index.*

Proof. Let H be an open subgroup for part (a). Then H is closed because $G = \cup_{g \in G} Hg$ and thereby $G \setminus H = \cup_{g \in G \setminus H} Hg$, a union of open sets Hg. In part (b), we assume that H is a closed finite-index subgroup. Partitioning G into a finite number of cosets of H, we obtain $G = \uplus_{i=1}^{n} Hg_i$ where $g_1, \ldots, g_n$ form a transversal and choose $g_1 = 1$ without loss of generality. This implies that $H = G \setminus \uplus_{i=2}^{n} Hg_i = \cap_{i=2}^{n} (G \setminus Hg_i)$ is a finite intersection of open sets, and thus it is open. To conclude part (c), we are left to show that under the extra compactness assumption, any open subgroup H has finite index. Consider the open cover $\{Hg : g \in G\}$ of G; we can choose a finite subcover by compactness, i.e. for some n and g_i, we have $G = \cup_{i=1}^{n} Hg_i$. Then the index of H is at most n. ∎

3.2. Inverse limits

The exposition in this section and the next is mostly based on [26, Sections 1.1, 2.1, 3.1, 3.2].

Let $I = (I, \preceq)$ be a directed partially-ordered set (poset), i.e. I is a set equipped with a binary relation $\preceq$ such that:
(a) $i \preceq i$, for $i \in I$;
(b) if $i \preceq j$ and $j \preceq k$ then $i \preceq k$, for $i, j, k \in I$;
(c) if $i \preceq j$ and $j \preceq i$ then $i = j$, for $i, j \in I$;
(d) if $i, j \in I$, there exists $k \in I$ such that $i, j \preceq k$.

An *inverse system* (*projective system*) of topological groups over a poset I consists of
(a) a collection $\{X_i : i \in I\}$ of topological groups indexed by the poset I;
(b) for every pair $i \succeq j$, a continuous homomorphism

$$\pi_{i,j} : X_i \to X_j$$

such that the following diagrams commute whenever $i \succeq j \succeq k$,

$$\begin{tikzcd}
X_i \arrow[dd, "\pi_{i,k}"'] \arrow[dr, "\pi_{i,j}"] & \\
& X_j \arrow[dl, "\pi_{j,k}"] \\
X_k &
\end{tikzcd}$$

i.e. $\pi_{i,k} = \pi_{j,k} \circ \pi_{i,j}$;

(c) $\pi_{i,i} = \mathrm{id}_{X_i}$ for all $i \in I$.

This system is denoted by $\{X_i, \pi_{i,j}, I\}$ or $\{X_i, \pi_{i,j}\}$ if I is unambiguous.

Let Y be a topological group and $\{X_i, \pi_{i,j}, I\}$ be an inverse system of topological groups. Let $\theta_i : Y \to X_i$ be a continuous homomorphism for each i. These homomorphims θ_i are said to be *compatible* if

$$\pi_{i,j} \circ \theta_i = \theta_j$$

whenever $i \succeq j$.

A topological group X with compatible continuous homomorphisms $\pi_i : X \to X_i$ is an *inverse limit* (*projective limit*) of an inverse system $\{X_i, \pi_{i,j}, I\}$, denoted by $\varprojlim_{i \in I} X_i$ (or $\varprojlim X_i$ if I is understood from the context), if the following universal property holds:

$$\begin{tikzcd}
Y \arrow[r, dashed, "\exists!\theta"] \arrow[dr, "\theta_i"'] & X \arrow[d, "\pi_i"] \\
& X_i
\end{tikzcd}$$

i.e. whenever Y is a topological group and $\{\theta_i : Y \to X_i\}_{i \in I}$ is a set of compatible continuous homomorphisms, there exists a unique continuous homomorphism $\theta : Y \to X$ such that

$$\pi_i \circ \theta = \theta_i$$

for all $i \in I$. The above two diagrams can be merged into the following commutative diagram whenever $i \succeq j$

$$\begin{tikzcd}
& Y \arrow[d, dashed, "\exists!\theta"] \arrow[ddl, bend right, "\theta_i"'] \arrow[ddr, bend left, "\theta_j"] & \\
& X = \varprojlim X_i \arrow[dl, "\pi_i"] \arrow[dr, "\pi_j"'] & \\
X_i \arrow[rr, "\pi_{i,j}"] & & X_j.
\end{tikzcd}$$

We say that θ is *induced* or *determined* by the compatible homomorphisms $\{\pi_i\}_{i \in I}$. The homomorphisms $\pi_i : X \to X_i$ are called *projections*. Note that, in general, they need not be surjective, but in our setting, they will be the natural projections onto quotient groups, which are surjective.

Proposition 3.4 (see [26, Proposition 1.1.1]). *Let $\{X_i, \pi_{i,j}, I\}$ be an inverse system of topological groups. Then there exists an inverse limit of this system that is unique up to a unique topological isomorphism.*

Proof. For the existence, we define X as the topological subspace of the Cartesian product $\prod_{i\in I} X_i$ consisting of those tuples $(x_i)_{i\in I}$ satisfying

$$\pi_{i,j}(x_i) = x_j$$

whenever $i \succeq j$; these are called *coherent tuples*. Observe that X is a subgroup of the group $\prod_i X_i$ since $\left(1_{X_i}\right)$ is coherent, and is closed under inversion and multiplication because each $\pi_{i,j}$ is a homomorphism. Since the multiplication map on the aforementioned Cartesian product $\prod_i X_i \times \prod_i X_i = \prod_i (X_i \times X_i) \to \prod_i X_i$ is made up of continuous multiplication maps in each component, itself is also continuous. By the same token, the inverse map on the product is continuous. Upon restricting to X, we conclude that X is a topological group with the topology induced from $\prod_i X_i$. Consider the continuous projection $p_i : \prod_{i\in I} X_i \to X_i$; their restrictions $\pi_i = p_i|_X$ are continuous homomorphisms which make the above diagram commute. Suppose that Y is a topological group and the maps $\theta_i : Y \to X_i$ are the compatible continuous homomorphisms. We define $\theta : Y \to X$ by letting $\theta(y) := (\theta_i(y))_{i\in I}$ a candidate for the induced continuous homomorphism. We see that

$$\theta_i(y) = \pi_i\left(\left(\theta_j(y)\right)_{j\in I}\right) = \pi_i \circ \theta(y),$$

thus the above diagram commutes. If θ' is another induced map, then

$$\theta'(y) = (\pi_i \circ \theta'(y))_{i\in I} = (\theta_i(y))_{i\in I} = \theta(y).$$

Hence, the induced map θ is unique. This concludes the existence part.

For the uniqueness, let (X, π_i) and (Y, θ_i) be two inverse limits of the inverse system $\{X_i, \pi_{i,j}, I\}$. By applying the universal property of inverse limits to X and Y, we then have the following commutative diagram where $\pi : X \to Y$ and $\theta : Y \to X$ are the unique induced continuous homomorphisms:

$$\begin{tikzcd}
X \arrow[r, dashed, "\pi"] \arrow[rr, dashed, bend left, "\mathrm{id}_X"] \arrow[dr, "\pi_i"'] & Y \arrow[r, dashed, "\theta"] \arrow[d, "\theta_i"] & X \arrow[dl, "\pi_i"] \\
& X_i. &
\end{tikzcd}$$

However, the identity homomorphism $\mathrm{id}_X : X \to X$ on X also makes this new diagram commute. Thus $\theta \circ \pi$ is the identity map on X by the uniqueness. Similarly, $\pi \circ \theta$ is the identity map on Y. Hence π is a topological isomorphism between X and Y since θ is the continuous inverse. ∎

As the inverse limit $\varprojlim X_i$ of a given inverse system of topological groups is unique up to topological isomorphism, we may realise $\varprojlim X_i$ as the explicit construction in the above proof, and write $X \cong \varprojlim X_i$ for other topological groups that satisfy the same universal property.

Recall that the product topology on $\prod_{i\in I} X_i$ is defined to be the weakest (coarsest) topology such that all the projections $p_i : \prod_{i\in I} X_i \to X_i$ are continuous. This induces a base of open sets consisting of finite intersections of sets $V_i = p_i^{-1}(U_i)$ for U_i open in X_i. Now, the basic open sets in $\varprojlim X_i$ of an inverse system $\{X_i, \pi_{i,j}, I\}$ consists of the finite intersections of

$$V_i \cap \varprojlim X_i = \pi_i^{-1}(U_i).$$

As I is directed, we may choose a base of open neighbourhoods of the identity consisting of just $\pi_i^{-1}(U_i) = V_i \cap \varprojlim X_i$ without the need for intersections. Suppose that $i \succeq i_1, \ldots, i_n$, then $\pi_{i_j} = \pi_{i,i_j} \circ \pi_i$ and hence

$$\bigcap_{j=1}^{n} \left(V_{i_j} \cap \varprojlim X_i\right) = \bigcap_{j=1}^{n} \pi_{i_j}^{-1}\left(V_{i_j}\right) = \bigcap_{j=1}^{n} \pi_i^{-1}\left(\pi_{i,i_j}^{-1}\left(V_{i_j}\right)\right) = \pi_i^{-1}(U)$$

with $U = \bigcap_{j=1}^{n} \pi_{i,i_j}^{-1}\left(V_{i_j}\right)$ open in X_i.

Proposition 3.5 (see [26, Proposition 1.1.2])**.** *Let $\{X_i, \pi_{i,j}\}$ be an inverse system of Hausdorff topological groups. Then $\varprojlim X_i$ is a closed subgroup of $\prod X_i$.*

Proof. To show that the inverse limit is closed in the product topology, we shall show that every point in the complement has an open neighbourhood that does not intersect the inverse limit; thus the union of such open neighbourhoods give the open complement of the inverse limit. For every $(x_i) \in \prod X_i \setminus \varprojlim X_i$, there exist $r, s \in I$ with $r \succeq s$ such that $\pi_{r,s}(x_r) \neq x_s$. Using the Hausdorff property of X_s, we choose two disjoint open neighbourhoods U and V of $\pi_{r,s}(x_r)$ and of x_s, respectively. Let $U' := \left(\pi_{r,s}\right)^{-1}(U)$ be an open neighbourhood of x_r in X_r. Consider the open neighbourhood $W := \prod V_i$ of (x_i) in $\prod X_i$ where $V_r = U'$, $V_s = V$, and $V_i = X_i$ otherwise. The open set W is disjoint from $\varprojlim X_i$ since $\pi_{r,s}(V_r) \cap V_s = \pi_{r,s}(U') \cap V = \emptyset$. Since this holds for any arbitrary element (x_i) outside of $\varprojlim X_i$, the set $\prod X_i \setminus \varprojlim X_i$ is open. ■

Proposition 3.6 (see [26, Proposition 1.1.3])**.** *Let $\{X_i, \pi_{i,j}\}$ be an inverse system of compact Hausdorff topological groups. Then $\varprojlim X_i$ is a compact Hausdorff topological group.*

Proof. It is also easy to see that taking products and subspaces preserves the Hausdorff property, so the inverse limit is Hausdorff. Next, the product of compact spaces is compact by Tychonoff's theorem. Since the inverse limit is a closed subspace of the product $\prod X_i$ by Proposition 3.5, it is also compact. ■

An inverse system $\{X_i, \pi_{i,j}\}$ is called a *surjective inverse system* if $\pi_{i,j}$ is surjective for every $i \succeq j$. The proof of the following lemma requires several lengthy constructions, so we omit it and instead refer the reader to [26, Corollary 1.1.6]:

Lemma 3.7 (see [26, Corollary 1.1.6])**.** *Let $\{X_i, \pi_{i,j}\}$ be a surjective inverse system of compact Hausdorff groups and X be a compact Hausdorff group with the compatible homomorphisms $\theta_i : X \to X_i$. Then the corresponding induced homomorphism $\pi : X \to \varprojlim X_i$ is surjective.*

Lastly, we should remark that the results in Section 3.2 are also valid more generally for inverse limits of topological spaces.

3.3. Profinite groups

A *profinite group* is the inverse limit $\varprojlim G_i$ of a surjective inverse system $\{G_i, \pi_{i,j}\}$ of finite groups G_i each endowed with the discrete topology.

Let G be a group and let $\mathcal{N}$ be a collection of finite-index normal subgroups that is filtered from below. This collection of subgroups $\mathcal{N}$ forms a base for open neighbourhoods of the identity in G. We refer to the corresponding topology as the *profinite topology generated by the collection* $\mathcal{N}$. Note that G is a topological group with respect to this topology.

Although it is tempting to say that a group G equipped with a profinite topology is the profinite group $\varprojlim G/N$, it is not the case unless G is compact and Hausdorff with respect to the profinite topology. We will see in Theorem 3.9 and Theorem 3.11 when this is the case.

The following result is based on the comment in [26, page 75]. We add the proof for the convenience of the reader.

Lemma 3.8. *A topological group G with the profinite topology generated by a collection $\mathcal{N}$ is Hausdorff if and only if $\cap_{N\in\mathcal{N}} N = 1$.*

Proof. Suppose that G is Hausdorff, then for every non-trivial element $g \in G$, there is a subgroup $N \in \mathcal{N}$ such that $Ng \cap N = \emptyset$; therefore $g \notin N$. Hence, the intersection of all sets in $\mathcal{N}$ must be trivial. For the converse, if G is not Hausdorff, there exist two distinct elements g and h such that $Ng \cap Nh \neq \emptyset$ for all subgroups $N \in \mathcal{N}$. Hence, $Ng = Nh$ and thereby $1 \neq gh^{-1} \in \bigcap_{N\in\mathcal{N}} N$. ∎

Let G be a group with a profinite topology generated by a collection $\mathcal{N}$. The *completion* of G with respect to this profinite topology is the profinite group given by the inverse limit of the inverse system $\{G/N, \pi_{N,K}, \mathcal{N}\}$ where $N \succeq K$ if $N \leq K$ and $\pi_{N,K}$ are the natural continuous projections $G/N \to G/K$.

The following result is modified from [26, Theorem 3.3.2], which was shown for the full profinite topology. The result is restated and proved for a more general profinite topology.

Theorem 3.9. *Let G be a topological group admitting a collection $\mathcal{N}$ of finite-index normal subgroups such that $\cap_{N\in\mathcal{N}} N = 1$. Then G is compact if and only if G is topologically isomorphic to its own completion, i.e.*

$$G \cong \varprojlim_{N\in\mathcal{N}} G/N.$$

Proof. The converse follows from Proposition 3.6, so it suffices to show the forward direction. Since the continuous surjective homomorphisms $\pi_N : G \to G/N$ are compatible with the surjective inverse system $\{G/N, \pi_{N,K}, \mathcal{N}\}$, they induce a continuous homomorphism

$$\theta : G \to \varprojlim_{N\in\mathcal{N}} G/N.$$

Using Lemma 3.8 along with Lemma 3.7, we deduce that θ must be surjective. Because $\pi_N \circ \theta = \theta_N$, we get

$$\ker(\theta) \leq \bigcap_{N\in\mathcal{N}} \ker(\pi_N \circ \theta) = \bigcap_{N\in\mathcal{N}} \ker(\theta_N) = \bigcap_{N\in\mathcal{N}} N = 1,$$

hence, θ is injective. Lastly, the continuous bijection θ is a homeomorphism since both spaces are compact and Hausdorff. ■

This means that a topological group G with a profinite topology is topologically isomorphic to its own completion whenever G is compact and Hausdorff. The following result gives a formula for the closure of a subgroup in a profinite topology; it is a modification of the observation in [26, Theorem 3.1.2(b)], which was shown for the full profinite topology.

Proposition 3.10. *Let H be a subgroup in a topological group G with a profinite topology generated by a collection $\mathcal{N}$. Then the closure of H in G is*

$$\overline{H} = \bigcap_{N \in \mathcal{N}} NH.$$

Proof. First, suppose that H is a closed subgroup of G. For each $x \in G - H$, there is a finite-index normal subgroup $N \in \mathcal{N}$ such that $Nx \subseteq G - H$ as the latter set is open. This means that $x \notin NH$ as otherwise, $x = nh$ and so $n^{-1}x = h$ for some $n \in N$ and $h \in H$. Therefore, $H = \cap_{N \in \mathcal{N}} NH$. Now, if H is an arbitrary subgroup, we have $\overline{H} = \cap_{N \in \mathcal{N}} N\overline{H}$. Since $NH = \cup_{h \in H} Nh$, the subgroup NH is open. It follows from Lemma 3.3(a) that NH is closed, for each $N \in \mathcal{N}$. Because $\cap_{N \in \mathcal{N}} NH$ is also a closed set containing H and is contained in $\cap_{N \in \mathcal{N}} N\overline{H}$, then $\overline{H} = \cap_{N \in \mathcal{N}} NH$. ■

For the following result, we modify the [26, Theorem 3.3.2(d)], which was proven for the full profinite topology.

Theorem 3.11. *Let G be a topological group with a profinite topology generated by a collection $\mathcal{N}$. If H is a subgroup of G, then*

$$\overline{H}/(\overline{H} \cap N) \cong H/(H \cap N)$$

and hence

$$\varprojlim_{N \in \mathcal{N}} \frac{\overline{H}}{\overline{H} \cap N} \cong \varprojlim_{N \in \mathcal{N}} \frac{H}{H \cap N}.$$

In other words, the completion of $\overline{H}$ with respect to the topology induced from G is topologically isomorphic to the completion of H with the induced topology from G.

Proof. First, we observe the following identity:

$$NH = N\left(\bigcap_{K \in \mathcal{N}} H\right) \subseteq N\left(\bigcap_{K \in \mathcal{N}} KH\right) \subseteq N(NH) \subseteq NH$$

whenever $N \in \mathcal{N}$ and $H \leq G$. Now, we can deduce the first isomorphism in the statement:

$$\overline{H}/(\overline{H} \cap N) \cong N\overline{H}/N = N\left(\bigcap_{K \in \mathcal{N}} KH\right)/N = NH/N \cong H/(H \cap N),$$

where the isomorphisms are given by the second isomorphism theorem and the first equality is given by Proposition 3.10. It follows directly from the above identification that the inverse limits are topologically isomorphic. ■

Chapter 4

Topological and measure-theoretic aspects of groups acting on rooted trees

From this chapter onwards T shall denote an infinite spherically homogeneous rooted tree, unless stated otherwise.

4.1. Topological properties of groups acting on rooted trees

The results in this section are widely known, and we refer to [9] as a reference. We provide the proofs for the convenience of the reader.

For a rooted tree T, let us consider its automorphism group $\mathrm{Aut}(T)$ as a topological group with the topology given by declaring the level stabilisers to be a neighbourhood basis of the identity. We call this the *congruence topology* on $\mathrm{Aut}(T)$. The induced topology on any subgroup $G \leq \mathrm{Aut}(T)$ has the level stabilisers $\mathrm{St}_G(n)$ in G as a neighbourhood basis of the identity. We call this the *congruence topology* on G.

Note that the level stabilisers are filtered from below because

$$\mathrm{St}_G(1) \geq \mathrm{St}_G(2) \geq \mathrm{St}_G(3) \geq \cdots \geq \mathrm{St}_G(n) \geq \cdots.$$

Moreover, all congruence quotients $G_n = G/\mathrm{St}_G(n)$ are finite. It is now plain that the congruence topology on G is a profinite topology. Observe that the congruence topology on any subgroup G satisfies

$$\bigcap_{n \in \mathbb{N}} \mathrm{St}_G(n) = 1$$

as the identity is the only automorphism that fixes every vertex in T. Hence, the group G is a Hausdorff topological group with respect to the congruence topology by Lemma 3.8.

Proposition 4.1. *Every closed subgroup $G \leq \mathrm{Aut}(T)$ with the congruence topology is topologically isomorphic to its own completion $\varprojlim G_n$.*

Proof. First, we prove that $G = \mathrm{Aut}(T)$ is compact. We claim that for every cover of $\mathrm{Aut}(T)$ consisting of cosets of level stabilisers, there is a number $n \in \mathbb{N}$ such that all the cosets of level stabilisers of level at most n in this covering cover $\mathrm{Aut}(T)$. Indeed, let us assume by contradiction that for every n, there is a coset $\mathrm{St}(n)g_n$ not covered by cosets of the level stabilisers of lower or equal levels in the covering. It is easy to see that $\{\mathrm{St}(n)g_n\}_{n\in\mathbb{N}}$ can be chosen to be a descending nested sequence of sets and have the intersection being a singleton $\{g\} \subseteq \mathrm{Aut}(T)$ since $\cap_{n\in\mathbb{N}} \mathrm{St}(n)$ is trivial. However, this means that g is not covered by the covering, hence a contradiction. Since open sets in $\mathrm{Aut}(T)$ are simply unions of cosets of level stabilisers in $\mathrm{Aut}(T)$, then $\mathrm{Aut}(T)$ is compact by our claim as every open cover admits a finite sub-cover. Now every closed subgroup G is compact in the compact group $\mathrm{Aut}(T)$, thus by Theorem 3.9, we get $G \cong \varprojlim G_n$. ∎

Proposition 4.2. *Let $G \leq \mathrm{Aut}(T)$. Then $G/\mathrm{St}_G(n) = \overline{G}/\mathrm{St}_{\overline{G}}(n)$ where $\overline{G}$ is the closure of G with respect to the congruence topology in $\mathrm{Aut}(T)$.*

Proof. This follows directly from Theorem 3.11. ∎

Proposition 4.3. *Let $G \leq \mathrm{Aut}(T)$. For each vertex $v \in T$, and $k \in \mathbb{N} \cup \{\infty\}$, the section map $\varphi_v^k : G \to \mathrm{Aut}(T_v^k)$ is continuous.*

Proof. Fix $k \in \mathbb{N} \cup \{\infty\}$, a group $G \leq \mathrm{Aut}(T)$, and a vertex $v \in T$ at level n. Recall that the cosets of the level stabilisers form a base of open sets in $\mathrm{Aut}(T_v^k)$. For any positive integers $m \leq k$ and $h \in \mathrm{Aut}(T_v^k)$, the preimages of basic open sets in $\mathrm{Aut}(T_v^k)$ are precisely

$$A := \left(\varphi_v^k\right)^{-1}\left(\mathrm{St}_{\mathrm{Aut}(T_v^k)}(m)h\right) \subseteq \mathrm{St}(n+m)A = \bigcup_{g \in A} \mathrm{St}(n+m)g \subseteq A,$$

where the last inclusion follows from $l|_v^k \in \mathrm{St}_{\mathrm{Aut}(T_v^k)}(m)$ whenever $l \in \mathrm{St}(n+m)$. Therefore, the preimages of open sets are open, and hence φ_v^k is continuous. ∎

We remark that the projections for the inverse limit $\varprojlim G_n$ are $\varphi_\emptyset^n : G \cong \varprojlim G_n \to G_n$.

4.2. Measure-theoretic properties of groups acting on rooted trees

We recall some definitions related to measure spaces and Haar measures on topological groups. For the full details, we refer the reader to [5].

A *measurable space* consists of a pair $(X, \mathcal{A})$ where X is a set, and $\mathcal{A}$ is a *σ-algebra*, i.e. a subcollection of $\mathcal{P}(X)$ satisfying the following conditions:
(a) $\emptyset, X \in \mathcal{A}$;
(b) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$;
(c) if $\{A_i\}_{i \in \mathbb{N}}$ is a countable collection of sets in $\mathcal{A}$, then $\cup_{i \in \mathbb{N}} A_i \in \mathcal{A}$.

We call the sets in $\mathcal{A}$ *measurable sets*. If $\mathcal{A}$ is understood by context, we may write simply X for the measurable space.

A *measure* $\mu : \mathcal{A} \to [0, \infty]$ is a set-function satisfying:
(a) $\mu(\emptyset) = 0$;
(b) if $\{A_i\}_{i \in \mathbb{N}}$ is a countable collection of disjoint sets in $\mathcal{A}$, then $\sum_{i \in \mathbb{N}} \mu(A_i) = \mu(\sqcup_{i \in \mathbb{N}} A_i)$.

We call a measurable space a *measure space* upon equipping it with a measure μ and denote it by $(X, \mathcal{A}, \mu)$, or simply (X, μ) if $\mathcal{A}$ is understood by context. We say that a measure space is a *probability space* if $\mu(X) = 1$. In this case, we call μ a *probability measure*.

We say that a property P holds *μ-almost everywhere* (*μ-almost surely*) or simply *almost everywhere* (*almost surely*) in X if $E := \{x \in X : P \text{ does not hold}\}$ is contained in a measurable set $F \in \mathcal{A}$ with measure zero.

When X and Y are two measurable spaces, a function $f : X \to Y$ is said to be *measurable* if all the preimages of measurable sets in Y are measurable in X, i.e. $f^{-1}(B)$ is measurable in X whenever B is measurable in Y. Note that the *characteristic function* χ_E of a measurable set E is measurable. For the definitions of integrable functions and integration against a measure, we refer the reader to [5, Chapter 2].

If X is a topological space, then we can choose the σ-algebra $\mathcal{A}$ to be the (smallest) σ-algebra generated by the open sets in X; this is called the *Borel σ-algebra on X*. Note that all open and closed sets are contained in the Borel σ-algebra, and if the space X is Hausdorff, then all compact sets are closed and hence measurable. If X and Y are topological spaces, then every continuous function $f : X \to Y$ is measurable. We take $\mathbb{R}$ with the Borel σ-algebra generated by the Euclidean metric on $\mathbb{R}$ as our canonical measure-theoretic structure on $\mathbb{R}$.

If X is Hausdorff and equipped with the Borel σ-algebra, we say that a measure μ is *regular* if

(a) for every compact subset K, we have $\mu(K) < \infty$;

(b) (outer regular) for every measurable subset A, we have

$$\mu(A) = \inf\{\mu(U) : A \subseteq U, \text{with } U \text{ open}\};$$

(c) (inner regular) for every open subset U, we have

$$\mu(U) = \sup\{\mu(K) : K \subseteq U, \text{with } K \text{ compact}\}.$$

If X is a compact Hausdorff topological group (compact group in short), then we say that μ is a *left Haar measure* if μ is regular and *left-translation invariant*, i.e. $\mu(xA) = \mu(A)$ for every Borel measurable set A and $x \in X$. Similarly, a *right Haar measure* is a regular measure that is *right-translation invariant*.

It can be shown that left and right Haar measures exist for every compact group, albeit with a lengthy proof; see [5, Theorem 9.2.2]. One can also show the uniqueness of a left (and right) Haar measure up to a constant multiple. Furthermore, the compactness assumption yields that every left Haar measure on a compact group is also a right Haar measure and thus there is a unique probability Haar measure; see [5, pages 293-295]. We will use the following result for calculating the measures of subgroups of compact groups:

Proposition 4.4. *Let G be a compact group and μ be the unique Haar probability measure on G. Then H is an open subgroup in G if and only if H is a closed subgroup with positive measure. Furthermore, if the above hold, then the measure of the subgroup H is given by*

$$\mu(H) = |G : H|^{-1}.$$

Proof. Suppose that H is an open subgroup, then it is closed and has finite index in G by Lemma 3.3(c). If $\mathcal{T}$ is a right transversal of H, then we have

$$1 = \mu(G) = \mu\left(\dot{\bigcup_{g\in\mathcal{T}}} Hg\right) = \sum_{g\in\mathcal{T}} \mu(Hg) = \sum_{g\in\mathcal{T}} \mu(H) = |G : H| \cdot \mu(H).$$

Hence, the measure of H is $\mu(H) = |G : H|^{-1} > 0$.

For the converse, suppose that H is a closed subgroup with positive measure. Let $\mathcal{T}$ be a right transversal of H in G and let $\mathcal{T}'$ be any finite subset of $\mathcal{T}$. Then

$$1 = \mu(G) = \mu\left(\dot{\bigcup_{g\in\mathcal{T}}} Hg\right) \geq \mu\left(\dot{\bigcup_{g\in\mathcal{T}'}} Hg\right) = \sum_{g\in\mathcal{T}'} \mu(Hg) = \sum_{g\in\mathcal{T}'} \mu(H) = \#\mathcal{T}' \cdot \mu(H).$$

Since $\mu(H) > 0$, then $\mu(H)^{-1} > 0$. If $\mathcal{T}$ were to be infinite, then we can choose $\mathcal{T}'$ to have cardinality $\lceil \mu(H)^{-1} \rceil + 1$, which yields a contradiction as $\#\mathcal{T}' \cdot \mu(H) > 1$. Hence $\mathcal{T}$ must be finite, i.e. $|G : H| < \infty$. The subgroup H is open according to Lemma 3.3(b). ■

Note that the result above is widely known, and we provide the proof for the convenience of the reader.

In the case of groups acting on rooted trees, if G is a subgroup of $\mathrm{Aut}(T)$, recall that the closure $\overline{G}$ of G in $\mathrm{Aut}(T)$ is a compact group by Proposition 4.1 and Theorem 3.9, and thus we can equip $\overline{G}$ with the Borel σ-algebra and the unique Haar probability measure. We denote the unique Haar probability measure on $\overline{G}$ by μ_G.

Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. Then, clearly G_n is a finite group. It is plain that $(G_n, \mathcal{P}(G_n))$ is a discrete measurable space, i.e. every subset of G_n is measurable. This σ-algebra is also the Borel σ-algebra generated by the discrete topology. Note that the normalised discrete (counting) measure is the unique Haar probability measure on this group, which we denote by μ_{G_n}.

Let X be a measure space with a measure μ and Y a measurable space. Given a measurable function $f : X \to Y$, we define the *pushforward measure* $f_{\#}\mu$ of μ by f on Y by

$$f_{\#}\mu(B) := \mu(f^{-1}(B))$$

for each B measurable in Y.

Let $n \in \mathbb{N}$ and recall that $\varphi_\emptyset^n : G \to G_n$ is a continuous map, hence measurable. Furthermore, the pushforward measure $\left(\varphi_\emptyset^n\right)_{\#}\mu_G$ coincides with μ_{G_n} on $\mathcal{P}(G_n)$. To see this, observe that if $A \subseteq G_n \leq \mathrm{Aut}(T^n)$, then

$$\begin{aligned}\mu_{G_n}(A) &= \frac{1}{|G_n|} \cdot \#A = |G : \mathrm{St}_G(n)|^{-1} \cdot \#A \\ &= \sum_A \mu_G(\mathrm{St}_G(n)) = \mu_G\Big(\left(\varphi_\emptyset^n\right)^{-1}(A)\Big) = \left(\varphi_\emptyset^n\right)_{\#}\mu_G(C_A).\end{aligned}$$

Therefore, the following equality holds as a result of [5, Theorem 2.6.8]:

$$\int_G (f \circ \varphi_\emptyset^n)\, d\mu_G = \int_{G_n} f\, d\Big(\left(\varphi_\emptyset^n\right)_{\#}\mu_G\Big) = \int_{G_n} f\, d\mu_{G_n} = \frac{1}{|G : \mathrm{St}_G(n)|} \sum_{g \in G_n} f(g)$$

for any function $f : G_n \to [-\infty, \infty]$ that may attain $+\infty$ or $-\infty$, but not both. We collect these facts in the following proposition.

Proposition 4.5. *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup, and $n \in \mathbb{N}$. For any set $A \subseteq G_n$ of n-patterns, we have*

$$\mu_{G_n}(A) = \frac{\#A}{|G : \mathrm{St}_G(n)|} = \left(\varphi_\emptyset^n\right)_{\#}\mu_G(A) = \mu_G\left(C_A^\emptyset\right),$$

where $\emptyset$ is the root of T. Moreover, if a function $f : G_n \to [-\infty, \infty]$ does not attain both $+\infty$ and $-\infty$, then

$$\int_G (f \circ \varphi_\emptyset^n)\, d\mu_G = \frac{1}{|G : \mathrm{St}_G(n)|} \sum_{g \in G_n} f(g).$$

Chapter 5

Fractality

5.1. Fractality

The exposition in this chapter starts from the notions of self-similarity, fractality, and super-strong fractality, which capture different degrees of similarity between actions on the subtrees and the whole tree. Self-similar groups were first studied as automata groups, dating back at least to the early 1980s. Nekrashevych introduced the term self-similar and fractal groups as these properties naturally arise in the study of iterated monodromy groups in complex dynamics; see [22]. Fractality also yields the measure-preserving property of certain dynamical systems; see [8, Theorem A]. Furthermore, in the view of dynamical systems, super-strongly fractal is another natural notion as it leads to probabilistic independence, ergodicity, and a strong mixing property; see [8, Lemma 4.2 and Theorem A].

Let T be a regular rooted tree. A group $G \leq \mathrm{Aut}(T)$ is said to be:

(a) *self-similar* if for every $v \in T$, we have

$$\varphi_v(G) \subseteq G;$$

(b) *fractal* if G is self-similar and for every $v \in T$, we have

$$\varphi_v(\mathrm{st}_G(v)) = G;$$

(c) *super-strongly fractal* if G is self-similar and for every $v \in T$, we have

$$\varphi_v(\mathrm{St}_G(|v|)) = G.$$

The first Grigorchuk group, the Gupta-Sidki p-groups for any odd primes p, and the iterated monodromy group of the complex polynomial $z^2 - 1$, also known as the Basilica group, are some examples of self-similar, fractal, and super-strongly fractal groups. Self-similarity can be easily verified as all the sections of the generators at the first-level vertices are contained in the original group. The verification for fractality and super-strong fractality involves the same observation, but with generators and their conjugates that fix the first level.

It is essential in these definitions that the tree is regular because a section of an element in $g \in G \leq \mathrm{Aut}(T)$ may not be identified with an element of $\mathrm{Aut}(T)$ otherwise. We remark that since self-similarity is assumed in the last two notions, we have

$$\varphi_v(\mathrm{st}_G(v)) = \varphi_v(G) \qquad \text{and} \qquad \varphi_v(\mathrm{St}_G(|v|)) = \varphi_v(G)$$

respectively. The above equations tell us that the fractality/super-strong fractality enables the vertex/level stabilisers to see the same sections at each $v \in T$ as the whole group G. However, observe that it is not necessary to require self-similarity to set these two equalities. This observation motivates us to define the following generalised notions of fractality and super-strong fractality.

Definition 5.1 (shifted fractal and shifted super-strongly fractal)**.** Let T be a spherically homogeneous rooted tree, not necessarily regular. A group $G \leq \mathrm{Aut}(T)$ is said to be *shifted fractal* if

$$\varphi_v(\mathrm{st}_G(v)) = \varphi_v(G)$$

for every $v \in T$ and *shifted super-strongly fractal* if

$$\varphi_v(\mathrm{St}_G(|v|)) = \varphi_v(G)$$

for every $v \in T$.

As $\varphi_v(\mathrm{St}_G(|v|)) \leq \varphi_v(\mathrm{st}_G(v)) \subseteq \varphi_v(G)$ for any vertex $v \in T$ and any group $G \leq \mathrm{Aut}(T)$, every shifted super-strongly fractal group is shifted fractal. Note that since φ_v is a homomorphism when restricted to $\mathrm{st}_G(v)$, if G is shifted fractal, then $\varphi_v(G)$ is also a group.

We should emphasise here that $\varphi_v(G)$ is not a subgroup of $\mathrm{Aut}(T)$, but rather of $\mathrm{Aut}(T_v)$. Hence, we do not require T to be regular. We call these new notions shifted as $\varphi_v(G)$ is no longer acting on T like elements of G, but the action is "shifted" to act on T_v as $\varphi_v(G)$ instead.

Note that each of these five definitions implies that the same condition holds for every finite-depth section and congruence quotient, e.g. if G is self-similar, then $\varphi_v^n(G) \subseteq G_n$ for every $v \in T$ and $n \in \mathbb{N}$. However, the converse only implies that the closure $\overline{G}$ has such a property due to Propositions 4.1 and 4.2.

For $G \leq \mathrm{Aut}(T)$, we shall write $G_v := \varphi_v(G)$ from this point onwards to simplify the notation. Furthermore, for any $v \in T$ and $m \in \mathbb{N}$, we note that the kernel $\ker \varphi_v^m$ is the kernel of φ_v^m restricted to $\mathrm{st}_G(v)$, i.e.

$$\ker \varphi_v^m := \{g \in \mathrm{st}_G(v) : g|_v^m = 1_m\}.$$

Recall that for a given set of m-patterns $A \subseteq (G_v)_m$, the cone over A in G_v is defined as $C_A^v := \{g \in G_v : g|_v^m \in A\}$. The following lemma shows that the cones are open.

Lemma 5.2. *Let $G \leq \mathrm{Aut}(T)$. For any $n \in \mathbb{N}$, the cones over the sets of n-patterns are open. Moreover, if $g \in G$, then $\mathrm{St}_G(n)g = C_{\{g|_\emptyset^n\}}$, i.e. all basic open sets are cones.*

Proof. We start with the second statement. The section of any element in the coset $\mathrm{St}_G(n)g$ at depth n is precisely $g|_\emptyset^n$, thus the cone contains the coset. Conversely, if $h \in C_{\{g|_\emptyset^m\}}$, then $h = hg^{-1}g$ where hg^{-1} fixes level n as $(hg^{-1})|_\emptyset^n = h|_\emptyset^n\, g^{-1}|_\emptyset^n = g|_\emptyset^n\, \left(g|_\emptyset^n\right)^{-1} = 1|_\emptyset^n$. Since the cosets of the level stabilisers are open, and an arbitrary cone is a union of cones of singleton sets, the first statement follows from the second. ■

5.2. Pushforward of measures and shifted fractality

In this section, we show that the unique Haar probability measure on G_v is simply the pushforward measure of the unique Haar probability measure on G by the section map φ_v whenever $G \leq \mathrm{Aut}(T)$ is a closed shifted fractal group. For self-similar groups, this reduces to the Haar probability measure being invariant under projection, as proven by Farina-Asategui in [8, Theorem A]

We start by generalising [8, Lemma 4.1(i)] to shifted fractal groups:

Lemma 5.3. *Let $G \leq \mathrm{Aut}(T)$ be shifted fractal. Then for any $v \in T$ at level $n \in \mathbb{N}$ and a set of m-patterns $A \subseteq (G_v)_m$, we have*

$$\#\varphi_\emptyset^{n+m}\Big((\varphi_v)^{-1}(C_A^v)\Big) = \#A \cdot |G : \mathrm{st}_G(v)| \cdot |\ker \varphi_v^m : \mathrm{St}_G(n+m)|.$$

Proof. Throughout this proof, we work in $\mathrm{Aut}(T^{n+m}) \cong \mathrm{Aut}(T)/\ \mathrm{St}(n+m)$ to simplify the notation. It suffices to show that for each $\sigma \in A$, every coset of $\mathrm{st}_G(v)$ has precisely $|\ker \varphi_v^m|$ distinct elements g in $(\varphi_v)^{-1}(C_A^v)$ such that

$$g|_v^m = \sigma. \tag{1}$$

Since G is shifted fractal, there exists $a \in \mathrm{st}_G(v)$ such that $a|_v^m = \sigma$.

First, we show that the intersection of each right coset of $\mathrm{st}_G(v)$ with $(\varphi_v)^{-1}(C_A^v)$ contains an element satisfying Equation 1. Fix an arbitrary right coset $\mathrm{st}_G(v)h$. Since

$$(h|_v^m)^{-1} = (h^{-1})|_{v^h}^m \in \varphi_{v^h}^m(G),$$

by shifted fractality, there exists $l \in \mathrm{st}_G\big(v^h\big)$ such that $l|_{v^h}^m = (h|_v^m)^{-1}$. Now

$$(ahl)|_v^m = a|_v^m\ h|_v^m\ l|_{v^h}^m = a|_v^m\ h|_v^m\ (h|_v^m)^{-1} = \sigma,$$

so the element ahl is contained in $(\varphi_v)^{-1}(C_A^v)$ and satisfies Equation 1. Furthermore, the element ahl is contained in the coset $\mathrm{st}_G(v)h$ as

$$v^{ahlh^{-1}} = v^{hlh^{-1}} = \big(v^h\big)^{h^{-1}} = v.$$

Next, we compute the number of elements in each coset of $\mathrm{st}_G(v)$ which satisfy Equation 1. For each right coset of $\mathrm{st}_G(v)$, we choose a coset representative g satisfying Equation 1. If $\tilde{g} \in (\varphi_v)^{-1}(C_A^v) \cap \mathrm{st}_G(v)g$ also satisfies Equation 1, we have $\tilde{g} = xg$ for some $x \in \mathrm{st}_G(v)$. We show further that x stabilises the m-truncated tree rooted at v as well, i.e. $x \in \ker \varphi_v^m$. For this, we use $x = \tilde{g}g^{-1} \in \mathrm{st}_G(v)$ and see that

$$x|_v^m = \tilde{g}|_v^m\ (g^{-1})|_{v^{\tilde{g}}}^m = \tilde{g}|_v^m\ (g^{-1})|_{v^g}^m = \tilde{g}|_v^m\ (g|_v^m)^{-1} = \sigma\sigma^{-1} = 1.$$

Hence, the subset of $(\varphi_v)^{-1}(C_A^v)$ that satisfies Equation 1 can be partitioned into the parts of the cosets of $\mathrm{st}_G(m)$ that lie in $(\varphi_v)^{-1}(C_A^v)$, each part containing precisely $|\ker \varphi_v^m|$ elements. ■

Instead of viewing Lemma 5.3 as a combinatorial counting result, we shall look at it from a measure-theoretic perspective. Indeed, the left-hand side in Lemma 5.3 resembles the pushforward of the Haar probability measure of a cone C_A^v whilst the right-hand side is related to the Haar probability measure on G_v. We make this intuition precise in the following theorem:

Recall that a closed subgroup $G \leq \mathrm{Aut}(T)$ is compact, since $\mathrm{Aut}(T)$ is compact and Hausdorff; thus, the group $G_v = \varphi_v(G)$ is compact as a continuous image of G. Hence, both G and G_v admit unique Haar probability measures μ_G and μ_{G_v} respectively.

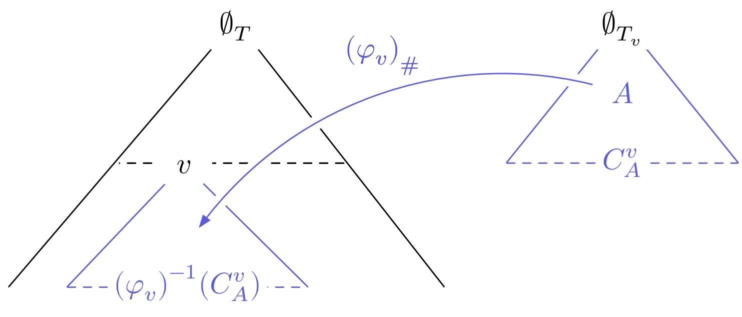

Figure 5: An intuition behind Theorem 5.4

Theorem 5.4 (shifted pushforward theorem)**.** *Let $G \leq \mathrm{Aut}(T)$ be a closed shifted fractal subgroup. Then, for any vertex $v \in T$ we have the equality of measures*

$$\mu_{G_v} = (\varphi_v)_\# \mu_G.$$

Proof. Let $m \in \mathbb{N}$, $v \in T$, and $A \subseteq (G_v)_m$ be a set of m-patterns. We make three observations. First, since $G_v = \varphi_v(\mathrm{st}_G(v))$ by shifted fractality, we get $\varphi_v^m(\mathrm{st}_G(v)) \cong G_v/\mathrm{St}_{G_v}(m)$. Second, since the map $\overline{\varphi}_v^m : \mathrm{st}_G(v)_{n+m} \to \mathrm{Aut}(T_v^m)$, induced from $\varphi_v^m : \mathrm{st}_G(v) \to \mathrm{Aut}(T_v^m)$, has image $\varphi_v^m(\mathrm{st}_G(v))$ and kernel $(\ker \varphi_v^m)_{n+m}$, the first isomorphism theorem yields

$$|\varphi_v^m(\mathrm{st}_G(v))| = |\mathrm{st}_G(v)_{n+m}| \ / \ |\ker \varphi_v^m : \mathrm{St}_G(n+m)|.$$

Note that $\mathrm{St}_G(n+m) \leq \ker \varphi_v^m$. Then, as the $(n+m)$th level stabiliser of $\mathrm{st}_G(v)$ is simply $\mathrm{St}_G(n+m)$, the third isomorphism theorem implies that

$$|G_{n+m}| \ / \ |\mathrm{st}_G(v)_{n+m}| = |G/\mathrm{st}_G(v)| = |G : \mathrm{st}_G(v)|.$$

Now, we deduce the statement of the theorem upon taking measures of a cone as follows:

$$\begin{aligned}
\mu_{G_v}(C_A^v) &= \frac{\#A}{|G_v : \mathrm{St}_{G_v}(m)|} \\
&= \frac{\#A}{|\varphi_v^m(\mathrm{st}_G(v))|} \\
&= \#A \cdot \frac{|\ker \varphi_v^m : \mathrm{St}_G(n+m)|}{|\mathrm{st}_G(v)_{n+m}|} \\
&= \#A \cdot |\ker \varphi_v^m : \mathrm{St}_G(n+m)| \cdot \frac{|G_{n+m}|}{|\mathrm{st}_G(v)_{n+m}|} \cdot \frac{1}{|G_{n+m}|} \\
&= \#A \cdot |\ker \varphi_v^m : \mathrm{St}_G(n+m)| \cdot |G : \mathrm{st}_G(v)| \cdot \frac{1}{|G_{n+m}|} \\
&= \frac{\#\varphi_\emptyset^{n+m}\big((\varphi_v)^{-1}(C_A^v)\big)}{|G_{n+m}|} \\
&= \mu_G\big((\varphi_v)^{-1}(C_A^v)\big),
\end{aligned}$$

where the first equality follows from Proposition 4.5, the second, third, and fifth equalities follow from the earlier observations, and the second-to-last equality is due to the computation in Lemma 5.3. Therefore, the pushforward $(\varphi_v)_{\#}\mu_G$ agrees with the Haar measure μ_{G_v} on every basic open set in G_v by Lemma 5.2.

Denote the Borel σ-algebra generated by the congruence topology on G_v by $\mathcal{B}(G_v)$. Let $\mathcal{A}$ be the sub σ-algebra of $\mathcal{B}(G_v)$ generated by the Borel measurable sets for which the pushforward $(\varphi_v)_{\#}\mu_G$ agrees with μ_{G_v}. We see that these two σ-algebras must agree as $\mathcal{A}$ contains all the basic open sets, thus all the open sets and $\mathcal{B}(G_v)$. Hence, the pushforward formula holds for all Borel measurable sets in G_v. ∎

The above result is analogous to [8, Theorem A(i)] by Fariña-Asategui for fractal groups.

5.3. Probabilistic independence and shifted super-strong fractality

Next, we shall show that shifted super-strongly fractality is closely linked to a measure-theoretic property: probabilistic independence. We start by generalising the counting result [8, Lemma 4.1(ii)] to the shifted super-strongly fractal case:

Lemma 5.5. *Let $G \leq \mathrm{Aut}(T)$ be shifted super-strongly fractal. Then, for any $n, m \in \mathbb{N}$, a vertex $v \in T$ at level n, we have*

$$\#\varphi_\emptyset^{n+m}\left(C_A \cap (\varphi_v)^{-1}(C_B^v)\right) = \#A \cdot \#B \cdot |\mathrm{St}_G(n) \cap \ker \varphi_v^m : \mathrm{St}_G(n+m)|,$$

for any sets of n-patterns $A \subseteq G_n$, and a set of m-patterns $B \subseteq (G_v)_m$.

Proof. Throughout this proof, we work in $\mathrm{Aut}(T^{n+m}) \cong \mathrm{Aut}(T)/\,\mathrm{St}(n+m)$ to simplify the notation. It is sufficient to show that for each pair $\sigma \in A$ and $\tau \in B$, there are exactly $|\mathrm{St}_G(n) \cap \ker \varphi_v^m|$ many distinct elements $g \in C_A \cap (\varphi_v)^{-1}(C_B^v)$ such that

$$g|_\emptyset^n = \sigma \qquad \text{and} \qquad g|_v^m = \tau. \tag{2}$$

First, we show the existence of such $g \in C_A \cap (\varphi_v)^{-1}(C_B^v)$ for each pair σ and τ. By definition of the cone C_A, there exists $h \in C_A \subseteq G$ such that $h|_\emptyset^n = \sigma$. Note that as there exists $t \in (\varphi_v)^{-1}(C_B^v) \subseteq G$ such that $t|_v^m = \tau \in B \subseteq (G_v)_m$, this leads to the fact that

$$(h|_v^m)^{-1}\tau = (h^{-1})|_{v^h}^m \; t|_v^m = (h^{-1}t)|_{v^h}^m \in (G_{v^h})_m.$$

By shifted super-strong fractality, we have $\varphi_{v^h}^m(\mathrm{St}_G(n)) = (G_{v^h})_m$. Therefore, there exists an element $l \in \mathrm{St}_G(n)$ such that

$$l|_{v^h}^m = (h|_v^m)^{-1}\tau \in (G_{v^h})_m.$$

Hence, the element $g := hl \in C_A \cap (\varphi_v)^{-1}(C_B^v)$ satisfies Equation 2 as

$$(hl)|_\emptyset^n = h|_\emptyset^n \; l|_\emptyset^n = \sigma \cdot 1 = \sigma \qquad \text{and} \qquad (hl)|_v^m = h|_v^m \; l|_{v^h}^m = h|_v^m \; (h|_v^m)^{-1}\tau = \tau.$$

For each pair σ, τ, we fix an element $g \in C_A \cap (\varphi_v)^{-1}(C_B^v)$ satisfying Equation 2. For any $x \in \mathrm{St}_G(n) \cap \ker(\varphi_v^m)$, the element $\tilde{g} := xg$ satisfies the same conditions as g since

$$\tilde{g}|_\emptyset^n = x|_\emptyset^n \; g|_\emptyset^n = 1 \cdot g|_\emptyset^n = \sigma \qquad \text{and} \qquad \tilde{g}|_v^m = x|_v^m \; g|_{v^x}^m = 1 \cdot g|_v^m = \tau.$$

Note that if x and y are two different elements in $\mathrm{St}_G(n) \cap \ker(\varphi_v^m)$, then $xg \neq yg$. On the other hand, if g and $\tilde{g}$ are two elements satisfying Equation 2, then $\tilde{g}g^{-1}$ is contained in $\mathrm{St}_G(n) \cap \ker(\varphi_v^m)$. Hence there are exactly $|\mathrm{St}_G(n) \cap \ker \varphi_v^m|$ many distinct elements satisfying Equation 2 in $C_A \cap (\varphi_v)^{-1}(C_B^v)$ for each pair σ and τ. ∎

As in Lemma 5.3, the equality in Lemma 5.5 can also be viewed from a probabilistic perspective where one side of the equation is the counting measure of an intersection of two cones, whilst the other side is a product of the sizes of two different sets of n- and m-patterns corresponding to the cones and some other quantities; see Figure 6. This formulation hints at a probabilistic independence between the cones C_A and C_B^v.

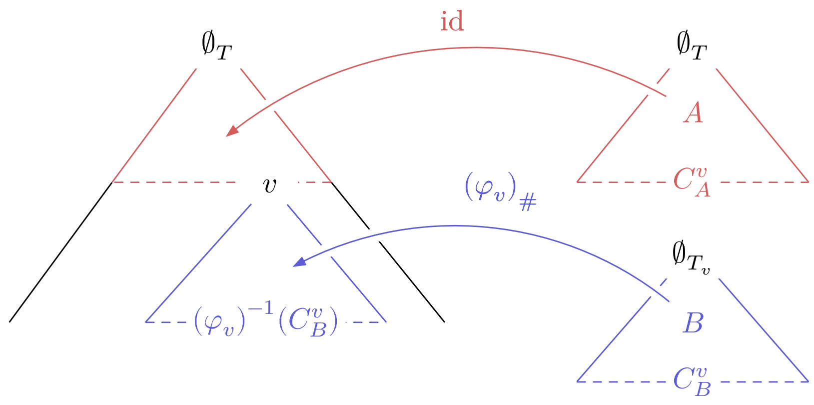

Figure 6: An intuition behind Theorem 5.6

It has been shown in [8, Lemma 4.2] that probabilistic independence can be obtained for super-strongly fractal groups. We will follow the same strategy as in [8] to establish it for shifted super-strongly fractal groups. We further show that it is also necessary that the group is shifted super-strongly fractal to obtain this independence, hence establishing an equivalence between shifted super-strong fractality and probabilistic independence:

Theorem 5.6 (shifted probabilistic independence theorem)**.** *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. Then G is shifted super-strongly fractal if and only if for any $n, m \in \mathbb{N}$ and a vertex $v \in T$ at level n, we have*

$$\mu_G\big(C_A \cap (\varphi_v)^{-1}(C_B^v)\big) = \mu_G(C_A) \cdot \mu_{G_v}(C_B^v),$$

for any sets $A \subseteq G_n$ of n-patterns and $B \subseteq (G_v)_m$ of m-patterns.

Proof. The forward direction can be deduced as follows:

$$\begin{aligned}
\mu_G\big(C_A \cap (\varphi_v)^{-1}(C_B^v)\big) &= \mu_{G_{n+m}}\Big(\varphi_\emptyset^{m+n}\big(C_A \cap (\varphi_v)^{-1}(C_B^v)\big)\Big) \\
&= \frac{\#\varphi_\emptyset^{m+n}\big(C_A \cap (\varphi_v)^{-1}(C_B^v)\big)}{|G_{n+m}|} \\
&= \frac{\#A \cdot \#B \cdot |\mathrm{St}_G(n) \cap \ker \varphi_v^m : \mathrm{St}_G(n+m)|}{|G_{n+m}|} \\
&= \frac{\#A}{|G_n|} \cdot \frac{\#B \cdot |\mathrm{St}_G(n) \cap \ker \varphi_v^m : \mathrm{St}_G(n+m)|}{|\mathrm{St}_G(n) : \mathrm{St}_G(n+m)|} \\
&= \frac{\#A}{|G_n|} \cdot \frac{\#B}{|\mathrm{St}_G(n) : \mathrm{St}_G(n) \cap \ker \varphi_v^m|} \\
&= \frac{\#A}{|G_n|} \cdot \frac{\#B}{|\varphi_v^m(\mathrm{St}_G(n))|}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\#A}{|G_n|} \cdot \frac{\#B}{|(G_v)_m|} \\
&= \mu_G(C_A) \cdot \mu_{G_v}(C^v_B).
\end{aligned}
$$

Note that the first, second, and last equalities follow from Proposition 4.5 and that the intersection $C_A \cap (\varphi_v)^{-1}(C_B)$ is the cone of $\varphi_\emptyset^{n+m}\big(C_A \cap (\varphi_v)^{-1}(C^v_B)\big) \subseteq G_{n+m}$. The third equality follows directly from Lemma 5.5. The fourth and fifth equalities follow from a direct application of the third isomorphism theorem. The third-to-last equality is deduced using the first isomorphism theorem on φ_v^m restricted to $\mathrm{St}_G(n)$. The second-to-last equality is given by the shifted super-strong fractality.

For the converse, we start by showing that $\mu_{G_v}(\varphi_v(\mathrm{St}_G(n))) = 1$. For any $m \in \mathbb{N}$, we set $B_m := \varphi_v^m(\mathrm{St}_G(n))$. Observe that $\big\{C^v_{B_m}\big\}_{m\in\mathbb{N}}$ is a descending sequence in $m \in \mathbb{N}$ and

$$
\begin{aligned}
\bigcap_{m\in\mathbb{N}} C^v_{B_m} &= \Big\{h \in G_v : h|^m_{\emptyset_{T_v}} \in \varphi_v^m(\mathrm{St}_G(n)), \text{for all } m \in \mathbb{N}\Big\} \\
&= \Big\{h \in G_v : h = h|_{\emptyset_{T_v}} \in \varphi_v(\mathrm{St}_G(n)), \text{for all } m \in \mathbb{N}\Big\} \\
&= \varphi_v(\mathrm{St}_G(n)),
\end{aligned}
$$

where the second equality holds because G_v is compact with respect to the congruence topology. For $A := 1_n$, we have $C_A = \mathrm{St}_G(n)$. We conclude that

$$
\begin{aligned}
\mu_{G_v}(\varphi_v(\mathrm{St}_G(n))) &= \mu_{G_v}\left(\bigcap_{m\in\mathbb{N}} C^v_{B_m}\right) \\
&= \lim_{m\to\infty} \mu_{G_v}\big(C^v_{B_m}\big) \\
&= \lim_{m\to\infty} \mu_G\big(C_A \cap (\varphi_v)^{-1}\big(C^v_{B_m}\big)\big)/\mu_G(C_A) \\
&= \mu_G\left(\bigcap_{m\in\mathbb{N}} C_A \cap (\varphi_v)^{-1}\big(C^v_{B_m}\big)\right)/\mu_G(C_A) \\
&= \mu_G\left(C_A \cap (\varphi_v)^{-1}\left(\bigcap_{m\in\mathbb{N}} C^v_{B_m}\right)\right)/\mu_G(C_A) \\
&= \mu_G\big(C_A \cap (\varphi_v)^{-1}(\varphi_v(\mathrm{St}_G(n)))\big)/\mu_G(C_A) = \mu_G(C_A)/\mu_G(C_A) = 1,
\end{aligned}
$$

where the third equality is given by the assumption in the theorem and the second-to-last equality follows from $C_A \subseteq (\varphi_v)^{-1}(\varphi_v(\mathrm{St}_G(n)))$.

We now obtain that μ_{G_v}-almost every element of G_v belongs to $\varphi_v(\mathrm{St}_G(n))$. As $\varphi_v(\mathrm{St}_G(n))$ is a continuous image of a compact set, it is compact in G_v. The group G_v is compact and so $\varphi_v(\mathrm{St}_G(n))$ is a closed subgroup in G_v. By Proposition 4.4, it follows that $\varphi_v(\mathrm{St}_G(n))$ is open and has index $|G_v : \varphi_v(\mathrm{St}_G(n))| = 1/\mu_{G_v}(\varphi_v(\mathrm{St}_G(n))) = 1$. Hence,

$$
\varphi_v(\mathrm{St}_G(n)) = G_v. \qquad \blacksquare
$$

In the following corollary, we show that Theorem 5.6 can be strengthened further. However, the version stated in the theorem will still be more convenient later.

Corollary 5.7. *Let us assume the same conditions as in Theorem 5.6. Then G is shifted super-strongly fractal if and only if for any $n \in \mathbb{N}$, and a vertex $v \in T$ at level n, we have*

$$\mu_G\Big(C_A \cap (\varphi_v)^{-1}(E)\Big) = \mu_G(C_A) \cdot \mu_{G_v}(E) = \mu_G(C_A) \cdot \mu_G\Big((\varphi_v)^{-1}(E)\Big),$$

for any sets $A \subseteq G_n$ of n-patterns and any measurable set E in G_v.

Proof. Assuming the first equality, the second equality in the statement follows from Theorem 5.4.

Theorem 5.6 yields the probabilistic independence formula

$$\mu_G\Big(C_A \cap (\varphi_v)^{-1}(E)\Big) = \mu_G(C_A) \cdot \mu_{G_v}(E)$$

for E a basic open set, i.e. a cone of a singleton set in G_v. For any fixed cone C_A, we show that the formula holds for any measurable set E. Denote the Borel σ-algebra generated by the congruence topology on G_v by $\mathcal{B}(G_v)$. Let $\mathcal{A}$ be the sub σ-algebra of $\mathcal{B}(G_v)$ generated by the collection

$$\Big\{E \in \mathcal{B}(G_v) : \mu_G\Big(C_A \cap (\varphi_v)^{-1}(E)\Big) = \mu_G(C_A) \cdot \mu_{G_v}(E)\Big\}.$$

We see that these two σ-algebras must agree as $\mathcal{A}$ contains all the basic open sets, thus all the open sets and $\mathcal{B}(G_v)$. Hence, the independence formula holds for all Borel measurable sets E in G_v. ∎

We note that we cannot replace C_A in Corollary 5.7 with any Borel measurable set in G because A should not impose constraints below level n. Therefore, one cannot replace C_A with any basic open set which is a coset of a level stabiliser of level larger than n. Intuitively, if we were able to replace C_A with a cone in G of k-patterns such that $k > n$, then this cone would overlap with the measurable set E in G_v. Hence, one should not expect probabilistic independence in this case.

Chapter 6

Fixed-point proportion

6.1. Fixed-point processes and martingales

In this section, we examine properties of the fixed-point process, a stochastic process counting the number of fixed points of an automorphism at each level of the tree. This process was first introduced by Jones in [16]. We follow his strategy to show that if the process is a martingale and satisfies a uniform bound on its conditional probability, then it converges to zero almost everywhere, i.e. such groups fix no points on the boundary of the tree almost surely.

A *(real) stochastic process* is a collection $\{Y_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions on a common probability space.

Let $G \le \mathrm{Aut}(T)$ be a closed subgroup. For every $n \in \mathbb{N}$, we define $X_n : G \to \mathbb{N}_0$ by

$$X_n(g) := \#\{v \in \mathcal{L}_n : v^g = v\}.$$

In other words, for $n \in \mathbb{N}$ and $g \in G$, the number $X_n(g)$ is simply the number of vertices fixed by g at level n. This function is measurable because, for $m > \#\mathcal{L}_n$, we have $X_n^{-1}(\{m\}) = \emptyset$, and for $m \le \#\mathcal{L}_n$, we have

$$\begin{aligned} X_n^{-1}(\{r\}) &= \{g \in G : g \text{ fixes } r \text{ vertices at level } n\} \\ &= \big(\varphi_\emptyset^n\big)^{-1}\big(\{g|_\emptyset^n \in G_n : g|_\emptyset^n \text{ fixes } r \text{ vertices at level } n\}\big). \end{aligned}$$

Indeed, as G_n is equipped with the discrete topology and $\varphi_\emptyset^n$ is continuous, so $X_n^{-1}(\{r\})$ is a Borel measurable set. These functions define a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ called the *fixed-point process* of G. For each $n \in \mathbb{N}$, the preimage $X_n^{-1}(\mathbb{N})$ is the set of all elements in G that fix at least one vertex at level n. Since if $g \in G$ fixes a vertex $v \in T$ at level n, it must fix the predecessor of v at level $n-1$ as well, we obtain the following descending sequence of nested sets:

$$X_1^{-1}(\mathbb{N}) \supseteq X_2^{-1}(\mathbb{N}) \supseteq \cdots \supseteq X_n^{-1}(\mathbb{N}) \supseteq \cdots.$$

Let us recall the following notions from probability theory. Let (X, μ) be a probability space. The *conditional probability* of a measurable set A given another measurable set B of positive measure is defined to be

$$\mu(A \mid B) := \mu(A \cap B)/\mu(B),$$

i.e. it is the normalised measure of the projection of A onto B as a subspace of the probability space X. If $f : X \to \mathbb{R}$ is a real measurable function (random variable), then the *conditional expectation* of f on a measurable set A of positive measure is given by

$$\mathbb{E}(f \mid A) := \frac{1}{\mu(A)} \int_A f \, d\mu.$$

Amongst stochastic processes, we are interested in the following class. A *martingale* is a stochastic process $\{Y_n\}_{n\in\mathbb{N}}$ in a measure space (X, μ) such that for all $n \in \mathbb{N}$ we have:
(a) $\mathbb{E}(Y_n) < \infty$;
(b) $\mathbb{E}(Y_n \mid Y_1 = t_1, \ldots, Y_{n-1} = t_{n-1}) = t_{n-1}$ for every $t_1, \ldots, t_{n-1} \in \mathbb{R}$ satisfying

$$\mu(Y_1 = t_1, \ldots, Y_{n-1} = t_{n-1}) > 0.$$

The following is an example of a martingale. Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be independent and identically distributed Bernoulli random variables on the σ-algebra $\{\emptyset, \{\text{tail}\}, \{\text{head}\}, \{\text{tail}, \text{head}\}\}$ with $Z_n(\text{tail}) = -1, Z_n(\text{head}) = 1$, and

$$\mu(Z_n = -1) = \mu(Z_n = 1) = 1/2$$

for each n. Each random variable represents the gain/loss from the nth round of an infinite sequence of independent games of a fair coin toss; a gambler gains 1 krona if the coin lands on a head and loses 1 krona otherwise. It is clear that each Z_n has a finite expectation. Since each toss is independent,

$$\mathbb{E}(Z_n \mid Z_1 = t_1, Z_2 = t_2, \ldots, Z_{n-1} = t_{n-1}) = t_{n-1} = 0.$$

Furthermore, the sum $Y_n = \sum_{k=1}^{n} Z_n$ representing the net gain/loss of a gambler also forms a martingale. This is because

$$\begin{aligned}
&\mathbb{E}(Y_n \mid Y_1 = t_1, \ldots, Y_{n-1} = t_{n-1}) \\
&\qquad = \frac{\sum_{k\in\{0,1\}} \big(t_{n-1} + (-1)^k\big) \cdot \mu\big(\, Y_1 = t_1, \ldots, Y_n = t_{n-1} + (-1)^k\big)}{\mu(\, Y_1 = t_1, \ldots, Y_{n-1} = t_{n-1})} \\
&\qquad = \sum_{k\in\{0,1\}} \big(t_{n-1} + (-1)^k\big) \cdot \frac{1}{2} \\
&\qquad = t_{n-1}.
\end{aligned}$$

In other words, a gambler is expected to have the same capital as in the previous round.

Theorem 6.1 (see [5, Theorem 10.4.8])**.** *If $\{Y_n\}_{n\in\mathbb{N}}$ is a non-negative martingale, then the limit*

$$\lim_{n\to\infty} Y_n(g)$$

exists almost everywhere with a finite value.

This is a special case of Doob's martingale convergence theorem.

As the fixed-point process is non-negative, if it is a martingale, then it converges almost everywhere. In fact, as it only takes discrete values, it becomes eventually constant almost everywhere. Jones has shown in [17, pages 128-129] that under the two conditions in Theorem 6.2, if the group $G \leq \mathrm{Aut}(T)$ is an iterated wreath product, the limit of $X_n(g)$ must be zero for almost every $g \in G$. A general proof for groups acting on regular rooted trees can be found in [10]. As we shall see below, the proof still holds without such a regularity assumption on the tree.

First, let us recall the following notation. For a collection of sets $\{E_n\}_{n\in\mathbb{N}}$ with $E_n \subseteq E$ for all $n \in \mathbb{N}$, we define

$$\liminf_n E_n := \bigcup_{n\in\mathbb{N}} \bigcap_{j\geq n} E_j.$$

Intuitively, this is the set of elements in E that are eventually contained in all the sets E_n for large enough indices n.

Theorem 6.2 (see [10, Theorem 3.2]). *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. Assume that:*
(a) *the fixed-point process $\{X_n\}_{n\in\mathbb{N}}$ of G is a martingale;*
(b) *for any $r > 0$, there exist $\varepsilon > 0$ and $m \in \mathbb{N}$, both depending on r, such that for almost every $n \in \mathbb{N}$ we have*

$$\mu_G(X_{n+m} = r \mid X_n = r) \leq 1 - \varepsilon.$$

Then, the fixed-point process of G is eventually zero for almost every $g \in G$.

Proof. By assumption (a) and Theorem 6.1, the fixed-point process of G is eventually constant for some constant r depending on g for almost all $g \in G$. Let

$$\Lambda := \{g \in G : X_n(g) \text{ becomes eventually constant}\}$$

and for every $r \geq 0$, let us further define

$$\Lambda_r := \{g \in G : X_n(g) \text{ becomes eventually } r\}.$$

It is clear that $\Lambda = \bigcup_{r\geq 0} \Lambda_r$. We also define, for every $r \geq 0$ and any $n \in \mathbb{N}$, the measurable subset $E_n^r := \{g \in \Lambda : X_n(g) = r\}$. One then has that $\Lambda_r = \liminf_n E_n^r$. With this notation, the inequality in condition (b) can be rewritten as $\mu_G(E_{n+m}^r \mid E_n^r) \leq 1 - \varepsilon$, as $\mu_G(\Lambda) = 1$.

We claim that for any $r > 0$ and any $m \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \mu_G(E_n^r) = \lim_{n\to\infty} \mu_G(E_{n+m}^r \cap E_n^r) = \mu_G(\Lambda_r).$$

Therefore for $r > 0$, if $\mu_G(\Lambda_r) > 0$, we get

$$\lim_{n\to\infty} \mu_G(E_{n+m}^r \mid E_n^r) = \lim_{n\to\infty} \frac{\mu_G(E_{n+m}^r \cap E_n^r)}{\mu_G(E_n^r)} = \frac{\mu_G(\Lambda_r)}{\mu_G(\Lambda_r)} = 1,$$

which contradicts assumption (b). Thus $\mu_G(\Lambda_r) = 0$ for every $r > 0$ and thereby

$$\mu_G(\Lambda_0) = \mu_G(\Lambda) = 1.$$

Let us prove the claim. We will show that $\lim_{n\to\infty} \mu_G(E_n^r) = \mu_G(\Lambda_r)$ and the other equality follows in the same manner. Intuitively, an element $g \in \Lambda$ such that either $X_n(g)$ converges to r or is already r should dwindle in measure to 0 as n grows larger. Indeed

$$\begin{aligned}
|\mu_G(\Lambda_r) - \mu_G(E_n^r)| &= |\mu_G((\Lambda_r \setminus E_n^r) \cup (\Lambda_r \cap E_n^r)) - \mu_G((E_n^r \setminus \Lambda_r) \cup (\Lambda_r \cap E_n^r))| \\
&= |\mu_G(\Lambda_r \setminus E_n^r) + \mu_G(\Lambda_r \cap E_n^r) - \mu_G(E_n^r \setminus \Lambda_r) - \mu_G(\Lambda_r \cap E_n^r)| \\
&= |\mu_G(\Lambda_r \setminus E_n^r) - \mu_G(E_n^r \setminus \Lambda_r)| \\
&\leq \mu_G(\Lambda_r \setminus E_n^r) + \mu_G(E_n^r \setminus \Lambda_r) \\
&= \mu_G(\Lambda_r \triangle E_n^r),
\end{aligned}$$

so if suffices to show that $\lim_{n\to\infty} \mu_G(\Lambda_r \triangle E_n^r) = 0$. Now, observe that

$$E_n^r \setminus \Lambda_r = \left\{g \in \Lambda : X_n(g) = r \text{ but } \lim_{n\to\infty} X_n(g) \neq r\right\} \subseteq \bigcupdot_{r'\in\mathbb{N}_0\setminus\{r\}} \Lambda_{r'} \setminus E_n^{r'}$$

for any $r \in \mathbb{N}_0$. Thus, we have $\Lambda_r \triangle E_n^r = (\Lambda_r \setminus E_n^r) \cup (E_n^r \setminus \Lambda_r) \subseteq \bigcup_{r'\in\mathbb{N}_0} \Lambda_{r'} \setminus E_n^{r'}$.

Since μ_G is a finite measure and $\Lambda_{r'} \setminus E_n^{r'}$ are pairwise disjoint, by the dominated convergence theorem on $\mathbb{N}_0$, we interchange the summation and the limit to obtain

$$\lim_{n\to\infty} \mu_G(\Lambda_r \bigtriangleup E_n^r) \leq \lim_{n\to\infty} \sum_{r'\in\mathbb{N}_0} \mu_G\Big(\Lambda_{r'} \setminus E_n^{r'}\Big) = \sum_{r'\in\mathbb{N}_0} \lim_{n\to\infty} \mu_G\Big(\Lambda_{r'} \setminus E_n^{r'}\Big).$$

Hence, it suffices to show that $\lim_{n\to\infty} \mu_G\Big(\Lambda_{r'} \setminus E_n^{r'}\Big) = 0$ for every non-negative integer r'. Let us write $F_k^{r'} := \cap_{l\geq k} E_l^{r'}$ for $k \in \mathbb{N}$ and $F_0^{r'} = \emptyset$. Note that $F_k^{r'} \subseteq E_k^{r'}$. Then

$$\Lambda_{r'} = \liminf_k E_k^{r'} = \bigcup_{k\in\mathbb{N}} F_k^{r'} = \dot{\bigcup_{k\in\mathbb{N}}} F_k^{r'} \setminus F_{k-1}^{r'} = F_n^{r'} \cup \left(\dot{\bigcup_{k\geq n+1}} F_k^{r'} \setminus F_{k-1}^{r'} \right).$$

One can deduce from the definition that $F_k^{r'} \setminus F_{k-1}^{r'} \subseteq \Lambda_{r'} \setminus E_{k-1}^{r'}$, and therefore

$$\bigcap_{n\in\mathbb{N}_0} \bigcup_{k\geq n+1} \Big(F_k^{r'} \setminus F_{k-1}^{r'}\Big) \subseteq \bigcap_{n\in\mathbb{N}_0} \bigcup_{k\geq n+1} \Big(\Lambda_{r'} \setminus E_{k-1}^{r'}\Big) = \Lambda_{r'} \setminus \liminf_k E_{k-1}^{r'} = \Lambda_{r'} \setminus \Lambda_{r'} = \emptyset.$$

We use the measure continuity and the above deduction to conclude the claim as follows

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \mu_G\Big(\Lambda_{r'} \setminus E_n^{r'}\Big) &\leq \lim_{n\to\infty} \mu_G\Big(\Lambda_{r'} \setminus F_n^{r'}\Big) = \lim_{n\to\infty} \mu_G\left(\dot{\bigcup_{k\geq n+1}} F_k^{r'} \setminus F_{k-1}^{r'} \right) \\ &= \mu_G\left(\bigcap_{n\in\mathbb{N}_0} \dot{\bigcup_{k\geq n+1}} F_k^{r'} \setminus F_{k-1}^{r'} \right) = \mu_G(\emptyset) = 0. \end{aligned}$$

∎

We proceed to find natural sufficient assumptions on a group that make the fixed-point process a martingale. The following theorem characterises when the fixed-point process of a group is a martingale. It was shown in [3] by Bridy, Jones, Kelsey, and Lodge in the case where T is regular. With minor adjustments, the result can be generalised to any spherically homogeneous rooted tree T:

Theorem 6.3 (see [3, Theorem 5.7]). *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. Then, the fixed-point process of G is a martingale if and only if for all $n \in \mathbb{N}$, the level stabiliser $\mathrm{St}_G(n-1)$ acts on T_v^1 transitively for any $v \in \mathcal{L}_{n-1}$.*

Proof. We make some comments before proceeding with the actual proof. Set

$$S := \{g \in G_n : g \text{ fixes } t_i \text{ vertices at level } i \text{ for } 1 \leq i \leq n-1\}.$$

We derive an alternative formula from the definition of the conditional expectation involved in condition (b) of the definition of martingales via the pushforward measures as follows

$$\begin{aligned} \mathbb{E}(X_n \mid X_1 = t_1, \ldots, X_{n-1} = t_{n-1}) &= \frac{1}{\mu_G(C_S)} \int_{C_S} X_n \, d\mu_G \\ &= \frac{1}{\mu_G(C_S)} \int_G \chi_{C_S} \cdot X_n \, d\mu_G \\ &= \frac{1}{\mu_G(C_S)} \int_G (\chi_S \cdot \#\,\mathrm{fix}) \circ \varphi_\emptyset^n \, d\mu_G \\ &= \frac{1}{\mu_G(C_S)} \int_{G_n} (\chi_S \cdot \#\,\mathrm{fix}) \, d\mu_{G_n} \end{aligned}$$

$$= \frac{1}{\mu_G(C_S)} \sum_{g \in S} \frac{\# \operatorname{fix}(g)}{|G : \operatorname{St}_G(n)|} = \frac{1}{\# S} \sum_{g \in S} \# \operatorname{fix}(g),$$

where we define $\operatorname{fix}(g) := \{w \in \mathcal{L}_n \subseteq T^n : w^g = w\}$ for $g \in G_n$ and the third-to-last equality follows from Proposition 4.5.

Let us begin with the backward direction. Firstly, condition (a) of martingales is obtained for every $n \in \mathbb{N}$ as

$$\mathbb{E}(X_n) = \int_G X_n \, d\mu_G = \sum_{r \in \mathbb{N}} r \mu_G(X_n = r) = \sum_{n=1}^{\# \mathcal{L}_n} r \mu_G(X_n = r) < \infty.$$

Now, let $v \in \mathcal{L}_{n-1}$. Assume that $\operatorname{St}_G(n-1)$ acts transitively on the 1-truncated tree T_v^1 rooted at v. We need to show that

$$\frac{1}{\# S} \sum_{g \in S} \# \operatorname{fix}(g) = t_{n-1}$$

for any $t_1, \ldots, t_{n-1} \in \mathbb{R}$ such that $\mu_G(X_1 = t_1, \ldots, X_{n-1} = t_{n-1}) > 0$. Each group $H_{n-1} := \operatorname{St}_G(n-1)/\operatorname{St}_G(n)$ acts trivially on $\mathcal{L}_{n-1}$, so S is invariant under multiplication by elements in H_{n-1}, whence S is a union of cosets of H_{n-1}. For $gH_{n-1} \subseteq S$, define

$$R := \{vx \in \mathcal{L}_n : v \in \mathcal{L}_{n-1},\ v^g = v, \text{ and } x \in T_v^1\}.$$

Note that as $g \in S$, we have $\# R = t_{n-1} \cdot d_n$; recall that d_n is the common number of descendants of every vertex at level $n-1$. For every $vx \in R$, there exists a unique $vy \in R$ such that $(vy)^g = vx$, i.e. $y = x^{(g|_v)^{-1}}$. Due to the transitivity of the action of H_{n-1} on T_v^1, the set

$$Q_{vx} := \{h \in H_{n-1} : (vy)^h = vx\}$$

is non-empty for every pair $x, y \in T_v^1$. Moreover Q_{vx} is a coset of $\operatorname{st}_{H_{n-1}}(vx)$ in H_{n-1}. By the orbit-stabiliser theorem, we obtain that

$$d_n = \# \operatorname{orb}_{H_{n-1}}(vx) = |H_{n-1} : \operatorname{st}_{H_{n-1}}(vx)| = \frac{|H_{n-1}|}{|\operatorname{st}_{H_{n-1}}(vx)|} = \frac{|H_{n-1}|}{|Q_{vx}|}.$$

Observe also that $Q_{vx} = \{h \in H_{n-1} : (vx)^{gh} = vx\}$, so we have

$$\begin{aligned} t_{n-1} = \# R \cdot d_n^{-1} &= \sum_{vx \in R} \frac{|Q_{vx}|}{|H_{n-1}|} = \sum_{vx \in R} \int_{H_{n-1}} \chi_{Q_{vx}} \, d\mu_{H_{n-1}} \\ &= \int_{H_{n-1}} \sum_{vx \in R} \chi_{Q_{vx}} \, d\mu_{H_{n-1}} = \int_{H_{n-1}} \# \operatorname{fix}(gh) \, d\mu_{H_{n-1}}(h) = \frac{1}{|H_{n-1}|} \sum_{h \in H_{n-1}} \# \operatorname{fix}(gh). \end{aligned}$$

If $\mathcal{T}$ is a transversal of H_{n-1} in S, then

$$t_{n-1} = \frac{1}{\# \mathcal{T}} \sum_{\mathcal{T}} t_{n-1} = \frac{1}{\# \mathcal{T} \cdot |H_{n-1}|} \sum_{g \in \mathcal{T}} \sum_{h \in H_{n-1}} \# \operatorname{fix}(gh) = \frac{1}{\# S} \sum_{g \in S} \# \operatorname{fix}(g),$$

as wanted.

For the forward direction, we make the following observation. If the action of $\operatorname{St}_G(n-1)$ on T_v^1 is transitive, the number k of H_{n-1}-orbits on T^n cannot exceed $d := d_1 \cdot d_2 \cdots d_{n-1}$.

Suppose that the action is not transitive on some T_v^1 for some vertex v at level $n-1$, then $k > d$. By Burnside's lemma, we have

$$\frac{1}{|H_{n-1}|}\sum_{h\in H_{n-1}} \#\operatorname{fix}(h) = k > d.$$

Since $S = H_{n-1}$ if and only if every $h \in S$ fixes all vertices at levels no greater than $n-1$, i.e. $X_i(h) = d_1 \cdot d_2 \cdots d_i$ for every $h \in S$ and $i = 1, \ldots, n$, we have

$$\mathbb{E}(X_n \mid X_1 = d_1, X_2 = d_1 \cdot d_2, \ldots, X_{n-1} = d) = \frac{1}{|H_{n-1}|}\sum_{h\in H_{n-1}} \#\operatorname{fix}(h) = k > d$$

proving that the fixed-point process of G is not a martingale. ∎

A group $G \leq \operatorname{Aut}(T)$ is said to be *level-transitive* if for each $n \in \mathbb{N}$, the group G acts transitively on $\mathcal{L}_n$, i.e. for every vertices $u, v \in \mathcal{L}_n$ there exists $g \in G$ such that $u^g = v$. We give a characterisation of level-transitivity of shifted super strongly fractal groups as follows:

Lemma 6.4. *Let $G \leq \operatorname{Aut}(T)$ be shifted super-strongly fractal. Then the following are equivalent:*
(a) *G is level-transitive;*
(b) *for every $n \in \mathbb{N}_0$, the nth level stabiliser acts transitively on every subtree rooted at level n, i.e. the subgroup $\operatorname{St}_G(n)$ acts transitively on T_v for every $v \in \mathcal{L}_n$;*
(c) *for every $n \in \mathbb{N}_0$, the subgroup $\operatorname{St}_G(n)$ acts transitively on T_v^1 for every $v \in \mathcal{L}_n$.*

Proof. Suppose that G is level-transitive. Fix $n \in \mathbb{N}_0$, and a vertex v at level n. Then for any two vertices $u, w \in T_v$ at the same level, there exists $g \in G$ such that $(vu)^g = vw$. Thus g stabilises the vertex v and $g_v : u \mapsto w$. As G is shifted super-strongly fractal, there exists $h \in \operatorname{St}_G(|v|)$ such that $\varphi_v(h) = \varphi_v(g)$. Hence (a) implies (b) since

$$(vu)^h = v^h u^{h|_v} = vu^{g_v} = vw.$$

It is trivial that (b) implies (c). Lastly, assuming condition (c), we will prove that G is transitive on every level by induction on the level n. It is plain that $G = \operatorname{St}_G(0)$ acts transitively on level 1. Suppose that G acts transitively on levels $1, 2, \ldots, n-1$, and let $v_1 u, v_2 w$ be any two vertices at level n in T with v_1 and v_2 at level $n-1$. By induction, there exists $g \in G$ sending v_1 to v_2, and so $u^{g|_{v_1}} \in T_{v_2}^1$. Now, by (c), we may pick $h \in \operatorname{St}_G(n-1)$ that sends $u^{g|_{v_1}}$ to $w \in T_{v_2}^1$. Hence, the transitivity of G on level n follows from

$$(v_1 u)^{gh} = \left(v_1^g u^{g|_{v_1}}\right)^h = \left(v_2 u^{g|_{v_1}}\right)^h = v_2\left(u^{g|_{v_1}}\right)^h = v_2 w. \qquad ∎$$

Lemma 6.5. *If $G \leq \operatorname{Aut}(T)$ is closed, shifted super-strongly fractal, and level-transitive, then the fixed point process $\{X_n\}_{n\in\mathbb{N}}$ is a martingale.*

Proof. This follows directly from Lemma 6.4 and Theorem 6.3. ∎

6.2. Fixed-point proportion

Motivated by several problems in arithmetic dynamics, we will be interested in the number of elements in a group $G \leq \operatorname{Aut}(T)$ that have many fixed points in T. However, as G is usually an infinite group, we need a notion that makes this rigorous.

Let $G \leq \operatorname{Aut}(T)$. The *fixed-point proportion* $\operatorname{FPP}(G)$ of G is given by

$$\mathrm{FPP}(G) := \lim_{n\to\infty} \frac{\#\{g \in G_n : g \text{ fixes a vertex at level } n\}}{|G_n|}.$$

Denote $F_n := \{g \in G_n : g \text{ fixes a vertex at level } n\}$, and observe that $F_{n+1} \subseteq F_n$. Note that as the sequence $\{\#F_n/\ |G_n|\}_{n\in\mathbb{N}}$ satisfies

$$\frac{\#F_{n+1}}{|G_{n+1}|} \leq \frac{\#F_{n+1} \cdot |\mathrm{St}_G(n) : \mathrm{St}_G(n+1)|}{|G_{n+1}|} = \frac{\#F_{n+1}}{|G_n|} \leq \frac{\#F_n}{|G_n|},$$

it is monotone decreasing and bounded below by 0, thereby $\mathrm{FPP}(G)$ exists for every subgroup $G \leq \mathrm{Aut}(T)$. Furthermore, as G_n and $\left(\overline{G}\right)_n$ are isomorphic by Theorem 3.11, then

$$\mathrm{FPP}(G) = \mathrm{FPP}\left(\overline{G}\right).$$

Thus, in what follows, we shall assume G is closed.

From the above fact about the fixed-point proportion of the group and its closure, we can derive another strategy for proving the existence of $\mathrm{FPP}(G)$. This will also lead to a more geometric interpretation of $\mathrm{FPP}(G)$ as we shall see below.

When G is closed, we recall that the fixed-point process of G is well defined and that

$$F_n = \{g \in G_n : g \text{ fixes a vertex at level } n\} = \varphi_\emptyset^n(X_n^{-1}(\mathbb{N})).$$

The following lemma relates this fact to the Haar probability measure on G. The statement of this lemma was given in [9], and we expand on the proof below:

Lemma 6.6. *Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup, then*

$$\#\{g \in G_n : g \text{ fixes a vertex at level } n\}/\ |G_n| = \mu_G(X_n^{-1}(\mathbb{N})).$$

Proof. Let $\mathcal{T}$ be a transversal of $N := \mathrm{St}_G(n)$ in G. Note that if an element in a coset of N fixes a vertex at level n, then all elements in that coset fix the same vertex. Therefore, the section $g|_\emptyset^n$ of $g \in \mathcal{T}$ is either not in F_n or is a unique element in F_n. From $G = \dot\bigcup_{g\in\mathcal{T}} Ng$, we have

$$\begin{aligned}
\mu_G(X_n^{-1}(\mathbb{N})) &= \mu_G\left(\dot\bigcup_{g\in\mathcal{T}} (Ng \cap X_n^{-1}(\mathbb{N}))\right) = \mu_G\left(\dot\bigcup_{\substack{g\in\mathcal{T} \\ g|_\emptyset^n \in F_n}} Ng\right) \\
&= \sum_{\substack{g\in\mathcal{T} \\ g|_\emptyset^n \in F_n}} \mu_G(Ng) = \sum_{\substack{g\in\mathcal{T} \\ g|_\emptyset^n \in F_n}} \mu_G(N) \\
&= \#F_n \cdot |G : N|^{-1} = \#\{g \in G_n : g \text{ fixes a vertex at level } n\}/\ |G_n|. \qquad \blacksquare
\end{aligned}$$

It follows from the descending property of $\{X_n^{-1}(\mathbb{N})\}_{n\in\mathbb{N}}$ and Lemma 6.6 that

$$\mathrm{FPP}(G) = \lim_{n\to\infty} \mu_G(X_n^{-1}(\mathbb{N})) = \mu_G\left(\bigcap_{n\in\mathbb{N}} X_n^{-1}(\mathbb{N})\right).$$

Note that $\cap_{n\in\mathbb{N}} X_n^{-1}(\mathbb{N})$ is measurable and the fixed-point proportion of G is simply the measure of this set. In fact, this set is heavily linked to the behaviour of the action of G on T as we shall see in the following result:

Note that we write ∂T for the boundary of T, i.e. the set of infinite rooted paths in T.

Lemma 6.7. *Let $g \in \mathrm{Aut}(T)$, then the following are equivalent:*
(a) *g fixes infinitely many vertices on T;*
(b) *g fixes at least a vertex on every level of T;*
(c) *g fixes an end $\gamma \in \partial T$.*

Proof. It is trivial that (c) implies (a).

Now, assuming (a), i.e. that g fixes infinitely many vertices on T, then for any level n, we can find a vertex v at a level deeper than n fixed by g as each level of T has finitely many vertices. Therefore, the vertex at level n lying on the path between v and the root must be fixed by g as well, hence condition (b).

For $n \in \mathbb{N}$, let P_n be the set of all finite or infinite rooted paths γ such that γ is fixed by g and γ passes through a vertex at level n. It is clear that P_n is contained in P_{n-1}. By assuming (b), there exists a vertex for each level n that is fixed by g; thereby, the unique path from this vertex to the root is fixed by g. Thus, each P_n is non-empty. The intersection $\bigcap_{n\in\mathbb{N}} P_n$ must be non-empty and consists only of infinite rooted paths, hence condition (c). ∎

We note that the statement of Lemma 6.7 (a) and (c) was given in [9]. We give the proof above for the convenience of the reader.

Hence, the fixed-point proportion of a closed subgroup G is simply the probability of finding an element in G fixing an infinite rooted path in T, i.e.

$$\mathrm{FPP}(G) = \mu_G\left(\bigcap_{n\in\mathbb{N}} X_n^{-1}(\mathbb{N})\right) = \mu_G(\{g \in G : g \text{ fixes some } \gamma \in \partial T\}).$$

Let us divert our attention to the main theorem of this section, which shows that the assumption in Lemma 6.5 is sufficient to satisfy condition (b) in Theorem 6.2 if the tree is bounded. Hence, the fixed-point proportion of such groups is zero. We say that a spherically homogeneous rooted tree T is *bounded* if the sequence $\{d_n\}_{n\in\mathbb{N}}$ is bounded. Furthermore, we say that the d-adic tree is the *regular completion* of a bounded tree T if $d = \max d_n$. We denote the regular completion by $\overline{T}$. Furthermore, we note that T can be embedded as a subtree of $\overline{T}$ with the same root. We state and prove the following results on the nature of this embedding.

Lemma 6.8. *Let T be a bounded tree, not necessarily regular. Then $\mathrm{Aut}(T)$ can be continuously embedded into $\mathrm{Aut}(\overline{T})$ by a homomorphism. Moreover, $\mathrm{Aut}(T)$ is a closed and compact subgroup of $\mathrm{Aut}(\overline{T})$.*

Proof. Let $\iota : T \to \overline{T}$ be a rooted embedding of T in its regular completion $\overline{T}$. This induces a natural homomorphism $\hat{\iota} : \mathrm{Aut}(T) \to \mathrm{Aut}(\overline{T})$ where the labels of $\hat{\iota}(g)$ are given by

$$\hat{\iota}(g)|_v^1 = \begin{cases} 1 & \text{if } v \in \overline{T} \setminus T; \\ g|_v^1 & \text{if } v \in T, \end{cases}$$

for each $g \in \mathrm{Aut}(T)$. It is plain that the kernel of $\hat{\iota}$ is trivial, hence it is an embedding. Furthermore, the congruence topology on $\mathrm{Aut}(T)$ is precisely the induced topology inherited from the congruence topology on $\mathrm{Aut}(\overline{T})$, thereby $\hat{\iota}$ is continuous. The closeness of $\mathrm{Aut}(T)$ in $\mathrm{Aut}(\overline{T})$ follows from $\mathrm{Aut}(T)$ being compact and $\mathrm{Aut}(\overline{T})$ being Hausdorff. ∎

We will only use the embedding above to bound the order of $\mathrm{Aut}(T_v^m)$. Note that the fixed-point proportions of G and $\hat{\iota}(G) \leq \mathrm{Aut}(\overline{T})$ may differ. In particular, since the image of the embedding $\hat{\iota}$ is precisely

$$\left\{g \in \mathrm{Aut}\left(\overline{T}\right) : g|_v^1 = 1 \text{ for } v \in \overline{T} \setminus T\right\},$$

if $d_n < \max d_m$ for some level n, then each element in the finite-index subgroup $\mathrm{St}_G(n)$ fixes an infinite path in $\overline{T}$. Hence G has a positive fixed-point proportion as a subgroup of $\mathrm{Aut}\left(\overline{T}\right)$, regardless of its fixed-point proportion as a subgroup of $\mathrm{Aut}(T)$. Bounded trees arise naturally in the study of specialisations of Galois groups of iterates of rational functions to post-critical points; see [20]. They also appear in the seminal work by Abért and Virág in [1] in the context of orbit trees.

The main result of this section is a generalisation of the main result of Fariña-Asategui and Radi in [10], where the regularity of the tree is not required, and the shifted counterpart of super-strong fractality is assumed:

Theorem 6.9. *Let T be a bounded tree and $G \leq \mathrm{Aut}(T)$ be shifted super-strongly fractal and level-transitive. Then*

$$\mathrm{FPP}(G) = 0.$$

Proof. Note that as the fixed-point proportions of G and $\overline{G}$ coincide, we may assume that G is a closed subgroup of $\mathrm{Aut}(T)$. It suffices to show that the fixed-point process of G is eventually zero for almost every $g \in G$ because

$$\mathrm{FPP}(G) = \mu_G\left(\bigcap_{n\in\mathbb{N}} X_n^{-1}(\mathbb{N})\right) = \mu_G\left(\lim_{n\to\infty} X_n > 0\right) = 1 - \mu_G\left(\lim_{n\to\infty} X_n = 0\right),$$

where the second equality follows from Lemma 6.5 and Theorem 6.1. To show that the fixed-point process is eventually zero almost everywhere, we simply need to prove that the conditions for Theorem 6.2 are satisfied.

Condition (a) of Theorem 6.2 is fulfilled by Lemma 6.5. For condition (b), we proceed as follows. Let us fix any $r > 0$, and choose $m := \lceil \log_2(r) \rceil + 1$ depending only on r. Now, for any $n \in \mathbb{N}$, we consider the subset of n-patterns

$$A_n := \{g \in G_n : g \text{ has exactly } r \text{ fixed points at } \mathcal{L}_n\} = \varphi_\emptyset^n(X_n^{-1}(\{r\})).$$

This means that $C_{A_n} = X_n^{-1}(\{r\}) = \{g \in G : X_n(g) = r\}$. Note that for any $a \in A_n$, there exists a vertex $v_a \in \mathcal{L}_n$ fixed by a as $r \geq 1$. By Theorem 5.6, we obtain that

$$\mu_G\left(C_{\{a\}} \cap \left(\varphi_{v_a}\right)^{-1}\left(C_{1_m}^{v_a}\right)\right) = \mu_G\left(C_{\{a\}}\right) \cdot \mu_{G_{v_a}}\left(C_{1_m}^{v_a}\right) = \mu_G\left(C_{\{a\}}\right) \cdot |\left(G_{v_a}\right)_m|^{-1},$$

as $G_{v_a} = \varphi_{v_a}(G) \leq \mathrm{Aut}\left(T_{v_a}\right)$ and $1_m = \{1\} \leq \left(G_{v_a}\right)_m$. The shifted super-strong fractality of G yields that $G_{v_a} = \varphi_{v_a}(\mathrm{St}_G(m))$. Observe that $\varphi_v^m(\mathrm{St}_G(n)) \leq \varphi_v^m(\mathrm{St}(n)) = \mathrm{Aut}(T_v^m)$ for any vertex v at level n. As $\mathrm{Aut}(T)$ can be embedded into $\mathrm{Aut}\left(\overline{T}\right)$ with $\overline{T}$ being the regular completion of T to the d-adic tree with $d := \max d_n$, we see that $|\mathrm{Aut}\left(T_{v_a}^m\right)| \leq |\mathrm{Aut}\left(\overline{T}_{v_a}^m\right)| = d^m$. We conclude that

$$|\left(G_{v_a}\right)_m| = |\varphi_{v_a}^m(\mathrm{St}_G(n))| \leq |\mathrm{Aut}\left(T_{v_a}^m\right)| \leq |\mathrm{Aut}\left(\overline{T}_{v_a}^m\right)| = d^m.$$

We fix $\varepsilon := d^{-m}$, which only depends on m and thus on r.

Now, for every $a \in A_n$, each element $g \in C_{\{a\}} \cap \left(\varphi_{v_a}\right)^{-1}\left(C_{1_m}^{v_a}\right)$ satisfies both that $X_{n+m}(g) \geq d_{n+1} \cdot d_{n+2} \cdots d_{n+m} \geq 2^m > r$ and $X_n(g) = r$. Therefore,

$$\begin{aligned}
\mu_G(X_{n+m} = r \mid X_n = r) &= 1 - \mu_G(X_{n+m} \neq r \mid X_n = r) \\
&\leq 1 - \mu_G(X_{n+m} > r \mid X_n = r) \\
&= 1 - \frac{\mu_G(X_{n+m} > r, X_n = r)}{\mu_G(X_n = r)} \\
&\leq 1 - \frac{\mu_G\left(\dot\bigcup_{a \in A_n} C_{\{a\}} \cap \left(\varphi_{v_a}\right)^{-1}\left(C_{1_m}^{v_a}\right)\right)}{\mu_G\left(C_{A_n}\right)} \\
&= 1 - \frac{\sum_{a \in A_n} \mu_G\left(C_{\{a\}} \cap \left(\varphi_{v_a}\right)^{-1}\left(C_{1_m}^{v_a}\right)\right)}{\mu_G\left(C_{A_n}\right)} \\
&= 1 - \frac{\sum_{a \in A_n} \mu_G\left(C_{\{a\}}\right) \cdot |\left(G_{v_a}\right)_m|^{-1}}{\mu_G\left(C_{A_n}\right)} \\
&\leq 1 - \frac{\sum_{a \in A_n} \mu_G\left(C_{\{a\}}\right) \cdot \varepsilon}{\mu_G\left(C_{A_n}\right)} \\
&= 1 - \frac{\mu_G\left(\dot\bigcup_{a \in A_n} C_{\{a\}}\right) \cdot \varepsilon}{\mu_G\left(C_{A_n}\right)} = 1 - \varepsilon.
\end{aligned}$$

Hence, the condition (b) in Theorem 6.2 is fulfilled with $m := \lceil \log_2(r) \rceil + 1$ and $\varepsilon := d^{-m}$, as they are both independent of n. ■

We remark that although the level transitivity of G seems to not be used explicitly in the proof of Theorem 6.9, it is crucial for the fixed-point process to be a martingale as seen in Theorem 6.3. In fact, Radi has shown in [25, Theorem B] that even an iterated wreath product $\cdots \wr P \wr P$ acting on the d-adic tree, with $P \leq S_d$, has fixed-point proportion 1 when every element of P has a fixed point.

We stress that the boundedness of T plays a big role in the proof, as otherwise, it is unclear whether one can obtain a uniform bound $1 - \varepsilon$. This motivates the following question:

Question. Let T be a spherically homogeneous rooted tree which is not bounded and let $G \leq \mathrm{Aut}(T)$ be shifted super-strongly fractal and level-transitive. Can $\mathrm{FPP}(G) > 0$?

Bibliography

[1] Abért, M. and Virág, B., Dimension and Randomness in Groups Acting on Rooted Trees., *J. Amer. Math. Soc.* **18** (2005), 157–192.

[2] Bourbaki, N., *General topology*, Springer, Berlin, 1989.

[3] Bridy, A., Jones, R., Kelsey, G. and Lodge, R., Iterated monodromy groups of rational functions and periodic points over finite fields, *Math. Ann.* **390** (2024), 439–475.

[4] Chou, C., Elementary amenable groups, *Illinois J. Math.* **3** (1980), 396–407.

[5] Cohn, D. L., *Measure theory*, 2nd edn, Birkhäuser/Springer, New York, 2013.

[6] Dark, R., A prime Baer group, *Math. Z.* **105** (1968), 294–298.

[7] Day, M. M., Amenable semigroups, *Ill. J. Math.* **I** (1957), 509–544.

[8] Fariña-Asategui, J., Cyclicity, hypercyclicity and randomness in self-similar groups, *Monatsh. Math.* **207** (2025), 275–292.

[9] Fariña-Asategui, J., Groups acting on rooted trees dimensions, subgroups, randomness and dynamics, PhD thesis in mathematics, Lund University, 2026.

[10] Fariña-Asategui, J. and Radi, S., On the fixed-point proportion of self-similar groups, *Bull. Lond. Math. Soc.*, to appear.

[11] Fariña-Asategui, J. and Radi, S., Fixed-point proportion of geometric iterated Galois groups, arXiv preprint: 2601.16173.

[12] Fariña-Asategui, J. and Radi, S., Random subgroups of branch groups, arXiv preprint : 2508.20082.

[13] Grigorchuk, R. I., On Burnside's problem for periodic groups, *Funktsional. Anal. i Prilozhen.* **14(I)** (1980), 53–54.

[14] Grigorchuk, R. I., Just infinite branch groups, in: New Horizons in pro-p Groups, *Birkhäuser Boston, MA* **I** (2000), 121–179.

[15] Grigorchuk, R. I. and Żuk, A., On a torsion-free weakly branch group defined by a three state automaton, *Internat. J. Algebra Comput.* **12** (2002), 223–246.

[16] Jones, R., Iterated Galois towers, their associated martingales, and the p-adic Mandelbrot set, *Compos. Math.* **143** (2007), 1108–1126.

[17] Jones, R., Galois representations from pre-image trees: an arboreal survey, *Pub. Math. Besancon.* (2013), 107–136.

[18] Jones, R. and Manes, M., Galois theory of quadratic rational functions, *Comment. Math. Helv.* **89** (2014), 173–213.

[19] Kesten, H., Full Banach mean values on countable groups, *Math. Scand.* **7** (1959), 146–156.

[20] Leung, C. and Petsche, C., The Minkowski dimension of the image of an arboreal Galois representation, arXiv preprint: 2512.18825.

[21] Milnor, J., Advanced Problems: 5603, *Amer. Math. Month.* **75(6)** (1968), 685–687.

[22] Nekrashevych, V., *Self-similar groups*, vol. 117, Amer. Math. Soc., Providence, Rhode Island, 2005.

[23] Neumann, J. von, Zur allgemeinen Theorie des Maßes, *Fundam. Math.* **13** (1929), 73–116.

[24] Odoni, R. W. K., The Galois theory of iterates and composites of polynomials, *Proc. London Math. Soc. (3)* **51** (1985), 385–414.

[25] Radi, S., A family of level-transitive groups with positive fixed-point proportion and positive Hausdorff dimension, arXiv preprint: 2501.00515.

[26] Ribes, L. and Zalesskii, P., *Profinite groups*, 2nd edn, Springer-Verlag, Berlin, 2010.

[27] Sukkumnoed, D., Independence of the fixed-point proportion and the Hausdorff dimension of branch groups, in preparation.

[28] Thilliasundaram, A., Burnside groups and groups acting on rooted trees, *Adv. Group Theory Appl.* **20** (2025), 121–161.

Bachelor’s Theses in Mathematical Sciences 2026:K6
ISSN 1654-6229

LUNFMA-4188-2026

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